Translation of the Kimberling's Glossary into barycentrics
"Le Glossaire de Pierre"
(v102-dvi5)

pldx

April 5, 2025

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Acknowledgement. This document began its life as a private copy of the Glossary accompanying the Encyclopedia of Clark Kimberling (1998-2024). This Glossary –as its name suggests– is organized alphabetically. As a never satisfied newcomer, I would have preferred a progression from the easiest to the hardest topics and I reordered this document in my own way. It is unclear whether this new ordering will be useful to someone else ! In any case a detailed index is provided, where (tentatively) the main entry is bolded.

Second point, the Kimberling's Glossary is written using trilinear coordinates. From an advanced point of view, these coordinates are neither better nor worse than the barycentric coordinates. Nevertheless, having some practice of the barycentrics, and none of the trilinears, I undertook to translate everything, from one system to another. In any case, this was a formative exercise, and this also puts the focus on the covariance/contra-variance properties that were subsequently systematized.

Drawings are the third point. Everybody knows –or should know– that geometry is not possible at all without drawings. Having no intention to pay royalties for using rulers and compasses, I turned to an open source software (kseg) in order to produce my own drawings. Thereafter, I have used Geogebra, together with pstricks. What a battle, but no progress without practice !

Subsequently, other elements have been incorporated from other sources, including materials about cubics, from the Gibert web site, and about Cremona transforms from the Déserti archive. Finally, original elements were also added. As it will appear at first sight, "pldx" is addicted to a tradition that requires a precisely specified universal space for each object to live in.

A second massive addiction of the author is computer algebra. Having at your fingertips a tool that gives the right answer to each and every expansion or factorization, and never lost the small paper sheet where the computation of the week was summarized is really great. Moreover, being constrained to explain everything to a computer helps to specify all the required details. For example, the "equivalence up to a proportionality factor" doesn't apply in the same way to a matrix whether this matrix describes a collineation, a triangle, a trigone or a set of incidence relations.

In this document, "beautiful geometrical proofs" are avoided as much as possible, since they are the most error prone. A safe proof of "*the triangle of contacts of the inscribed circle and the triangle of the mid-arcs on the circumcircle admit the insimilicenter of the two circles as perspector*" is :

`ency(persp(matcev(vX(7)),matucev(vX(1))))` $\mapsto$ `56`

where the crucial point is `ency`, i.e. a safe implementation of the Kimberling's search key method to explore the database.

To summarize, the present document is rather a "derived work", where the elements presented are not intended to be genuine, apart perhaps from the way to assemble the ingredients and cook them together.

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Chapter 1

Introduction

Many changes have occurred in the way we are doing geometry, from the old ancient times of Euclid and Apollonius. Most of them are related to yet another way of performing "automated" computing of properties, rather than relying on intuition to find "beautiful geometric proofs". Many individuals have contributed to this long process, and attributing a given discovery to a given individual is not an easy task (Coolidge, 1940).

The most eminent milestones along this long road are the individuals who have summarized the discoveries of their time into an efficient way of writing down the questions to solve (Fourrey, 1907). Each time, the new way of writing was appearing as doing the job by itself and providing the required answers through something like a least action trajectory. Heroes are no more required, replaced by computing power.

Writing numbers and calculations in a tractable manner is associated with Al-Khwarizmi and his *Algebra* (825). Using coordinates (x, y) to describe points and compute their geometrical properties (as well as the exponent notation for polynomials) is associated with Descartes and his *Géometrie* (1637). Using homogeneous coordinates $x : y : z$ to implement the principle of continuity when dealing with objects that escape to infinity is associated with Moebius and his *barizentrische Calcul* (1827).

More recently, the very idea of stamping an hash-code on each noteworthy point involved in Triangle Geometry and then practice some kind of computer aided inventory management (Kimberling, 1998-2024) has changed the practice of geometers. This idea has emerged from a more general trend, where

> barycentrics are understood to ***define*** points, lines, circles, triangle centers, etc., and zero determinants are understood to ***define*** collinearity and concurrence. Doing that way, triangle geometry, formally speaking, is much more general than the study of a single Euclidean triangle. In the formal treatment, sometimes called *transfigured triangle geometry*, the symbols a, b, c are regarded as algebraic unknowns, so that points, defined as functions of a, b, c, are not the usual points of a two-dimensional plane. When (a, b, c) are real numbers restricted by the "triangle inequalities" for sidelengths, the resulting geometry is traditional triangle geometry (Kimberling, 1998).

When possible, computed proofs are given that use formal computing tools. This kind of proof is deprecated by several authors. Nevertheless, these proofs are the easiest since all the messy job is done by a computer and are also the safest. A construction that sounds like a "beautiful geometrical proof" is too often invalid due to some hidden exception. During a computerized proof, exceptions are appearing as multiplicative factors, according to the polynomial model :

$$conclusion \times exceptions = hypothesis$$

To quote the Knuth's foreword to Petkovsek et al. (1996) :

> Science is what we understand well enough to explain it to a computer. Art is everything else we do. During the past several years important parts of mathematics has been transformed from an Art to a Science.

1.1 Special remark for French natives

Lorsque vous écrivez pour un public américain, mieux vaut commencer par montrer que le sujet est suffisamment intéressant pour mériter du temps et de la peine. Lorsque vous écrivez pour un public français, mieux vaut commencer par montrer que l'auteur dispose d'une hauteur de vue suffisante. Si vous percevez les choses de cette façon, le Chapitre 4 est le bon endroit par où commencer.

1.2 Basic objects : points and lines

Fact 1.2.1. *You can safely build temples and pyramids by drawing reduced maps, and then rescale your measurements to the real world.*

Remark 1.2.2. Beside this founding property the reader is supposed to have heard about some basic facts concerning numbers. There are sheep. We can eat some of them and count the others. And so, we get $\mathbb{N}$. Over the centuries, $\mathbb{Q}, \mathbb{Z}, \mathbb{R}$ have been obtained from geometric considerations. One can follow this process in Stillwell (2010), while it's result has been summarized in Artin (1957).

Definition 1.2.3. In this book, $\mathbb{R}$ and $\mathbb{C}$ are **defined** as axiomatic objects.

Definition 1.2.4. The abbreviation "etc." will stand for "et cyclically". When an "A-object" has been defined as $F(A, B, C)$, then the B-object is $F(B, C, A)$, obtained by the cyclic permutation $ABC \mapsto BCA$, and the C-object is, likewise, $F(C, A, B)$. Example: If A' is the point where lines AP and BC meet, and B' and C' are defined cyclically, then B' is where lines BP and CA meet, and C' is where lines CP and AB meet.

Definition 1.2.5. Equality. We will use the **simeq** sign (i.e. $\simeq$) to denote an equality up to a non vanishing multiplier, and restrict the use of the **equal** sign (i.e. $=$) to a strong equality. In other words

$$x = y \text{ means } x - y = 0 \text{ while } x \simeq y \text{ means } (\exists k \neq 0)\,(k\,x - y = 0)$$

Moreover we will use the special signs **doteq** (i.e. $\doteq$) and **simdoteq** (i.e. $\dot{\simeq}$) to denote "definitional equalities", in order to emphasize the fact that these equalities aren't equations: they are introducing new objects.

Definition 1.2.6. A **point** is an element of $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^3\right)$. To tell the same thing more simply, a point is represented by a **column** of three numbers (the barycentrics of the point), not all of them being zero, and such a column is dealt "in a projective manner", i.e. up to a proportionality factor. An efficient way to write such a projective column is the colon notation :

$$P \dot{\simeq} p : q : r \qquad \text{meaning that} \qquad P \simeq \begin{pmatrix} p \\ q \\ r \end{pmatrix} \simeq \begin{pmatrix} k\,p \\ k\,q \\ k\,r \end{pmatrix} \forall k \in \mathbb{R} \setminus \{0\}$$

Definition 1.2.7. A **line** is an element of the dual of the point space. To tell the same thing more simply, a line is represented by a **row** of three numbers (the barycentrics of the line), not all of them being zero, and such a row is dealt "in a projective manner", i.e. up to a proportionality factor. A line will be described as :

$$\Delta \dot{\simeq} \begin{pmatrix} \rho & \sigma & \tau \end{pmatrix} \simeq \begin{pmatrix} \lambda\rho & \lambda\sigma & \lambda\tau \end{pmatrix} \forall \lambda \in \mathbb{R} \setminus \{0\}$$

Notation 1.2.8. In all these definitions, property $p^2 + q^2 + r^2 \neq 0$ and $\rho^2 + \sigma^2 + \tau^2 \neq 0$ are ever intended. Colon notation will *ever* be restricted to inline equations describing columns, and *never* be used for rows. Anyway, such a notation would be hopeless when dealing with matrices.

Definition 1.2.9. Incidence relations. We will say that a line $\Delta \dot{\simeq} (p, q, r)$ contains a point $U \dot{\simeq} u : v : w$, or that Δ goes through U, or that U belongs to Δ when their dot product vanishes, i.e. :

$$U \in \Delta \Longleftrightarrow pu + qv + rw = 0$$

Remark 1.2.10. It is clear that the incidence relation is projective, i.e. holds for any choice of the proportionality factors.

Definition 1.2.11. The **collinearity** of three points $P \doteq p : q : r$, $U \doteq u : v : w$, $X \doteq x : y : z$ is defined by the following determinant equation :

$$\begin{vmatrix} x & p & u \\ y & q & v \\ z & r & w \end{vmatrix} = 0 \tag{1.1}$$

When $P \neq U$, the set of all the X that satisfies (1.1) is what is usually called the line PU.

Definition 1.2.12. A **triangle** is an ordered set of three non collinear points. Its natural representation is an invertible square "matrix of columns", where each column is defined up to a proportionality factor (right action of a diagonal matrix). On the contrary, a "may be degenerate triangle" is a matrix such that (i) columns are not proportional to each other and (ii) rank is at least two. When at least two points are equal, the triangle is "totally degenerate".

Remark 1.2.13. Without explicit permission, a triangle is not allowed to be degenerate, while totally degenerate triangles are (quite ever) to be avoided.

Definition 1.2.14. The **concurrence** of three lines $\Delta_1 \doteq (d, e, f)$, $\Delta_2 \doteq (p, q, r)$ and $\Delta_3 \doteq (u, v, w)$ is defined by the following determinant equation :

$$\begin{vmatrix} d & e & f \\ r & s & t \\ u & v & w \end{vmatrix} = 0 \tag{1.2}$$

When $\Delta_1 \neq \Delta_2$, the set of all the Δ_3 that satisfies (1.2) is usually called the pencil generated by the two lines.

Definition 1.2.15. A **trigone** is a set of ordered three non concurrent lines. Its natural representation is an invertible square "matrix of rows", where each row is defined up to a proportionality factor (left action of a diagonal matrix).

Proposition 1.2.16. *The reciprocal matrix of a triangle is a trigone and conversely. Adjoint matrices can be used instead of inverses due to proportionality. Relation* $\mathcal{T}^{-1} \cdot \mathcal{T} = \mathrm{Id}$ *is nothing but the incidence relations :* $A' \notin B'C'$, $A' \in A'B'$, $A' \in A'C'$ *and cyclically.*

Remark 1.2.17. It should be noticed that a tetra-angle defines an hexa-gone, while an tetra-gone (quadrilateral) defines an hexa-angle : $n = n\,(n-1)\,/2$ holds only when $n = 3$.

Definition 1.2.18. The **line at infinity** $\mathcal{L}_b$ is the locus of points $x : y : z$ such that $x + y + z = 0$, so that any point out of $\mathcal{L}_b$ can be described by a triple such that $x + y + z = 1$. Using only this representation would discard $\mathcal{L}_b$ and is nothing but the usual affine geometry.

Definition 1.2.19. Barycentric basis. Special points $A = 1 : 0 : 0$, $B = 0 : 1 : 0$, $C = 0 : 0 : 1$ are usually identified with the vertices of a triangle in the euclidian[1] plane, so that all (barycentric) points can be mapped onto the euclidian plane (completed with the appropriate line at infinity). The side lengths of this triangle are denoted $BC = a$, $CA = b$ and $AB = c$. Since a triangle is not a two-angle, none of the a, b, c are allowed to vanish.

1.3 Figures using geogebra

1.3.1 Alternatives

The very idea to pay something for using Pythagoras theorem seems terrific, and therefore using a free software like *geogebra* appears as a requirement. The following remarks may nevertheless have some historical value.

[1]This kind of plane is to be named as euclidian rather than as Euclidean, since the very idea to compute figures instead of drawing computations is as far as possible from the thoughts and practices of the historical figure who wrote the celebrated *Elements*.

ps, pstricks PostScript was a proprietary language. But it became a de facto standard for talking to printers of any brand, and was finally released to the open source status.

maple proprietary. Moreover not sufficiently versatile.

kseg ***kseg*** (KSEG, 1999-2006) was our initial best choice, providing *.ps images. But the maintenance stopped near 2006. Figures were fine in the Linux version, but -constructions were badly saved, requiring the use of the win$ version (through wine) for writing the s.

1.3.2 Various versions of geogebra

1. `https://wiki.geogebra.org/en/Keyboard_Shortcuts`
2. /opt/geogebra-old/geogebra/geogebra
   --v GeoGebra 5.0.309.0 20 December 2016 Java 1.8.0_121-64bit
3. /usr/share/geogebra-classic/GeoGebra
   --v error!!!
   Version: 6.0.666.0-offline (21 September 2021)
4. /usr/share/geogebra/geogebra
   --v GeoGebra 5.0.755.0 17 January 2023 Java 1.8.0_121-64bit
   213686 Dec 2016 geogebra_cas.jar
   054109 Jan 2023 geogebra_cas.jar si court !!! surprise !!!

1.3.3 Using Geogebra

From 2014, more and more figures of the present Glossary were drawn using Geogebra.

1. Nowadays, exporting a graphical output is easy. In the old ancient times, the best output was obtained as a *.pdf file, resulting into a page, requiring a manual `pdfcrop` to the bounding box.
2. Obtaining the second intersection of two objects requires the following incantation :
   `Element[Remove[{Intersect[BBB,CCC]}, {A}], 1]`
3. A command like
   `X186 = TriangleCenter[A, B, C, 186]`
   can be used when $n < 3053$ (Geogebra 5.0.309.0-3D, tested 2016-12-12). Moreover
   `indexof(P, sequence(Trianglecenter(A,B,C,j),j,1,10))` allows to identify P among the points X(j) known to geogebra (the other points generate `(?,?)` and everything stands at the right place).

1.3.3.1 Construction protocol

Export the construction protocol in *.html, with nocolor, nopicture, only three columns (name&type,definition,value). And then `extract.sh` will display something to insert into a LyX equation.

1.3.3.2 3D geometry

Triangle	ABC	Polygon(A, B, C)
Segment	AB	Segment[A, B, ABC]
Circle	f	Circle(A, 4.5, ABC)
Point	D	($x(H)$, $y(H)$, 5)
Pyramid	$ABCD$	Pyramid(ABC, D)
Segment	BD	Segment[B, D, $ABCD$]
Triangle	ABD	Polygon(A, B, D, $ABCD$)
Line	j	PerpendicularLine[P, AB, ABC]
Line	k	PerpendicularLine[P, ABC]
Plane	o	PerpendicularPlane[P, BD]

- circle f is drawn in the ABC plane
- line j is parallel to plane ABC, through point P and perpendicular to line AB
- line k is perpendicular to plane ABC
- segment AB is an element of ABC, not an "independent" object.

1.3.3.3 Macros

1. Dealing with macros is not so easy. As a rule: ever suppress any non-required sub-macro when archiving a macro from its source file. Sometimes, you have to unzip the *.ggb file and modify it with a text editor.
2. When transmitting a complex point P to a macro, a safe method is to create, inside the macro, a copy of this point using `Tocomplex`

1.4 Type-keeping and type-crossing functions

Remark 1.4.1. The type-keeping/type-crossing properties are better understood when they are described in terms of collineations. Chapter 16 will be devoted to this topic. The aim of the current Section is only to provide some useful tools as soon as possible.

Definition 1.4.2. Trilinears and **barycentrics**. Triangle people splits into a barycentric tribe and a trilinear tribe. The trilinear tribe thinks that trilinears, i.e. $p : q : r$[2] are better looking than barycentrics and redefine everything according to their preferences. The barycentric tribe thinks that barycentrics, i.e. $p : q : r$[3] are better looking than trilinears and redefine everything according to their preferences.

Remark 1.4.3. Trilinears can be measured directly on the figure, since they are the directed distances to the sidelines. When compasses were actual compasses and not a button to click over, using trilinears was a must. Nowadays, the existence –and the persistence– of both systems can be used for an interesting renewal of the Capulet against Montague story, as in http://mathforum.org/kb/message.jspa?messageID=1091956. But this could also be used to gain a better insight over many point-transforms used in the Triangle Geometry.

Definition 1.4.4. Vectors are covariant, while forms are contravariant. Therefore, coordinates that measure a vector are forms and are contravariant. At the same time, coordinates that measure a form are covariant. In other words, $p : q : r$ is contravariant, while $[p, q, r]$ is covariant.

Definition 1.4.5. We will say that a function $P \mapsto f(P) \,:\, p : q : r \mapsto u : v : w$ is type-keeping or type-crossing or type scrambling according to :

$$\begin{cases} \text{type keeping} & when \quad f(\alpha p : \beta q : \gamma r) = \alpha u : \beta v : \gamma w \\ \text{type crossing} & when \quad f(\alpha p : \beta q : \gamma r) = \frac{u}{\alpha} : \frac{v}{\beta} : \frac{w}{\gamma} \\ \text{type scrambling} & \text{otherwise} \end{cases}$$

For a function of several variables, global type-keeping means :

$$f(\alpha p{:}\beta q{:}\gamma r,\, \alpha u{:}\beta v{:}\gamma w) = \alpha x{:}\beta y{:}\gamma z \quad when \quad f(p{:}q{:}r,\, u{:}v{:}w) = x{:}y{:}z$$

Remark 1.4.6. An object that is intended to describe a point has to be contravariant. An object that is intended to describe a line has to be covariant, while relationships like collinearity (1.1) and concurrence(1.2) have to be invariant. Therefore a function whose input and output are points has to be type-keeping. In the same way, a function whose input and output are lines has to be type-keeping. On the contrary, a function whose entries are points and output are lines has to be type-crossing. In the same way, a function whose entries are lines and output are points has to be type-crossing. These facts are the reasons why both tribes, using barycentrics or trilinears, are proceeding to the same geometry.

[2]someone from the barycentric tribe would write them $p/a : q/b : r/c$, since she would use $p : q : r$ for the barycentrics

[3]someone from the trilinear tribe would write them $ap : bq : cr$, since she would use $p : q : r$ for the trilinears

Definition 1.4.7. Barycentric multiplication is the multiplication component by component of the barycentrics of two points. This operation is denoted :

$$P \underset{b}{*} X \doteq p\,x : q\,y : r\,z \tag{1.3}$$

Component-wise multiplication of trilinears would be another possibility. This is the way of doing of the trilinear tribe.

Definition 1.4.8. Barycentric division is the division component by component of the barycentrics of two points. This operation is denoted :

$$P \underset{b}{\div} X \doteq \frac{p}{x} : \frac{q}{y} : \frac{r}{z} \tag{1.4}$$

Component-wise division of trilinears would be another possibility. This is the way of doing of the trilinear tribe.

Remark 1.4.9. These transforms are introduced here to provide an easy description of some other transforms. The study of their geometrical meaning is postponed to Chapter 18. For our present needs, we only need to remark that both :

$$X \mapsto X \underset{b}{*} P \underset{b}{\div} U \quad \textit{and} \quad X \mapsto P \underset{b}{*} U \underset{b}{\div} X$$

are globally type-keeping transforms, that can be used to obtain points from points, or lines from lines.

Definition 1.4.10. Sqrtdiv. Let $F \doteq f : g : h$ be a fixed point, and U a moving point, restricted to avoid the sidelines of ABC. The mapping defined by :

$$sqrtdiv_F\,(U) \doteq U_F^{\#} \doteq \frac{f^2}{u} : \frac{g^2}{v} : \frac{h^2}{w} \tag{1.5}$$

is globally type-keeping and describes a pointwise transform, whose fixed points are the four $\pm f : \pm g : \pm h$. This map $U \mapsto U_F^{\#}$ is exactly the same as $U \mapsto U_P^{*}$ defined by $U_P^{*} \doteq P \underset{b}{\div} U$ and $P \doteq F \underset{b}{*} F$. The second form is often used, introducing a fictitious point $P \doteq f^2 : g^2 : h^2$. This will be studied at length at Section 18.4.

Remark 1.4.11. Using # instead of $*$ in this context is already the way of doing of the cubics' people (Ehrmann and Gibert, 2005) : fixed points of the transform (F and its relatives) have a clearer geometrical meaning than P. On the other hand, when P crosses the borders of ABC, the coordinates of point F become imaginary, and the configuration is less visual.

Remark 1.4.12. The converse operation of *sqrtdiv* would be *sqrtmul* defined as $(U, X) \mapsto \sqrt{ux} : \sqrt{vy} : \sqrt{wz}$ but this map is multivalued. When U, X are triangle centers and ux, vy, wz are perfect squares, it makes sense to fix signs so that F is also a triangle center.

1.5 Duality between point and lines

Does equation $pu + qv + rw = 0$ means $P \in \Delta_U$ or $U \in \Delta_P$? Without any further indication, one cannot decide which is the point and which is the line. This is called duality. If you want to be specific, you have to say :

$$\begin{bmatrix} u & v & w \end{bmatrix} \cdot \begin{bmatrix} p \\ q \\ r \end{bmatrix} = 0 \quad \text{or} \quad \begin{bmatrix} p & q & r \end{bmatrix} \cdot \begin{bmatrix} u \\ v \\ w \end{bmatrix} = 0$$

and remember how points/lines are mapped into columns/rows. In any case, points aren't lines and columns aren't rows. An efficient formulation of incidence axioms must recognize this elementary fact.

Definition 1.5.1. The **wedge** operator is the universal factorization of the determinant. This means that wedge of two columns is a row, while wedge of two rows is a column. One has :

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \wedge \begin{pmatrix} u \\ v \\ w \end{pmatrix} \simeq (qw - rv,\, ru - pw,\, pv - qu)$$

$$(p, q, r) \wedge (u, v, w) \simeq \begin{pmatrix} qw - rv \\ ru - pw \\ pv - qu \end{pmatrix}$$

Proposition 1.5.2. *When $P \neq U$, the barycentrics of the line PU are provided by operation $P \wedge U$. As it should be, this operation is commutative and is type-crossing.*

Proof. The wedge of two points cannot be $0 : 0 : 0$ when the points are different, therefore $\Delta = P \wedge U$ defines a line. By definition, we have :

$$\left(\begin{pmatrix} p \\ q \\ r \end{pmatrix} \wedge \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{vmatrix} p & u & x \\ q & v & y \\ r & w & z \end{vmatrix}$$

and the conclusion comes from the fact that inclusion of a line into another implies equality. Type-crossing is obvious from stratospheric reasons... but can also be checked on the components (up to a global $\alpha\beta\gamma$ factor). When dealing with lines, the same argument shows that "line wedge line" is a point. □

Proposition 1.5.3. *When line Δ_{12} is given by points P_1 and P_2 (with $P_1 \neq P_2$) and line Δ_3 is given by its barycentrics then either both lines are equal or their intersection M is given by :*

$$\Delta_{12} \cap \Delta_3 \simeq (\Delta_3 \cdot P_1)\, P_2 - (\Delta_3 \cdot P_2)\, P_1$$

Proof. Call M this object. It is clear that $M \in P_1P_2$. And we can check that $\Delta_3 \cdot M = 0$. Another proof is that this $\simeq$ is in fact a component-wise identity. □

Proposition 1.5.4. *Suppose that lines Δ_{12} and Δ_{34} are respectively defined by points P_1, P_2 and points P_3, P_4. Then either both lines are equal or their intersection M is given by :*

$$M \doteq (P_1 \wedge P_2) \wedge (P_3 \wedge P_4) \simeq P_2 \det [P_1P_3P_4] - P_1 \det [P_2P_3P_4]$$

Proof. Obvious from the previous proposition and the definition $\det [P_1P_3P_4] = (P_3 \wedge P_4) \cdot P_1$. Another proof is that this $\simeq$ is in fact a component-wise identity. □

Definition 1.5.5. The **wedge point** $X_\wedge$ **of a line** is what is obtained by a simple transposition of the barycentrics. This way of doing is based on a misperception of the wedge operation since $PU \doteq (P \wedge U)$ is a line (row) and not a point (column). When written in trilinears, this object don't look good. Not without reason.

Definition 1.5.6. The **Weisstein point** X_W **of a line** is what is obtained when applying the same misperception to trilinears. Applied to line PU, this leads to the **crossdifference** of P and U. When written in barycentrics :

$$X_W \doteq \text{crossdiff}\,(P, U) \simeq a^2\,(qw - rv) : b^2\,(ru - pw) : c^2\,(pv - qu)$$

this object don't look good. Not without reason.

A better founded concept must lead to a type-crossing transform.

Definition 1.5.7. The **tripole** of a line and the **tripolar** of a point is what is obtained by "transpose and reciprocate". Clearly, the one-to-one correspondence between pole and polar is lost when a coordinate vanishes (line through a vertex, or point on a sideline).

Remark 1.5.8. A less stratospheric definition of the tripolar is given in Definition 3.4.3.

name	2020	line	tripo	${}^t\Delta$	L	∞	$\infty^\perp$	$\perp$po	2014
Euler	4877	X_2-X_3	X_{648}	X_{525}	L_{647}	X_{30}	X_{523}	X_{125}	404
Nagel	1303	X_1-X_2	X_{190}	X_{514}	L_{649}	X_{519}	???	???	142
Infinity	993	X_{30}-X_{511}	X_2	X_2	L_6				269
Brocard	981	X_3-X_6	X_{110}	X_{850}	L_{523}	X_{511}	X_{512}	X_{115}	229
Bevan	740	X_1-X_3	X_{651}	???	L_{650}	X_{517}	X_{513}	X_{11}	128
?	421	X_1-X_6	X_{100}	X_{693}	L_{513}	X_{518}	X_{3309}	???	92
Soddy	300	X_1-X_7	X_{658}	X_{3239}	X_{657}	X_{516}	X_{514}	X_{1565}	?
van Aubel	205	X_4-X_6	X_{107}	X_{3265}	L_{520}	X_{1503}	X_{525}	X_{1562}	40
	192	X_1-X_4	X_{653}	X_{6332}	L_{652}	X_{515}	X_{522}	???	64
Fermat	137	X_2-X_6	X_{99}	X_{523}	L_{512}	X_{524}	X_{1499}	???	97
Lemoine	132	X_{187}-X_{237}	X_6	X_{76}	L_2	X_{512}	X_{511}	X_{1513}	37
antiorthic	125	X_{44}-X_{513}	X_1	X_{75}	L_1	X_{513}	X_{517}	X_{1512}	92
orthic	118	X_{230}-X_{231}	X_4	X_{69}	L_3	X_{523}	X_{30}	X_{1514}	29
Longchamps	102	X_{325}-X_{523}	X_{76}	X_6	L_{32}	X_{523}	X_{30}	X_{1531}	30
	72	X_3-X_8	X_{13136}	???	L_{3310}	X_{952}	X_{900}	X_{3259}	14
Gergonne	69	X_{241}-X_{514}	X_7	X_8	L_{55}	X_{514}	X_{516}	X_{1541}	21
model		X_1-X_1	X_1	X_1	X_1	X_1	X_1	X_1	

Table 1.1: Some well-known lines

Remark 1.5.9. When applying "transpose and reciprocate", both tribes are thinking they are acting "their way", and are talking about "trilinear pole" and "barycentric pole". But the result is the same since reciprocation of barycentrics (aka isotomic conjugacy, Section 3.4) acts over $X_\wedge$ while reciprocation of trilinears (aka isogonal conjugacy, Section 9.2) acts over X_W. To summarize (using later introduced concepts) :

$$X_\Lambda = {}^t\Delta \;;\; X_W = {}^t\Delta \underset{b}{*} X_6 \;;\; \text{tripole} = isotom\,(X_\Lambda) = isogon\,(X_K) \tag{1.6}$$

Remark 1.5.10. When tripole is at infinity, the line is tangent to the Steiner in-ellipse (cf Example 12.11.1).

Example 1.5.11. Table 1.1 describes some well-known lines. For example, the Euler line goes through X_2(centroid) and X_3(circumcenter) . Its equation is $\sum x\,(b^2-c^2)(b^2+c^2-a^2) = 0$. Formally, center $(b^2-c^2)(b^2+c^2-a^2)$ is $X_{525} = {}^t\Delta$, while center $a^2\,(b^2-c^2)(b^2+c^2-a^2)$ is $X_{647} = X_W$. This center has been used sometimes to describe lines, leading to $Euler = L_{647}$. The next column (∞) gives the infinity point while the remaining two columns give the later defined orthopoint ($\infty^\perp$) and orthopole ($\perp$pole).

Proposition 1.5.12. *Tripole and tripolar, being correctly typed, are constructible (Figure 1.1). Start from P. Draw AP and obtain $A' = AP \cap BC$. Construct $A'' \in BC$ so that division $BCA'A''$ is harmonic (Section 3.2). Act cyclically and obtain B'' and C''. Then $A''B''C''$ are collinear, and the line they define is nothing but the tripolar of P. (and are named* tripo *in Table 1.1).*

Remark 1.5.13. One of the most important consequence of all these duality formulas is the rock-solid equality giving the intersection of two lines each of them defined by two points :

$$(PQ \cap RS) = (P \wedge Q) \wedge (R \wedge S)$$

Proposition 1.5.14. *A symmetric parametrization in ρ, σ, τ of the points $U \doteq u : v : w$ that lie on line $\Delta \doteq [p, q, r]$ is :*

$$U \simeq q\tau - r\sigma : r\rho - p\tau : p\sigma - q\rho \tag{1.7}$$

A symmetric parametrization in ρ, σ, τ of the points $U \doteq u : v : w$ that lie on line $\Delta \doteq$ tripolar (P) where $P \doteq p : q : r$ is :

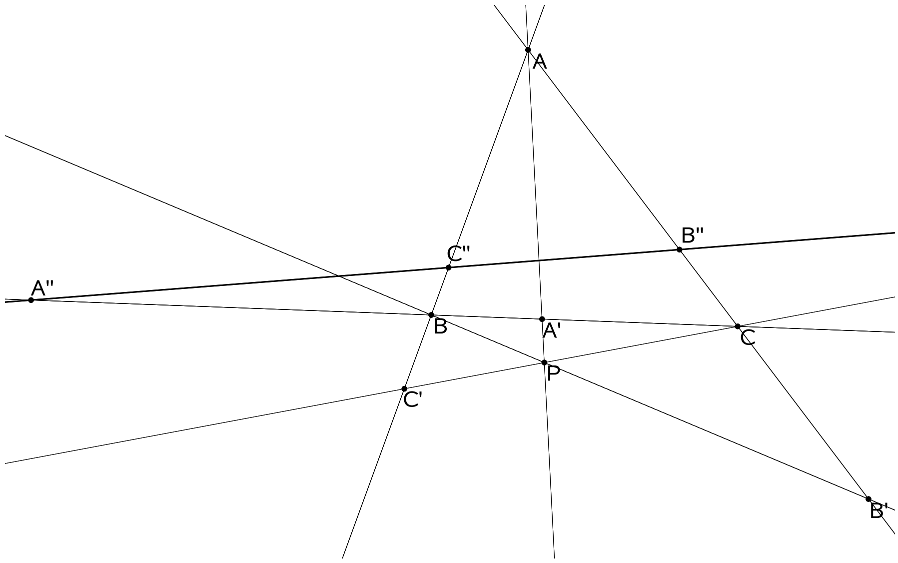


Figure 1.1: Point P and line A"B"C" are the tripolars of each other.

$$U \simeq p\,(\tau - \sigma) : q\,(\rho - \tau) : r\,(\sigma - \rho) \tag{1.8}$$

Proof. The first formula is $U \simeq \Delta \wedge [\rho, \sigma, \tau]$, defining a point on a line as the (projective) intersection of this line with another one. The second one is obvious. □

Proposition 1.5.15. *(Spoiler) Line* tripolar$\,(P)$ *is the locus of the tripoles U of the tangents to the inconic $IC\,(P)$. Moreover, the contact point of* tripolar$\,(U)$ *with the conic is* $Q \simeq U \underset{b}{*} U \underset{b}{*} P$.

Proof. This is the right place for the assertion, but not for its proof... Postponed to Proposition 12.8.11. □

Remark 1.5.16. ***Caveat : A triangle is not a conic***. When U lies on the polar of P wrt a conic, then P lies on the polar of U wrt the same conic. When U lies on *tripolar* (P) (the polar of P wrt triangle ABC) we have :

$$\frac{u}{p} + \frac{v}{q} + \frac{w}{r} = 0$$

and this isn't a commutative relation.

1.6 Isoconjugacy has moved

See Chapter 18, chiefly Section 18.4.

Chapter 2

Central objects

Centrality is a key notion. Emphasis on this concept was put by the founding paper of Kimberling (1998). The corresponding definitions have been tailored so that a central point is something like $I = X(1)$ or $G = X(2)$ or $O = X(3)$ etc., while a central triangle is something like ABC itself or JKL (the triangle of the excenters).

2.1 Triangle centers

When a concept emerges, various attemps are tested. Twenty years later, the most efficient ones take place in the "favored" list, while the others have to be deprecated. No criticism is implied here.

2.1.1 Favored concepts

Three points defines six triangles when taking the order into account. But there exists only one orthocenter. A central point is something that behaves like that. Moreover, geometrical theorems are not supposed to change when the king's foot shortens (not to speak of what happens when the king itself is shortened) : barycentric functions have to be homogeneous. This leads to the following definition.

Definition 2.1.1. A **triangle center** (or a **central line**) is a point (or a line) of the form

$$\begin{pmatrix} f(a,b,c) \\ f(b,c,a) \\ f(c,a,b) \end{pmatrix}$$

where f is a nonzero function satisfying two conditions:

1. f is homogeneous in a, b, c; i.e., there is a real number h such that $f(\lambda a, \lambda b, \lambda c) = \lambda^h f(a,b,c)$ for all (a,b,c) in the domain of f;

2. f is symmetric in b and c; i.e., $f(a,c,b) = f(a,b,c)$.

Definition 2.1.2. Bicentric points. When condition (2) is relaxed, we have to consider the two different points

$$\begin{pmatrix} f(a,b,c) \\ f(b,c,a) \\ f(c,a,b) \end{pmatrix}, \begin{pmatrix} f(a,c,b) \\ f(b,a,c) \\ f(c,b,a) \end{pmatrix}$$

They are called bicentric, together comprising a *bicentric pair*. Example: the Brocard points, $\omega^+ = a^2b^2 : b^2c^2 : c^2a^2$ and $\omega^- = c^2a^2 : a^2b^2 : b^2c^2$ (cf Proposition 7.11.1).

Definition 2.1.3. A strong center is a triangle center whose defining function belongs to $\mathbb{C}\left[a^2, b^2, c^2\right]$. Coordinates of such a point only depend on the coordinates of the vertices.

Definition 2.1.4. A **rational center** is a triangle center whose defining function belongs to $\mathbb{C}\left[a^2, b^2, c^2, S\right]$, where S is the area of triangleABC. When the coordinates of the vertices are rational, the coordinates of the center are made of rational quantities and fixed numbers.

Remark 2.1.5. Rational points that are not of the strong kind are in fact occuring by pairs, depending on the sign chosen for S (i.e. depending from the orientation of the triangle wrt the orientation of the plane). Example:

$$\begin{aligned} X(15) &\simeq a^2\left(S_a + 2/3\, S\sqrt{3}\right) : b^2\left(S_b + 2/3\, S\sqrt{3}\right) : c^2\left(S_c + 2/3\, S\sqrt{3}\right) \\ X(16) &\simeq a^2\left(S_a - 2/3\, S\sqrt{3}\right) : b^2\left(S_b - 2/3\, S\sqrt{3}\right) : c^2\left(S_c - 2/3\, S\sqrt{3}\right) \end{aligned}$$

Definition 2.1.6. A **weak center** is a triangle center whose defining function belongs to $\mathbb{C}\left[a, b, c, S\right]$ without being a rational center.

Theorem 2.1.7. ***Lemoine transforms***. *When using barycentrics, the identity together with the three transforms $a \mapsto -a$, $b \mapsto -b$ and $c \mapsto -c$ form a Klein group. Nowadays, they are called the Lemoine transforms, while Lemoine himself called them the "*transformations continues*" (see Lemoine, 1891). Spoiler: when using Lubin-2, these transforms are obtained by $\alpha \mapsto -\alpha$ or $\beta \mapsto -\beta$ or $\gamma \mapsto -\gamma$.*

Proof. The fact that $L_a \circ L_b = L_c$ comes from the homogeneity required for the formulas of interest. Remember: a theorem is a proposition with the biggest consequences, not necessarily something difficult to prove. □

Remark 2.1.8. Rational points (strong or not) are invariant by the Lemoine transforms, while the weak centers are replicated, leading to what is called a set of four **extraversions**. Obviously, the incenter and its three excenters were the pattern used to shape this concept.

Theorem 2.1.9. ***Klein transforms***. *When using barycentrics $x : y : z$, the identity together with the three transforms $x \mapsto -x$, $y \mapsto -y$ and $z \mapsto -z$ form a Klein group. They are called the Klein transforms (see Lemoine, 1891) and a group of four points $\pm f : \pm g : \pm h$ is called a **Klein quadrangle**. Spoiler. Any three points of such a set are the vertices of the anticevian triangle of the remaining one, while triangle ABC is the diagonal triangle of the quadrangle (see Proposition 3.4.15.*

Proof. Here, the fact that $L_a \circ L_b = L_c$ comes from the very definition of a projective space. □

Remark 2.1.10. When function $f(a, b, c)$ is a times an even polynomial, then both transforms give the same result. A list of such points is given at the introduction of points X(7001) - X(7373).

2.1.2 Deprecated concepts

Definition 2.1.11. A **transcendental center** is a triangle center X that cannot be defined as X $= f(a, b, c) : f(b, c, a) : f(c, a, b)$ using an algebraic function f. Examples: X(359) and X(360).

Definition 2.1.12. A **major center** is a triangle center X for which there exists a function f of the angles such that $X = f(A) : f(B) : f(C)$. Examples: $X(1)$, $X(2)$, $X(3)$, $X(4)$, $X(6)$. Major centers solve certain problems in functional equations (Kimberling, 1993; **?**).

Consider two examples, $X(9)$ and $X(37)$, of which first trilinears are $b + c - a$ and $b + c$, respectively. It is not clear from these trilinears that $X(9)$ is a major center, whereas $X(37)$ is not. Indeed, $X(9)$ also has first trilinear $cot(A/2)$, so that $X(9)$ is a major center, but there remains this problem: how to establish that $X(37)$ and others are not major. In April, 2008, Manol Iliev found a criterion for a triangle center to be not a major center (**reference missing**). He applied his test to the first 3236 triangle centers in ETC and found that exactly 292 of them are major, as listed Table 2.1.

Definition 2.1.13. Angular Lemoine transform.

$$A \mapsto -A\ ;\ B \mapsto \pi - A\ ;\ C \mapsto \pi - C\ ;\ S \mapsto -S\ ;\ R \mapsto R\ ;\ r_0 \leftrightarrow r_a$$

1	2	3	4	6	7	8	9	13	14	15
16	17	18	19	24	25	31	32	33	34	35
36	41	47	48	49	50	55	56	57	61	62
63	68	69	75	76	77	78	79	80	85	91
92	93	94	158	173	174	179	184	186	188	200
202	203	212	215	219	220	222	236	255	258	259
264	265	266	269	273	278	279	281	289	298	299
300	301	302	303	304	305	312	317	318	319	320
323	326	328	331	340	341	345	346	348	357	358
359	360	365	366	371	372	378	393	394	400	470
471	472	473	479	480	483	485	486	491	492	506
507	508	509	554	555	556	557	558	559	560	561
562	563	571	577	601	602	603	604	605	606	607
608	728	738	847	999	1000	1028	1049	1077	1081	1082
1085	1088	1092	1093	1094	1095	1096	1102	1106	1115	1118
1119	1123	1124	1127	1128	1129	1130	1131	1132	1133	1134
1135	1136	1137	1139	1140	1143	1147	1151	1152	1250	1251
1253	1259	1260	1264	1265	1267	1270	1271	1274	1321	1322
1327	1328	1335	1336	1395	1397	1398	1399	1407	1411	1435
1442	1443	1488	1489	1496	1497	1501	1502	1583	1584	1585
1586	1593	1597	1598	1599	1600	1659	1748	1802	1804	1807
1820	1847	1857	1870	1917	1928	1969	1973	1974	1989	1993
1994	2003	2006	2052	2066	2067	2089	2151	2152	2153	2154
2160	2161	2165	2166	2174	2175	2207	2212	2289	2306	2307
2323	2351	2361	2362	2477	2671	2672	2673	2674	2962	2963
2964	2965	3043	3076	3077	3082	3083	3084	3092	3093	3179
3200	3201	3205	3206	3218	3219					

Table 2.1: Major centers

2.2 Central triangle

Definitions of this Section are tailored so that triangle ABC itself as well as later defined (Section 3.4) cevian and anticevian triangles are central objects. The corresponding matrices are, columnwise, as follows :

$$\mathcal{C}_P \simeq \left(\begin{array}{ccc} 0 & p & p \\ q & 0 & q \\ r & r & 0 \end{array}\right), \mathcal{A}_P \simeq \left(\begin{array}{ccc} -p & p & p \\ q & -q & q \\ r & r & -r \end{array}\right)$$

Definition 2.2.1. *Suppose that f, g are two homogeneous functions having the same degree of homogeneity. One of them (but not both) can be the zero function. Then the (f, g)central triangle* is defined as :

$$[A', B', C'] \simeq \left(\begin{array}{ccc} f\,(a,b,c) & g\,(a,c,b) & g\,(a,b,c) \\ g\,(b,c,a) & f\,(b,c,a) & g\,(b,a,c) \\ g\,(c,b,a) & g\,(c,a,b) & f\,(c,a,b) \end{array}\right) \tag{2.1}$$

Example 2.2.2. Triangle ABC is $(1, 0)$ while $\mathcal{C}_P$ is $(0, f)$ and $\mathcal{A}_P$ is $(-f,\ f)$ when point P is defined by center function f.

Proof. We have $P = f(a, b, c) : f(b, c, a) : f(c, a, b)$, together with $f\,(a, b, c) = f\,(a, b, c)$. □

Example 2.2.3. As a summary, we have:

$$\boxed{\mathcal{T}\left(bc,c^2\right)} = \begin{pmatrix} \dfrac{bc}{b^2+bc+c^2} & \dfrac{c^2}{a^2+ca+c^2} & \dfrac{b^2}{a^2+ab+b^2} \\ \dfrac{c^2}{b^2+bc+c^2} & \dfrac{ca}{a^2+ca+c^2} & \dfrac{a^2}{a^2+ab+b^2} \\ \dfrac{b^2}{b^2+bc+c^2} & \dfrac{a^2}{a^2+ca+c^2} & \dfrac{ab}{a^2+ab+b^2} \end{pmatrix}$$

Notation 2.2.4. Among the functions often used, we have:

S	*area* (ABC). Not twice the area, as used in ETC.
S_a, S_b, S_c	the Conway symbols. $S_a = \left(b^2 + +c^2 - a^2\right)/2$
$\cot\omega$	where ω is the Brocard angle, $\cot\omega = \left(a^2+b^2+c^2\right)/2S$
s	$(a+b+c)/2$, so that $2s$ is the perimeter.

Proposition 2.2.5. *Symmetric (metric) functions. An* (a,b,c,S) *expression that is symmetric in* (a,b,c) *can be expressed as a function of* s, R, r_0 *where* s *is the semiperimeter, while* R, r_0 *are the* ***oriented*** *radiuses of the circum- and in-scripted circles (as given in* $r_0 = S/s$; $R = abc/4S$, *so that* $R\,r_0 > 0$*).*

Proof. Use the symmetric functions $S = r_0\,s$; $a+b+c = 2\,s$; $ab+bc+ca = D$; $abc = P$ and then substitute

$$P = 4\,s\,r_0\,R\ ;\ D = 4\,R\,r_0 + r_0^2 + s^2$$

Caveat: quantities R, r_0 are often perceived as positive but, here, they have to carry the orientation of the triangle. □

2.3 Symbolic substitution

Definition 2.3.1. Symbolic substitution. Suppose $p(a,b,c)$, $q(a,b,c)$, $r(a,b,c)$ are functions of a,b,c, all of the same degree of homogeneity. As the transfigured plane consists of all functions of the form $X = x(a,b,c) : y(a,b,c) : z(a,b,c)$, the substitution indicated by

$$a \mapsto p(a,b,c), b \mapsto q(a,b,c), c \mapsto r(a,b,c)$$

maps the transfigured plane into itself.

Remark 2.3.2. Such a substitution may have no clear geometric meaning, as suggested by the name, *symbolic* substitution. On the other hand, symbolic substitutions are of geometric interest because they map lines to lines, conics to conics, cubics to cubics, and they preserve incidence.

Example 2.3.3. The symbolic substitution $(a,b,c) \mapsto (1/a, 1/b, 1/c)$ maps every triangle center to a triangle center, every pair of bicentric points to a pair of bicentric points, every circumconic to a circumconic, etc. However, when $(a,b,c) = (3,4,5)$, for example, then a,b,c are sidelengths of an euclidian triangle, but $1/a$, $1/b$, $1/c$ are not.

Symbolic substitutions were introduced in Kimberling (2007).

Chapter 3

Cevian stuff

3.1 Centroid stuff

Definition 3.1.1. The **reflection** of point $U = u : v : w$ in point $P = p : q : r$ (not at infinity) is the point X such that :

$$X \simeq \left(\begin{array}{c} (p-q-r)u + 2p(v+w) \\ (q-p-r)v + 2q(u+w) \\ (r-p-q)w + 2r(u+v) \end{array} \right) \tag{3.1}$$

Proof. We want to obtain $X = 2P - U$ when P, U, X are finite and in normalized form. When P is finite and $U \in \mathcal{L}_b$ then $X = U$ (OK). Taking $P \in \mathcal{L}_b$ would result into $X = P$ for any value of U, not an acceptable result. □

Stratospheric proof. X is obtained under the action described by :

$$2 \left(\left(\begin{array}{c} p \\ q \\ r \end{array} \right) \cdot \mathcal{L}_b \right) - \left(\mathcal{L}_b \cdot \left(\begin{array}{c} p \\ q \\ r \end{array} \right) \right) \cdot \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)$$ □

Definition 3.1.2. Complement and **anticomplement** are inverse transforms defined so that the complement of a vertex of triangle ABC is the middle of the opposite side. In other words (using barycentrics),

$$\begin{array}{rcl} complement\,(U) & \doteq & (3\,G - U)\,/2 \simeq q + r : p + r : p + q \\ anticomplement\,(Q) & \doteq & (3\,G - 2\,Q) \simeq -p + q + r : p - q + r : p + q - r \end{array}$$

According to Court, p. 297, the term *complementary point* dates from 1885, and the term *anti-complementary point* dates from 1886.

Definition 3.1.3. The **medial triangle** $\mathcal{C}_2$ is the complement of triangle ABC. The A-vertex of $\mathcal{C}_2$ is the middle of segment BC, and cyclically. In other words :

$$\mathcal{C}_2 \simeq \left(\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array} \right)$$

Definition 3.1.4. The **antimedial triangle** $\mathcal{A}_2$ is the anticomplement of triangle ABC (and is also called the anticomplementary triangle). Each sideline of $\mathcal{A}_2$ contains a vertex of ABC and corresponding sidelines of both triangles are parallel. In other words :

$$\mathcal{A}_2 \simeq \left(\begin{array}{ccc} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{array} \right)$$

These triangles are the model used to define the next coming cevian and anticevian triangles (they are the triangles related to the centroid X_2).

3.2 Cross-ratio and fourth harmonic

Remark 3.2.1. This section about $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^2)$ is set here, i.e far away from the chapter about $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^3)$, in order to emphasize the fact that $2 \neq 3$: both sets are different and they will be used for very different things. The later, i.e. $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^3)$ will be used to enlarge the description of points or lines already provided by $\mathbb{P}_{\mathbb{R}}(\mathbb{R}^3)$, while the former, i.e. $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^2)$, will be used to parametrize various families of objects.

Definition 3.2.2. The Riemann sphere is the set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. It can be identified with the set $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^2)$ of the $\mathbf{Z} : \mathbf{T}$, i.e. the set of the couples $(\mathbf{Z}, \mathbf{T})$ when treated projectively, i.e. "up to a factor".

Definition 3.2.3. The cross-ratio of four (different) parameters $z_j \in \mathbb{C}$ is defined as

$$\text{cross\_ratio}_{\mathbb{C}}(z_1, z_2, z_3, z_4) \doteq \frac{(z_4 - z_2)(z_3 - z_1)}{(z_4 - z_1)(z_3 - z_2)}$$

This definition can be extended to the Riemann sphere by the formula :

$$\text{cross\_ratio}\,(z_1 : t_1; z_2 : t_2; z_3 : t_3; z_4 : t_4) \doteq \text{cross\_ratio}_{\mathbb{C}}\left(\frac{z_1}{t_1}, \frac{z_2}{t_2}, \frac{z_3}{t_3}, \frac{z_4}{t_4}\right)$$

with the usual rules to resolve indeterminacies.

Remark 3.2.4. Exactly this same quantity is introduced as $\frac{z_1 - z_3}{z_1 - z_4} \div \frac{z_2 - z_3}{z_2 - z_4}$, i.e. as (z_4, z_3, z_2, z_1) by (Pedoe, 1970, p 212) and (Schwerdtfeger, 1962, p. 35). The intent of these authors was to emphasize the definition as a ratio of ratios. Our intent here is to emphasize the choice made among the six possibilities described just below.

Proposition 3.2.5. *An homography ψ is defined as a mapping of $\overline{\mathbb{C}}$ which preserves the birapport. As a result, ψ can be written as*

$$z \mapsto \psi(z) \doteq \frac{\mathrm{a}\, z + \mathrm{b}}{\mathrm{c}\, z + \mathrm{d}} \qquad \text{where a, b, c, d} \in \mathbb{C} \text{ and ad} - \text{bc} \neq 0$$

Such transform can also be seen as an element of $\mathbb{PGL}_{\mathbb{C}}(\mathbb{C}^2)$.

Proof. Write $\dfrac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)} = \dfrac{(\widehat{z} - \widehat{z}_2)(\widehat{z}_3 - \widehat{z}_1)}{(\widehat{z} - \widehat{z}_1)(\widehat{z}_3 - \widehat{z}_2)}$ and solve in $\widehat{z}$. □

Remark 3.2.6. Some French writers are using 'homography' as a synonym to collineations acting on $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^3)$. This is as stupid as possible. At High Schools, students have acquired (or should have acquired) some behavioral skills about these ψ functions acting on the complex plane $\mathbb{C}$. When introducing new concepts, new names should be used, in order to preserve what is already acquired.

Proposition 3.2.7. *An involution is an homography ψ such that $\psi \circ \psi = \mathrm{Id}$ while $\psi \neq \mathrm{Id}$. Three characterizations are:*

1. $\mathrm{a} + \mathrm{d} = 0$
2. *it exists z_1, z_2 such that $\psi(z_1) = z_2$, $\psi(z_2) = z_1$, $z_1 \neq z_2$.*
3. *it exists three different points such that* $\det_3 [z_j \psi(z_j), z_j + \psi(z_j), 1] = 0$

Proof. (1) one has $z - \psi^2(z) = (a + d)\, \dfrac{cz^2 + (d - a) z - b}{c(a + d) z + bc + d^2}$
(2) one has $cz_1 z_2 - az_1 + dz_2 = b$; $cz_1 z_2 - az_2 + dz_1 = b$. Substracting, one gets

$$(z_1 - z_2)(a + d) = 0$$

(3) requites the existence of a, b, c such that $cz_j \psi(z_j) + (z_j + \psi(z_j))\, dz_2 - b = 0$. □

Proposition 3.2.8. *Group of the cross-ratio. The cross ratio remains unchanged under the action of the bi-transpositions like* $[a, b, c, z] \mapsto [b, a, z, c]$. *Therefore the action of* $\mathfrak{S}_4$ *generates only 6 values for the cross-ratio. We have :*

a	b	c	z	a	a
b	c	a	b	z	b
c	a	b	c	c	z
z	z	z	a	b	c
k	$\dfrac{k-1}{k}$	$\dfrac{1}{1-k}$	$1-k$	$\dfrac{k}{k-1}$	$\dfrac{1}{k}$

Special cases are $\{\exp(+i\pi/3), \exp(-i\pi/3)\}$ *(equilateral triangle and one of its centers)* $\{1, 0, \infty\}$ *(two points are equal) and* $\{-1, 2, 1/2\}$ *(harmonicity) .*

Proof. Direct examination. □

Definition 3.2.9. Consider the linear projective family defined from two fixed rows (or columns) X, Y (where $X \not\simeq Y$ is assumed)

$$\Delta = \{ zX + tY | (z, t) \neq (0, 0)\}$$

The cross-ratio of four elements of Δ is defined as the cross-ratio of their parameters, i.e.

$$\text{cross\_ratio}\ (z_j X + t_j Y) = \text{cross\_ratio}_{\mathbb{C}}\ (z_j/t_j)$$

Theorem 3.2.10. *The cross-ratio of four elements of a linear projective family (assuming at least three different elements) is intrinsic, i.e. doesn't depend of the columns (or rows) chosen to describe the family. When four different collinear objects* P, Q, R, S *are given, then non zero multipliers* p, q, r, s *and a constant* λ *can be found that satisfy the hard equalities :*

$$\begin{aligned} r\,R &= p\,P + q\,Q \\ s\,S &= \lambda\,p\,P + q\,Q \end{aligned}$$

Moreover quantity λ *depends only on the four objects and their order. This quantity is the former defined cross-ratio of the 4-uple.*

Proof. To prove the existence, we write $P = z_1 X + t_1 Y$, etc and solve the resulting set of 4 equations.The value obtained for λ is the former defined cross-ratio "using X, Y". The uniqueness of λ comes from the uniqueness, up to a proportionality factor, of $r : p : q$ and $s : \lambda p : q$ since P, Q is yet another generating family. □

Corollary 3.2.11. *When the family is parametrized as* $P_j = k_j\,X + (1-k_j)\,Y$, *then* $\lambda = \text{cross\_ratio}_{\mathbb{C}}\,(k_j)$.

Proof. By definition, $\lambda = \text{cross\_ratio}\ (k_j/(1-k_j))$. But $\text{cross\_ratio}_{\mathbb{C}}\,()$ is invariant under $z \mapsto z/(1-z)$. □

Corollary 3.2.12. *Let the* P_j *be given by their barycentrics* $p_j : q_j : r_j$. *Assuming that two of the* p_j *are not* 0, *then* $\lambda = \text{cross\_ratio}\ (p_j/(p_j + q_j + r_j))$.

Proof. This holds only if the points are on the same line, but not the infinity line ! □

Definition 3.2.13. Four points A, B, J, K on an ordinary straight line form an **harmonic division** when :

$$\frac{\overline{AJ}}{\overline{BJ}} \div \frac{\overline{AK}}{\overline{BK}} = -1$$

Since cross-ratio is a projective invariant, this relationship is carried along collineations.

Proposition 3.2.14. *Let be given three members* P, Q, R *of a linear projective family with, at least* $P \neq Q$. *Then it exists exactly one member* S *of the family such that* $\text{cross\_ratio}\ (P, Q, R, S) = -1$. *This object is called the* **fourth harmonic** *of the first three.*

Proof. Cross-ratio is an homographic function of parameter k_S, and therefore bijective between λ and k_S. □

Construction 3.2.15. *Let A, P, B be three aligned distinct points, and E be a point external to this green line (see Figure 3.1). Division (A, B, P, Q) will be harmonic if the pencil (EA, EB, EP, EQ) is harmonic. To this end, we draw the blue line Δ i.e. the parallel to EA through B. Then M is $DP \cap \Delta$ and N the reflection of M into C. The division (∞_Δ, B, M, N) is obviously harmonic. And we obtain Q as $EN \cap AB$.*

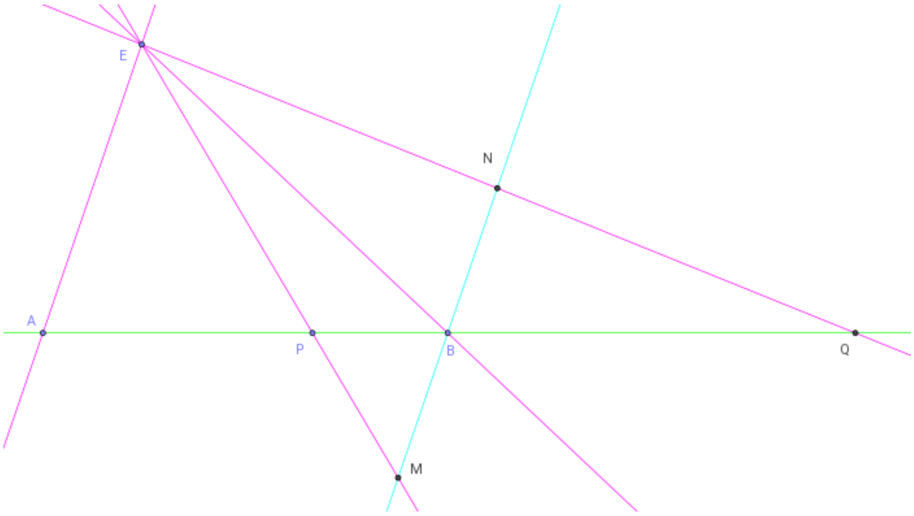


Figure 3.1: Obtain Q from A,P,C and auxiliary D

Remark 3.2.16. In the Cartesian plane, the fourth harmonic is also the reflection of the third point into the circle having the first two as diameter.

Remark 3.2.17. Conic cross-ratio is described in Section 12.10.

3.3 About combos

Definition 3.3.1. combos. When $P \simeq p : q : r$ and $U \simeq u : v : w$ are points at finite distance and $f = f(a, b, c)$, $g = g(a, b, c)$ are nonzero homogeneous functions having the same degree of homogeneity, then the (f, g) combo of P and U, denoted as $f \times P + g \times U$, is :

$$f \times P + g \times U \simeq \frac{P}{\mathcal{L}_b \cdot P}\, f\,(a, b, c) + \frac{U}{\mathcal{L}_b \cdot U}\, g\,(a, b, c)$$

Remark 3.3.2. Written that way, one has not to discuss if barycentrics or trilinears are used.

Proposition 3.3.3. *With the same hypotheses, points P, U, $fP + gU$, $fP + hU$ are collinear and*

$$\text{cross\_ratio}\ (P,\ U,\ fP + gU,\ fP + hU) = h/g$$

As a special case, $f\ P \pm g\ U$ are harmonic conjugate wrt P, U.

Proposition 3.3.4. *When f, g, h are homogeneous symmetric functions all of the same degree of homogeneity, and $X, X', X"$ are triangle centers, then $f\ X + g\ X' + h\ X"$ is a triangle center. Conversely, given three non collinear triangle centers, any fourth triangle center is a combo of the first three, using symmetric functions as coefficients.*

Proof. This amounts to say that, in $\mathbb{R}^3$, any invertible matrix defines a basis of the space. □

Remark 3.3.5. Part of the time, normalization is useless. Knowing that $X(482) \simeq s \times nX(1) + (r + 4R) \times nX(7)$ can be required, but using $X(482) \simeq vX(1) + 4S\ vX(7)$ (where vX is what is given in the table) can be sufficient.

Proposition 3.3.6. *The columns of a matrix $\boxed{\mathcal{T}}$ describing the vertices of a central triangle have to be normalized in such a way that $\mathcal{L}_b \cdot \boxed{\mathcal{T}} \simeq \mathcal{L}_b$. Such matrices form a group under multiplication. And then $\boxed{\mathcal{T}} \cdot P$ is (another) triangle center when P is a triangle center.*

3.4 Cevian, anticevian, cocevian triangles

Definition 3.4.1. Cevian triangle. Let P be a point not on a sideline of ABC. The lines AP, BP, CP are the *cevians* of P. Let $A_p = AP \cap BC$. Define B_p and C_p cyclically. Triangle $A_pB_pC_p$ is called the cevian triangle of triangle ABC.

$$\text{cevian}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 & p & p \\ q & 0 & q \\ r & r & 0 \end{pmatrix} \tag{3.2}$$

Example 3.4.2. Examples of cevian triangles are given in Table 3.1.

Proposition 3.4.3. ***Cocevian triangle****. Let $P = p : q : r$ be a point not on a sideline of ABC, let $A_PB_PC_P$ be its cevian triangle and define T_A as $BC \cap B_PC_P$,* etc. *Then points T_A, T_B, T_C are aligned on a line which is called the* **tripolar line** *of P, while the (degenerate) triangle T_A, T_B, T_c itself is called the cocevian triangle of P. One has:*

$$\text{cocevian}\,(p : q : r) \simeq \begin{pmatrix} 0 & p & -p \\ -q & 0 & q \\ r & -r & 0 \end{pmatrix} \tag{3.3}$$

$$\text{tripolar}\,(p : q : r) \simeq \left[\frac{1}{p}, \frac{1}{q}, \frac{1}{r}\right] \simeq [qr, rp, pq] \tag{3.4}$$

Proof. Direct computation. □

Remark 3.4.4. (Spoiler) Point P is the *perspector* of triangles ABC and $A_PB_PC_P$, i.e. their (degenerate) vertex triangle (see Definition 3.8.1). While tripolar$\,(P)$ is the *perspectrix* of ABC and $A_PB_PC_P$, i.e. their (degenerate) sideline trigone (see Definition 3.8.4).

Definition 3.4.5. Anticevian triangle. Let P be a point not on a sideline of ABC. The anticevian of P is the triangle $P_AP_BP_C$ such that ABC is the cevian triangle of P wrt $P_AP_BP_C$.

Example 3.4.6. Examples of anticevian triangles are given in Table 3.1.

Construction 3.4.7. *Let P be a point not on the sidelines of ABC. Draw $A_p \doteq AP \cap BC$,* etc *(the cevian triangle of P) and then draw $T_A \doteq BC \cap B_PC_P$,* etc *(the cocevian triangle of P. Then we have $P_A = BT_B \cap CT_C$,* etc *(the anticevian triangle of P)*

Proof. Compute all these points and obtain

$$P \simeq \begin{pmatrix} p \\ q \\ r \end{pmatrix} ; \text{cev} = \begin{pmatrix} 0 & p & p \\ q & 0 & q \\ r & r & 0 \end{pmatrix} ; \text{cocev} \simeq \begin{pmatrix} 0 & p & -p \\ -q & 0 & q \\ r & -r & 0 \end{pmatrix} ; \text{anticev} \simeq \begin{pmatrix} -p & p & p \\ q & -q & q \\ r & r & -r \end{pmatrix} \tag{3.5}$$

Then one can verify the relations $A \in P_BP_C$, etc (ABC is inscribed in $P_AP_BP_C$), $P \in AP_A$, etc (P is the perspector of ABC and $P_AP_BP_C$), $T_A = BC \cap P_BP_C$, etc ($T_AT_BT_C$ is the perspectrix of ABC and $P_AP_BP_C$). □

Remark 3.4.8. (Spoiler) Point P is the *perspector* of triangles ABC and $P_AP_BP_C$, i.e. their (degenerate) vertex triangle (see Definition 3.8.1). While tripolar$\,(P)$ is the *perspectrix* of ABC and $P_AP_BP_C$, i.e. their (degenerate) sideline trigone (see Definition 3.8.4).

Proposition 3.4.9. *We have three sets of aligned points whose divisions are harmonic, namely*

$$\begin{aligned} \text{cross\_ratio}\,(A, A_P, P, P_A) &= -1 \\ \text{cross\_ratio}\,(B, C, A_P, T_a) &= -1 \\ \text{cross\_ratio}\,(P_B, P_C, A, T_a) &= -1 \end{aligned}$$

Proof. We have formally: $(0 : q : r) \pm (p : 0 : 0) = (p : q : r)\,, (-p : q : r)$ while $(0 : q : 0) \pm (0 : 0 : r) = (0 : q : r)\,, (0 : q : -r)$ and $(p : -q : r) \pm (p : q : -r) = (2p : 0 : 0)\,, (0 : -2q : +2r)$ □

cevian				bary (p)	G	O	I
incentral	24.3	incenter	X(1)	a			
medial	24.5	centroid	X(2)	1	X(2)	X(5)	X(10)
orthic	24.7	H-center	X(4)	$\tan A$	X(51)	X(5)	X(4)
intouch	24.12	Gergonne	X(7)	$(b+c-a)^{-1}$			
extouch	24.13	Nagel	X(8)	$b+c-a$	X(210)	X(1158)	
anticevian				bary	G	O	I
excentral	24.4	incenter	X(1)	a	X(165)	X(40)	X(164)
antimedial	24.6	centroid	X(2)	1	X(2)	X(4)	X(8)
tangential	24.8	Lemoine	X(6)	a^2	X(154)	X(26)	X(3)
other							
Fuhrmann	24.15						
Brocard triangles	24.9						
Hexyl	24.14				X(3576)	X(1)	
star	24.16				X(3817)	X(946)	?

Table 3.1: Some well-known triangles

3.4.1 Well-known triangles

3.4.2 Isotomic and reciprocal conjugacies

Proposition 3.4.10. *Equation3.6 gives the condition for an inscribed triangle to be the Cevian triangle of some point P.*

$$\begin{pmatrix} 0 & p_2 & p_3 \\ q_1 & 0 & q_3 \\ r_1 & r_2 & 0 \end{pmatrix} \text{ is Cevian } \Longleftrightarrow p_2\,q_3\,r_1 - p_3\,q_1\,r_2 = 0 \tag{3.6}$$

$$\text{and then } p : q : r = r_1 p_2 : q_1 r_2 : r_1 r_2$$

An absolutely hopeless formula, but nevertheless more "obviously symmetric" is :

$$p : q : r = \sqrt[3]{p_2^2\,p_3^2\,q_1\,r_1} \;:\; \sqrt[3]{q_3^2\,q_1^2\,p_2\,r_2} \;:\; \sqrt[3]{r_1^2\,r_2^2\,p_3\,q_3}$$

Proof. Line AP_1 is $(0,\, r_1,\, -q_1)$ and cyclically. These lines are concurrent when their determinant vanishes. Their common point is then given by any column of the adjoint matrix, or the cubic root of their product. □

Proposition 3.4.11. Isotomic conjugacy. *Suppose $U = u : v : w$ is a point not on a sideline of ABC. Take the cevians $A_U \doteq AU \cap BC$,* etc *and then reflect $A_U B_U C_U$ about the midpoints of sides BC, CA, AB, respectively, to obtain points A', B', C'. Then lines AA', BB', CC' are concurrent. Their common intersection is called the isotomic conjugate of U. The corresponding barycentrics are :*

$$\text{isotom}\,(u : v : w) = \frac{1}{u} : \frac{1}{v} : \frac{1}{w} \tag{3.7}$$

Proof. Immediate computation. □

Remark 3.4.12. The fixed points of this transform are the gravity center and its relatives, so that $isot\,(U) = U_G^{\#}$ while $U \underset{b}{*} isot\,(U) =$X(2). It should be noticed that $X\,(2)$ plays together the role of points F and P of Definition 1.4.10.

Proposition 3.4.13. *Reciprocal conjugacy*. *Suppose $\Delta = [\rho, \sigma, \tau]$ is a line not through a vertex of ABC. Let $A_\Delta = BC \cap \Delta$,* etc *the traces of Δ on the sidelines. Reflect them about the* *corresponding midpoint and obtain* $T_A = B + C - A_\Delta$, etc. *The points* T_A, T_B, T_C *are aligned, defining a line called the reciprocal of the given one since we have:*

$$\operatorname{recip}([\rho,\sigma,\tau]) \simeq \left[\frac{1}{\rho},\frac{1}{\sigma},\frac{1}{\delta}\right]$$

Proof. Immediate computation. □

Remark 3.4.14. For a given $U \simeq u : v : w$, we have

$$(\text{tripolar} \circ \text{isotom})(U) = [u, v, w] = (\text{recip} \circ \text{tripolar})(U)$$

Proposition 3.4.15. *Anticevian quadrangle. (1) Triangle* ABC *is inscribed in triangle* $P_A P_B P_C$*. (2)* ABC *is the cevian triangle of* P *wrt the anticevian triangle. (3) Anticevian triangle of point* P_A *wrt* ABC *is* $P P_C P_B$ *(the two remaining points are permuted). See also Theorem 2.1.9.*

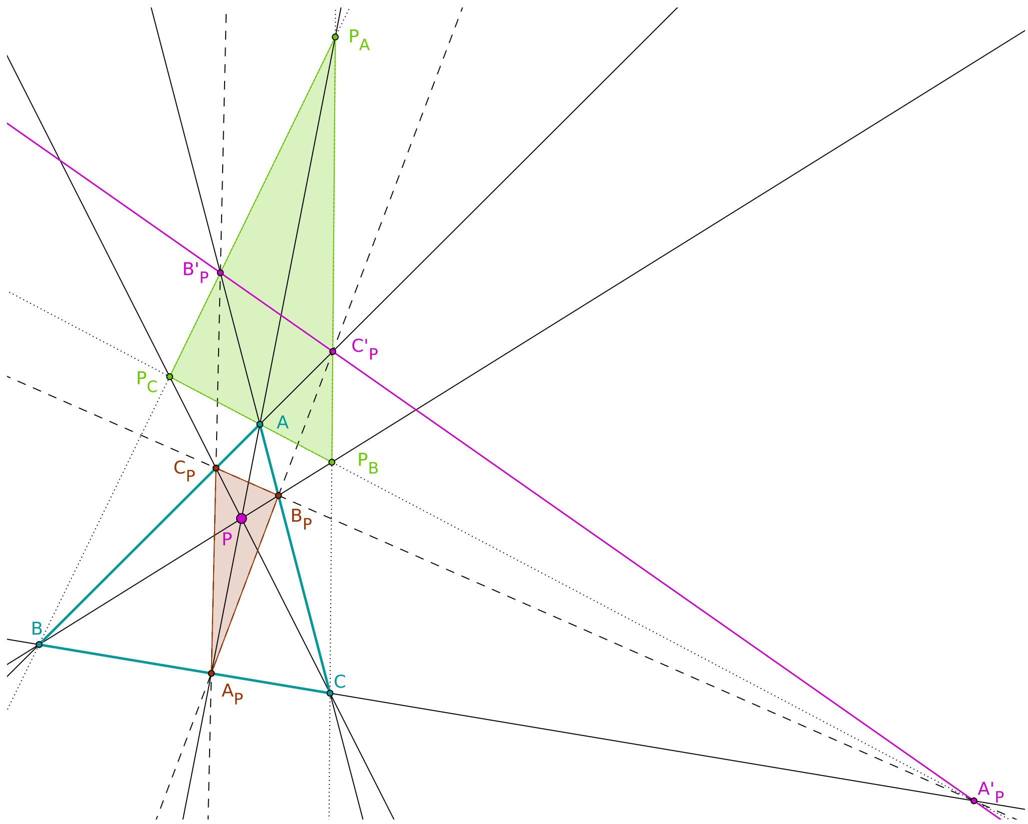


Figure 3.2: Cevian, anticevian and cocevian triangles

3.5 Transversal lines, Menelaus and Miquel theorems

Theorem 3.5.1 (Ceva)**.** *Let* $A' \in BC$, $B' \in CA$ *and* $C' \in AB$ *be three points on the sidelines of triangle* ABC*, but different from the vertices. Then lines* AA', BB', CC' *are concurrent if and only if :*

$$\frac{\overline{AB'}}{\overline{CB'}}\,\frac{\overline{BC'}}{\overline{AC'}}\,\frac{\overline{CA'}}{\overline{BA'}} = -1$$

Proof. The usual proof uses Menelaus theorem. Another proof, using determinants, is given below. □

Proposition 3.5.2 (Genuine Menelaus theorem)**.** *Let* $A' \in BC$, $B' \in CA$ *and* $C' \in AB$ *be three points on the sidelines of triangle* ABC*, but different from the vertices. Then* A', B', C' *are collinear if and only if :*

$$\frac{\overline{AB'}}{\overline{CB'}}\,\frac{\overline{BC'}}{\overline{AC'}}\,\frac{\overline{CA'}}{\overline{BA'}} = +1$$

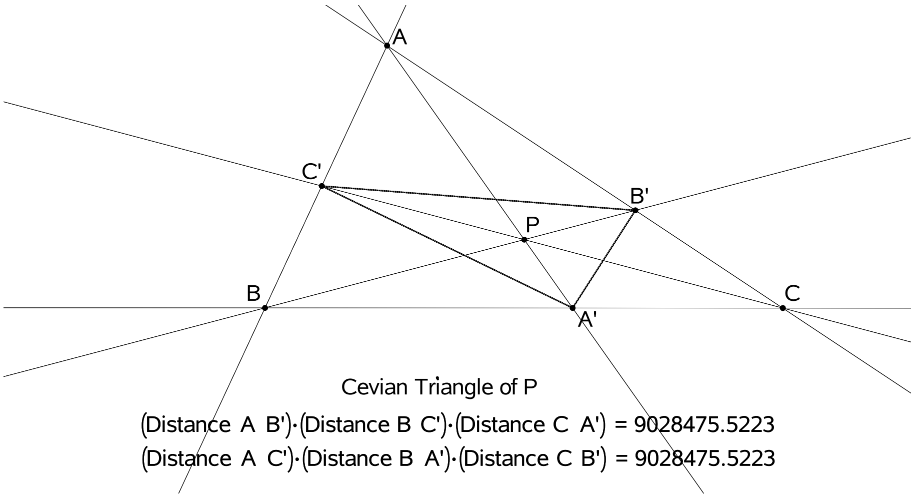


Figure 3.3: Ceva's Theorem

Proof. Let us parametrize the situation by $A' = k_a B + (1 - k_a)C$, etc. Alignment is described by :

$$\begin{vmatrix} 0 & 1-k_b & k_c \\ k_a & 0 & 1-k_c \\ 1-k_a & k_b & 0 \end{vmatrix} = 0$$

while $A' - B = (1 - k_a)(C - B)$ and $A' - C = k_a (B - C)$. □

Example 3.5.3. The cevian of P verify this formula.

Proposition 3.5.4 (Miquel theorem)**.** *Let $A' \in BC$, $B' \in CA$ and $C' \in AB$ be three points on the sidelines of triangle ABC, but different from the vertices. Then circles $AB'C'$, $A'BC'$, $A'B'C$ are passing through a same point M_q, the Miquel point of $A'B'C'$ wrt ABC.*

Proof. Equation of circle $AB'C'$ is :

$$a^2 y\, z + b^2 z\, x + c^2 x\, y - (x + y + z)\left(y\, k_c\, c^2 + z\,(1 - k_b)\, b^2\right)$$

Therefore their last common point is :

$$M_q \simeq \begin{pmatrix} a^2\left(k_a\,(k_a - 1)\, a^2 + (k_a - 1)\,(k_b - 1)\, b^2 + c^2 k_c k_a\right) \\ b^2\left(k_a k_b\, a^2 + k_b\,(k_b - 1)\; b^2 + (k_b - 1)\,(k_c - 1)\, c^2\right) \\ c^2\left((k_c - 1)\,(k_a - 1)\, a^2 + b^2 k_b k_c + k_c\,(k_c - 1)\, c^2\right) \end{pmatrix}$$

Since this expression is symmetric, the point M_q is also on the third circle. □

Theorem 3.5.5 (Extended Menelaus theorem)**.** *Let $A' \in BC$, $B' \in CA$ and $C' \in AB$ be three points on the sidelines of triangle ABC, but different from the vertices. All the following are necessary and sufficient conditions for A', B', C' to be collinear :*
(i) the Menelaus condition : $((1 - k_a)(1 - k_b)(1 - k_c) + k_a k_b k_c) = 0$
(ii) the midpoints $M_a = (A + A')/2$, $M_b = (B + B')/2$, $M_c = (C + C')/2$ are on the same line (the so-called Newton line of the quadrilateral)
*(iii) the **Miquel point** M_q of $A'B'C'$ is on the circumcircle of ABC.*
*(iv) –spoiler– the **Clawson-Schmidt homography** application Ψ in $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ by $A \mapsto A'$, $B \mapsto B'$, $C \mapsto C'$ is involutory .*

Proof. (i) obvious ; (ii) determinant ; (iii) condition for $M \in \Gamma$ is the Menelaus condition times an ugly factor that can be written as :

$$\left(2\, a^2 k_a + 2\, b^2 k_b + 2\, c^2 k_c - c^2 - b^2 - a^2\right)^2 + 16\, S^2$$

(iv) Condition for Ψ to be involutory is $\det_3 [1,\, z_A + z'_A,\, z_A\, z'_A] = 0$. This results into Is_4 times the Menelaus condition (see Section 15.4 for notations and more details). □

3.6 Tripolar centroid

Proposition 3.6.1. *When P is neither a vertex nor $X(2)$, the centroid of the cocevian triangle is well defined (perhaps on $\mathcal{L}_b$), and called the tripolar centroid of P (Stothers, 2003b).*

$$TG\,(P) = p(q-r)(2p-q-r) : q(p-r)(p-2q+r) : r(q-p)(p+q-2r)$$

Remark 3.6.2. When $q = r$ then $TG\,(P) = 0 : 1 : -1$ (at infinity on BC). When all p, q, r are different, $TG\,(P)$ is a finite point.

Proposition 3.6.3. *When all p, q, r are different, then it exists exactly another point that shares the same $TG\,(P)$, namely :*

$$other\,(P) = \frac{q+r-2\,p}{rp+qp-2\,qr} : \frac{r+p-2\,q}{qp+qr-2\,rp} : \frac{p+q-2\,r}{qr+rp-2\,qp}$$

Proof. Direct computation. When eliminating k, u, v in $TG\,(P) = k\,TG\,(U)$, special cases are p, $q-r$, u, $v-w$, $qw-rv$, $2p-q-r$ and cyclically. For all named points X, it happens that $other\,(X)$ is not named. □

Example 3.6.4. Points $X(1635)$ to $X(1651)$ are defined that way. Examples include :

1	1635	98	1640	263	2491	525	1650
3	1636	99	1641	512	1645	648	1651
4	1637	100	1642	513	1646	957	3310
6	351	105	1643	514	1647	1002	665
7	1638	190	1644	523	1648	1022	244
8	1639	262	3569	524	1649	2394	125

3.7 Cross-triangle

Definition 3.7.1. Crosstriangle. The crosstriangle of two given triangles $\mathcal{T}_1 = A_1B_1C_1$ and $\mathcal{T}_2 = A_2B_2C_2$ is defined as the triangle $\mathcal{T}_4 = B_1C_2 \cap B_2C_1$, $C_1A_2 \cap C_2A_1$, $A_1B_2 \cap A_2B_1$. Its name comes from the fact we are crossing the vertices of the homologue sidelines.

3.8 Perspectivity

Definition 3.8.1. Vertex trigone, **vertex triangle**. Let $\mathcal{T}_1 = A_1B_1C_1$ and $\mathcal{T}_2 = A_2B_2C_2$ be two triangles (i.e. two ordered sets of three points). Their vertex trigone $\mathcal{T}_3^*$ is the set of three lines $\mathfrak{A}_3 = A_1A_2$, $\mathfrak{B}_3 = B_1B_2$, $\mathfrak{C}_3 = C_1C_2$, while their vertex triangle is the set of points $A_3 = \mathfrak{B}_3 \cap \mathfrak{C}_3$, etc. The vertex triangle is the dual of the vertex trigone.

Remark 3.8.2. Exclude the case where all points are on the same line, the rank of the vertex trigone is either 3 or 2, while the rank of the vertex triangle is either 3 or 1 (adjoint matrix).

Definition 3.8.3. Perspector. Let $\mathcal{T}_1 = A_1B_1C_1$ and $\mathcal{T}_2 = A_2B_2C_2$ be two triangles. When the vertex trigone degenerates, i.e. when the lines $\mathfrak{A}_3, \mathfrak{B}_3, \mathfrak{C}_3$, concur at some point P, this point is called the *perspector* (replacing *center of perspective*) of the (ordered) triangles.

Definition 3.8.4. Sideline triangle, sideline trigone. Let $\mathfrak{T}_1 = \mathfrak{A}_1\mathfrak{B}_1\mathfrak{C}_1$ and $\mathfrak{T}_2 = \mathfrak{A}_2\mathfrak{B}_2\mathfrak{C}_2$ be two trigones (i.e. two ordered sets of three lines). Their sideline triangle $\mathcal{T}_4$ is the set of three points $A_4 = \mathfrak{A}_1 \cap \mathfrak{A}_2$, $B_4 = \mathfrak{B}_1 \cap \mathfrak{B}_2$, $C_4 = \mathfrak{C}_1 \cap \mathfrak{C}_2$, while their sideline trigone is the set of lines B_4C_4, C_4A_4, A_4B_4 (i.e. the dual of the sideline triangle).

Remark 3.8.5. Excluding the case where all lines are through the same point, the rank of the sideline triangle is either 3 or 2, while the rank of the sideline trigone is either 3 or 1 (adjoint matrix).

Definition 3.8.6. *Perspectrix*. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two **triangles**. When the sideline triangle of $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$ degenerates, i.e. when points A_4, B_4, C_4 are on the same line, this line is called the perspectrix (replacing *axis of perspective*) of the triangles.

Theorem 3.8.7. ***[Desargues]*** . *When none of the triangles $\mathcal{T}_1$ and $\mathcal{T}_2$ are degenerate, the existence of a perspector is equivalent to the existence of a perspectrix.*

Proof. In this context, trigone $\mathfrak{T}_3$ is called the Desargues trigone and triangle $\mathcal{T}_4$ is called the Desargues triangle. The result comes from

$$\det \mathcal{T}_4 = \det \mathcal{T}_1 \, \det \mathcal{T}_2 \, \det \mathfrak{T}_3$$

and such a lack of symmetry is better understood when considering the dual formulas:

$$\begin{aligned} \det {}^t homol\,(\mathcal{T}_1^*, \mathcal{T}_2^*) &= \det \mathcal{T}_1 \, \det \mathcal{T}_2 \det homol\,(\mathcal{T}_1, \mathcal{T}_2) \\ \det homol\,(\mathfrak{T}_1^*, \mathfrak{T}_2^*) &= \det \mathfrak{T}_1 \, \det \mathfrak{T}_2 \det {}^t homol\,(\mathfrak{T}_1, \mathfrak{T}_2) \end{aligned} \quad \square$$

Example 3.8.8. Let $\mathcal{T}_1$ be the reference triangle ABC and $\mathcal{T}_2$ the cevian triangle $A_P B_P C_P$ of a point P. Then

1. $\mathfrak{T}_3 = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$ is the set of the three cevian lines AP, etc while $\mathfrak{T}_3^* = P \cdot {}^t P$: this triangle degenerates into three times the point P. Therefore P is the perspector of both triangles.

2. $\mathcal{T}_4 = \begin{pmatrix} 0 & p & -p \\ -q & 0 & q \\ r & -r & 0 \end{pmatrix}$ is the cocevian triangle. Compared to $\mathcal{T}_2 = \begin{pmatrix} 0 & p & p \\ q & 0 & q \\ r & r & 0 \end{pmatrix}$, this $\mathcal{T}_4$ is the set of the associated cross-ratio points, while $\mathcal{T}_4^*$ degenerates into three times the tripolar line tripolar $P \doteq [qr,\, rp,\, pq]$. Therefore this line is the perspectrix of both triangles.

Example 3.8.9. Let $\mathcal{T}_1$ be the triangle and $\mathcal{T}_2$ the anticevian triangle $P_A P_B P_C$ of a point P. Let us recall that cross_ratio $(A, A_p, P, P_A) = -1$. Then again perspector is P and perspectrix is tripolar P.

Exercise 3.8.10. Spoiler: for any P, the pedal triangle of P is in perspective with $\boxed{\mathcal{M}_b}$, the triangle whose vertices are the directions of the altitudes AH, BH, CH. The perspector is P itself. Since vertices of $\boxed{\mathcal{M}_b}$ are on $\mathcal{L}_b$, the flatness of the sideline triangle was granted and caries no additional information.

Proposition 3.8.11. Perspectivity kit. *Such a kit is defined as $p : q : r : u : v : w$, i.e. six numbers up to a common proportionality factor, none of them being 0. This amounts to give points $P \simeq p : q : r$, $U \simeq u : v : w$ (none on an ABC-sideline) together with a synchronization factor $k = (p+q+r)\,/\,(u+v+w)$. Exchanging P and U changes k in $1/k$. We define triangle $\mathcal{T}_0$ as ABC and triangle $\mathcal{T}_1$ and $\mathcal{T}_2$ by :*

$$\mathcal{T}_1 \simeq \begin{pmatrix} u & p & p \\ q & v & q \\ r & r & w \end{pmatrix} \; ; \; \mathcal{T}_2 \simeq \begin{pmatrix} p & u & u \\ v & q & v \\ w & w & r \end{pmatrix}$$

1. *$\mathcal{T}_0$ and $\mathcal{T}_1$ are perspective wrt to P, $\mathcal{T}_0$ and $\mathcal{T}_2$ are perspective wrt U, while $\mathcal{T}_2, \mathcal{T}_3$ admit $P+U$ as perspector*

2. *Each triangle $\mathcal{T}_1$, $\mathcal{T}_2\, \mathcal{T}_3$ is the cross-triangle of the other two.*

3. *All pair of triangles have the same Desargues triangle, defining the same perspectrix:*

$$\Delta \simeq \left[\frac{1}{p-u} \; ; \; \frac{1}{q-v} \; ; \; \frac{1}{r-w}\right]$$

4. *Reciprocally, any triangle in perspective with ABC can be parametrized that way.*

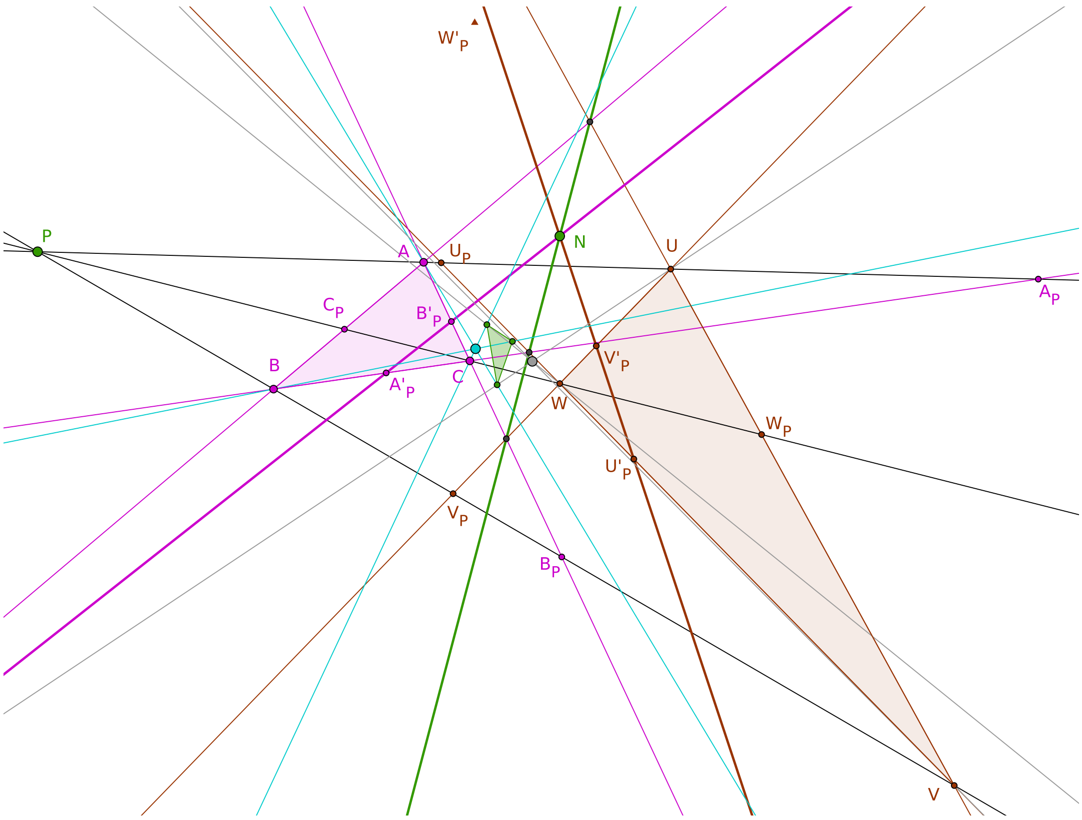


Figure 3.4: Triangles ABC and UVW that admit P as perspector.

Proposition 3.8.12. *When triangle* $\mathcal{T}_0 = ABC$ *and triangle* $\mathcal{T}$ *admit* $P \simeq p : q : r$ *and* tripolar $\Delta \simeq [qr, rp, pq]$ *as perspector and perspectrix, then it exists* $k \in \mathbb{R} \cup \{\infty\}$ *such that*

$$\mathcal{T}_k \simeq \begin{pmatrix} kp & p & p \\ q & kq & q \\ r & r & kr \end{pmatrix} \tag{3.8}$$

Proof. Start from preceding proposition and write that Δ is tripolar P rather than another line. □

Proposition 3.8.13. *Suppose additionally that line PU doesn't go through any of the vertices* A, B, C*, define*

$$Q \simeq \begin{pmatrix} (qw - rv)\,(p - u)\,p \\ (ru - pw)\,(q - v)\,q \\ (pv - qu)\,(r - w)\,r \end{pmatrix} ;\ V \simeq \begin{pmatrix} (qw - rv)\,(p - u)\,u \\ (ru - pw)\,(q - v)\,v \\ (pv - qu)\,(r - w)\,w \end{pmatrix}$$

and consider the triangles $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2$*, the cevian triangles of* P*:* $\mathcal{T}_{0P}, \mathcal{T}_{1P}, \mathcal{T}_{2P}$ *and the cevian triangles of* U*:* $\mathcal{T}_{0U}, \mathcal{T}_{1U}, \mathcal{T}_{2U}$*.*
Then: (1) P is the perspector of each pair : $(\mathcal{T}_0, \mathcal{T}_1)$, $(\mathcal{T}_{0P}, \mathcal{T}_{1P})$, $(\mathcal{T}_0, \mathcal{T}_{0P})$, $(\mathcal{T}_0, \mathcal{T}_{1P})$, $(\mathcal{T}_1, \mathcal{T}_{0P})$, $(\mathcal{T}_1, \mathcal{T}_{1P})$*;*
(2) Perspector of $(\mathcal{T}_1, \mathcal{T}_2)$ *is* $p + u : q + v : r + w$*.*
(3) $Q \neq V$ *and line* $\Delta = QV$ *is the perspectrix of each pair :* $(\mathcal{T}_0, \mathcal{T}_1)$, $(\mathcal{T}_{0P}, \mathcal{T}_{1P})$, $(\mathcal{T}_1, \mathcal{T}_2)$, $(\mathcal{T}_0, \mathcal{T}_2)$, $(\mathcal{T}_{0U}, \mathcal{T}_{2U})$ *;*
(4) Moreover, the perspectrices of $(\mathcal{T}_0, \mathcal{T}_{0P})$, $(\mathcal{T}_0, \mathcal{T}_{1P})$, $(\mathcal{T}_1, \mathcal{T}_{0P})$, $(\mathcal{T}_1, \mathcal{T}_{1P})$ *are going through Q (and symmetrically for U).*

Proof. The perspectrix of $(\mathcal{T}_0, \mathcal{T}_{0P})$ is $\Delta_P = \begin{bmatrix} qr & pr & pq \end{bmatrix}$, the tripolar of P and we have :

$$\begin{array}{lll} \mathcal{T}_0 & \mathcal{T}_{1P} & pqr\,\Delta_1 - \det \mathcal{T}_2\,\Delta_P \\ \mathcal{T}_1 & \mathcal{T}_{0P} & 2\,pqr\,\Delta_1 - \det \mathcal{T}_2\,\Delta_P \\ \mathcal{T}_1 & \mathcal{T}_{1P} & 3\,pqr\,\Delta_1 - \det \mathcal{T}_2\,\Delta_P \end{array}$$

□

Remark 3.8.14. Behavior wrt isogonal transform is examined in Proposition 18.6.2.

3.9 Cevian nests

Definition 3.9.1. Cevian nest. Suppose $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ are triangles and that $\mathcal{T}_1$ is inscribed in $\mathcal{T}_2$ and $\mathcal{T}_2$ is inscribed in $\mathcal{T}_3$. If any two of the tree triangles are perspective, it is well-known that each is perspective to the third : $\mathcal{T}_1$ is a cevian triangle of $\mathcal{T}_2$ for some point P, $\mathcal{T}_3$ is an anticevian triangle of $\mathcal{T}_2$ for some point U and $\mathcal{T}_1$, $\mathcal{T}_3$ are perspective wrt some point X. Such configuration is called a cevian nest.

Remark 3.9.2. Dyslexic readers – and the author among them – are advised to organize their memories around the P/U cevadivision, which gives the perspector between the cevian of P and the anticevian of U.

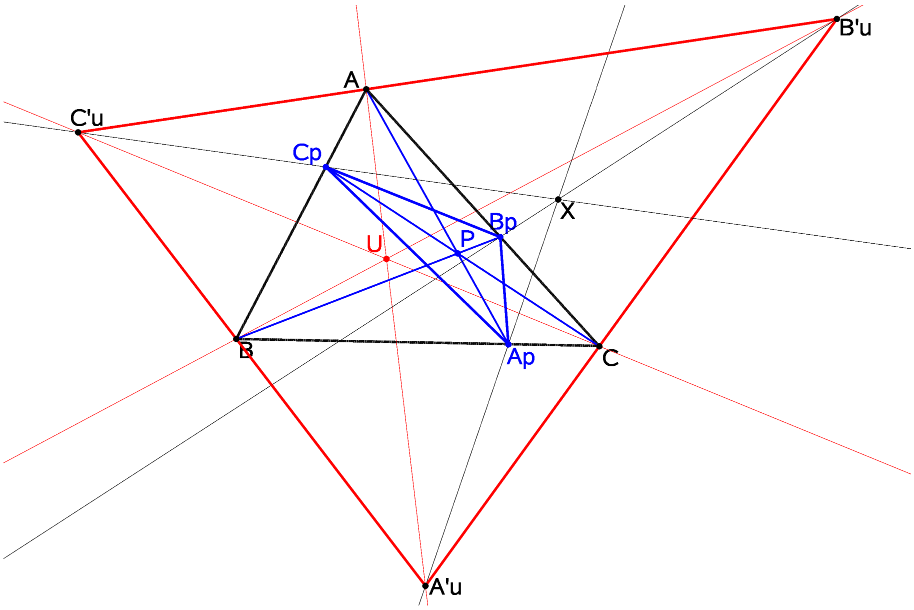


Figure 3.5: X is cevadiv (P,U) while P is cevamul(U,X)

Proposition 3.9.3. *Map $P = P\,(U, X)$ giving the perspector of $\mathcal{T}_1$, $\mathcal{T}_2$ from the other two perspectors of a cevian nest is symmetric, while –for a given P– map $X \longleftrightarrow U$ is involutory.*

Proof. Given the vertices S_i $(i = 1, 2, 3)$ of triangle $\mathcal{T}_3$ and perspector U, the vertices S_i $(i = 4, 5, 6)$ of triangle $\mathcal{T}_2$ are obtained by $S_4 = (S_1 \wedge U) \wedge (S_2 \wedge S_3)$ and cyclically. Process can be iterated, obtaining vertices S_i $(i = 7, 8, 9)$ of $\mathcal{T}_1$. Then X is obtained as $X = (S_1 \wedge S_7) \wedge (S_2 \wedge S_8)$. It can be checked that substituting U by X (and keeping everything else unchanged) leads back to U, proving the second part. The first part follows immediately. □

When triangle ABC belongs to such a nest, three possibilities can occur. The corresponding operations are summarized in Table 3.2, where "mul" stands for multiplication (giving P) and "div" for the converse operation (giving the missing one from U, X). The Kimberling's name is also given.

3.10 The cross case (aka case I, cev of cev)

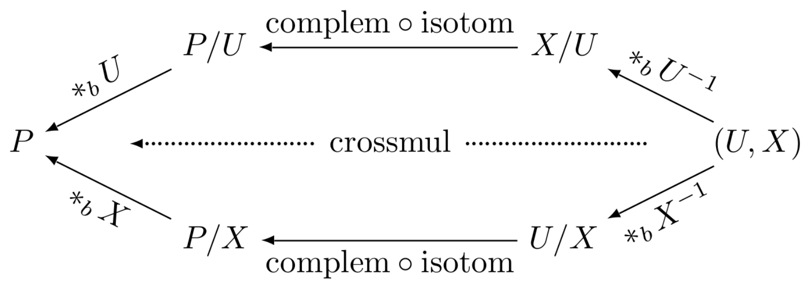
Definition 3.10.1. Crossmul(U,X). As in Table 3.2 (I), let $\mathcal{T}_3$ (the biggest triangle) be ABC, $\mathcal{T}_2 = A_U B_U C_U$ the (usual) cevian triangle of U and $\mathcal{T}_1 = A'B'C'$ the triangle inscribed in $\mathcal{T}_2$ obtained by $A' = AX \cap B_U C_U$ and cyclically. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ have a perspector (P), and mapping $(U, X) \mapsto P$ is called cross-multiplication.

small $\mathcal{T}_1$	medium $\mathcal{T}_2$	large $\mathcal{T}_3$	X, P	Kimberling
$\mathcal{C}_P$ wrt $\mathcal{T}_2$	$\mathcal{C}_U$	ABC	$X = \operatorname{crossdiv}(P, U)$	cross − conj
			$P = \operatorname{crossmul}(U, X)$	cross − point
$\mathcal{C}_P$	ABC	$\mathcal{A}_U$	$X = \operatorname{cevadiv}(P, U)$	ceva − conj
			$P = \operatorname{cevamul}(U, X)$	ceva − point
ABC	$\mathcal{A}_P$	$\mathcal{A}_U$ wrt $\mathcal{T}_2$	$X = \operatorname{sqrtdiv}(P, U)$	
			$P = \operatorname{sqrtmul}(U, X)$	

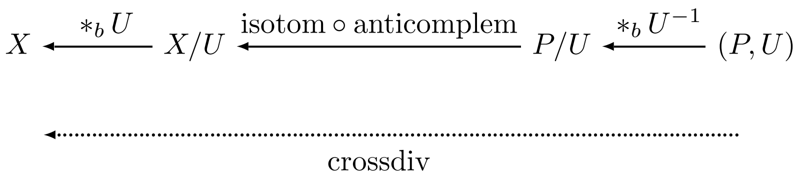
All these operations are (globally) type-keeping, since they transform points into constructible points.

Table 3.2: Three cases of cevian nets

Definition 3.10.2. Crossdiv(P,U). As in Table 3.2 (I), let $\mathcal{T}_3$ (the biggest triangle) be ABC, $\mathcal{T}_2 = A_U B_U C_U$ the (usual) cevian triangle of U and $\mathcal{T}_1 = A'B'C'$ the cevian triangle of P wrt $\mathcal{T}_2$, obtained by $A' = A_U P \cap B_U C_U$ and cyclically. Then $\mathcal{T}_1$ and $\mathcal{T}_3$ have a perspector (X), and mapping $(P, U) \mapsto X$ is called cross-division.



$$\operatorname{crossmul}(u : v : w,\ x : y : z) = (v\,z + w\,y)\,u\,x : (u\,z + w\,x)\,v\,y : (u\,y + v\,x)\,w\,z \tag{3.9}$$



$$\operatorname{crossdiv}(p : q : r,\ u : v : w) = \frac{u}{quw + ruv - pvw} : \frac{v}{pvw + ruv - quw} : \frac{w}{pvw + quw - ruv} \tag{3.10}$$

Figure 3.6: crossmul, crossdiv

Proposition 3.10.3. *Computing rules of crossmul and crossdiv are given (using barycentrics) in Figure 3.6. Map $(U, X) \mapsto P$ is commutative and behaves like ordinary multiplication (eponymous property). Map $(P, U) \mapsto X$ behaves wrt crossmul like division behaves wrt ordinary multiplication (eponymous property).*

Proof. Barycentrics $p : q : r$ are defining point P' with respect to triangle $\mathcal{T}_3$. Call $P : Q : R$ its barycentrics with respect to triangle $\mathcal{T}_2$, so that $(p : q : r) = \boxed{\mathcal{T}_2}(P : Q : R)$. Then :

$$\mathcal{T}_1 \simeq \begin{pmatrix} 0 & u & u \\ v & 0 & v \\ w & w & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & P & P \\ Q & 0 & Q \\ R & R & 0 \end{pmatrix} \cdot \begin{pmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{pmatrix} \simeq \begin{pmatrix} \dfrac{u\,(Q+R)}{QR} & \dfrac{u}{P} & \dfrac{u}{P} \\ \dfrac{v}{Q} & \dfrac{v\,(P+R)}{PR} & \dfrac{v}{Q} \\ \dfrac{w}{R} & \dfrac{w}{R} & \dfrac{w\,(P+Q)}{PQ} \end{pmatrix}$$

where the diagonal matrix had been chosen to "synchronize" the columns of triangle $\mathcal{T}_2$. Then Proposition 3.8.11 shows that $\mathcal{T}_3$ and $\mathcal{T}_1$ are perspective wrt point $u/P : v/Q : w/R$. □

Remark 3.10.4. The cross-multiplication and cross-division were introduced in Kimberling (1998), using the names **crosspoint** and "**cross conjugacy**", together with the notation $X = C(P,U)$ and also $X = PcU$. In our opinion, a name that emphasizes the properties is more efficient.

The factorization given can be interpreted in terms of isoconjugacies (see Chapter 18). First compute what happens when U =X(2) and obtain complem ∘ isotom. Then transmute this map by the transform $\underset{b}{*}U$ that fixes A, B, C and sends G =X(2) onto U.

Proposition 3.10.5. *The crossmul P of U, X is also the point of concurrence of (1) the line through points $AX \cap BU$ and $AU \cap BX$, (2) the line through points $BX \cap CU$ and $BU \cap CX$, (3) the line through points $CX \cap AU$ and $CU \cap AX$.*

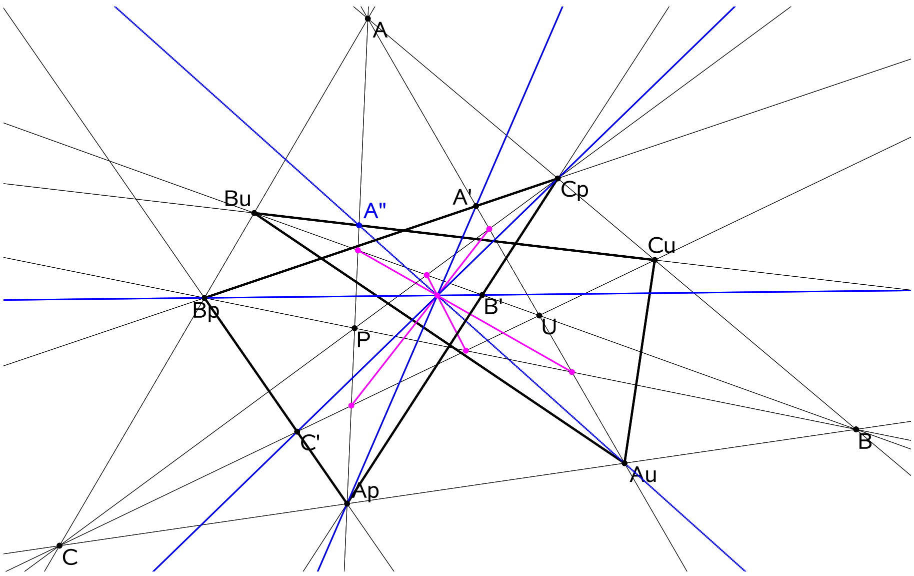


Figure 3.7: P is crossmul of U and X

3.11 The ceva case (aka case II, cev and acev)

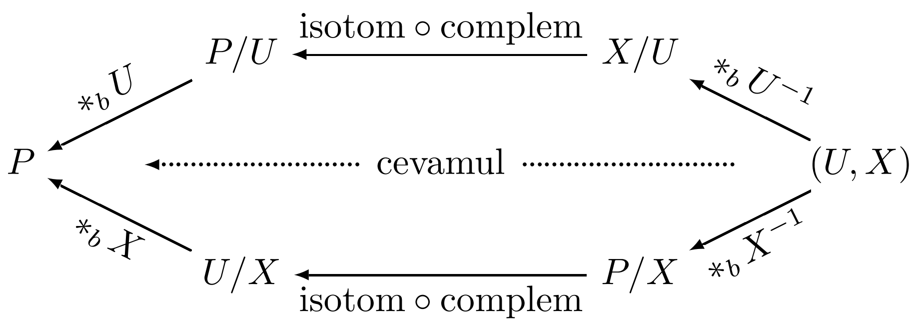
Definition 3.11.1. Cevamul(U,X). As in Table 3.2 (II), let $\mathcal{T}_2$ (the middle triangle) be ABC, $\mathcal{T}_3 = U_A U_B U_C$ the anticevian triangle of U and $\mathcal{T}_1 = A'B'C'$ be the triangle inscribed in $\mathcal{T}_2 = ABC$ obtained by $A' = U_A X \cap BC$ and cyclically. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ have a perspector (i.e. $\mathcal{T}_1$ is the cevian triangle of some point P), and mapping $(U, X) \mapsto P$ is called ceva-multiplication.

Definition 3.11.2. Cevadiv(P,U). As in Table 3.2 (II), let $\mathcal{T}_2$ (the middle triangle) be ABC, $\mathcal{T}_3 = U_A U_B U_C$ the anticevian triangle of U and $\mathcal{T}_1 = A_P B_P C_P$ be the cevian triangle of P, obtained (as usual) by $A_P = AP \cap BC$ and cyclically. Then $\mathcal{T}_1$ and $\mathcal{T}_3$ have a perspector (X), and mapping $(P, U) \mapsto X$ is called ceva-division.

Proposition 3.11.3. *Computing rules of cevamul and cevadiv are given (using barycentrics) in Figure 3.8. Map $(U, X) \mapsto P$ is commutative i.e. :*

$$\begin{cases} X \text{ is the perspector of } \mathcal{C}_P \text{ and } \mathcal{A}_U \\ U \text{ is the perspector of } \mathcal{C}_P \text{ and } \mathcal{A}_X \end{cases}$$

and behaves like ordinary multiplication (eponymous property). Map $(P, U) \mapsto X$ behaves wrt crossmul like division behaves wrt ordinary multiplication (eponymous property).



$$\text{cevamul}\,(u:v:w,\ x:y:z) = \\ (uz+wx)(uy+vx) : (vz+wy)\,(uy+vx) : (vz+wy)\,(uz+wx) \quad (3.11)$$

$$\text{cevadiv}\,(p:q:r,\ u:v:w) = \\ u(-q\,r\,u + r\,p\,v + p\,q\,w) : v(q\,r\,u - r\,p\,v + p\,q\,w) : w(q\,r\,u + r\,p\,v - p\,q\,w) \quad (3.12)$$

Figure 3.8: cevamul, cevadiv

Proposition 3.11.4. *(Spoiler) Seen as an $V \mapsto X$ map, $U \mapsto$ cevamul (U, V) is a Cremona transform, whose points of indeterminacy are the anticevian vertices of U and the exceptional locus is the union of the U-anticevian sidelines. Seen as an $U \mapsto X$ map, $U \mapsto$ cevadiv (P, U) is a Cremona transform, whose points of indeterminacy are the cevian vertices of P and the exceptional locus is the union of the P-cevian sidelines. Seen as an $P \mapsto X$ map, $P \mapsto$ cevadiv (P, U) is a Cremona transform, whose points of indeterminacy are the ABC vertices and the exceptional locus is the union of the ABC sidelines.*

Proof. Direct examination. □

Remark 3.11.5. The ceva-multiplication and ceva-division were introduced in Kimberling (1998), using the names **cevapoint** and "**ceva conjugacy**", together with the notation $X = P \copyright U$. In our opinion, a name that emphasizes the properties is more efficient.

Proposition 3.11.6. *Suppose $U = u : v : w$ and $X = x : y : z$ are distinct points, neither lying on a sideline of ABC. Let $\mathcal{A}_x = X_A X_B X_C$ and $\mathcal{A}_u = U_A U_B U_C$ be the anticevian triangles of X and U (wrt ABC). Define A' as $U_A X \cap X_A U$ and B', C' cyclically (Figure 3.5). Then triangle $A'B'C'$ is inscribed in ABC and is, in fact, the cevian triangle of $P =$ cevamul (U, V).*

Proof. Direct computation. □

Construction 3.11.7 (Floor Van Lamoen (2003/10/17))**.** *Point* cevamul(U, X) *can be constructed from the cevian triangles : let $A_U B_U C_U$ be the cevian triangle of U, and $A_X B_X C_X$ the cevian triangle of X. Define :*

$A_{ux} = A_u C_x \cap A_x B_u$	$A_{xu} = A_u B_x \cap A_x C_u$
$B_{ux} = B_u A_x \cap B_x C_u$	$B_{xu} = B_u C_x \cap B_x A_u$
$C_{ux} = C_u B_x \cap C_x A_u$	$C_{xu} = C_u A_x \cap C_x B_u$

Proposition 3.11.8. *Then, as seen in Figure 3.9, triangle ABC is perspective to both triangles A_{ux}, B_{ux}, C_{ux} and A_{xu}, B_{xu}, C_{xu}, and the perspector in both cases is the* cevamul (U, X).

Proof. Direct computation. □

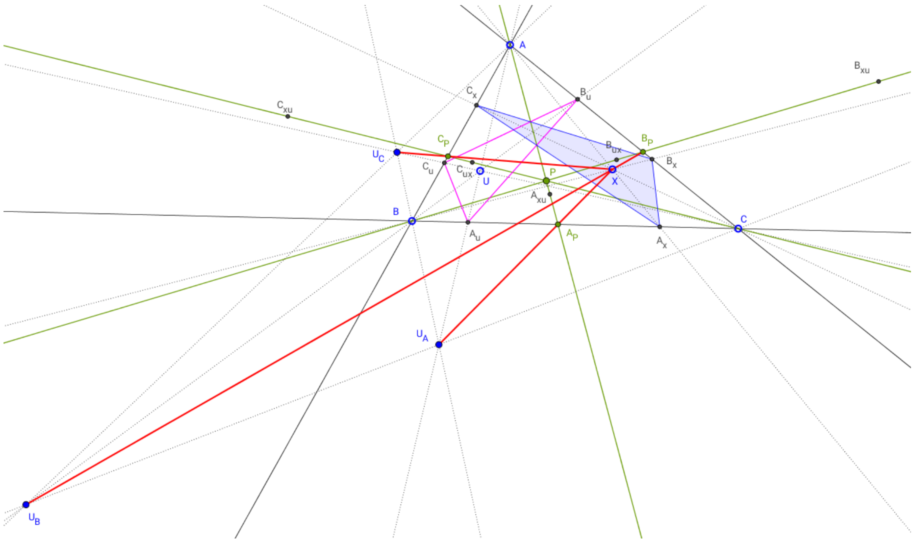


Figure 3.9: Lamoen's construction of P=cevamul(U,X)

3.12 The square case (aka case III, acev of acev)

Construction 3.12.1. ***sqrtdiv(F,U)***. *As in Table 3.2 (III), let $\mathcal{T}_1$ (the smallest triangle) be ABC, $\mathcal{T}_2 = F_A F_B F_C$ the anticevian triangle of F and $\mathcal{T}_3$ be the anticevian triangle of U wrt $\mathcal{T}_2$. Then $\mathcal{T}_1$ and $\mathcal{T}_3$ have a perspector X and mapping $(F,U) \mapsto X$ is called sqrtdiv. We have formula :*

$$sqrtdiv_F\,(U) \doteq U_F^{\#} \simeq \frac{f^2}{u} : \frac{g^2}{v} : \frac{h^2}{w}$$

Remark 3.12.2. This construction is not so easy as it seems to be. In fact, drawing an anticevian triangle requires a knowledge of the barycentrics (and this is no more a construction!), or some conics, or a forest of lines that are equivalent to the drawing of the conics themselves. One can also start from the cevian triangle and use some fourth harmonics, that are equivalent to inversions into some circles.

Definition 3.12.3. sqrtmul(U,X). The inverse mapping of *sqrtdiv* should be the mapping *sqrtmul* $(U,X) \mapsto P$ "defined" by :

$$p : q : r = \pm\sqrt{ux} : \pm\sqrt{vy} : \pm\sqrt{wz}$$

... but (1) the solution is not unique and (2) in fact, the problem cannot even be stated clearly.

3.13 Danneels perspectors

Definition 3.13.1. First Danneels perspector. Let $\mathcal{T}_1 = A_U B_U C_U$ be the cevian triangle of a point $U = u : v : w$. Let L_A be the line through A parallel to $B_U C_U$, and define L_B and L_C cyclically. The lines L_A, L_B, L_C determine a triangle $\mathcal{T}_2$ perspective to $\mathcal{T}_1$ (and in fact homothetic, with factor 2). The corresponding perspector is $\mathrm{DP}_1\,(U)$, the first Danneels perspector (#11037) of U. Using barycentrics :

$$\mathrm{DP}_1\,(U) = u^2(v+w) : v^2(w+u) : w^2(u+v) \tag{3.13}$$

Proof. Compute (from left to right) the row $B_U \wedge C_U \wedge \mathcal{L}_b \wedge A$ and cyclically. Obtain a matrix describing a trigone and takes the adjoint to obtain $\mathcal{T}_2$. Then compute the perspector. One can also remark that $\mathcal{T}_2$ is the anticevian triangle of :

$$X = u(v+w) : v(u+w) : w(u+v)$$

and obtain $\mathrm{DP}_1\,(U)$ as *cevadiv* (U/X) (homothetic property is obvious... and useless to compute the perspector). □

Remark 3.13.2. This point is named $D(U)$ in ETC. No name of only one letter! Moreover this conflicts with the Maple's derivation operator.

Proposition 3.13.3. *Point $G = X(2)$ is invariant under DP_1. Moreover, G, U and $\mathrm{DP}_1(U)$ are ever collinear. For example, Euler line is globally invariant.*

Proof. Check that $\det(X_2, U, \mathrm{DP}_1(X_2 + \lambda U)) = 0$. □

Proposition 3.13.4. *When $\mathrm{DP}_1(X) = G$ then either $X = G$ or X lies on the Steiner circumellipse.*

Proof. Write $\mathrm{DP}_1(X) = k\,G$ and solve. Except from $X = G$, $xy + yz + zx = 0$ is obtained. □

Proposition 3.13.5. *When $\mathrm{DP}_1(U) \neq G$, i.e. when $U \neq G$ and U not on the circumSteiner, it exists two other points that verify $\mathrm{DP}_1(X) = \mathrm{DP}_1(U)$. Using barycentrics, we have :*

$$\begin{aligned} X &\simeq \begin{pmatrix} v + w - 2\,u - W\,\frac{1}{(u+w)(u+v)} \\ w + u - 2\,v - W\,\frac{1}{(u+v)(v+w)} \\ u + v - 2\,w - W\,\frac{1}{(u+w)(v+w)} \end{pmatrix} \quad \text{where} \\ W^2 &= (u+v)(v+w)(w+u)\left(u^2(v+w) + v^2(w+u) + w^2(u+v) - 6\,uvw\right) \end{aligned}$$

Proof. Direct computation, assuming $xy + yz + zx \neq 0$. The main difficulty is to re-obtain a symmetric expression after elimination of k, z and resolution on y. □

Example 3.13.6. Here is a list of pairs (I, J) of named points such that $\mathrm{DP}_1(X(I)) = X(J)$:

1	42	20	3079	189	1422	1370	455
3	418	25	3080	264	324		
4	25	30	3081	366	367		
5	3078	69	394	651	222		
6	3051	75	321	653	196		
7	57	100	55	1113	25		
8	200	110	184	1114	25		

2, *on Steiner* : $190, 290, 648, 664, 666, 668, 670, 671, 886, 889, 892,$
$903, 991, 1121, 1494, 2479, 2480, 2481, 2966, 3225, 3226, 3227, 3228$

Definition 3.13.7. Second Danneels perspector. Suppose $\mathcal{T}_1 = A_U B_U C_U$ is the cevian triangle of a point U. Let L_{AB} be the line through B parallel to $A_U B_U$, and let L_{AC} be the line through C parallel to $A_U C_U$. Define $A' = L_{AB} \cap L_{AC}$ and B', C' cyclically. Finally obtain $A" = BB' \cap CC'$ and $B"$, $C"$ cyclically. It happens that triangle $\mathcal{T}_2 = A"B"C"$ is perspective to $\mathcal{T}_1 = A_U B_U C_U$. The corresponding perspector is $\mathrm{DP}_2(U)$, the second Danneels perspector of U (Danneels, 2006). Using barycentrics :

$$\mathrm{DP}_2(U) = u(v-w)^2 : v(w-u)^2 : w(u-v)^2 \tag{3.14}$$

Proof. Compute (left to right) $L_{AB} = A_U \wedge B_U \wedge \mathcal{L}_b \wedge B$, L_{AC} accordingly, then $A' = L_{AB} \wedge L_{AC}$ and B', C' cyclically. Obtain $A" = (B \wedge B') \wedge (C \wedge C')$ and $B"$, $C"$cyclically. See that $\mathcal{T}_2 = A"B"C"$ is the anticevian triangle of point :

$$X \doteq u(v-w) : v(w-u) : w(u-v)$$

and obtain $\mathrm{DP}_2(U)$ as *cevadiv* (U, X). □

Proposition 3.13.8. *The circumconic through $A, B, C, U, isot(U)$ admits $\mathrm{DP}_2(U)$ as center and*

$$u\left(v^2 - w^2\right) : v\left(w^2 - u^2\right) : w\left(u^2 - v^2\right)$$

as perspector. And therefore isotomic conjugates have the same second Danneels' perspector.

Proof. Immediate computation. □

Example 3.13.9. List of $(U, U^*, \mathrm{DP}_2(U))$:

U	U^*	DP	U	U^*	DP	U	U^*	DP
1	75	244	37	274	3121	394	2052	3269
3	264	2972	42	310	3122	519	903	1647
4	69	125	57	312	2170	524	671	1648
6	76	3124	81	321	3125	536	3227	1646
7	8	11	94	323	2088	538	3228	1645
9	85	3119	98	325	868	2394	2407	1637
10	86	3120	99	523	1649	2395	2396	2491
20	253	122	100	693	3126	2397	2401	3310
30	1494	1650	200	1088	2310	2398	2400	676

and of points without named isotomic :

$$[43, 3123], [88, 2087], [694, 2086], [1022, 1635], [1026, 2254], [2421, 3569]$$

Chapter 4

The French Touch

This chapter is translated from an article originally published under the title "Géométrie projective pour agrégatifs" (Douillet, 2012). One cannot totally exclude that some provocative tone was used here or there.

Our friend keeps repeating that the current program of the recruitment competition for French Second Degree teachers (agrégation de Mathématiques) does not contain the elements necessary to understand the geometries of Moebius, Kimberling or Morley, that is to say the different versions of the plane projective geometry. I rather disagree with this point of view. Here are some elements to support my own opinion.

4.1 The "not so flying plane"[1]

1. Definition. We call rantanplan the "quad ruled paper sheet" which is placed in front of a schoolchild. We make a (red) cross somewhere (at the intersection of two gridlines), and we write "Here you are". When this takes up too much space, we write "O". And we say "this is the origin".

2. Definition. You obtain the point "tchik, tchik, tchik, kling, tchouk, tchouk, bang, A" by placing yourself in O, with the margin behind you (the gaze is then directed along the horizontal lines of the papersheet). And you move a square forward (tchik), and another square forward (the second tchik), and again another square forward (the third tchik). Then you do a quarter turn (kling) and now, the gaze is directed along the vertical lines of the papersheet. And you move one square forward (tchouk), then another square forward (the second tchouk), you put down the pencil and make a cross (bang). And you write the name of the point, A.

3. Axiom (Archimedes + Thales + Cantor). The preceding definitional schema gives access to all the points of the rantanplan.

4. Definition, Borel notation. When there are "a lot" of tchiks and tchouks, we count them and we note $A = 3 + 2i$. The "i" is used to denote the kling.

5. Foundational experiment. The candidate is at the blackboard and stands in front of the door. Then she makes a quarter turn, in the direction of the Jury. Then she makes another quarter turn, in the direction of the window. As a result, the candidate has made a half turn and all (horizontal) directions have been reversed. This is noted $i^2 = -1$.

6. Scholie. Suppose the previous scene is observed by two Vice-Presidents of the Jury, one placed on the floor above, the other placed on the floor below, and further assume that the President of the Jury has a sufficient authority to impose on his assessors the use of watches whose hands turn in the same "clockwise" direction. Then one of them will see $A = 3 + 2i$, and the other will see $A = 3 - 2i$.

7. Hands out. The above experiment can be reproduced without such a grandiose staging. It is enough that the pupil looks at his rantanplan not from above, but from below, by transparency. So "everything starts turning the other way".

[1] Tentative translation of the French joke: "le rantanplan"

8. Definition. Normalized coordinates. In order obtain an intrinsic object, we define the Normalized Coordinates of the point A by:

$$\zeta_{norm}\left(A\right)=\left(\begin{array}{c}3+2i\\ 1\\ 3-2i\end{array}\right)$$

We note above what is seen by the Vice president who is on the floor above and we note below what is seen by the Vice president who is on the floor below. And one can even imagine that the "1" between the two serves to unify the points of view of the two Vice-Presidents.

9. Scholie. When the two Vice-Presidents enter into a full open war (to become President of the next year Jury), the "unifying" point of view cannot be maintained, and it is advisable to replace this "1=unity" by a multiple, leading to the following definition.

10. Definition. Superior Coordinates. Coordinates proportional to the Normalized Coordinates are called Superior Coordinates[2]. This will be noted using the sign $\simeq$ (simeq) and we have, for example, the relation

$$\zeta_A\simeq\left(\begin{array}{c}6+4i\\ 2\\ 6-4i\end{array}\right)$$

In the wide outer world, where jokes about the ENS are likely to fail miserably, ζ_A is called the Morley coordinates of A.

11. Notation. The Morley coordinates of the current point of the plane are written $\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}$ (big zed, big tea, big zeta). Each candidate to the aggregation knows (or at least should know) that an algebraic variable is nothing else than a mark-a-place in the writing of polynomials which are finite series of multi-indexed coefficients. What could be the complex conjugate of a mark-a-place?

4.2 Algorithmic in the rantanplan

1. Theorem. The fraction field $\mathbb{K}$ of an integral ring $\mathbb{A}$ is constructed by identifying among themselves all the couples of $\mathbb{A}\times\mathbb{A}\setminus(0,0)$ which share the same alignment with the origin $(0,0)$. Whatever it has been said, this projectification $\mathbb{K}=\mathbb{P}_{\mathbb{A}}\left(\mathbb{A}^2\right)\setminus\{\infty\}$ is and remains on the Agrégation program, and when you have that, you have quite everything else.

2. Softer version. In order to not frighten the school children, we can also say that when two points of the rantanplan are aligned with the origin, the coordinates are proportional, and the cross difference is zero, i.e. $x_Ay_B-x_By_A=0$.

3. Thales theorem. When three points are three in number and aligned, the abscissa variations and ordinate variations are proportional. This can be written as:

$$\left(x_C-x_A\right)\left(y_B-y_A\right)-\left(x_B-x_A\right)\left(y_C-y_A\right)=0$$

4. Definition (slope). The equation of a line is the condition for a third point (x,y) to be aligned with two given points (two is two: the given points must be different). Once reorganized, the above expression can be written $y=px+m$, where p is the slope, i.e. the proportionality ratio between the Δy and the Δx.

5. Scholie. Beyond its limitations, the formula $y=px+m$ has the immense merit of characterizing the direction of a straight line by its slope, and even of characterizing straight lines as being the curves that keep going in a straight line, i.e. curves with a constant slope.

6. Algorithmic version of the Thales theorem. The stratospheric point of view consists in summarizing the Thales theorem by "we develop, we reorganize and we obtain $ax+by+c=0$".

[2] Alluding to the École Normale Supérieure

The algorithmic point of view consists in being interested in computations, to the point of trying to facilitate them. We have

$$a = -y_B + y_A,\ b = x_B - x_A,\ c = x_A y_B - y_A x_B$$

In other words, the number c is the cross difference of the x and the y. The result is known: the other two are also cross differences when using the "ever one" quantity. In other words, the equation of the line passing through two points is computed using:

$$A \wedge B = \begin{pmatrix} x_A \\ y_A \\ 1 \end{pmatrix} \wedge \begin{pmatrix} x_B \\ y_B \\ 1 \end{pmatrix} = [y_A \times 1 - y_B \times 1,\ x_B \times 1 - x_A \times 1,\ x_A y_B - y_A x_B]$$

7. Duality. The cross-differences are put in a row because they characterize a straight line. It's not just a joke "line / row". We deliberately note the lines differently from the points ... quite simply because the lines are not points and the points are not lines[3]. If wanted, we could note the points in row, and the lines in column[4]. We could even exchange all the points with all the lines. We just have to find a way not to mix the two kinds of things.

8. From an advanced point of view, the equation of a line is written using a determinant and the wedge operator is the universal factorization of this multilinear operator:

$$\begin{vmatrix} x_A & x_B & x \\ y_A & y_B & y \\ 1 & 1 & 1 \end{vmatrix} = \left(\begin{pmatrix} x_A \\ y_A \\ 1 \end{pmatrix} \wedge \begin{pmatrix} x_B \\ y_B \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

But, despite the possibility of such a stratospheric description, the fact remains that the cross difference method is a practical and fast way to calculate lines and intersections. This method can be taught and used well before learning any theory on large determinants. It is the same for the algorithm of the ordinary decimal division: it can be taught and used well before learning any theory on the limited expansions of a quotient of uniformly converging series.

9. Let us insist heavily on this fundamental point. The formula for subtracting fractions: $\dfrac{a}{b} - \dfrac{c}{d} = \dfrac{a\,d - b\,c}{b\,d}$ is only saying that (a, b) and $(a\,d, b\,d)$ are aligned with the origin. This is projective geometry. More precisely, projective geometry is nothing more than that: drawing the fractions that we want to subtract, then expand them to the same denominator. There is no reason to make a molehill out of such an elementary thing.

10. Example. We want to calculate $E = AB \cap CD$ with $A = 3 + 4i, B = 2 - 5i, C = -1 + i, D = 1 - i$. We have successively:

$$\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} = \begin{bmatrix} 9 & -1 & -23 \end{bmatrix}$$

$$\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 9 & -1 & -23 \end{bmatrix} \wedge \begin{bmatrix} 3 & 2 & -1 \end{bmatrix} = \begin{pmatrix} 47 \\ -60 \\ 21 \end{pmatrix} \simeq \begin{pmatrix} 47/21 \\ -20/7 \\ 1 \end{pmatrix}$$

[3]As the French joke says, les points ne sont point des droites, et les droites ne sont droite des points.
[4]Cross over the Chanel, they behave like that.

11. Example (continued). Using Morley's affixes leads to:

$$\begin{pmatrix} 3+4\,i \\ 1 \\ 3-4\,i \end{pmatrix} \wedge \begin{pmatrix} 2-5\,i \\ 1 \\ 2+5\,i \end{pmatrix} = \left[\begin{array}{ccc} -1+9\,i & -46\,i & 1+9\,i \end{array}\right]$$

$$\begin{pmatrix} -1+2\,i \\ 1 \\ -1-2\,i \end{pmatrix} \wedge \begin{pmatrix} 1-i \\ 1 \\ 1+i \end{pmatrix} = \left[\begin{array}{ccc} 2+3\,i & -2\,i & -2+3\,i \end{array}\right]$$

$$\left(\begin{array}{c} \left[\begin{array}{ccc} -1+9\,i & -46\,i & 1+9\,i \end{array}\right] \wedge \\ \left[\begin{array}{ccc} 2+3\,i & -2\,i & -2+3\,i \end{array}\right] \end{array}\right) = \begin{pmatrix} +120+94\,i \\ 42\,i \\ -120+94\,i \end{pmatrix} \simeq \begin{pmatrix} \dfrac{47}{21} - \dfrac{20}{7}\,i \\ 1 \\ \dfrac{47}{21} + \dfrac{20}{7}\,i \end{pmatrix}$$

Since complex conjugacy is an automorphism, the third component ends up being the conjugate of the first, since this was the case when starting. For now, the $\overline{\mathcal{Z}}$ component is merely a concession made to the fans of intrinsic constructions. Let's show how to use it for a good reason.

4.3 Thales antiquadratic form

1. Fundamental formula for affine spaces. To arrive at B, we start from some A, then we follow the path that goes from A up to B . We have:

$$B = A + (B - A)$$

2. Definition. The vector $\overrightarrow{AB}$ is what we get by subtracting the normalized coordinates of A from the normalized coordinates of B.

3. Scholie. Subtracting the upper coordinates would not give a well-defined object, since the upper coordinates of A can be multiplied by a factor different from that of the upper coordinates of B.

4. Proposition. When we consider the vectors $\overrightarrow{AB}$ for what they are, i.e. exactly defined objects (and not defined up to a factor), their set $\overrightarrow{\mathcal{V}}$ forms a vector space of dimension 2.

5. Definition. If we assume that the vector $\overrightarrow{AB}$ is non-zero, but if we disregard its size, only its direction remains, and we get a new kind of points, i.e. the points with $\mathbf{T} = 0$. The set of the points which verify this equation is the line $[0; 1; 0]$. We write it $\mathcal{L}_z$.

6. Proposition. The point $k\overrightarrow{AB}$ belongs to line AB. In fact, this point is nothing else than $AB \wedge \mathcal{L}_z$. Proof: this result is obvious from the coordinates. The basic reason is that the subtraction of fractions also uses cross differences: don't we have $\mathbb{K} = \mathbb{P}_{\mathbb{A}}\left(\mathbb{A}^2\right) \setminus \{\infty\}$?

7. Operator W. This is the operator which takes a straight line as input and gives its point at infinity as output. Using Morley's affixes, we have:[5]

$$\Delta \wedge \mathcal{L}_z \simeq \boxed{W_z} \cdot {}^t\Delta \qquad \text{where } \boxed{W_z} = \begin{pmatrix} 0 & 0 & -\boldsymbol{i} \\ 0 & 0 & 0 \\ +\boldsymbol{i} & 0 & 0 \end{pmatrix}$$

8. Theorem (Thales). Two lines are parallel when the point at infinity of one of them belongs to the other. This is expressed by:

$$\Delta_1 \parallel \Delta_2 \Longleftrightarrow \Delta_1 \cdot \boxed{W_z} \cdot \Delta_2 = 0$$

[5]When only dealing with $\Delta \wedge \mathcal{L}_z$, we could have simplified by $\boldsymbol{i}$. But there are other uses of $\boxed{W_z}$ where this factor will be required.

4.4 Pythagoras quadratic form

1. Theorem (pons caballorum). The squared norm of a vector is computed using $|z|^2 = z\,\overline{z}$. For vectors of type $\overrightarrow{AB}$, this translates into:

$$\left|\overrightarrow{AB}\right|^2 = {}^t\overrightarrow{AB} \cdot \boxed{\mathrm{Pyth}_z} \cdot \overrightarrow{AB} \qquad \text{where } \boxed{\mathrm{Pyth}_z} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

2. Proposition. Let P a point at finite distance. Then there are two linear transformations ψ such that

   (a) $\psi(P) = 0 : 0 : 0$

   (b) $\psi\left(\overrightarrow{\mathcal{V}}\right) = \overrightarrow{\mathcal{V}}$

   (c) for all $V \in \overrightarrow{\mathcal{V}}$, $\langle \psi(V) \mid V \rangle = 0$ while $\langle \psi(V) \mid \psi(V) \rangle = \langle V \mid V \rangle$.

   Their characteristic polynomial is $\mu^3 + \mu$ and we have:

$$\boxed{\psi} = i \begin{pmatrix} +1 & -z/t & 0 \\ 0 & 0 & 0 \\ 0 & +\overline{z}/t & -1 \end{pmatrix}$$

   the other possibility being the opposite of the $\boxed{\psi}$ operator .

3. Definition. Operator $\boxed{\mathrm{OrtO}_z}$. The action of one of the operators ψ_P on the vector 1: 0: 1 is the vector + i: 0: -i. Therefore it corresponds to a direct rotation of a quarter turn for the observer from above (i.e. **Z**). The choice $P = O$ leads to the operator:

$$\boxed{\mathrm{OrtO}_z} = i \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

   and we have $\boxed{\mathrm{OrtO}_z} = -2\,{}^t\boxed{W_z} \cdot \boxed{\mathrm{Pyth}_z}$.

4. Proposition. Considered as acting up to a factor, each of the operators ψ –and $\boxed{\mathrm{OrtO}_z}$ among them– sends a point at infinity on the point at infinity representing the orthogonal direction (orthopoint).

5. Remark. When $U \in \mathcal{L}_z$, then $\Delta_U \doteq {}^tU \cdot \boxed{\mathrm{Pyth}_z}$ is the line of points V such as ${}^tU \cdot \boxed{\mathrm{Pyth}_z} \cdot V = 0$. The point at infinity of Δ_U is $U' = \boxed{W_z} \cdot {}^t\Delta_U = \boxed{W_z} \cdot \boxed{\mathrm{Pyth}_z} \cdot U$. It is therefore convenient to choose P in O, so that $\boxed{\psi}$ is proportional to $\boxed{W_z} \cdot \boxed{\mathrm{Pyth}_z}$.

6. Operator $\mathcal{M}$ (orthodir). This is the operator which takes a straight line as input and gives the orthopoint of its point at infinity as output. Using Morley's affixes, we have:

$$\mathit{orthodir}(\Delta) = \boxed{\mathcal{M}_z} \cdot {}^t\Delta \qquad \text{where } \boxed{\mathcal{M}_z} = \boxed{\mathrm{OrtO}_z} . \boxed{W_z} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

7. Theorem (Pythagoras). Two lines are perpendicular when the orthodir of one of them belongs to the other. This is expressed by:

$$\Delta_1 \perp \Delta_2 \Longleftrightarrow \Delta_1 \cdot \boxed{\mathcal{M}_z} \cdot \Delta_2 = 0$$

4.5 Tangent of an angle between two lines

1. Theorem. The tangent of the oriented angle determined by the lines Δ_1 and Δ_2 is obtained by dividing the antisymmetric form of Thales by the symmetric form of Pythagoras. In other words, we have:
$$\tan(\Delta_1,\,\Delta_2)=\frac{\Delta_1\cdot\boxed{W_z}\cdot{}^t\Delta_2}{\Delta_1\cdot\boxed{\mathcal{M}_z}\cdot{}^t\Delta_2}$$

2. Remark. One of the interests of this formula is to provide a *projective* expression, i.e. an expression that resists to all these "definitions up to a factor". This is clear for the factors involving the Δ_j. But this is also true for the transformation matrices. Indeed $\boxed{W_z}$ and $\boxed{\mathcal{M}_z}$ are of the same type, namely "line to point", and are both transformed according to the same $X\mapsto aller\cdot X\cdot{}^t aller$ paradigm.

3. Elementary proof (the shorter, the better). We have:
$$([p+i\,;\,2m\,;\,p-i]\ ,\ [q+i\,;\,2n\,;\,q-i])\mapsto\frac{q-p}{1+p\,q}$$

4. Everybody knows that tangents are making a group, which satisfies:
$$\tan(\Delta_1,\,\Delta_3)=\frac{\tan(\Delta_1,\,\Delta_2)+\tan(\Delta_2,\,\Delta_3)}{1-\tan(\Delta_1,\,\Delta_2)\,\tan(\Delta_2,\,\Delta_3)}$$
with the usual homographic rules to manage the point $\infty\in\overline{\mathbb{C}}$. And this applies even when the lines are not visible, and the tangents are complex numbers.

5. Note that the visible points at infinity of $\mathbb{P}_\mathbb{C}\left(\mathbb{C}^3\right)$, in other words the directions, are written $z:0:\overline{z}$ and therefore can be normalized to $\omega^2:0:1$. An angle between lines is represented by its double, which is an angle between vectors, so that the later can be represented by a point on the trigonometric circle.

6. Interpretation. When the President of the Jury raises his head to contemplate the infinity, he forms his opinion by dividing the points of view of his two Vice-Presidents, according to the formula $\omega^2=z/\overline{z}$. And then, each of the quarter turns multiplies ω^2 by -1. So intrinsic, says the President !

4.6 Down with flechi-flecha !

Remark 4.6.1. The physical space doesn't know about some point which would be central and privilegied, contrary to what happens in a vector space. A better model is provided by the so-called "affine spaces", where all points can be taken as a "random observer".

Definition 4.6.2. An **affine combination** of vectors is a linear combination whose sum of coefficients is equal to 1. An affine space is some non empty set $\mathcal{E}$ closed under such affine combinations. Obviously, this requires the possibility to compute the said barycenters, and therefore $\mathcal{E}$ needs to be a subset of some vector space $\widehat{E}$.

Notation 4.6.3. As usual, operations in $\mathcal{E}$ are noted $+$, $-$, $\cdot$. ; but, sometimes, il could be useful to use another set of notations for the operations used in some external vector space. We will use the $\oplus$, $\ominus$, $\odot$ symbols.

Proposition 4.6.4. *When $\mathcal{E}$ is an affine subset of $\widehat{E}$, the set $\overrightarrow{\mathcal{V}}\doteq\left\{\overrightarrow{BC}\doteq C\ominus B\mid B,C\in\mathcal{E}\right\}$ is a vectorial subset of $\widehat{E}$, while $\mathcal{E}$ can be written as $A\oplus\overrightarrow{\mathcal{V}}$ for any observer $A\in\mathcal{E}$.*

Proof. Since $\mathcal{E}\neq\emptyset$, it exists some $A\in\mathcal{E}$. Therefore, $\overrightarrow{0}\in\overrightarrow{\mathcal{V}}$ is obtained as $A-A$. If $u_j\doteq C\ominus B\in\overrightarrow{\mathcal{V}}$, then $D_j\doteq A\oplus u_j=A\oplus C\ominus B$ is the barycenter of three points of $\mathcal{E}$ so that $D_j\in\mathcal{E}$. Thus each $u_j\in\overrightarrow{\mathcal{V}}$ is an $D_j\ominus A$ for some D_j and the same fixed A. Then $ku=k\,(D\ominus A)=(kD\oplus(1-k)\,A)\ominus A=D_k-A\in\overrightarrow{\mathcal{V}}$. Moreover
$$ku_1\oplus(1-k)\,u_2=k\,(M_1-A)\oplus(1-k)\,(M_2-A)=(kM_1\oplus(1-k)\,M_2)\ominus A\in\overrightarrow{\mathcal{V}}\qquad\square$$

Definition 4.6.5. A French flechi-flecha is a pair $\left(\mathcal{E},\overrightarrow{\mathcal{V}}\right)$ where $\overrightarrow{\mathcal{V}}$ is a vector space acting freely over the non empty set $\mathcal{E}$. In other words, paradigms are reversed and Proposition 4.6.4 is taken as a definition, while Definition 4.6.2 becomes a property.

Definition 4.6.6. Since the French flechi-flecha despises the set $\widehat{E}$, barycenters have to be defined otherwise. Using the obvious fact that the set of all the $\mathcal{E}\hookrightarrow\overrightarrow{\mathcal{V}}$ functions is a $\mathbb{K}$ vector space, the so-called **Leibniz functions** are members of

$$\mathfrak{F}\doteq span\left\{\lambda_A \mid A\in\mathcal{E}\right\}\qquad\text{where } \lambda_A:\mathcal{E}\hookrightarrow\overrightarrow{\mathcal{V}},\ M\mapsto\overrightarrow{AM}$$

Proposition 4.6.7. *Define the mass μ of some element of $\mathfrak{F}$ by $\mu\left(\sum k_A\lambda_A\right)\doteq\sum k_A$ then*

1. *Function μ is well defined and is a linear map $\mathfrak{F}\mapsto\mathbb{K}$.*
2. *When $f\in\mathfrak{F}$ and $\mu(f)=0$, then f is constant*
3. *When $f\in\mathfrak{F}$ and $\mu(f)\neq 0$, it exists an unique $G\in\mathcal{E}$ (the so-called barycenter) such that $f=\mu(f)\cdot\lambda_G$*

Proof. (1-2). Suppose that your French flechi-flecha $\mathcal{E}$ contains at least two different points P,Q. Then, for any decomposition of f,

$$f(Q)-f(P)=\sum k_B\lambda_B(Q)-\sum k_B\lambda_B(P)=\sum k_B\left(\overrightarrow{BQ}-\overrightarrow{BP}\right)=\left(\sum k_B\right)\overrightarrow{PQ}$$

(3). Assume $\mu(f)\neq 0$ and define $G_P=P-f(P)\div\mu(f)$. Then $G_P=G_Q$ by the previous relation, together with $f(G)=f(P)-\overrightarrow{f}(f(P))\div\mu(f)=\overrightarrow{0}$. □

Theorem 4.6.8. *Using the French flechi-flecha, i.e. reverting the paradigms to obtain Definition 4.6.5 is only stupid. Indeed, knowledge of the pair $\left(\mathcal{E},\overrightarrow{\mathcal{V}}\right)$ is sufficient to determine*

1. *a vector space $\widetilde{E}$ isomorph to $\mathcal{E}\times\mathbb{K}$,*
2. *a linear form $\mu:\widetilde{E}\hookrightarrow\mathbb{K}$,*
3. *an affine injection $j:\mathcal{E}\hookrightarrow\widetilde{E}$ such that $j(\mathcal{E})=\left\{x\in\widetilde{E}|\mu(x)=1\right\}$*
4. *and then $\left\{x\in\widetilde{E}\mid\mu(x)=0\right\}$ is an isomorphic copy of $\overrightarrow{\mathcal{V}}$ (Lelong-Ferrand, 1986, p.104).*

You can even obtain a canonical vector space $\widehat{E}$ where $\overrightarrow{\mathcal{V}}$ and $\mathcal{E}$ are actually included as genuine substructures, not just up to an isomorphism (Nekovar, 2007, p. 10).

Nekovar. . Use self-explanatory notations, i.e. $u,v\in\overrightarrow{\mathcal{V}},x,y\in\mathcal{E},s,t\in\mathbb{K}^*$.

- Define $\widehat{E}$ as $\overrightarrow{\mathcal{V}}\cup(\mathbb{K}^*\times\mathcal{E})$. Therefore $\widehat{x}\in\widehat{E}$ is either some u or some pair $q(s,x)$, where $s\neq 0$.
- Define μ as $\mu(u)=0$ while $\mu(sx)=s\neq 0$ otherwise.
- Enforce $q(1,x)=x$ and define $\odot$, the scalar multiplication in $\widehat{E}$, by $0\odot\widehat{x}=\overrightarrow{0}$, $s\odot u=su$ and $t\odot q(s,x)=q(ts,x)$.
- Define $\oplus$, the addition in $\widehat{E}$, by the necessary formulas
  - $u\oplus v=u+v$; $x\oplus u=x+u$;
  - $q(s,x)\oplus u=q\left(s,x+s^{-1}u\right)$
  - $q(-s,x)\oplus q(+s,y)=s\cdot\overrightarrow{xy}$
  - $q(s,x)\oplus q(t,y)=q\left(s+t,x+\dfrac{t}{s+t}\overrightarrow{xy}\right)$
- And now, it "only" remains to verify that these operations define a vector space structure over the set $\widehat{E}$, while h is linear.

□

Lelong-Ferrand. The algebraic burden can be alleviated by using

$$\widetilde{E} = \left\{ g_u = (M \mapsto u) \,|u \in \overrightarrow{\mathcal{V}} \right\} \bigcup \left\{ s \cdot \lambda_x | s \in \mathbb{K}^+ ,\, x \in \mathcal{E} \right\}$$

and μ as defined before. But this way of doing cannot be described as "canonical" since what is obtained is not $\mathcal{E}$ itself, but only an isomorphic copy of this space. □

Remark 4.6.9. The French flechi-flecha was a must during the "new maths" period in France. But now, we have:

Theorem 4.6.10. *There is no such thing like an "intrinsic affine space". (Polo, 2015, chap3, p.65)*

Chapter 5

Teaching Geometry to a computer

5.1 Maple

1. Catch `maple 2023, Linux` from the net.
2. Update: `/opt/maple2023/maple2023/update/update.ini` contient la valeur de `encyfile = "$ipse/docs/Cherche/Geometry/maple/ency48.m"`
3. nouvelle procedure `kiestu()` qui donne 18 ou 2023 selon le cas.
4. modifier `mkici` pour initialiser $\sin(\omega), \cos(\omega)$ avant de calculer $\tan(\omega)$.
5. modifier `latexx` pour avoir `latex(*,writeto=*,*)` .

5.2 The random observer

When dealing with triangle geometry, we have to organize the coexistence of three kinds of objects. We have vectors, we have points and we have 3-tuples. A vector describes a translation of the "true plane". A vector has a direction but also a length and therefore is not defined "up to a proportionality factor". The set of all these vectors is a 2-dimensional vector space $\overrightarrow{\mathcal{V}}$ (more about it in what follows). A point is either an element of the "true plane", i.e. an ordinary point at finite distance, or a point at infinity describing the direction of some line. Such points have therefore to be described "up to a proportionality factor" by a column that belongs to a given copy of $\mathbb{P}_\mathbb{R}(\mathbb{R}^3)$. In the same vein, lines are described "up to a proportionality factor" by a row that belongs to another copy $\mathbb{P}_\mathbb{R}(\mathbb{R}^3)$.

And we need to talk with our computer in order to let it compute all the required results. These computations are done using 3-tuples and tools acting over 3-tuples so that computations must be described using $\mathbb{R}^3$ rather than using $\overrightarrow{\mathcal{V}}$ or $\mathbb{P}_\mathbb{R}(\mathbb{R}^3)$. It remains to optimize the coexistence of these three points of view

Notation 5.2.1. In the plane we are dealing with, an affine description of a point at finite distance P is a 3-tuple (ξ, η, ζ) where $\zeta = 1$ is assumed. The semantic of these coordinates is the pre-existence of some random observer, that uses a Cartesian frame (ξ, η) to describe what is occurring before her eyes.

5.3 Working out an example

Let us take an example and begin with an informal approach (cf http://www.les-mathematiques.net/phorum/read.php?8,585414,586289#msg-586289). We have a point P, given by a column, and two triangles T_1, T_2 given by the columns of their vertices :

$$P = \frac{1}{2}\begin{pmatrix} 8 \\ 15 \\ 2 \end{pmatrix} \; ; \; T_1 = \begin{pmatrix} 7 & 3 & -2 \\ 9 & 9 & -3 \\ 1 & 1 & 1 \end{pmatrix} \; ; \; T_2 = \begin{pmatrix} -9 & 19 & 6 \\ 1 & -15 & 11 \\ 1 & 1 & 1 \end{pmatrix}$$

As it should be, each column verifies $\zeta = 1$, which is the equation of the affine plane $\mathcal{E}$ when seen as a subspace of $\mathbb{R}^3$.

We define $\boxed{W}$ as the matrix that transforms the matrix T of a triangle (P_j) into the matrix of the sideline vectors $\left(\overrightarrow{P_{j+1}P_{j+2}}\right)$ of this triangle (indices are taken modulo 3 so that $P_4 = P_1$ etc). And now, we compute $\boxed{\mathcal{K}} = {}^t\boxed{W} \cdot {}^tT \cdot T \cdot \boxed{W}$ for both triangles. We have :

$$\boxed{W} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\boxed{\mathcal{K}_1} = \begin{pmatrix} 169 & -189 & 20 \\ -189 & 225 & -36 \\ 20 & -36 & 16 \end{pmatrix} = \begin{pmatrix} a^2 & -S_c & -S_b \\ -S_c & b^2 & -S_a \\ -S_b & -S_a & c^2 \end{pmatrix}$$

$$\boxed{\mathcal{K}_2} = \begin{pmatrix} 845 & -65 & -780 \\ -65 & 325 & -260 \\ -780 & -260 & 1040 \end{pmatrix} = \begin{pmatrix} \alpha^2 & -S_\gamma & -S_\beta \\ -S_\gamma & \beta^2 & -S_\alpha \\ -S_\beta & -S_\alpha & \gamma^2 \end{pmatrix}$$

Here we use a, b, c for the sidelengths of the first triangle and α, β, γ for the second one. The Conway's symbols $S_a = \left(b^2 + c^2 - a^2\right)/2$ etc. are defined accordingly. The letter used to name the matrix $\boxed{\mathcal{K}}$ has been chosen by reference to the Al-Kashi formula. Clearly, row $(1, 1, 1)$ belongs to the kernel of matrix $\boxed{\mathcal{K}}$. The characteristic polynomial of this matrix is $\chi(\mu) = \mu^3 - \left(a^2 + b^2 + c^2\right)\mu^2 + 12\, S^2 \mu$. One eigenvalue is $\mu = 0$ and the other two are real (from symmetry of $\boxed{\mathcal{K}}$) and positive.

Remark 5.3.1. Matrix $\boxed{\mathcal{M}_b}$ used to compute the orthodir of a line is nothing but $\boxed{\mathcal{M}_b} = \boxed{\mathcal{K}}/8S^2$.

A more "stratospheric", but nevertheless equivalent, definition for these Al-Kashi matrices would be :

$$\boxed{\mathcal{K}} \doteq {}^t\boxed{W} \cdot {}^tT \cdot \boxed{Pyth_3} \cdot T \cdot \boxed{W}$$

where $\boxed{Pyth_3}$ describes any $\mathbb{R}^3$-quadratic form that embeds $(\xi, \eta) \mapsto \xi^2 + \eta^2$, the quadratic form of the ordinary affine euclidian plane. This embedding quadratic form depends on three arbitrary parameters since 6 coefficients are needed for dimension three, while only 3 are needed for dimension two.

5.4 An involved observer

Now, we will describe how things are looking when the observer is no more a random ☼ but rather an actor of the play. For example, we can take triangle T_1 as a new vector basis inside vector space $\mathbb{R}^3$ and calculate everything again using this new basis. From :

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = T_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

it is clear that condition $\zeta = 0$ that shows that a 3-tuple belongs to $\overrightarrow{\mathcal{V}}$ becomes now $x + y + z = 0$. Defining $\mathcal{L}_b = (1, 1, 1)$ this can be rewritten as $\mathcal{L}_b \cdot {}^t(x, y, z) = 0$. Seen "up to a proportionality factor" this will gives $\mathcal{L}_b \cdot (x : y : z) = 0$, i.e. the condition for the point $x : y : z \in \mathbb{P}_\mathbb{R}\left(\mathbb{R}^3\right)$ to lie on the line at infinity. But this is not our purpose for the moment.

The $\mathbb{R}^3$-metric is now described by matrix ${}^tT_1 \cdot \boxed{Pyth_3}.T_1$. This matrix depends in turn on three arbitrary parameters. In fact, any other matrix that can be written as :

$${}^tT_1 \cdot \boxed{Pyth_3}.T_1 + U.\mathcal{L}_b + {}^t(U.\mathcal{L}_b) \qquad \text{where } \boxed{Pyth_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

using an arbitrary column U will be just as well to calculate the Pythagoras of $\overrightarrow{\mathcal{V}}$-vector. A zero diagonal gives a more nice looking matrix, and is also more efficient for computing. Therefore we define the matrix $\boxed{\mathrm{Pyth}_b}$ by this property and we obtain :

$$\boxed{\mathrm{Pyth}_b} = -\frac{1}{2}\begin{pmatrix} 0 & 16 & 225 \\ 16 & 0 & 169 \\ 225 & 169 & 0 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{pmatrix}$$

Now, matrix $\boxed{\mathcal{K}_1}$ can be computed using :

$$\boxed{\mathcal{K}_1} = {}^t\boxed{W} \cdot \boxed{\mathrm{Pyth}_b} \cdot \boxed{W} = \begin{pmatrix} 169 & -189 & 20 \\ -189 & 225 & -36 \\ 20 & -36 & 16 \end{pmatrix}$$

while the coordinates of the other triangle and the extra point are transformed according to :

$$\mathcal{T}_2 = T_1^{-1} \cdot T_2 = \frac{1}{24}\begin{pmatrix} -52 & 156 & 13 \\ 60 & -180 & 15 \\ 16 & 48 & -4 \end{pmatrix} \; ; \; P_{[1]} = T_1^{-1} \cdot P = \frac{1}{32}\begin{pmatrix} 13 \\ 15 \\ 4 \end{pmatrix}$$

And now, matrix $\boxed{\mathcal{K}_2}$ can be computed using :

$$\boxed{\mathcal{K}_2} = {}^t\boxed{W} \cdot {}^t\mathcal{T}_2 \cdot \boxed{\mathrm{Pyth}_b} \cdot \mathcal{T}_2 \cdot \boxed{W} = \begin{pmatrix} 845 & -65 & -780 \\ -65 & 325 & -260 \\ -780 & -260 & 1040 \end{pmatrix}$$

The quadratic form $\boxed{Pyth_2}$ can be obtained as above, or directly as:

$$\boxed{Pyth_2} = -\frac{1}{2}\begin{pmatrix} 0 & \gamma^2 & \beta^2 \\ \gamma^2 & 0 & \alpha^2 \\ \beta^2 & \alpha^2 & 0 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & 1040 & 325 \\ 1040 & 0 & 845 \\ 325 & 845 & 0 \end{pmatrix}$$

5.5 From an involved observer to another one

All the preceding computations are standard ones. As long as points have no dependence relations all together, nothing else can be done. On the contrary, when points are constructed from each other, another point of view is more powerful. Taking ABC as reference triangle, and describing all points by their barycentric coordinates, we obtain $P = 13 : 15 : 4$, $A' = -13 : 15 : 4$, $B' = 13 : -15 : 4$, $C' = 13 : 15 : -4$. If this is only a random coincidence, nothing more can be said.

On the contrary, if these points are really defined wrt $\boxed{\mathcal{T}_1}$ as $P = a : b : c$, $A' = -a : b : c$, $B' = a : -b : c$, $C' = a : b : -c$, then things get more interesting. We have :

$$P_{[2]} = T_2^{-1} \cdot P = b + c - a : c + a - b : a + b - c$$

$$\boxed{\mathcal{K}_2} = \begin{pmatrix} \dfrac{4\,a^2bc}{(a-b+c)\,(b+a-c)} & \dfrac{-2\,abc}{b+a-c} & \dfrac{-2\,abc}{a-b+c} \\ \dfrac{-2\,abc}{b+a-c} & \dfrac{4\,ab^2c}{(b+c-a)\,(b+a-c)} & \dfrac{-2\,abc}{b+c-a} \\ \dfrac{-2\,abc}{a-b+c} & \dfrac{-2\,abc}{b+c-a} & \dfrac{4\,c^2ab}{(b+c-a)\,(a-b+c)} \end{pmatrix}$$

so that : $\left\{ \alpha^2 = k\,\dfrac{4\,a^2bc}{(a-b+c)\,(b+a-c)},\ \text{etc} \right\}$

Since barycentrics are defined only "up to a proportionality factor", the former set of equations has only to be solved in k, b, c. This gives the ratios between the quantities a, b, c. One gets :

$$(a : b : c) \simeq \left(\alpha^2\left(\beta^2+\gamma^2-\alpha^2\right) : \beta^2\left(\gamma^2+\alpha^2-\beta^2\right) : \gamma^2\left(\beta^2+\alpha^2-\gamma^2\right)\right)$$

so that :

$$P_{[2]} = \frac{1}{\beta^2 + \gamma^2 - \alpha^2} : \frac{1}{\gamma^2 + \alpha^2 - \beta^2} : \frac{1}{\alpha^2 + \beta^2 - \gamma^2}$$

This results identifies the incenter of the reference triangle ABC as the orthocenter of the excentral triangle. In the same vein, the circumcenter of ABC can be identified as the nine-points center of $A'B'C'$. More details on this specific relation will be given in Subsection 24.7.

5.6 Reducing up to a factor

Maple 5.6.1. Reducing "up to a factor" is a key feature in computer-aided projective geometry.

```
REDUCE := proc (qui_ :: {Matrix, Vector, list}) ; local qui
qui0:=convert(qui_, set) \ {0}
if qui0 = {} then return qui_ end if
qui:=FActor(convert(qui_, Vector)/qui0[1])
lili:=convert(qui, list)
nunu, dede:=map(simplify@numer, lili), map(simplify@denom, lili)
fac:=(lcm@op)(dede)
if type(nunu, list(numeric)) then
   fac:=fac/(igcd@op)(nunu)
else
   fac:=fac/(gcdd@op)(nunu)
end if
qui:=qui * fac
quii:=quii * fac
if type(qui_, Vector) then
   return qui
else if type(qui_, list) then
   return convert(qui, list)
else
   return LTr(Matrix(ColDim(qui_), RowDim(qui_), convert(qui, list)))
end if
```

Listing 5.1: The reduce procedure.

```
REDUCOL := proc (ma :: Matrix)
Matrix([seq(reduce(Column(ma, j)), j = 1..ColDim(ma))])
```

Listing 5.2: The reducol procedure

```
REDUROW := proc (ma :: Matrix)
< seq(reduce(Row(ma, j)), j = 1..RowDim(ma)) >
```

Listing 5.3: The redurow procedure

```
WEDGE := proc (pp, pu)   ; local p, q, r, u, v, w, tmp
p, q, r, u, v, w := op(convert(pp, list)), op(convert(pu, list))
tmp := [q * w − r * v, −p * w + r * u, p * v − q * u]
if type(pp, Vector[row])  then
  return Vector(tmp)
else if  type(pp, Vector)  then
  return Vector[row](tmp)
end if
tmp
```

Listing 5.4: The wedge procedure

5.7 packages

logic.m		443	*geogebra.m*	4172
latex.m		834	*geo4D.m*	5318
geo2c.m	cartesian	971	*tensor.m*	6067
hyperbolic.m		1257	*vector.m*	8632
pldx.m	bootstrap	1268	*lfit.m*	10656
homogr.m		1988	*cevian.m*	14749
gone.m		2172	*encydb.m*	15936
birap.m		3174	*faisceaux.m*	15981
map2alg.m		3905	*lubin.m*	16280
circles.m		4085		
dessin.m		3056	*tcurv.m*	8374
quadlat2.m		7166	*quadlat.m*	8897
relire_sk.m		7223325		

Chapter 6

Maple procedures about searchkeys

Remark 6.0.1. At 2024-01-02, there were 61097 points subject to the curse of having coordinates in the Kimberling's database. Too much for continuing to recompile the web pages and extract the formal coordinates of these points.

From 2019, I have decided to limit my "formal database" to the $n \leq 6809$ first points and to the 59 points having complex coordinates. In fact, only four of the later, namely the $\{8072, 8073, 11065, 11066\}$ have been added.

On the other hand, the Kimberling (1998-2024) database can be used to obtain the "numerical barycentrics" of all the $6809 < n < 61098$ remaining points (except from the 59 complex ones), allowing numerical explorations using triangle $6, 9, 13$.

6.1 Procedure mkalgo

The `map2alg:-mkalgo` procedure creates something at `ipse/docs/maple/latex/proc.tex`
Then you go into `ipse/docs/maple/latex` and run `proc-maker.bat`
Then you type `M-b d` in LyX. This creates the required Algo box.

```
CIRCLE3 := proc( )
if ColDim(Args[1]) = 3 then
   return Procname(Column(Args[1], 1..3))
end if
wedge3 (seq ((FActor@reduce@Ver)(jj), jj = Args))
‘if‘ (%[4] = 0, FActor(%), FActor (%/%[4]))
```

Listing 6.1: The circle3 procedure

manual: algorithmic before and after ; Ditto1()$\mapsto$% ; LColumn $\mapsto$ Column ; gestion des @ ;

6.2 Standardized barycentrics

Proposition 6.2.1. ***Standardized barycentrics*** *are defined as follows :*

$$(x, y, z) \mapsto \begin{cases} (x, y, z) \times \frac{1}{x+y+z} & \text{if } x + y + z \neq 0 \\ (x, y, z) \times \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) & \text{otherwise} \end{cases}$$

They are defined for all points, except from the directions of the sidelines and the points $-1 \pm \boldsymbol{i}\sqrt{3} : -1 \mp \boldsymbol{i}\sqrt{3} : 2$.

Remark 6.2.2. This quantity is projectively defined... but depends on a preference towards the barycentrics, rather than towards the trilinears.

Remark 6.2.3. The two exceptional points of the former proposition **are not** the so called umbilics, defined at Subsection 14.1.2.

Algorithm 6.2.4. NORMALIZE. This procedure ALG. 6.2 returns a normalized numerical vector (using triangle $6, 9, 13$) from what is given on entry (a vector, a list or an expression).

Memory from the past: the former procedure `kicety` was combining this `normalize` procedure with the `make_a_key` instructions (which are now a part of procedure `dichot`).

```
NORMALIZE := proc (pu) ; local u, v, w, tmp, tmp1, tmp2
if type(pu, Vector) then tmp := subs (ency__, pu)
else if type(pu, list) then tmp := Vector(subs(ency__, pu))
else tmp := Vector(subs (ency__ [rot3](pu)))
end if
indets(tmp)
if % <> {} then Error("unresolved indets in normalize", %) end if
u, v, w := tmp[1], tmp[2], tmp[3]
if evalf (abs(u + v + w)/(abs(u) + abs(v) + abs(w))) < .1E − 8 then
   return xpurge(evalf (tmp * (1/u + 1/v + 1/w)))
else
   return xpurge(evalf (tmp/(u + v + w)))
end if
```

Ensure: Decides if $M \in \mathcal{L}_b$ and returns a normalized, numerical vector, in $\mathbb{C}^3$.

LISTING 6.2: The normalize procedure

6.3 Numerical values

Definition 6.3.1. The **Kimberling's search key** has been defined as the directed distance between X and the BC sideline of the reference triangle $a = 6$, $b = 9$, $c = 13$ (for an ordinary point) and as $x' \times (1/x' + 1/y' + 1/z')$ for a point at infinity (caveat: here $x' : y' : z'$ are trilinears !). When using barycentrics, this quantity is given by:

$$\begin{aligned} x : y : z \notin \mathcal{L}_b,\ key &= \frac{x}{x+y+z} \times \frac{2S}{a} \\ x : y : z \in \mathcal{L}_b,\ key &= \frac{x}{a} \times \left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z} \right) \end{aligned} \tag{6.1}$$

The actual value of factor $bc \div 2R = 2S/a$ is $4\sqrt{35}/3$. Reference values of this search key are provided by the ETC (Kimberling, 1998-2024).

Definition 6.3.2. The **patched search key** has been introduced to deal with points like X(5000) that would otherwise have a non real searchkey. When $x/\left(x+y+z\right) \notin \mathbb{R}$, then

$$\text{key} \doteq \arg \left(\frac{x}{x+y+z} \right) \in [-\pi,\ +\pi]$$

If one uses Maple, $\arg(z)$ is obtained by evalf $(\arctan(\Im z,\ \Re z))$. Using $\arctan(\Im z / \Re z)$ would be wrong when $\Re z < 0$. And `evalf` is required to avoid symbolic π.

Example 6.3.3. The key for the first umbilic, i.e. $S_b - 2\,\boldsymbol{i}S : S_a + 2\,\boldsymbol{i}S : -c^2$ (see Subsection 14.1.2 for more details about the umbilics) is obtained by:

$$\left(\frac{1}{S_b - 2\,\boldsymbol{i}S} + \frac{1}{S_a + 2\,\boldsymbol{i}S} - \frac{1}{c^2} \right) \begin{pmatrix} S_b - 2\,\boldsymbol{i}S \\ S_a + 2\,\boldsymbol{i}S \\ -c^2 \end{pmatrix} \mapsto \begin{pmatrix} 0.9541237489\cdots & -0.3042530621\cdots\boldsymbol{i} \\ 1.089086127\cdots & +1.034633282\cdots\boldsymbol{i} \\ -2.043209876\cdots & -0.7303802201\cdots\boldsymbol{i} \end{pmatrix}$$

$$\text{key} = -0.30868862910087033702$$

Fact 6.3.4. *When repeating the same process with triangles* $9, 13, 6$ *and* $13, 6, 9$, *we obtain* key_a, key_b, key_c. *Then either* $\sum a\, key_a = 0$ *(point at infinity) or* $\sum a\, key_a = 2S = 8\sqrt{35}$ *(ordinary point)... or anything else for a may be complex point. About key conflicts:*

1. At 2009-08-27, the minimal distance between two search keys was $4.8\,E-7$.

2. At 2017-01-11, this minimal distance was $2.95\ E-8$. This can only become smaller as more points are added to the database. Therefore, one has to be careful when computing a search key, in order to face the possibility of huge canceling terms. Using Maple `Digits:=20` seems to be safe.

3. At 2017-12-27, point have appeared that share the same $A-key$.

older	667	3239	3616	3617	3635	3875	5592
newer	9780	8834	14078	15224	15519	7292	10896

   Thus we can use $key_a + X \times 1E-19$ to build the unique key required by a dichitomy, and then use $1E-13$ as the blur limit for recognition of a key. And go back using a process that gives either X or $[X_1, X_2]$.

4. At 2020-3-30, there were 24 pairs of points that share the same $A-key$. Moreover, there were now 4 pairs of points that have the same 6-9-13 barycentrics:

nn	3635	4098	4691	22166
alt	15519	24150	21267	22266
x	2	110	14	54
y	5	209	11	51
z	9	285	7	47

   Therefore, a safe identification requires a special treatment for these points.

6.3.1 The new reliresk

Algorithm 6.3.5. RELIRESK. From 2019, the numerical database is pre-compiled in Maple format and is simply loaded as is (or even not loaded at all). We have fac47 $= 8\sqrt{35}$, while current values (2024-01-02) are $smax = 53090$; $siz\_enc = 53024$. As an example, we have

$$enc\_sort\,[20086] = [0.12500000000000000006623134309, (15519, 3635)]$$

```
RELIRESK := proc ( ) ; global fac47, smax, siz_enc, sk, fk, enc_sort
read("$ipse/docs/Cherche/Geometry/maple/reliresk/pas_toujours/reliresk.m")
"numerical database imported"
Ensure: sk[j] = [x, y, z] ; fk[j] ∈ {0, 1, 2} ; enc_sort[J] = [key, qui]
```

Listing 6.3: The reliresk procedure

6.4 The new buildsk

Let `Ketc=https://faculty.evansville.edu/ck6/encyclopedia/` and download the sources

1. Create and go into `ipse/docs/Cherche/Geometry/ETC_20xx/`

2. Download `Ketc/ETC.html` and `Ketc/ETCPartn.html` as `ETCPartNN.html` for NN = 1..36

3. Download `Ketc/Search$kki.html` for `kki in 6_9_13 9_13_6 13_6_9` and then

4. grep "<tr align=right>" $qui | sed -e "s¶<tr align=right><td>¶¶; s¶</td><td>¶;¶; s¶</td>.*¶¶" > $kki.csv

5. All required Maple procedures are stored in the `relire_sk` package.

Algorithm 6.4.1. BUILDSK. Start from the searchkeys given by ETC. Use them to identify the type of the point and set `fk[j]`. Then compute their normalized values and set `sj[k]`. Set the searchkey `xk[j]` as `sk[j][1]`$+j*1E-19$. Read the special file that deals with the few points using complex values (and overwrite what needs to be overwriten).

And then, compute the sorted version `enc_sort` so that a key is linked to one or more values (duplicated lines are not deleted !).

$$\begin{bmatrix} 0.367331791787001425\,68 & 5409 \\ 0.3673469387755102707 & 667,9780 \\ 0.36734693877551118208 & 667,9780 \\ 0.3674078636432206616 & 7914 \end{bmatrix}$$

```
BUILDSK := proc (qqqq := 20)                                                  2024-01-03
global fac47, smax, sk, xk, fk, siz_enc, enc_sort
local source, fd, tmp1, tmp2, qrand, j6, j9, j13, js, jj, j, stamp, laps, tmp, lili
source := ipse/docs/Cherche/Geometry/ETC_2023/
Digits := 30 ; stamp := time() ; fac47 := evalf(8 * sqrt(35), 30)
qrand := rand(1000..9999) * 1E − 24
sk := table() ; fk := table() ; xk := table()
try close(fd) catch : end try
fd := open(cat(source,"6_9_13.csv"), READ)
for j to qqqq  do
    tmp := readline(fd) ;
    if % = 0  then Break()  end if
    tmp1, tmp2 := op(sscanf(tmp,"%d; %a")) ; sk[tmp1] := 6 * tmp2
end for
smax := −1 + j ; close(fd)
fd := open(cat(source,"9_13_6.csv"), READ)
for j to smax  do
    tmp := readline(fd) ;
    if % = 0  then  Break()  end if
    tmp1, tmp2 := op(sscanf(tmp,"%d; %a")) ; sk[tmp1] := sk[tmp1], 9 * tmp2
end for
close(fd) ; fd := open(cat(source,"13_6_9.csv"), READ)
for j to smax  do
    tmp := readline(fd) ;
    if % = 0  then  Break()  end if
    tmp1, tmp2 := op(sscanf(tmp,"%d; %a")) ;
    sk[tmp1] := sk[tmp1], 13 * tmp2
end for
close(fd)
for jj to smax  do
    j6, j9, j13 := sk[jj] ; js := j6 + j9 + j13
    if abs(js) < .1e − 12  then
        fk[jj] := 1 ; js := 1/j6 + 1/j9 + 1/j13 ; sk[jj] := [j6 * js, j9 * js, j13 * js]
    else if  abs(js − fac47) < .1e − 12  then
        fk[jj] := 0; sk[jj] := [j6/js, j9/js, j13/js]
    else
        fk[jj] := 2 ; sk[jj] := [j6/6, j9/9, j13/13]
    end if
    xk[jj] := sk[jj][1] + qrand()
end for
return smax
Ensure: tables sk, xk, fk  are set ; fac47, smax, siz_enc, enc_sort
```

Listing 6.4: Procedure buildsk


```
BUILDENCSORT := proc( )                                                    2024-01-03
Require: sk_plex requires the formal coordinates of the complex points
global fac47, smax, siz_enc, enc_sort, sk, xk, fk
local fd, j, jj, jjj, jk, tmp, ttmp, enc_tmp
fd := open("ipse/public_html/etc/sk_plex.csv", READ)
for j to 20000 do
    ttmp := readline(fd)
    if % = 0 then Break() end if
    SubstituteRec(ttmp," ","", "\";\"",", " ", "\"","")
    tmp := sscanf(%,"%d%a%a%a%a") ; jj := tmp[1] ; sk[jj] := tmp[2..4] ; xk[jj] := tmp[−1]
end for
close(fd) ; printf("read %d corrections", j − 1) ; print()
enc_tmp := Array(sort([seq]([xk[jj], jj], jj = 1..smax))) ; jj := 1
for j to smax while jj <= smax do
    jk := NULL
    for jjj from jj to smax do
        if .1e − 9 < enc_tmp[jjj, 1] − enc_tmp[jj, 1] then Break() end if
        jk := jk, jjj
    end for
    enc_tmp[j, 1] := enc_tmp[jj, 1]
    enc_tmp[j, 2] := seq(enc_tmp[jjj, 2], jjj = jk) ; jj := 1 + op(−1, [jk])
end for
siz_enc := j − 1 ; enc_sort := SubMatrix(enc_tmp, 1..siz_enc, 1..2)
return siz_enc
Ensure: create enc_sort ; its size is siz_enc < smax.
```

Listing 6.5: The buildencsort procedure

6.5 Complex points

Algorithm 6.5.1. build_sk_plex. The file t6913.csv is collected from ETC, and is not supposed to change on a daily basis. Required corrections are to be stored somewhere else, namely in the sk_plex file. These corrections are computed from the barycentrics stored in fdat. It uses normalize and the key produced by dichot.

When writing these corrections, we have to be tricky, since Maple doesn't have a native procedure for printing complex numbers. One can see that files t6913.csv and sk_plex are not read/written the same way. The first one has a very strong syntax, and can be read by a customized sscanf. The second one is more convoluted (there is no specific format for reading complex numbers). Spaces and nothing else !

Algorithm 6.5.2.

```
BUILDSK := proc(qqqq := 20000)
Ensure: Initializes the following global variables
global fac47, smax, siz_enc, sk, fk, enc_sort
local j, jj, j6, j9, j13, js, ttmp, enc_tmp, tmp, xk, fd
Digits := 30 ; fac47 := evalf(8 * sqrt(35)) ; sk, xk, fk := table(), table(), table()
fd := open("/server_root/etc/t6913.csv", READ)#2017 - 12 - 26
for smax to qqqq do
   ttmp := readline(fd)
   if % = 0 then Break() end if
   jj, j6, j9, j13 := op( sscanf(%, "\"%d\";\"%f\";\"%f\";\"%f"))
   j6, j9, j13 := 6 * j6, 9 * j9, 13 * j13 ; js := j6 + j9 + j13
   if abs(js) < 1E - 12 then
      fk[jj] := 1 ; js := 1/j6 + 1/j9 + 1/j13
      sk[jj] := [j6 * js, j9 * js, j13 * js] ; xk[jj] := %[1]
   else if abs(js - fac47) < 1E - 12 then
      fk[jj] := 0 ; sk[jj] := [j6/js, j9/js, j13/js] ; xk[jj] := %[1]
   else
      fk[jj] := 2 ; sk[jj] := [j6/js, j9/js, j13/js] ; xk[jj] := %[1]
   end if
end for
smax := smax - 1 ; siz_enc := smax
close(fd) ; print("t6913 has been read : %d items", smax)
fd := open("/server_root/etc/sk_plex.csv", READ)
for j to 10 do
   tmq := readline(fd) ;
   if % = 0 then Break() end if
   SubstituteRec(ttmp, " ","", "\";\""," ", "\"",""); tmp:= sscanf(%, "%d %a %a %a %a");
   jj := tmp[1] ; xk[jj] := tmp[-1] ; sk[jj] := tmp[2..4]
end for
close(fd) ; printf("read %d corrections", j)
enc_tmp := [seq]([xk[jj] + 1E - 18 * jj, jj], jj = 1..smax) ;
enc_sort := Array(sort(enc_tmp))
for jj to siz_enc - 1 do
   if 1E - 9 < enc_sort[1 + jj, 1] - enc_sort[jj, 1] then Next() end if
   tmp := enc_sort[jj, 2], enc_sort[1 + jj, 2] ;
   enc_sort[1 + jj, 2] := tmp ; enc_sort[jj, 2] := tmp
end for
```

LISTING 6.6: The writesk procedure

```
BUILD_SK_PLEX := proc; local j, theplex, lefichier, fd, tmp, tmp2, tmp3, qq
theplex := seq(`if`(fk[j] = 2, j, NULL), j = 1..6802)
fd := open("$ipse/public_html/etc/sk_plex.csv", WRITE)
for j in theplex[1..5] do
   tmp := map(cut, normalize(parse(fdat[j])), 20)
   tmp1 := op(convert(tmp, list))
   tmp2 := (op@map)(op@[Re, Im], convert(tmp, list))
   dichot(tmp[1]) ; tmp3 := cut(key + j * 1E - 19, 20)
   try:
   fprintf(fd, ""%d";"%24.20f";"%24.20f";"%24.20f";"%24.20f"\n", j, tmp1, tmp3)
   catch:
   fprintf(fd, cat(""%d"", ";"%24.20f%+24.20f*i"" $3, ";"%24.20f"\n" ), j, tmp2, tmp3)
   end try
end for
close(fd)
```

LISTING 6.7: The build_sk_plex procedure

6.5.1 Procedures ency and dichot

```
ENCY := proc qui ; local tm1, tm2, tmpd, norqui, rep
Require: receive a vector, or a list or an algebraic expression to be rotated
Ensure: fails with '?' or proves the existence and unicity of a specific match with the entry
  Digits := max(Digits, 20) ; norqui := normalize(qui) ; tmpd := dichot(norqui[1])
  if tmpd[1] = '?' then return '?' end if
  [seq]([j, max( map(abs, norqui − normalize(sk[j])) )], j = tmpd[2])
  rep := [seq]('if'(j[2] < .1e − 11, j[1], NULL), j = %)
  if rep = [] then return '?' end if
  return op(rep)
```

LISTING 6.8: The ency procedure

```
DICHOT := proc qui ; global key ; local u, o, m ensu, enso, ensm, eps
Require: receive qui ∈ cc, the normalized x-barycentric of some point
Ensure: fails with left<key<right or returns the key and either a point or a collision_list
  if Im(qui) = 0 then key := qui else key := argument(qui) end if
  ϵ := .1e − 7 ; u, o := 1, siz_enc
  ensu, enso := enc_sort[u][1], enc_sort[o][1]
  if enso < key then
    if |enso − key| < ϵ then return enc_sort[o] else return ?, [enso, key, infinity] end if
  end if
  if key < ensu then
    if |ensu − key| < ϵ then return enc_sort[u] else return '?', [−infinity, key, ensu] end if
  end if
  while 1 < o − u do
    m := floor(o/2 + u/2) ; ensm := enc_sort[m][1]
    if ensm < key then u := m ; ensu := ensm else o := m ; enso := ensm end if
  end while
  if |ensu − key| < ϵ then return enc_sort[u]
  else if |enso − key| < ϵ then return enc_sort[o]
  else return ?, [ensu, key, enso]
  end if
```

LISTING 6.9: The dichot procedure

*** descriptions of procedures like localize should be moved here ***

6.6 morley

Definition 6.6.1. The **Morley's search key** is the search key associated with a point defined by its Morley affix. Details and formula are provided in the corresponding chapter. See Proposition 15.4.10.

Chapter 7

Euclidian structure using barycentrics

Barycentric coordinates were intended to describe affine properties, i.e. properties that remains when points are moved freely. Therefore describing euclidian[1] properties when using barycentrics is often presented as contradictory. We will show that, on the contrary, all the required properties can be described simply. The key fact is that orthogonality only depends on the directions of lines so that all what is really needed is a bijection that sends each point at infinity onto the point that characterizes the orthogonal direction.

7.1 More about the cartesian projective plane

Definition 7.1.1. The cartesian projective plane $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^3\right)$ is what is obtained when using $x : y : 1$ to describe the usual Cartesian points (x, y). As a result the line by two points A, B is obtained as $A \wedge B$, while the point on two lines δ, ϵ is obtained as $\delta \wedge \epsilon$.

Definition 7.1.2. When two lines are parallel in the cartesian plane, their intersection is some $x : y : 0$ point, which belongs to a new special line, namely $\mathcal{L}_c \simeq [0, 0, 1]$ (the cartesian line at infinity).

7.2 Embedded euclidian vector space

Definition 7.2.1. Given two ordinary (at finite distance) points $P \simeq x_1 : y_1 : t_1$ and $U \simeq x_2 : y_2 : t_2$ of $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^3\right)$, the **embedded vector** from P to U is defined as :

$$\overrightarrow{vec}\,(P,\,U) \underset{c}{=} \frac{1}{t_2}\left(\begin{array}{c} x_2 \\ y_2 \\ t_2 \end{array}\right) - \frac{1}{t_1}\left(\begin{array}{c} x_1 \\ y_1 \\ t_1 \end{array}\right)$$

This quantity is projective since replacing P, U with $k_p P, k_u U$ doesn't change the result. This can be generalized as

$$\overrightarrow{vec}\,(P,\,U) \doteq \frac{U}{\mathcal{L}_\infty \cdot U} - \frac{P}{\mathcal{L}_\infty \cdot P} \tag{7.1}$$

When applying a collineation Ψ to the points, the collineation Ψ^{-1} is applied to the line $\mathcal{L}_\infty$, the coefficients remain invariant and $\overrightarrow{vec}\,(P,\,U) \mapsto \Psi \cdot \overrightarrow{vec}\,(P,\,U)$.

Proposition 7.2.2. *Embedded vectors belong to the vector plane* $\overrightarrow{\mathcal{V}}$: $\{X \mid \mathcal{L}_\infty \cdot X = 0\}$*, seen as a subspace of the Cartesian (non projective) vector space* $\mathbb{R}^3$*. These vectors obey to the usual Chasles rule :*

$$\overrightarrow{vec}\,(P_1,\,P_3) = \overrightarrow{vec}\,(P_1,\,P_2) + \overrightarrow{vec}\,(P_2,\,P_3)$$

[1] Let us recall that computing figures, instead of figuring computations, is as far as possible of what the historical Euclide was doing in his Elements (see Euclid, fl. 300BC). Using the adjective *Euclidean* to describe the modern *euclidian* geometry would only be misleading.

and space $\overrightarrow{\mathcal{V}}$ *is isomorphic to the usual vector space* $\mathbb{R}^2$ *where* $\overrightarrow{P_1P_2} = (x_2 - x_1,\, y_2 - y_1)$ *when assuming* $t_1 = t_2 = 1$.

Proof. Obvious from definition. Nevertheless, a key property for what follows. □

Fact 7.2.3. *When using* (x, y, t) *coordinates, the metric of the usual euclidian plane* $\overrightarrow{\mathcal{V}}_c$ *is described by matrix :*

$$\boxed{\mathrm{Pyth}_c} \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.2}$$

In fact, any other matrix defined as :

$$\boxed{\mathrm{Pyth}_c} + U \cdot \mathcal{L}_c + {}^t(U \cdot \mathcal{L}_c) \tag{7.3}$$

can be used to define the metric of vector space $\overrightarrow{\mathcal{V}}_c$ *since* $t = 0$ *is assumed.*

7.3 Lengths and areas in the Cartesian plane

Notation 7.3.1. In this section, T_0 describes the reference triangle ABC in the cartesian affine plane (where $\mathcal{L}_c = [0, 0, 1]$, and T_x is the normalised description of a generic triangle (P_j) where indices are dealt modulo 3, i.e. $P_4 = P_1$, etc. In other words :

$$T_0 = \begin{vmatrix} \xi_a & \xi_b & \xi_c \\ \eta_a & \eta_b & \eta_c \\ 1 & 1 & 1 \end{vmatrix},\ T_x = \begin{vmatrix} \xi_\alpha & \xi_\beta & \xi_\gamma \\ \eta_\alpha & \eta_\beta & \eta_\gamma \\ 1 & 1 & 1 \end{vmatrix} \tag{7.4}$$

We will use $|BC| = a$, $|P_2P_3| = \alpha$, etc together with $S_a = \left(b^2 + c^2 - a^2\right)/2$, $S_\alpha = \left(\beta^2 + \gamma^2 - \alpha^2\right)/2$, etc. In other words,

$$(\xi_a - \xi_b)^2 + (\eta_a - \eta_b)^2 = c^2, \text{ etc} \tag{7.5}$$

Definition 7.3.2. Matrix W. The matrix $\boxed{W_0}$ is defined by :

$$\boxed{W_0} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \tag{7.6}$$

Proposition 7.3.3. *When a matrix* T_x*, as defined in* (7.4)*, gives the vertices* P_j *of a triangle, then* $T_x \cdot \boxed{W_0}$ *gives the (cartesian) sideline vectors* $\overrightarrow{P_{j+1}P_{j+2}}$ *of this triangle (using modulo 3 indices).*

Proof. This is obvious from $\overrightarrow{P_jP_{j+1}} = P_{j+1} - P_j$ that holds when P_j are 3-tuples in the $t = 1$ plane. □

Lemma 7.3.4. ***Matrix K.*** *We have the following Al-Kashi formula :*

$$\boxed{\mathcal{K}_x} = {}^t\boxed{W_0} \cdot {}^tT_x \cdot \boxed{\mathrm{Pyth}_c} \cdot T_x \cdot \boxed{W_0} = \begin{pmatrix} \alpha^2 & -S_\gamma & -S_\beta \\ -S_\gamma & \beta^2 & -S_\alpha \\ -S_\beta & -S_\alpha & \gamma^2 \end{pmatrix} \quad \text{where } \boxed{\mathrm{Pyth}_c} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Proof. Diagonal elements are $\left\langle \overrightarrow{BC} \mid \overrightarrow{BC} \right\rangle$, etc and the others are $\left\langle \overrightarrow{BC} \mid \overrightarrow{CA} \right\rangle = -\left\langle \overrightarrow{CB} \mid \overrightarrow{CA} \right\rangle$, etc. □

Fact 7.3.5. *The area* S_x *of triangle* T_x *is given by the well-known formula*

$$S_x = \frac{1}{2} \begin{vmatrix} \xi_a & \xi_b & \xi_c \\ \eta_a & \eta_b & \eta_c \\ 1 & 1 & 1 \end{vmatrix} \tag{7.7}$$

Moreover, we have the Heron formula :

$$\begin{aligned} S_x^2 &= \frac{1}{16}(+\alpha+\beta+\gamma)(-\alpha+\beta+\gamma)(+\alpha-\beta+\gamma)(+\alpha+\beta-\gamma) \\ &= \frac{1}{4}\left(S_a S_b + S_b S_c + S_c S_a\right) \end{aligned} \tag{7.8}$$

Proof. The first formula gives the oriented area of a triangle, which cannot be obtained from the Heron formula. On the other hand, Heron formula can be obtained in many ways, from simple identification to the more sophisticated :

$$4S^2 = |AB|^2\,|AC|^2 \sin^2 A = b^2c^2 - \left\langle \overrightarrow{AB} \mid \overrightarrow{AC} \right\rangle^2 = b^2c^2 - S_a^2$$ □

7.4 Lengths and areas in the barycentric plane

Definition 7.4.1. Barycentrics coordinates wrt the triangle $T_0 \doteq ABC$ formed by three points at finite distance are what is obtained by using

$$\begin{pmatrix} x \\ y \\ t \end{pmatrix} \mapsto \begin{pmatrix} p \\ q \\ r \end{pmatrix} \doteq \boxed{T_0}^{-1} \cdot \begin{pmatrix} x \\ y \\ t \end{pmatrix} \tag{7.9}$$

Proposition 7.4.2. *Let matrix $\boxed{\mathcal{T}}$ be giving the barycentric coordinates of the vertices $P_i = p_i : q_i : r_i$ of a triangle $\mathcal{T}$. Then area of $\mathcal{T}$ is given by :*

$$S \times \frac{\det \boxed{\mathcal{T}}}{\prod (p_i + q_i + r_i)} \tag{7.10}$$

where S is the area of the reference triangle.

Proof. This formula combines (7.7) with the usual properties of determinants. The denominator appearing at (7.10) enforces the required invariance wrt multiplicative factors acting on barycentrics, and also recognizes the fact that only triangles with finite vertices have an area. □

Remark 7.4.3. A key point is that formula (7.10) is of first degree in S : once the orientation of the reference triangle is chosen, all the orientations of the other triangles are fixed.

Theorem 7.4.4. Pythagoras theorem. *When using barycentrics as defined at (7.9), we have*

$$\mathcal{L}_b = [1, 1, 1] \;\; ; \;\; \boxed{\mathrm{Pyth}_b} = \frac{1}{2} \begin{pmatrix} 0 & -c^2 & -b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{pmatrix} \tag{7.11}$$

As a result, the transformation $\overrightarrow{PQ} \in \mathbb{R}^2 \mapsto \overrightarrow{vec}(P, Q) \in \overrightarrow{\mathcal{V}}_b$ defines an isomorphism of euclidian spaces, while the squared distance between two (finite) points of the triangle plane is given by :

$$\begin{aligned} |PU|^2 &= {}^t\overrightarrow{vec}(P, U) \cdot \boxed{\mathrm{Pyth}_b} \cdot \overrightarrow{vec}(P, U) \\ \left|\overrightarrow{(\rho, \sigma, \tau)}\right|^2 &= -\left(a^2\sigma\tau + b^2\tau\rho + c^2\rho\sigma\right) \end{aligned} \tag{7.12}$$

Proof. The usual change of basis formulae applyed to the quadratic form $\boxed{\mathrm{Pyth}_c}$ are leading to

$${}^tT_0 \cdot \boxed{\mathrm{Pyth}_c} \cdot T_0 + U \cdot \mathcal{L}_b + {}^t(U \cdot \mathcal{L}_b)$$

And then U is chosen to obtain a zero diagonal. One can also identify the coefficients in order to obtain

$$\langle \overrightarrow{vec}(A, B) \mid \overrightarrow{vec}(A, B) \rangle = c^2 \;\; (\text{etc}) \;\; \text{and} \;\; \langle \overrightarrow{vec}(A, B) \mid \overrightarrow{vec}(A, C) \rangle = S_a \;\; (\text{etc})$$ □

7.5 About circumcircle and infinity line

Definition 7.5.1. The **power** of a point $X = x : y : z$ (at finite distance) with respect to the circle Ω centered at P with radius R defined by :

$$power\left(\Omega,\ X\right) \doteq \left|PX\right|^2 - R^2$$

Theorem 7.5.2. *The* ***power formula*** *giving the Ω-power of any point $X = x : y : z$ from the power at the three vertices of the reference triangle is :*

$$\begin{aligned} power\left(\Omega,\ X\right) &= \frac{ux + vy + wz}{x + y + z} - \frac{a^2yz + b^2xz + c^2xy}{\left(x + y + z\right)^2} \\ \text{where } u &= power\left(\Omega,\ A\right),\ \text{etc} \end{aligned} \tag{7.13}$$

Proof. Use (7.1) to obtain $\overrightarrow{PX}$ and then Theorem 7.4.4 to obtain $power\left(\Omega,\ X\right)$. Substitute $y = z = 0$ to obtain u, etc. Then a simple subtraction leads to the required result. □

Definition 7.5.3. The standard equation of the circumcircle is defined as :

$$\Gamma_{std}\left(x, y, z\right) \doteq -\frac{a^2yz + b^2xz + c^2xy}{x + y + z} \tag{7.14}$$

Proposition 7.5.4. *The equation of any circle can be written as :*

$$\begin{aligned} \Omega\left(x, y, z\right) \doteq \left(ux + vy + wz\right) + \Gamma_{std}\left(x, y, r\right) &= 0 \\ \text{where } u = power\left(\Omega,\ A\right),\ \text{etc} \end{aligned} \tag{7.15}$$

where $x : y : z$ is the generic point and Γ_{std} is the standard equation of the circumcircle, as defined just above.

Proof. Obvious from (7.13) and $power\left(\Gamma,\ A\right) = 0$, etc. □

Remark 7.5.5. "Can be written" must be understood as "when required, multiply by $x+y+z$ and use polynomials". For more details, see Chapter 13.

Corollary 7.5.6 (Heron)**.** *Center and radius of the circumcircle are :*

$$\begin{aligned} X_3 &= a^2\left(b^2 + c^2 - a^2\right) : b^2\left(c^2 + a^2 - b^2\right) : c^2\left(a^2 + b^2 - c^2\right) \\ R^2 &= \frac{a^2b^2c^2}{\left(a + b + c\right)\left(a + b - c\right)\left(b + c - a\right)\left(c + a - b\right)} \end{aligned}$$

Computed Proof. Direct elimination from $\left\{\left|XA\right|^2 = R^2,\ \text{etc}\right\}$ and (7.1,7.12). □

Proposition 7.5.7. *For $t \in \mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^2\right)$, point*

$$U \simeq 1 : t : -1 - t \tag{7.16}$$

belongs to $\mathcal{L}_b$. On the contrary, the point

$$V \doteq \frac{a^2}{1} : \frac{b^2}{t} : \frac{-c^2}{1 + t} \tag{7.17}$$

belongs to the circumcircle, inducing a rational parametrization of this curve, and a bijection $\mathcal{L}_\infty \longleftrightarrow \Gamma$.

Proof. Quite obvious and nevertheless so useful. Spoiler: this is another encounter with the isogonal conjugacy $u : v : w \mapsto a^2/u : b^2/v : c^2/w$. □

Remark 7.5.8. Information conveyed by a 3-tuple like (7.1) is multiple. A first part is the direction of line PU, described –up to a proportionality factor– by the point $\rho : \sigma : \tau \in \mathcal{L}_b$. Another part is the squared length $\left|PU\right|^2$ given by (7.12). In this formula, circumcircle appears as the conic that defines how lengths are computed in each direction.

7.6 Orthogonality

Proposition 7.6.1. *Let $V \in \mathbb{P}_{\mathbb{R}}(\mathbb{R}^3)$ be the point at infinity of a line given by its barycentrics Δ. Then we have*

$$V \doteq \Delta \wedge \mathcal{L}_b \simeq \boxed{W_b} \cdot ({}^t\Delta) \qquad \text{where } \boxed{W_b} \doteq \frac{1}{2S}\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \tag{7.18}$$

Proof. This assertion is nothing but the very definition of the $\wedge$ operator, while the $1/2S$ coefficient is not involved in this property. □

Proposition 7.6.2. *A point at infinity $V \in \mathcal{L}_b$ defines a direction of lines. The point $V^\perp \in \mathcal{L}_b$ that defines the orthogonal direction is called the orthopoint of V. And we have:*

$$V^\perp \simeq \boxed{\mathrm{OrtO}_b} \cdot V \qquad \text{where } \boxed{\mathrm{OrtO}} \simeq {}^t\boxed{W_b} \cdot \boxed{\mathrm{Pyth}_b}$$

Proof. Here again, the coefficient is not involved in the property, which simply results from substituting $V \simeq 1 : t : -1-t$ into

$${}^tV \cdot \boxed{\mathrm{Pyth}_b} \cdot {}^t\boxed{W_b} \cdot \boxed{\mathrm{Pyth}_b} \cdot V = 0$$

□

Remark 7.6.3. As discussed in the next section, there are many "formal rules" like:

$$\begin{aligned} V^\perp &= \frac{+1}{4S}\,\mathcal{L}_b \wedge {}^t\!\left(V \underset{b}{\div} X_4\right) \\ &\simeq {}^t(V \wedge X_4) \underset{b}{*} X_4 \\ \boxed{OrtH} &= \frac{1}{2S}\begin{bmatrix} 0 & -S_b & +S_c \\ +S_a & 0 & -S_c \\ -S_a & +S_b & 0 \end{bmatrix} \end{aligned}$$

They all provide the same result when $V \in \overrightarrow{\mathcal{V}}$.

Theorem 7.6.4. ***Orthopoint transform*** . *When exactly defined as*

$$\boxed{\mathrm{OrtO}_b} \doteq {}^t\boxed{W_b} \cdot \boxed{\mathrm{Pyth}_b} = \frac{1}{4S}\begin{pmatrix} c^2-b^2 & -a^2 & a^2 \\ b^2 & a^2-c^2 & -b^2 \\ -c^2 & c^2 & b^2-a^2 \end{pmatrix} \tag{7.19}$$

the matrix $\boxed{\mathrm{OrtO}_b}$ describes the $+90°$ rotation in the $\overrightarrow{\mathcal{V}}$ space, while it's transpose matrix $\boxed{W_b} \cdot \boxed{\mathrm{Pyth}_b}$ describes the $-90°$ rotation. Spoiler: Matrix $\boxed{\mathrm{OrtO}_b}$ provides $\Omega_y \mapsto +\boldsymbol{i}\Omega_y$, $\Omega_x \mapsto -\boldsymbol{i}\Omega_x$. See the next coming Proposition 7.6.7.

Proof. We have $\mathrm{Charpoly}\left(\boxed{\mathrm{OrtO}_b}, \mu\right) = \mu^3 + \mu = \mu(\mu - \boldsymbol{i})(\mu + \boldsymbol{i})$. The factor at (7.19) was precisely chosen to enforce the conservation of length, as it could be easily checked. We have indeed

$${}^tV \cdot \boxed{\mathrm{Pyth}_b} \cdot V = {}^tV \cdot {}^t\boxed{\mathrm{OrtO}_b} \cdot \boxed{\mathrm{Pyth}_b} \cdot \boxed{\mathrm{OrtO}_b} \cdot V$$

when $V \in \overrightarrow{\mathcal{V}}$. It remains to prove the orientation. Go back to the ordinary Cartesian coordinates by(7.4), substitute the squared sidelengths using (7.5), and obtain :

$$\boxed{T_0} \cdot \boxed{\mathrm{OrtO}_b} \cdot \boxed{T_0}^{-1} = \begin{pmatrix} 0 & -1 & * \\ 1 & 0 & * \\ 0 & 0 & * \end{pmatrix} = \boxed{\mathrm{OrtO}_c}$$

□

Proposition 7.6.5. *The* **orthodir** *U of any line Δ (except from the line at infinity) is defined as the orthopoint of $\Delta \wedge \mathcal{L}_b$. It can be computed as $U = \boxed{\mathcal{M}_b} \cdot {}^t\Delta$ where :*

$$\boxed{\mathcal{M}_b} \doteq \boxed{\mathrm{OrtO}_b} \cdot \boxed{W_b} = {}^t\boxed{W_b} \cdot \boxed{\mathrm{Pyth}_b} \cdot \boxed{W_b} = \frac{1}{4S^2}\begin{pmatrix} a^2 & -S_c & -S_b \\ -S_c & b^2 & -S_a \\ -S_b & -S_a & c^2 \end{pmatrix} \tag{7.20}$$

Proof. This comes directly from the orthopoint formula. Normalization factor $1/4S^2$ is useless here, but will be required when measuring, i.e. computing angles or distances. Moreover, one can check that, for example, the first column gives the direction of the first altitude (orthogonal to sideline BC). □

Remark 7.6.6. Characteristic polynomial of $S \times \boxed{\mathcal{M}_b}$ is : $\chi(\mu) = \mu^3 + \mu^2 S_\omega / S + 3\mu$ and it can be checked that its left null space is $[1,1,1]$, the row associated with $\mathcal{L}_b$: for any column X, $\boxed{\mathcal{M}_b} \cdot X \in \mathcal{L}_b$ (as it should be).

Proposition 7.6.7. ***Isotropic lines.*** *A line $[f,g,h]$ that satisfies $\Delta \cdot \boxed{\mathcal{M}_b} \cdot {}^t\Delta = 0$ is orthogonal to itself, whence the name 'isotropic' given to these lines. This equation can be factored:*

$$\begin{aligned} \Delta \cdot \boxed{\mathcal{M}_b} \cdot {}^t\Delta &\doteq \frac{1}{2S}\left(a^2 f^2 + b^2 g^2 + c^2 h^2 - 2\,S_a\,gh - 2\,S_b\,hf - 2\,S_c\,fg\right) \\ &\simeq (\Delta \cdot \Omega_x) \times (\Delta \cdot \Omega_y) \end{aligned}$$

so that an isotropic line is a line that goes through one of the so-called ***umbilics*** *of the plane, whose barycentrics are:*

$$\Omega_x \simeq \begin{pmatrix} S_b - 2iS \\ S_a + 2iS \\ -c^2 \end{pmatrix} \; ; \; \Omega_y \simeq \begin{pmatrix} S_b - 2iS \\ S_a + 2iS \\ -c^2 \end{pmatrix}$$

Computed Proof. The possibility of this factorization comes from $\det \boxed{\mathcal{M}_b} = 0$. But a formal computing tool rather factorizes in the $\mathbb{Q}$ field, and won't guess to use $\mathbb{Q}\left(\sqrt{-S^2}\right)$! □

Proof. Let us proceed by undetermined coefficients wrt f,g,h. Eliminating $u : v : w$ in the equation:

$$\Delta \cdot \boxed{\mathcal{M}_b} \cdot {}^t\Delta = (fp + gq + hr)\,(fu + gv + hw)$$

leads to $u = a^2/p$, $v = b^2/q$, $w = c^2/r$ (i.e. these points are isogonal conjugate of each other). And then solving in q, r leads to the given values. The appearance of these results is rather not symmetric... but using

$$\Omega_x \simeq \begin{pmatrix} \left(b^4 + c^4 - a^2 b^2 - a^2 c^2\right) a^2 + 4\boldsymbol{i}\left(b^2 - c^2\right) a^2 S \\ \left(c^4 + a^4 - a^2 b^2 - b^2 c^2\right) b^2 + 4\boldsymbol{i}\left(c^2 - a^2\right) b^2 S \\ \left(a^4 + b^4 - a^2 c^2 - b^2 c^2\right) c^2 + 4\boldsymbol{i}\left(a^2 - b^2\right) c^2 S \end{pmatrix}$$

is rather too convoluted to be useful. □

Remark 7.6.8. Spoiler: all the isotropic lines form a 'degenerate tangential conic' (see Proposition 12.4.2). Therefore, the adjoint of matrix $\boxed{\mathcal{M}_b}$ is ${}^t\mathcal{L}_b \cdot \mathcal{L}_b$ (the "all ones" matrix) and describes the line Ω_x, Ω_y, i.e. the $\mathcal{L}_b$ line. Rank is one (since $\boxed{\mathcal{M}_b}$ has rank $n-1$), so that product $\mathrm{Adjoint}\left(\boxed{\mathcal{M}_b}\right) \cdot \boxed{\mathcal{M}_b}$ is the null matrix.

7.7 Angles between straight lines

Proposition 7.7.1. *Let P, U_1, U_2 be three points at finite distance, such that $\overrightarrow{PU_i} \neq \overrightarrow{0}$. Then :*

$$\begin{aligned} |PU_1| \,.\, |PU_2| \,.\, \sin\left(\overrightarrow{PU_1}, \overrightarrow{PU_2}\right) &= 2S\,\frac{\det(P, U_1, U_2)}{(p+q+r)\prod(u_i + v_i + w_i)} \\ |PU_1| \,.\, |PU_2| \,.\, \cos\left(\overrightarrow{PU_1}, \overrightarrow{PU_2}\right) &= 2S\,\frac{(P \wedge U_1) \cdot \boxed{\mathcal{M}_b} \cdot {}^t(P \wedge U_2)}{(p+q+r)^2 \prod(u_i + v_i + w_i)} \end{aligned}$$

Proof. The sin formula comes from (7.10), while the cos formula can be obtained by using (7.12) into $|PU_1|^2 + |PU_2|^2 - |U_1U_2|^2$ and rearranging. □

Theorem 7.7.2. *Let P be a point at finite distance, and U_1, U_2 two other points (at finite distance or not). Then the angle between straight lines PU_1, PU_2 is characterized by its* ***tangent****, according to :*

$$\tan\left(\widehat{PU_1,\ PU_2}\right) = (p+q+r)\,\frac{\det(P, U_1, U_2)}{(P\wedge U_1)\cdot\boxed{\mathcal{M}_b}\cdot{}^t(P\wedge U_2)} \tag{7.21}$$

When U_1, U_2 are at infinity, the angle between all the lines having the given directions can be computed as :

$$\tan_\infty\left(\widehat{U_1,\ U_2}\right) = \frac{2S\,(v_1w_2 - w_1v_2)}{(v_1w_2 + w_1v_2)\,S_a + w_1w_2b^2 + v_1v_2c^2}$$

Proof. The key point here is that formula (7.21) is square-root free. Extension to U_i at infinity is obtained by continuity after cancellation of the $(u_i + v_i + w_i)$. Formula $\tan_\infty$ is not formally symmetric ($P = A$ has been used). But the u_i are nevertheless present since $u_i = -v_i - w_i$. □

Theorem 7.7.3. Tangent of two lines. *If the triangle plane is oriented according to $\left(\widehat{AB,\ AC}\right) = +A$, then the oriented angle from line Δ_1 to line Δ_2 is characterized by :*

$$\tan\left(\widehat{\Delta_1,\ \Delta_2}\right) = \frac{\Delta_1\cdot\boxed{W_b}\cdot{}^t\Delta_2}{\Delta_1\cdot\boxed{\mathcal{M}_b}\cdot{}^t\Delta_2} \tag{7.22}$$

where $\boxed{W_b}$ and $\boxed{\mathcal{M}_b}$ are exactly as given in (7.6) and (7.20) (i.e. not up to a proportionality factor).

Proof. Simple use of $\Delta_1 \wedge \Delta_2$, $\mathcal{L}_b \wedge \Delta_1$, $\mathcal{L}_b \wedge \Delta_2$ in (7.21). Among other things, this formula tells us that $\vartheta = 0$ when each line contain the point at infinity of the other, while $|\vartheta| = \pi/2$ when each line contains the orthopoint of the other (formula is anti-symmetric). □

Stratospheric proof. Start from the affine space $\mathcal{E}$. Equations of both lines are $a_j\xi + b_j\eta + c_j = 0$ and their angle is given by

$$\tan\left(\widehat{\Delta_1,\ \Delta_2}\right) = \frac{a_1b_2 - a_2b_1}{a_1a_2 + b_1b_2} = \frac{\det(\mathcal{L}_3,\ D_1,\ D_2)}{D_1\cdot\boxed{Orth_3}\cdot{}^tD_2}$$

where $\mathcal{L}_3 = [0, 0, 1]$ and $\boxed{Orth_3}$ is the matrix of quadratic form $\xi^2 + \eta^2$ –precisely this one, without any of the extra terms used in Proposition 7.2.3. Taking now ABC for basis, we have $\Delta_j = D_j \cdot T_0^{-1}$, inducing a factor $1/2S$ in the numerator. Let us now compare the following two expressions :****

$${}^tT_0^{-1} = \frac{1}{2S}\begin{bmatrix} \eta_b - \eta_c & -\eta_a + \eta_c & \eta_a - \eta_b \\ -\xi_b + \xi_c & \xi_a - \xi_c & -\xi_a + \xi_b \\ \xi_b\eta_c - \xi_c\eta_b & -\xi_a\eta_c + \xi_c\eta_a & \xi_a\eta_b - \xi_b\eta_a \end{bmatrix}$$

$$T_0\cdot\boxed{W_b} = \begin{bmatrix} -\xi_b + \xi_c & \xi_a - \xi_c & -\xi_a + \xi_b \\ -\eta_b + \eta_c & \eta_a - \eta_c & -\eta_a + \eta_b \\ 0 & 0 & 0 \end{bmatrix}$$

Due to the specific value of $\boxed{Orth_3}$, we can replace ${}^tT_0^{-1}$ by $T_0\cdot\boxed{W_b}/2S$ in the change of basis formulas and obtain $\boxed{Orth_3} \mapsto \boxed{\mathcal{K}}/(2S)^2$. Using the orthodir matrix (instead of the Al-Kashi one) leads to a formula without remaining factors. □

Theorem 7.7.4. Cosinus of two lines. *The non-oriented angle between lines Δ_1 and Δ_2 is characterized by :*

$$\cos(\Delta_1, \Delta_2) = \frac{\Delta_1\cdot\boxed{\mathcal{M}_b}\cdot{}^t\Delta_2}{\sqrt{\Delta_1\cdot\boxed{\mathcal{M}_b}\cdot{}^t\Delta_1}\sqrt{\Delta_2\cdot\boxed{\mathcal{M}_b}\cdot{}^t\Delta_2}} \tag{7.23}$$

namex	ψ	#	barycentrics	#	inclusive	#	cartesian
$\mathcal{L}_b$	2		$[1;1;1]$	(15.1)	$[0;1;0]$		linfz
pinf $\boxed{W_b}$	3	(7.6)	$\dfrac{1}{4S}\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$	(15.7)	$\begin{pmatrix} 0 & 0 & -\boldsymbol{i} \\ 0 & 0 & 0 \\ +\boldsymbol{i} & 0 & 0 \end{pmatrix}$		$\dfrac{1}{2}\begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
pythagoras $\boxed{\mathrm{Pyth}_b}$	5	(7.11)	$\dfrac{1}{2}\begin{pmatrix} 0 & -c^2 & -b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{pmatrix}$	(15.5)	$\dfrac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
orthopoint $\boxed{\mathrm{OrtO}}$	4	(7.19)	$\dfrac{1}{4S}\begin{pmatrix} c^2-b^2 & -a^2 & a^2 \\ b^2 & a^2-c^2 & -b^2 \\ -c^2 & c^2 & b^2-a^2 \end{pmatrix}$	(15.8)	$i\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$		$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
orthodir $\boxed{\mathcal{M}}$	3	(7.20)	$\dfrac{1}{8S^2}\begin{pmatrix} a^2 & -S_c & -S_b \\ -S_c & b^2 & -S_a \\ -S_b & -S_a & c^2 \end{pmatrix}$	(15.9)	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
tangent of $\left(\overbrace{\Delta_1, \Delta_2}\right)$		(7.22)	$\dfrac{\Delta_1 \cdot \boxed{W_b} \cdot {}^t\Delta_2}{\Delta_1 \cdot \boxed{\mathcal{M}_b} \cdot {}^t\Delta_2}$	(15.10)	$\dfrac{\Delta_1 \cdot \boxed{W_z} \cdot {}^t\Delta_2}{\Delta_1 \cdot \boxed{\mathcal{M}_z} \cdot {}^t\Delta_2}$		idem
circles $\boxed{\mathcal{Q}}$			$\frac{-1}{8S^2}\begin{bmatrix} a^2 & -S_c & -S_b & -a^2S_a \\ -S_c & b^2 & -S_a & -b^2S_b \\ -S_b & -S_a & c^2 & -c^2S_c \\ -a^2S_a & -b^2S_b & -c^2S_c & a^2b^2c^2 \end{bmatrix}$	(14.9) (19.3)	$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{bmatrix}$
$\boxed{\mathrm{OrtO}} = 2\,{}^t\boxed{W} \cdot \boxed{\mathrm{Pyth}}$; $\boxed{\mathcal{M}} = 2\,{}^t\boxed{W} \cdot \boxed{\mathrm{Pyth}} \cdot \boxed{W}$ ********							
transform $bar \mapsto zin$: (1) point: $X \mapsto \boxed{Lu}X$; (2) line: $X \mapsto X\boxed{Lu^{-1}}$ $\boxed{\mathrm{b\Phi m}} = \left[[R\alpha, R\beta, R\gamma], [1,1,1], \left[\frac{R}{\alpha}, \frac{R}{\beta}, \frac{R}{\gamma}\right]\right]$ (3) line to point $\boxed{Lu}X^t\boxed{Lu}$; (4) point to point $\boxed{Lu}X\boxed{Lu^{-1}}$; (5) quad form ${}^t\boxed{Lu^{-1}}X\boxed{Lu^{-1}}$							

Table 7.1: All these matrices

Proof. This is nothing but the general formula in an Euclidian space. Nevertheless, this result can be checked using Proposition 7.7.1. □

7.8 Rotations in the barycentric plane

Lemma 7.8.1. *When M is an ordinary point, then $\pi_1 \doteq (M \cdot \mathcal{L}_b) / (\mathcal{L}_b \cdot M)$ and $\pi_2 \doteq \boxed{1} - \pi_1$ are projectors.*

Lemma 7.8.2. *Then $\pi_1 \cdot \boxed{\mathrm{OrtO}} = 0$, and therefore $\pi_2 \cdot \boxed{\mathrm{OrtO}} = \boxed{\mathrm{OrtO}}$*

Proposition 7.8.3. *The matrix of the rotation with angle ϑ and center M is:*

$$\boxed{rot} \doteq \pi_1 + \sin\vartheta \boxed{\mathrm{OrtO}} \cdot \pi_2 + \cos\vartheta\, \pi_2$$

Proof. We obviously have $\boxed{rot} \cdot M = M$. From $\pi_2 \cdot \Omega_y = \Omega_y$, we have $\boxed{rot} \cdot \Omega_y = (\cos\vartheta + \boldsymbol{i}\sin\vartheta)\, \Omega_y$, etc, as it should be. □

7.9 Distance from a point to a line

Definition 7.9.1. The distance from a point P to a line Δ is the lower bound of the distance from P to a point U that belongs to Δ. By continuity, this bound is attained and is equal to the distance of P to its orthogonal projection P_0 on Δ.

Theorem 7.9.2. ***Distance from point** P **to line** Δ is given by :*

$$\operatorname{dist}(P, \Delta) \underset{b}{=} \frac{\Delta \cdot P}{(\mathcal{L}_b \cdot P)\sqrt{\Delta \cdot \boxed{\mathcal{M}_b} \cdot {}^t\Delta}} \tag{7.24}$$

where $\mathcal{L}_b = [1, 1, 1]$ and $\boxed{\mathcal{M}_b}$ is as given in (7.20) (not up to a proportionality factor).

Proof. Project P into Q and compute $\sqrt{|PQ|^2}$. □

Remark 7.9.3. Formula (7.24) is invariant when barycentrics of P or Δ are modified by a proportionality factor. Denominators are enforcing the fact that P is supposed to be finite, and Δ is not supposed to be an isotropic line.

Exercise 7.9.4. Check this formula by computing $\operatorname{dist}(A, BC)$.

Proposition 7.9.5. *The distance between two parallel lines is defined as the distance from one of the lines to a point of the other. We have the formula:*

$$\operatorname{dist}(\Delta, \Delta + \mu\mathcal{L}_b) = |\mu| \frac{\sqrt{2S}}{\sqrt{\Delta \cdot \boxed{\mathcal{M}_b} \cdot {}^t\Delta}}$$

Proof. Apply (7.24) with $(1/f : t/g : -(1+t)/h) \in \Delta$. □

7.10 More about the Pyth matrix

As stated in Proposition 7.2.3, any matrix

$$\boxed{\mathrm{Pyth}_U} \doteq \boxed{\mathrm{Pyth}_b} + \frac{1}{2}\left(U \cdot \mathcal{L}_b + {}^t(U \cdot \mathcal{L}_b)\right)$$

can be used to define the euclidian metric of the $\overrightarrow{\mathcal{V}}$ vector space. Let us discuss this "degree of freedom" and note $U \simeq u : v : w$.

Proposition 7.10.1. *When applied to points $X \in \mathbb{P}_{\mathbb{C}}(\mathbb{C}^3)$, matrix $\boxed{\mathrm{Pyth}_U}$ describes a visible circle $\mathcal{C}_U$, characterized by power $(\mathcal{C}_U, A) = u$, etc.*

Proof. Obvious from the equation ${}^{t}X \cdot \boxed{\mathrm{Pyth}_U} \cdot X = 0$ and the power formula (7.13). $\square$

Remark 7.10.2. Spoiler: the Veronese representation of circle $\mathcal{C}_U$ is therefore $u : v : w : 1$. Remember:

$$\underset{b}{Ver}\,(x:y:z)\cdot(u:v:w:1)=0 \Longleftrightarrow -\left(a^2yz+b^2zx+c^2xy\right)+(x+y+z)\,(ux+vy+wz)=0$$

And therefore, any circle can be used: the center and the radius of $\mathcal{C}_U$ can be chosen at will.

Example 7.10.3. The choice $U = 0 : 0 : 0$ leads to $\mathcal{C}_0 = \Gamma = (O, R)$, which is nothing else than the circumscribed circle.

Example 7.10.4. Requiring a diagonal form for $\boxed{\mathrm{Pyth}_U}$ leads to

$$\boxed{\mathrm{Pyth}_H} \doteq \left[\begin{array}{ccc} S_a & 0 & 0 \\ 0 & S_b & 0 \\ 0 & 0 & S_c \end{array}\right]$$

that uses the polar circle $\mathcal{C}_H = \left(H\ ;\ \sqrt{-S_aS_bS_c} \div 2S\right)$ as isotropic circle (see Section 13.7).

Example 7.10.5. Spoiler: the choice given by the change of basis formula applied to the Morley formula $\mathbf{Z}\overline{\mathcal{Z}} = 0$ leads to the $\mathcal{C}_M = (O; 0) = (O)$, the point circle centered at X(3):

$$\mathcal{C}_M = \frac{1}{2}\left[\begin{array}{ccc} 2R^2 & 2R^2-c^2 & 2R^2-b^2 \\ 2R^2-c^2 & 2R^2 & 2R^2-a^2 \\ 2R^2-b^2 & 2R^2-a^2 & 2R^2 \end{array}\right]$$

Proposition 7.10.6. *Let $P = (p, q, r) \in \mathbb{R}^3$ be a 3-uple that does not belongs to $\overrightarrow{\mathcal{V}}$. Then it exists a linear transform ψ such that (i) $\psi(P) = 0$, (ii) $\psi\left(\overrightarrow{\mathcal{V}}\right) = \overrightarrow{\mathcal{V}}$ (iii) for all $V \in \overrightarrow{\mathcal{V}}$, $\langle \psi(V) \mid \psi(V)\rangle = \langle V \mid V\rangle$ while $\langle \psi(V) \mid V\rangle = 0$. Then its characteristic polynomial is $\chi(\mu) = \mu^3 + \mu$ and we have :*

$$Orth\,(P) \quad = \quad \frac{1}{2S\,(p+q+r)}\left(\begin{array}{ccc} qS_b-rS_c & -\left(ra^2+pS_b\right) & qa^2+pS_c \\ rb^2+qS_a & rS_c-pS_a & -\left(pb^2+qS_c\right) \\ -\left(qc^2+rS_a\right) & pc^2+rS_b & pS_a-qS_b \end{array}\right)$$

The opposite of this matrix is the only other solution to the problem.

Computed Proof. Assertions $M \cdot P = 0$, $\mathcal{L}_b \cdot M = 0$ and $\langle \psi(V) \mid V\rangle = 0$ when $V = x : y : -x-y$ gives nine equations. Elimination leads to the given matrix, up to a coefficient. Then $\langle \psi(V) \mid \psi(V)\rangle = \langle V \mid V\rangle$ gives the coefficient. Division by $p+q+r$ enforces the fact that P is at finite distance. $\square$

Proposition 7.10.7. *For any U, matrix $\boxed{OrtU} \doteq \frac{1}{2S}\,{}^{t}\boxed{W_b}\cdot\boxed{\mathrm{Pyth}_U}$ is equal to $Orth\,(P)$ where P is the center of the isotropic circle $\mathcal{C}_U$. Thus, matrices $Orth\,(P)$ and $Orth\,(U)$ relative to finite points P, U are related by the "translation" formula :*

$$Orth(P) = Orth\,(U)\cdot\left(1-\frac{1}{p+q+r}\left(\begin{array}{c} p \\ q \\ r \end{array}\right)\cdot\mathcal{L}_b\right)$$

Proof. Acting at infinity, both matrices induces a quarter-turn. At finite distance, it remains only to move the kernel to the right place. $\square$

Remark 7.10.8. Any $Orth\,(P)$ matrix describes a quarter-turn in the euclidian plane $\overrightarrow{\mathcal{V}}$. In order to provide a better perception of this result, let O =X(3), H =X(4), put $W_3^2 = 16\,|OH|^2\,S^2 = \sum_3 a^6 - \sum_6 a^4b^2 + 3a^2b^2c^2$ and consider the linear transform (collineation) whose matrix is $\phi = [X\,(3)\,,\,X\,(30)\,/4S,\,X\,(523)]$. When computing $|OH|^2$, vector $\overrightarrow{OH}$ is involved and therefore X(30) (the direction of the Euler line), while X(523) is known to be the direction orthogonal to the Euler line. We have :

$$\phi = \begin{pmatrix} a^2\left(b^2+c^2-a^2\right) & \left[2\,a^4-\left(b^2-c^2\right)^2-a^2\left(b^2+c^2\right)\right]\div 4S & b^2-c^2 \\ b^2\left(c^2+a^2-b^2\right) & \left[2\,b^4-\left(c^2-a^2\right)^2-b^2\left(c^2+a^2\right)\right]\div 4S & c^2-a^2 \\ c^2\left(a^2+b^2-c^2\right) & \left[2\,c^4-\left(a^2-b^2\right)^2-c^2\left(a^2+b^2\right)\right]\div 4S & a^2-b^2 \end{pmatrix}$$

$$^t\phi\cdot\boxed{\mathrm{Pyth}_b}\cdot\phi = \begin{pmatrix} -16\,a^2c^2b^2S^2 & 0 & 0 \\ 0 & W_3^2 & 0 \\ 0 & 0 & W_3^2 \end{pmatrix}$$

$$\phi^{-1}\cdot\boxed{\mathrm{OrtO}}\cdot\phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$^t\phi\cdot{}^t\boxed{\mathrm{OrtO}}\cdot\boxed{\mathrm{Pyth}_b}\cdot\boxed{\mathrm{OrtO}}\cdot\phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & W_3^2 & 0 \\ 0 & 0 & W_3^2 \end{pmatrix}$$

Remark 7.10.9. Mind the signs in $\boxed{\mathcal{M}_b}$! The following matrix

$$\begin{pmatrix} a^2 & S_c & S_b \\ S_c & b^2 & S_a \\ S_b & S_a & c^2 \end{pmatrix}$$

describes the triangle of the midpoints of the altitudes. And also the Longchamps circle 13.8 (see 12.22.17).

7.11 Brocard points and the sequel

7.11.1 Some results

Proposition 7.11.1. Brocard points. *It exists exactly one point ω^+ and one point ω^- such that :*

$$\begin{array}{ccccc} \angle(A\omega^+,\,AC) & = & \angle(B\omega^+,\,BA) & = & \angle(C\omega^+,\,CB) \\ \angle(AB,\,A\omega^-) & = & \angle(BC,\,B\omega^-) & = & \angle(CA,\,C\omega^-) \end{array}$$

They are given by $\omega^+ = a^2b^2 : b^2c^2 : c^2a^2$ and $\omega^- = c^2a^2 : a^2b^2 : b^2c^2$. Moreover, when defined exactly *that way, both angles are equal. This quantity is called the Brocard angle and one has :*

$$\cot\omega = \frac{a^2+b^2+c^2}{4S} \tag{7.25}$$

Proof. Equating the tangents of the angles and eliminating, one obtains a third degree equation with one simple real root and tho others that involves $\sqrt{-S^2}$. As it should be Brocard points are isogonal conjugates of each other. □

Proposition 7.11.2. *cot versus Conway.* *We have the following equalities between Conway symbols and some cotangents:*

$$[S_a, S_b, S_c, S_\omega] = [2\,S\cot A,\ 2\,S\cot B,\ 2\,S\cot C,\ 2\,S\cot\omega]$$

$$\cot\left(\frac{A}{2}\right) = \frac{bc+S_a}{2S}\ ;\ \cot\left(\frac{B}{2}\right) = \frac{ac+S_b}{2S}\ ;\ \cot\left(\frac{C}{2}\right) = \frac{ab+S_c}{2S}$$

Proof. First three are obvious, the ω one is given just above. This proves the Volenec (2005) formula :

$$\cot\omega = \cot A + \cot B + \cot C$$ □

.

Remark 7.11.3. A subsection about the so-called Brocard triangles is located at Subsection 24.9.

Remark 7.11.4. The ETC points on the Brocard line are 9 (Mittenpunkt), 512 (at infinity), 881, 882, 2524, 2531.

Lemma 7.11.5. *Since* $\tan\omega > 0$ *and* $|\omega| \leq \pi/6$*, we have :*

$$\cos(\omega) = \frac{c^2+a^2+b^2}{2\sqrt{a^2b^2+a^2c^2+b^2c^2}}\,,\ \sin(\omega) = \frac{2S}{\sqrt{a^2b^2+a^2c^2+b^2c^2}} \tag{7.26}$$

Fact 7.11.6. *Each Brocard point is at the intersection of three isogonal circles, according to :*

$$\begin{aligned} \angle\left(\omega^+B,\,\omega^+C\right) &= \angle(BA,\,BC) \\ \angle\left(\omega^-B,\,\omega^-C\right) &= \angle(CB,\,CA) \end{aligned}$$

Proposition 7.11.7. Neuberg circles. *The locus of A when* B, C, ω *are given is a circle (Neuberg circle of vertex A). Barycentric equation, center and radius are :*

$$a^2yz + b^2xz + c^2xy - a^2(x+y+z)(y+z) = 0$$

$$N_a \simeq \begin{pmatrix} a^2\left(c^2+a^2+b^2\right) \\ \left(a^2+b^2\right)c^2 - b^4 - a^4 \\ \left(a^2+c^2\right)b^2 - c^4 - a^4 \end{pmatrix} \simeq \begin{pmatrix} -a\cos(\omega) \\ b\cos(C+\omega) \\ c\cos(B+\omega) \end{pmatrix}$$

$$\rho_A = \frac{a}{2}\sqrt{\cot^2\omega - 3}$$

Proof. Straightforward computation, replacing b^2 by $|A'C|^2$ etc. The form given proves the circular shape. □

Proposition 7.11.8. Tarry point. *Triangle* $N_aN_bN_c$ *is perspective with ABC and perspector is the Tarry point X(98)=*

$$\frac{1}{a^2b^2+a^2c^2-b^4-c^4} : \frac{1}{b^2c^2+b^2a^2-c^4-a^4} : \frac{1}{c^2a^2+c^2b^2-a^4+b^4}$$

Conversely, Neuberg center N_a *is common point of the A cevian of* X_{98} *and the perpendicular bisector of side BC, etc.*

Proof. Direct computation. □

Proposition 7.11.9. *The Steiner angles* $\omega_1 > \omega_2$ *are defined as follows.* $2\omega_1$ *is the maximal value of A when* ω *is given and* $2\omega_2$ *is the minimal value. We have the following relations :*

$$\begin{gathered} \cot 2\omega_j + 2/\cot\omega_j = \cot\omega \\ \cot\omega_1 = \cot\omega - \sqrt{\cot^2\omega - 3} \\ \cot\omega_2 = \cot\omega + \sqrt{\cot^2\omega - 3} \\ \sin(2\omega_j + \omega) = 2\sin\omega \\ \omega + \omega_1 + \omega_2 = \pi/2 \end{gathered}$$

Proof. First formula comes from (7.25) (at extremum, triangle ABC is isosceles). This gives a second degree equation whose discriminant $\cot^2\omega - 3$ is non negative, and last formula comes from $\cot(\omega_1+\omega_2)$ depends on sum and product of the $\cot\omega_j$. □

Proposition 7.11.10. *Any Neuberg circle is viewed from another vertex under angle* 2ϑ *where :*

$$\cos\vartheta = 2\sin\omega = \sin(2\omega_j + \omega)$$

Proof. The polar of B cuts circle N_a in two points T_1, T_2 (equation of second degree, $\Delta = a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2$). And we have $2\vartheta = \angle(BT_1,\,BT_2)$. A better choice is $T_0 = midpoint(T_1,\,T_2)$ and $\vartheta = \angle(BT_1,\,BT_0)$ ... taking orientation into account ! □

7.11.2 Results related to the Kiepert RH

This section has moved to Proposition 13.22.2.

7.11.3 Spoiler: study of the Neuberg pencil

Let us consider the level curves of the Brocard angle $\omega(M,B,C)$ when vertex M moves in the ABC plane. Barycentric coordinates are surely not the best system here, and using the Veronese map is probably not required. Therefore, this subsection should rather be considered as a spoiler related to the chapter devoted to pencils of cycles.

1. Let us note $M \simeq x:y:z$, $|MB| = \gamma$, $|MC| = \beta$ and $u/v = \tan\omega$. Then

$$\frac{u}{v} = \frac{a^2+\beta^2+\gamma^2}{4\,S_M} \qquad \text{where } \beta^2 = \frac{z\,(x+z)\,a^2 + x\,(x+z)\,c^2 - b^2 zx}{(x+y+z)^2},\ \text{etc.}\ ;\ S_M = \frac{x\,S}{x+y+z}$$

2. It can be seen that numerator of $u/v - \cdots$ collects nicely, leading to equation

$$v \times \mathcal{C}\,(x,y,z) - 4\,uS \times (x\,(x+y+z)) = 0$$

3. In the second term, one recognizes that $x\,(x+y+z)$ describes the line BC seen as a cycle, i.e. completed by the line at infinity. On the other hand, the conic $\mathcal{C}_0\,(x,y,z)$ can be recognized as the circle whose equation is the column : $V \simeq S_\omega : a^2 : a^2 : 1$. And therefore is centered at $\boxed{\underset{b}{\mathcal{Q}}} \cdot V \simeq 0:1:1:$ and has radius $\sqrt{{}^tV \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot V} = ia\sqrt{3}/2$.

4. Thus the set of the level curves is the pencil of cycles generated by the two special cases, i.e. the Neuberg pencil. Since $\mathcal{C}_0$, the cycle centered on the radical axis, is virtual, the pencil is an *isotomic* pencil, whose limit points $P_\pm$ are on the real circle associated with $\mathcal{C}_0$: these points are the vertices of the equilateral triangles whose basis is BC.

5. From $u^2/(u^2+v^2) = \cos(\omega_M)^2$, we have $u = v\ \cot(\omega_M)$ so that the level curves are the circles:

$$\begin{aligned} V &\simeq S_\omega - 2\,S\ \cot(\omega)\ :\ a^2\ :\ a^2\ :\ 1 \\ \boxed{\underset{b}{\mathcal{Q}}} \cdot V &\simeq a\ \cos(\omega) : -b\ \cos(\omega+C) : -c\ \cos(\omega+B) : -abc\ \cos(A+\omega) \\ \sqrt{{}^tV \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot V} &= \frac{1}{2}\ a\ \sqrt{\cot^2\omega - 3} \end{aligned}$$

6. And therefore, $\mathcal{C}_0$ is the (virtual) locus of vertices M such that $\omega_M = 90°$.

7.11.4 Spoiler: Brocard angle of a pedal triangle

1. Consider $M \simeq x:y:z$ and its pedal triangle. Apart from a common factor $1 \div (2R\ (x+y+z))^2$, area and squared sidelengths are:

$$S_M = S\left(a^2yz + b^2zx + c^2yx\right)\ ;\ a_M^2 = a^2\left(z\,(y+z)\,b^2 + y\,(y+z)\,c^2 - a^2yz\right)\ ;\ etc.$$

Thus the level curves of the Brocard angle are given by

$$\frac{u}{v} = \cot(\omega_M) = \frac{a_M^2 + b_M^2 + c_M^2}{4S_M}$$

2. As before, the numerator collects nicely and the locus is given by:

$$v\ \mathcal{C}\,(x,y,z) - 4u\ S\left(a^2yz + b^2zx + c^2yx\right) = 0$$

3. This locus is a circle, whose Veronese image V, center E and radius ρ are:

$$V \simeq \begin{pmatrix} b^2c^2v \\ c^2a^2v \\ a^2b^2v \\ 2\,uS+S_\omega\,v \end{pmatrix}$$

$$E \simeq \begin{pmatrix} a^2\,(2\,Sv+S_a\,u) \\ b^2\,(2\,Sv+S_b\,u) \\ c^2\,(2\,Sv+S_c\,u) \end{pmatrix} \simeq \begin{pmatrix} a\cos(A-\omega_M) \\ b\cos(B-\omega_M) \\ c\cos(C-\omega_M) \end{pmatrix} \;;\; \rho = \frac{R\sqrt{\cot^2(\omega_M)-3}}{\cot(\omega_M)+\cot(\omega)}$$

4. One sees that $v=0$ describes the circumcircle. Projections of M are aligned on the Simson line, so that $\omega_M=0$.

5. One sees that $\omega_M=\pm 30°$, i.e. $v=\cot(\omega_M)=\pm\sqrt{3},\; u=1$ characterizes two points ($\rho=0$). They are X(15), X(16), the **isodynamic points**. No other pedal triangle is equilateral.

6. The locus characterized by $\omega_M=\omega_{ABC}$ is centered at X(182). One identifies the 3-6-Brocard circle. The locus characterized by $\omega_M=-\omega_{ABC}$ is a straight line ($\rho=\infty$). Therefore, this line is the perpendicular bisector of segment X(15),X(16): the line X(187),X(512).

7.12 Orthogonal projector onto a line

Proposition 7.12.1. *The matrix π_Δ of the **orthogonal projector** onto line $\Delta \simeq [p,q,r]$ –not the line at infinity – is given by :*

$$\pi_\Delta = \mathrm{Id} - \frac{\boxed{\mathcal{M}_b}\cdot {}^t\Delta\cdot\Delta}{\Delta\cdot\boxed{\mathcal{M}_b}\cdot {}^t\Delta} \tag{7.27}$$

where matrix $\boxed{\mathcal{M}_b}$ is defined by (7.20).

Proof. Let $\overrightarrow{\delta}=\boxed{\mathcal{M}_b}\cdot {}^t\Delta$ be the orthodir of Δ and P a generic point of Δ. Then $\pi_\Delta=\mathrm{Id}-\left(\overrightarrow{\delta}\cdot\Delta\right)\div\left(\Delta\cdot\overrightarrow{\delta}\right)$. Then $\pi_\Delta(P)=P$ since $\left(\overrightarrow{\delta}\cdot\Delta\right)\cdot P=\overrightarrow{\delta}\cdot(\Delta\cdot P)=0$. On the other hand, one has $\left(\overrightarrow{\delta}\cdot\Delta\right)\cdot\overrightarrow{\delta}=\overrightarrow{\delta}\cdot\left(\Delta\cdot\overrightarrow{\delta}\right)$ so $\pi_\Delta\left(\overrightarrow{\delta}\right)=0$. As a result, $\pi_\Delta\left(P+\lambda\overrightarrow{\delta}\right)=P$, as it should be. □

Proposition 7.12.2. *The matrix σ_Δ of the **orthogonal reflection** wrt line $\Delta \simeq [p,q,r]$ –not the line at infinity – is given by :*

$$\sigma_\Delta = \mathrm{Id} - 2\,\frac{\boxed{\mathcal{M}_b}\cdot {}^t\Delta\cdot\Delta}{\Delta\cdot\boxed{\mathcal{M}_b}\cdot {}^t\Delta} \tag{7.28}$$

where matrix $\boxed{\mathcal{M}_b}$ is defined by (7.20).

Proof. Obvious from the preceding proof. □

Proposition 7.12.3. Cosine of a projection. *Consider line $\Delta_1 \simeq [p,q,r]$ and use $\overrightarrow{e_1}=(q-r,r-p,p-q)$ as unit vector for this direction. Consider also line $\Delta_2 \simeq [u,v,w]$ and use $\overrightarrow{e_2}=(v-w,w-u,u-v)$ as unit vector for that other direction. Then orthogonal projection π onto Δ_1 transforms Δ_2-vectors into Δ_1-vectors according to :*

$$\pi\left(\overrightarrow{e_2}\right) = \overrightarrow{e_1}\,\frac{\Delta_1\cdot\boxed{\mathcal{M}_b}\cdot {}^t\Delta_2}{\Delta_1\cdot\boxed{\mathcal{M}_b}\cdot {}^t\Delta_1}$$

Proof. Formula is homogeneous, as it should be. Vectors $\overrightarrow{e_i}$ are not normalized, that the reason why this formula is square-root free. Formula $\boxed{\mathcal{M}_b}=2\,{}^t\boxed{W_b}\cdot\boxed{\mathrm{Pyth}_b}\cdot\boxed{W_b}$ indicates that scaling factor can be interpreted in terms of a "cosine of projection". □

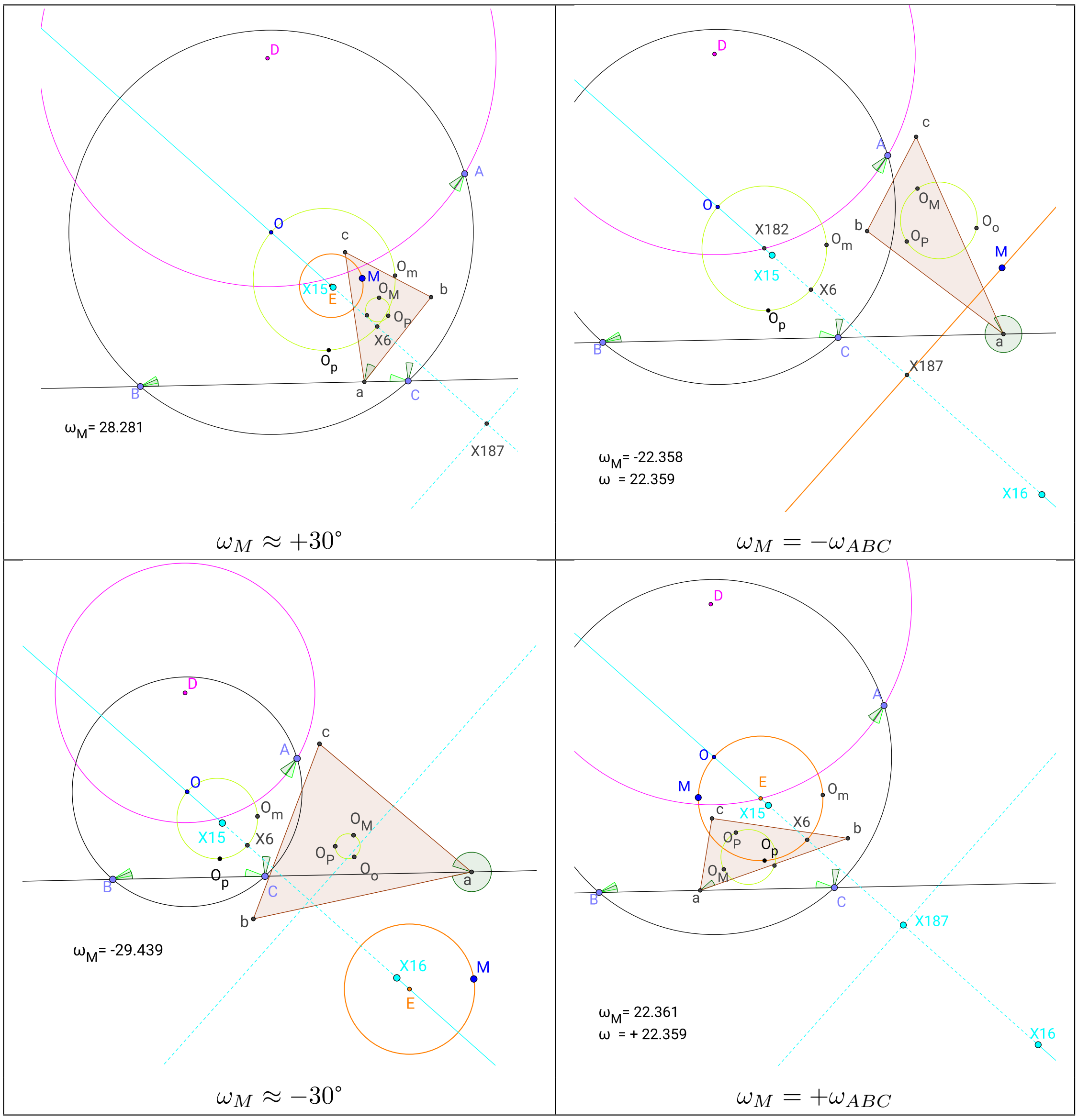


Figure 7.1: Level curves of the Brocard angle of the pedal triangle of point M

Proposition 7.12.4. *Consider the* ***homothety*** *of center $P = p : q : r$ (not on the infinity line) and ratio k (not 0 !). Then points U are transformed as $U \mapsto h(P,k) \cdot U$ while lines Δ are transformed as $\Delta \mapsto \Delta \cdot h(P, 1/k)$ where :*

$$h(P,k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1-k}{k(p+q+r)} \begin{bmatrix} p & p & p \\ q & q & q \\ r & r & r \end{bmatrix} \simeq k + (1-k)\,\frac{P \cdot \mathcal{L}_b}{\mathcal{L}_b \cdot P} \tag{7.29}$$

Proof. Applied to column P, the last formula gives P. Applied to a vector $\overrightarrow{V}$, this gives $k\overrightarrow{V}$ since a vector is defined by $\mathcal{L}_b \cdot \overrightarrow{V} = 0$. Another method: we want $U \mapsto X$ such that, given U, we have $(X - P) = k(U - P)$. Expressed in barycentrics, this leads to :

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \simeq \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \frac{(1-k)(u+v+w)}{k(p+q+r)} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

Property about lines comes from the fact that action over lines and action over points are inverse of each other. □

7.13 The hortocenter romance

The three perpendicular bissectors of a triangle intersect at a single point. The proof is well known. Note them $\mu(B,C)$, etc and define P as $\mu(A,B) \cap \mu(B,C)$. Then $PA = PB$ together with $PB = PC$ (from Pythagoras theorem). This implies $PA = PC$ and proves $P \in \mu(A,C)$.

And now, define $A' = B + C - A$, etc. The $\mu(B'C')$, etc intersect at some point H, while $A = (B' + C')$, so that $\mu(B'C')$ is also the altitude issued from A and perpendicular to $BC \parallel B'C'$. As a result, we have proven that the three altitudes of a triangle intersect at a single point. This enlighthening proof appears to be a recent one... while the property itself is not listed in the Greek geometry books (Bogomolny, 2015).

In this section, we will explore what happens when this result is taken as an axiom.

Definition 7.13.1. Orthopoints. Let us take a triangle ABC in the projective plane and choose the point $H \simeq p : q : r$ as the "rightfull center" together with choosing the line $\mathcal{L}_b \doteq [f, g, h]$ as the line at infinity. Points $H_a \doteq (HA) \cap (BC)$, etc are called the cevian feets of H. And then directions $D_a \doteq (BC) \cap \mathcal{L}_b$ and $T_a \doteq (AH) \cap \mathcal{L}_b$ are called orthopoints of each other (this relation will be extended later into an involution of $\mathcal{L}_b$).

Definition 7.13.2. Holy Garden and hortocenter. Let R_a be the fourth-harmonic of H wrt H_a, T_a, etc. Then A, B, C, R_a, R_b, R_c are co-conic. Let us call this conic Γ the holly garden, i.e. the most beautiful circle. Its perspector is the Lemoine point (and therefore is usually named K). Let us draw the polar of T_a. It goes through D_a. The three polars intersect at a single point (the pole of $\mathcal{L}_b$). This point is the center of the holy garden, i.e. the hortocenter. Let us call it O, since the orthocenter is called H.

Proof. One has

$$H_a, T_a, R_a \simeq \begin{pmatrix} 0 \\ q \\ r \end{pmatrix}, \begin{pmatrix} -qg - rh \\ qf \\ fr \end{pmatrix}, \begin{pmatrix} -p(qg+rh) \\ q(2fp+qg+rh) \\ r(2fp+qg+rh) \end{pmatrix}$$

$$K, O \simeq \begin{pmatrix} p(qg+rh) \\ q(rh+pf) \\ r(pf+qg) \end{pmatrix}, \begin{pmatrix} gh(qg+rh) \\ hf(rh+pf) \\ fg(pf+qg) \end{pmatrix}$$

□

Definition 7.13.3. Gravity center. By O, the hortocenter, let us draw the line OT_a having the same direction as (HA) and obtain the points $G_a \doteq (BC) \cap (OT_a)$, etc. The triangle $G_aG_bG_c$ is perspective with ABC, defining a point G. To avoid confusions with the Garden Center O, let us call G as the Gravity center. It occurs that O, G, H are colinear, defining the celebrated Euler line.

Proof. One has:

$$G \simeq \left(\begin{array}{c} gh \\ fh \\ fg \end{array} \right) = \mathrm{tripolar}\left(\mathcal{L}_b\right) \;\; ; \;\; Euler \simeq \left[f\left(qg - rh\right), g\left(rh - pf\right), h\left(pf - qg\right) \right]$$ □

Definition 7.13.4. Isogonal conjugacy and orthopoint transform. It happens that $G \underset{b}{*} K = O \underset{b}{*} H$. Let us extend this result to the whole plane and define the isogonal conjugacy as $M^* = G \underset{b}{*} K \underset{b}{\div} M$. (Spoiler: this is a quadratic Cremona transform). Moreover, we can see that the pairs (T_a^*, D_a^*), etc are aligned with O. Let us use this property to extend the orthopoint relation to the whole $\mathcal{L}_b$, i.e.

$$N = \mathrm{orthopoint}\left(M\right) \Longleftrightarrow \det\left|O, M^*, N^*\right| = 0$$

This leads to: $N \simeq \boxed{OrtH} \cdot M$ where $\boxed{OrtH} \simeq \left[\begin{array}{ccc} 0 & -\dfrac{pr}{f} & \dfrac{pq}{f} \\ \dfrac{rq}{g} & 0 & -\dfrac{pq}{g} \\ -\dfrac{rq}{h} & \dfrac{pr}{h} & 0 \end{array} \right]$

Proof. Write the system $\det\left|O, M^*, N^*\right| = \mathcal{L}_b \cdot M = \mathcal{L}_b \cdot N = 0$ and solve in r, q', r'. Substitute into $p' : q' : r'$, re-introduce r and take the gradient wrt M. One can remark that $\ker \boxed{OrtH}$ is $H \simeq p : q : r$ while $\ker {}^t\boxed{OrtH} = \mathcal{L}_b$. □

Proposition 7.13.5. *The characteristic polynomial of* $\boxed{OrtH}$ *is:*

$$\chi_{\boxed{OrtH}}\left(X\right) = X^3 + \frac{pqr\left(fp + qg + rh\right)}{fgh} X$$

Non vanishing roots can be taken as $\pm i$ *or as* ± 1 *according to the signum of* $pqr\left(fp + qg + rh\right) \div fgh$ *(as it should be, this signum behaves projectively).*

Chapter 8

Brief extension to 3D spaces

Previous chapters were dealing with 2D spaces (planes), represented as the projective of a 3D vector space. In this chapter, we are using a 4D vector space to describe a 3D geometric space.

8.1 Basic results

Definition 8.1.1. A 3D point is a projective column in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$, most of the time noted as:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \overrightarrow{X} \\ x \end{pmatrix}$$

Definition 8.1.2. We will say that four points X_j are coplanar when $\det_{1..4}\left(X_j\right) = 0$.

Theorem 8.1.3 (Universal factoring)**.** *Given four columns, we have :*

$$\det\left(X_1 X_2 X_3 X_4\right) = {}^t X_1 \cdot \left(X_2 \underset{6}{\wedge} X_3\right) \cdot X_4 \qquad \textit{where}$$

$$\left(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ t_2 \end{pmatrix} \underset{6}{\wedge} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \\ t_3 \end{pmatrix}\right) \doteq \begin{pmatrix} 0 & t_3\, z_2 - t_2\, z_3 & t_2\, y_3 - t_3\, y_2 & y_2\, z_3 - y_3\, z_2 \\ t_2\, z_3 - t_3\, z_2 & 0 & t_3\, x_2 - t_2\, x_3 & x_3\, z_2 - x_2\, z_3 \\ t_3\, y_2 - t_2\, y_3 & t_2\, x_3 - t_3\, x_2 & 0 & x_2\, y_3 - x_3\, y_2 \\ y_3\, z_2 - y_2\, z_3 & x_2\, z_3 - x_3\, z_2 & x_3\, y_2 - x_2\, y_3 & 0 \end{pmatrix}$$

Proof. Operator det is multilinear: this ensures the existence of the central matrix. The actual values of the coefficients are obtained by partial derivatives. □

Definition 8.1.4. Matrix $\boxed{\Delta} \simeq \left(X_2 \underset{6}{\wedge} X_3\right)$ is anti-symmetric, and thus depends on 6 parameters. We call it the punctual matrix. Two concurent matrix notations are used in the litterature:

$$\boxed{\Delta} = \begin{pmatrix} 0 & D_{1,2} & -D_{1,3} & D_{1,4} \\ -D_{1,2} & 0 & D_{2,3} & D_{2,4} \\ +D_{1,3} & -D_{2,3} & 0 & D_{3,4} \\ -D_{1,4} & -D_{2,4} & -D_{3,4} & 0 \end{pmatrix} = \left(\begin{array}{ccc|c} 0 & B_z & -B_y & E_x \\ -B_z & 0 & B_x & E_y \\ B_y & -B_x & 0 & E_z \\ \hline -E_x & -E_y & -E_z & 0 \end{array}\right) = \left(\begin{array}{c|c} \mathcal{B} & \overrightarrow{E} \\ \hline -\overrightarrow{E} & 0 \end{array}\right)$$

Notation 8.1.5. Beside the **electromagnetic notation** given above, there is another representation, called the Plucker column representation. We have:

$$col\left(\boxed{\Delta}\right) = \begin{pmatrix} D_{2,3} \\ D_{1,3} \\ D_{1,2} \\ D_{1,4} \\ D_{2,4} \\ D_{3,4} \end{pmatrix} = \begin{pmatrix} B_x \\ B_y \\ B_z \\ E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} \overleftarrow{B} \\ \overrightarrow{E} \end{pmatrix} \tag{8.1}$$

Remark 8.1.6. Caveat: (1) D_{12} is B_z, not B_x ; (2) a minus sign is used at place $[1,3]$ (3) $\mathcal{B}$ is the antisymmetric matrix such that $\mathcal{B}\cdot\overrightarrow{X}=\overrightarrow{X}\times\overrightarrow{B}=(X_x:X_y:X_z)\wedge(B_x:B_y:B_z)$... mind the order ! (4) $\mathcal{E}$ is defined by $\mathcal{E}\cdot\overleftarrow{Y}=\overleftarrow{Y}\times\overrightarrow{E}$.

Remark 8.1.7. When $X_2\neq X_3$, the points collinear with X_2,X_3 are the elements of $\ker\boxed{\Delta}$. Moreover, the characteristic polynomial is :

$$\chi_\Delta(\lambda)=\lambda^4+\lambda^2\sum_6 D_{jk}^2+\left(\sum_3 D_{1,j}D_{k,l}\right)^2$$

so that $\sum_3 D_{1,j}D_{k,l}=E_xB_x+E_yB_y+E_zB_z$ is null.

Proposition 8.1.8. *When $X\notin\Delta$, the row characterizing the plane containing both point X and line Δ is obtained as $\Pi={}^tM\cdot\boxed{\Delta}$.*

Proposition 8.1.9. *Beside the punctual matrix, let us define the planar matrix $\boxed{\Delta^*}$ so that the column characterizing the point common to plane $\mathfrak{P}$ and line Δ is given by $X=\boxed{\Delta^*}\cdot{}^t\mathfrak{P}$ (obviously, $\Delta\not\subset\mathfrak{P}$ is assumed). Then, using notations of (8.1),we have:*

$$\boxed{\Delta^*}=\begin{pmatrix}0&D_{3,4}&-D_{2,4}&D_{2,3}\\-D_{3,4}&0&D_{1,4}&D_{1,3}\\D_{2,4}&-D_{1,4}&0&D_{1,2}\\-D_{2,3}&-D_{1,3}&-D_{1,2}&0\end{pmatrix}=\left(\begin{array}{ccc|c}0&E_z&-E_y&B_x\\-E_z&0&E_x&B_y\\E_y&-E_x&0&B_z\\\hline -B_x&-B_y&-B_z&0\end{array}\right)=\left(\begin{array}{c|c}\mathcal{E}&\overrightarrow{B}\\\hline -\overleftarrow{B}&0\end{array}\right)$$

$$col\left(\boxed{\Delta^*}\right)=\begin{pmatrix}D_{1,4}\\D_{2,4}\\D_{3,4}\\D_{2,3}\\D_{1,3}\\D_{1,2}\end{pmatrix}=\begin{pmatrix}E_x\\E_y\\E_z\\B_x\\B_y\\B_z\end{pmatrix}=\begin{pmatrix}\overleftarrow{E}\\\overrightarrow{B}\end{pmatrix}$$

Proof. This result is obtained by solving $\mathfrak{P}\cdot X=0$, $\boxed{\Delta}\cdot X=\overrightarrow{0}$ and then generically expressing X as $\boxed{\Delta^*}\cdot{}^t\mathfrak{P}$. □

Proposition 8.1.10. *Given two (different) planes $\mathfrak{P}_1,\mathfrak{P}_2$, the punctual matrix $\boxed{\Delta}$ of their diedral line can be obtained directly as:*

$$\boxed{\Delta}\simeq{}^t\mathfrak{P}_1\cdot\mathfrak{P}_2-{}^t\mathfrak{P}_2\cdot\mathfrak{P}_1$$

Given two (different) points P_1,P_2, the planar matrix $\boxed{\Delta^*}$ of line P_1P_2 can be obtained directly as:

$$\boxed{\Delta^*}\simeq P_1\cdot{}^tP_2-P_2\cdot{}^tP_1$$

Remember that $\boxed{\Delta}$ describes a punctual line, while $\boxed{\Delta^*}$ describes a planar line.

Proof. These objects are clearly projective objects and antisymmetric matrices of rank 2. The first one satisfies $\boxed{\Delta}\cdot X=0:0:0:0$ as soon as $\Pi_1\cdot X=\Pi_2\cdot X=0$. The second one satisfies $Y\cdot\boxed{\Delta^*}=[0,0,0,0]$ as soon as $Y\cdot P_1=Y\cdot P_2=0$. □

Proposition 8.1.11. *The so-called **Klein quadratic form** defined by*

$$\begin{aligned}\text{Klein}\left(\boxed{\Delta}_1,\boxed{\Delta}_2\right)&=\text{trace}\left(\boxed{\Delta}_1\cdot\boxed{\Delta^*}_2\right)\\&=D_{1,2}E_{3,4}+D_{1,3}E_{2,4}+D_{2,3}E_{1,4}+D_{1,4}E_{2,3}+D_{2,4}E_{1,3}+D_{3,4}E_{1,2}\\&={}^t col\left(\boxed{\Delta^*}_1\right)\cdot col\left(\boxed{\Delta}_2\right)=\overleftarrow{C}\cdot\overrightarrow{E}+\overleftarrow{B}\cdot\overrightarrow{F}\end{aligned}$$

is indeed a quadratic form. Then Klein $\left(\boxed{\Delta}_{23}, \boxed{\Delta}_{14}\right)$ *is exactly equal to* $\det(X_1X_2X_3X_4)$ *when* $\boxed{\Delta}_{14} = \left(X_1 \underset{6}{\wedge} X_4\right)$ *and* $\boxed{\Delta}_{23} = \left(M_2 \underset{6}{\wedge} M_3\right)$. *And therefore,* Klein $\left(\boxed{\Delta}_1, \boxed{\Delta}_2\right) = 0$ *is the condition for the two lines to be coplanar.*

Maple 8.1.12. In order to obtain independent points on a line Δ then take $\boxed{\Delta^*}$ and call `map(reduce, ColumnSpace(%))`. This gives a list of two columns.

Proposition 8.1.13. *Using the electromagnetic notation, we have the following formulas:*

$$\begin{array}{rl}
\text{line through two points} & \left(X \underset{6}{\wedge} Y\right) = \begin{pmatrix} \overleftarrow{B} \\ \overrightarrow{E} \end{pmatrix} = \begin{pmatrix} y\overrightarrow{X} - x\overrightarrow{Y} \\ \overrightarrow{X} \times \overrightarrow{Y} \end{pmatrix} \\
\text{line by a point } X \text{ and a direction } \overrightarrow{W} & \left(X \underset{6}{\wedge} \left(\overrightarrow{W} : 0\right)\right) = \begin{pmatrix} x\overrightarrow{W} \\ \overrightarrow{W} \times \overrightarrow{X} \end{pmatrix} \\
\text{plane through a line and a point} & {}^tX \cdot \boxed{\Delta} = \left[x\,\overrightarrow{E} + \overrightarrow{X} \times \overleftarrow{B}\ ;\ \overrightarrow{E} \cdot \overrightarrow{V}\right] \\
\text{incident punctual lines } \begin{pmatrix} \overleftarrow{B} \\ \overrightarrow{E} \end{pmatrix}, \begin{pmatrix} \overleftarrow{C} \\ \overrightarrow{F} \end{pmatrix} & \begin{pmatrix} \overrightarrow{E} \times \overrightarrow{F} \\ \overleftarrow{B} \cdot \overrightarrow{F} \end{pmatrix} = X \text{ ... or vanishes} \\
\text{assuming } \overleftarrow{C} \cdot \overrightarrow{E} + \overleftarrow{B} \cdot \overrightarrow{F} = 0 & \left[\overleftarrow{B} \times \overleftarrow{C}\ ;\ \overleftarrow{B} \cdot \overrightarrow{F}\right] = \mathfrak{P} \text{ ... or vanishes}
\end{array}$$

Proof. Direct examination. In the last two lines, one can check that $\mathfrak{P} \cdot X = 0$. Caveat: in some special cases, X or $\mathfrak{P}$ can vanish. □

Exercise 8.1.14. Compute by your-self, and check the following assertions. Four points P_1, P_2, P_3, P_4 and a collineation $\mathcal{A}$ are given. Consider them as written, i.e. not to a proportionality factor.

$$\boxed{P} = \begin{bmatrix} 53 & -58 & 0 & 76 \\ -97 & -9 & -16 & 40 \\ -74 & 32 & -43 & 45 \\ 79 & -34 & 58 & 39 \end{bmatrix} ;\ \det\boxed{P} = 24803093\ ;\ \boxed{\mathcal{A}} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix} ;\ \det\boxed{\mathcal{A}} = 4$$

Let Q_1, Q_2, Q_3, Q_4 be the images of the P_j by the collineation. We have:

$$\boxed{\mathcal{A}^{-1}} = \frac{1}{4}\begin{bmatrix} +2 & -1 & 0 & 1 \\ -2 & -1 & 0 & 1 \\ 0 & -2 & 0 & -2 \\ -4 & +2 & -4 & 2 \end{bmatrix} ;\ \boxed{Q} \doteq \boxed{\mathcal{A}}\,\boxed{P} = \begin{bmatrix} 150 & -49 & 16 & 36 \\ 118 & 35 & 59 & -161 \\ -155 & 51 & -31 & -120 \\ 30 & -99 & 27 & 71 \end{bmatrix} ;\ \det\boxed{Q} = 99212372$$

Using the $\left(\underset{6}{\wedge}\right)$ operator, we compute line $P_{23} = \left(P_2 \underset{6}{\wedge} P_3\right)$ and $Q_{23} = \left(Q_2 \underset{6}{\wedge} Q_3\right)$:

$$P_{23} = \begin{bmatrix} 0 & 394 & 1066 & 899 \\ -394 & 0 & -3364 & -2494 \\ -1066 & 3364 & 0 & 928 \\ -899 & 2494 & -928 & 0 \end{bmatrix} ;\ Q_{23} = \begin{bmatrix} 0 & -1692 & -6786 & -4094 \\ 1692 & 0 & 261 & -703 \\ 6786 & -261 & 0 & -3451 \\ 4094 & 703 & 3451 & 0 \end{bmatrix}$$

And also the planes $P_{123} = \bigwedge_3 (P_1, P_2, P_3)$ and Q_{123} :

$$P_{123} = [46081,\ -31028,\ 309494,\ 220893]\ ;\ Q_{123} = [-729354, -192255, -883572, -162149]$$

Proposition 8.1.15. *Action of a collineation. We have the following properties:*

1. ${}^tP_1 \cdot P_{23} \cdot P_4 = P_{123} \cdot P_4 = \det\boxed{P}$, ${}^tQ_1 \cdot Q_{23} \cdot Q_4 = Q_{123} \cdot Q_4 = \det Q = \det\boxed{P}\ \det\boxed{\mathcal{A}}$
2. $Q_{23} = \det\boxed{\mathcal{A}}\ \boxed{{}^t\mathcal{A}^{-1}} \cdot P_{23} \cdot \boxed{\mathcal{A}^{-1}}$
3. $Q_{123} = \det\boxed{\mathcal{A}}\ P_{123} \cdot \boxed{\mathcal{A}^{-1}}$

4. $P_{123} \wedge P_{234} = \det P \ \operatorname{dual}(P_{23})$; ${}^tP_{123} \cdot P_{234} - {}^tP_{234} \cdot P_{123} = \det P \ P_{23}$

5. *The same* dual $\left(\overrightarrow{B}, \overleftarrow{E}\right) = \epsilon \left(\overleftarrow{E}, \overrightarrow{B}\right)$ *is used for* P *and* Q.

Exercise 8.1.16. Loosely speaking and using Cartesian coordinates: a line goes through $(-1,1,5)$ and is incident with line $D_1 : \{x+y=2,\ x-y=2\}$ together with line $D_2 :\ (1,3,1)+\lambda(2,-2,1)$. We have two planes $\mathfrak{P}_{11} \simeq [1,1,0,-2]$, $\mathfrak{P}_{12} \simeq [1,-1,0,-2]$ and three points $P_0 \simeq -1:1:5:1$, $P_{21} \simeq 1:3:1:1$, $P_{22} \simeq 2:-2:1:0$. Thus $D_1 = \operatorname{dual}\left(\left(\mathfrak{P}_{11} \underset{6}{\wedge} \mathfrak{P}_{12}\right)\right)$, $D_2 = \left(P_{21} \underset{6}{\wedge} P_{22}\right)$, and we have to solve the set of equations :

$$\Delta \cdot P_0 = 0:0:0:0,\ \phi(\Delta, D_1) = \phi(\Delta, D_2) = 0$$

After expansion, we obtain:

$$\begin{Bmatrix} -6\,B_y - 3\,E_z \ ; \ -B_y - B_x + E_z \ ; \ B_z + 5\,B_x + E_y \ ; \ B_z - 5\,B_y + E_x \ ; \\ E_x - E_y - 5\,E_z \ ; \ -8\,B_z + B_y + 5\,B_x - 2\,E_x + 2\,E_y - E_z \end{Bmatrix}$$

so that $\overleftarrow{B}, \overrightarrow{E} = -3:1:1:4:14:-2$. The incidence points are: $P_4 \simeq 2:0:4:1$ (who clearly belongs to both planes) and $P_5 \simeq 5:-1:3:1$ (obtained with $\lambda = 2$). Finally, P_0, P_4, P_5 are aligned, since P_4 is the middle of the other two.

One can also obtain the rows that describes the plane through P_0, D_1, the plane through P_0, D_2 and obtain Δ using Proposition 8.1.10.

8.2 Euclidian cartesian metric

Notation 8.2.1. In the "usual" cartesian space, points are noted $\overrightarrow{M} : m \doteq M_x : M_y : M_z : m$, and the metric is described by:

$$\mathcal{L}_{4c} \simeq [0,0,0,1] \ ; \ \boxed{\mathrm{Pyth}_{4c}} = \boxed{\mathcal{M}_{4c}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The c in the $4c$ index is "cartesian", while indices 4 are spared for a later use (tetrahedron barycentric space).

Proposition 8.2.2. *Assume that* $\mathfrak{P}$ *is not the plane at infinity. Then the orthogonal projection of point* $P \simeq X:Y:Z:T$ *on* $\mathfrak{P} \simeq [E,F,G,H]$ *is given by:*

$$\begin{aligned} pr\,(P, \mathfrak{P}) &= \left(E^2+F^2+G^2\right) \begin{bmatrix} X \\ Y \\ Z \\ T \end{bmatrix} - (EX+FY+GZ+HT) \begin{bmatrix} E \\ F \\ G \\ 0 \end{bmatrix} \\ &= \left(\mathfrak{P} \cdot \boxed{\mathcal{M}_{4c}} \cdot {}^t\mathfrak{P}\right)(P) - (\mathfrak{P} \cdot P)\left(\boxed{\mathcal{M}_{4c}} \cdot {}^t\mathfrak{P}\right) \end{aligned}$$

and therefore

$$dist\,(P, \mathfrak{P}) = \frac{\mathfrak{P} \cdot P}{\mathcal{L}_b \cdot P} \frac{1}{\sqrt{\left(\mathfrak{P} \cdot \boxed{\mathcal{M}_4} \cdot {}^t\mathfrak{P}\right)}}$$

Proof. Express that the generic $M \in \mathfrak{P}$ is generated by the three points $[H:0:0:-E], [0:H:0:-F], [0:0:H:-G]$. Then compute $|PM|^2$, minimize and substitute. □

Proposition 8.2.3. *The orthogonal projection of point* $P \simeq X:Y:Z:T$ *on* $\Delta = \left[\overrightarrow{B}, \overrightarrow{E}\right]$ *is given by:*

$$pr\,(P, \Delta) \underset{4c}{=} T \begin{pmatrix} B_zE_y - B_yE_z \\ B_xE_z - B_zE_x \\ B_yE_x - B_xE_y \\ B_x^2 + B_y^2 + B_z^2 \end{pmatrix} + (XB_x + YB_y + ZB_z) \begin{pmatrix} B_x \\ B_y \\ B_z \\ 0 \end{pmatrix}$$

Proof. Minimize

$$\left| t\,\mathrm{nor}\begin{pmatrix} 0 \\ -E_z \\ +E_y \\ -B_x \end{pmatrix} + (1-t)\,\mathrm{nor}\begin{pmatrix} +E_z \\ 0 \\ -E_x \\ -B_y \end{pmatrix} - \mathrm{nor}\begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix} \right|^2$$

□

Proposition 8.2.4. *It exists a unique point situated at equal distance from four given points when (1) they are all at finite distance (2) they are not coplanar.*

Proof. Consider the four $U_j \simeq x_j : y_j : z_j : t_j$. The computation is straightforward, and the denominator of R^2 is $\left(\prod t_j\right)^2 \times \det\left(U_j\right)^2$. Nevertheless, the length of the formal formula is roughly equal to 470.000 ! □

8.3 Rotations in the 3D Euclidean space

In this section, the usual Cartesian coordinates $x : y : z$ are used to describe the usual 3D Euclidean space, $\mathcal{E}_3$ using the usual metric $|v| = \sqrt{x^2+y^2+z^2}$.

Proposition 8.3.1. *In the Euclidean space $\mathcal{E}_3$, the orthogonal projector onto the line directed by the (not zero) vector $V \doteq {}^t[f,g,h]$ is given by*

$$\boxed{\pi^{\|}} = \frac{1}{f^2+g^2+h^2}\,V \cdot {}^tV$$

On the contrary, the matrix

$$\boxed{\omega} \doteq \frac{1}{\sqrt{f^2+g^2+h^2}}\begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix}$$

*describes "project onto $V^{\perp}$ and quarter-turn", while the projector itself is given by $\pi^{\perp} = -\omega^2$. Thus the matrix of the **3D-rotation** of angle τ around axis V is given by:*

$$\boxed{\rho} = \boxed{\mathrm{Id}} + \sin\tau\,\boxed{\omega} + (1-\cos\tau)\,\boxed{\omega}^2 \tag{8.2}$$

Proof. One can see that $\omega^2 + \omega^4 = 0$. Thus $-\omega^2$ is a projector. And the rest follows. One can check that $\chi_\omega = X\left(X^2+1\right)$, while $\chi_\rho = (X-1)\left(X^2 - 2X\cos\tau + 1\right)$. □

8.4 Euclidian metric in the tetrahedron space

Proposition 8.4.1. *The matrix describing the metric in the **tetrahedron** $A_0B_0C_0D_0$ space is*

$$\boxed{\mathrm{Pyth}_4} = \frac{-1}{2}\begin{bmatrix} 0 & c^2 & b^2 & A^2 \\ c^2 & 0 & a^2 & B^2 \\ b^2 & a^2 & 0 & C^2 \\ A^2 & B^2 & C^2 & 0 \end{bmatrix}$$

where $a \doteq B_0C_0$ and $A = A_0D_0$ (in this context, vertices are noted with index 0 while the sidelength are noted with bare letters). And the matrix giving the direction orthogonal to a given plane is symmetric matrix whose fourth column is:

$$col\left(\boxed{\mathcal{M}_4}, 4\right) \doteq \delta_D = \begin{bmatrix} 2\,A^2a^2 + a^4 - \left(B^2+C^2+b^2+c^2\right)a^2 - \left(b^2-c^2\right)\left(B^2-C^2\right) \\ 2\,B^2b^2 + b^4 - \left(C^2+A^2+a^2+c^2\right)b^2 - \left(c^2-a^2\right)\left(C^2-A^2\right) \\ 2\,C^2c^2 + c^4 - \left(A^2+B^2+a^2+b^2\right)c^2 - \left(a^2-b^2\right)\left(A^2-B^2\right) \\ (b+a+c)(a+b-c)(a+c-b)(b+c-a) \end{bmatrix} \tag{8.3}$$

Proposition 8.4.2. *The squared volume of the reference tetrahedron is*

$$\begin{aligned} V^2 &= \frac{1}{144}\left(-a^2b^2c^2 + \sum_3 a^2\left[b^2\left(A^2+B^2\right) - A^2a^2 + \left(A^2-B^2\right)\left(C^2-A^2\right)\right]\right) \\ &= \frac{1}{36}\,\mathcal{L}_b \cdot adjoint\left(\boxed{\mathrm{Pyth}_4}\right) \cdot {}^t\mathcal{L}_b = \frac{1}{2304}\left|\delta_D\right|^2 \div S_D^2 \end{aligned}$$

Proof. Cf. the Heron's formula: $\mathcal{L}_b \cdot adjoint\left(\boxed{\mathrm{Pyth}_b}\right) \cdot {}^t\mathcal{L}_b = 4S^2$. □

Proposition 8.4.3. *Let* medAB *be the plane perpendicular to line AB at $(A+B)/2$, i.e. the perpendicular bissector of the edge. Let* cutAB *be parallel to* medAB *through $(C+D)/2$. Then the six* med *planes intersect at a point Ω called the circumcenter of the tetrahedron. Its barycentrics wrt $ABCD$ are:*

$$\Omega \underset{b}{\simeq} \left[\begin{array}{c} a^2A^2\left(B^2+C^2-a^2\right)+b^2B^2\left(a^2+C^2-B^2\right)+c^2C^2\left(a^2+B^2-C^2\right)-2\,a^2B^2C^2 \\ a^2A^2\left(C^2+b^2-A^2\right)+b^2B^2\left(A^2+C^2-b^2\right)+c^2C^2\left(A^2+b^2-C^2\right)-2\,A^2b^2C^2 \\ a^2A^2\left(B^2+c^2-A^2\right)+b^2B^2\left(A^2+c^2-B^2\right)+c^2C^2\left(A^2+B^2-c^2\right)-2\,A^2B^2c^2 \\ a^2A^2\left(b^2+c^2-a^2\right)+b^2B^2\left(a^2-b^2+c^2\right)+c^2C^2\left(a^2+b^2-c^2\right)-2\,a^2b^2c^2 \end{array}\right]$$

The circumscribed sphere is described by the $\boxed{\mathrm{Pyth}_4}$ *quadratic form while its radius R is given by:*

$$R^2 = (Aa+Bb+Cc)(Bb+Cc-Aa)(Cc+Aa-Bb)(Aa+Bb-Cc) \div (24V)^2$$

Moreover, the six cut *planes intersect at a point M called the Monge point of the tetrahedron. And we have $2(\Omega+M) = A+B+C+D$.*

Proof. Obtain medAB as the X-gradient of $XA^2 - XB^2$, etc and take the wedge of the three planes medAB, medBC, medCD. The result is symmetric, proving the existence of Ω. Then compute R from the Pythagoras formula (the obtained value is symmetric, as it should be). Finaly, compute cutAB, etc by solving $(\mathrm{med}AB + x\mathcal{L}_4)\cdot(C+D) = 0$, etc and obtain M using $\bigwedge_3$. □

Proposition 8.4.4. *The distance between a point P and a plane $\mathfrak{P}$ is given by:*

$$\mathrm{dist}\left(P,\mathfrak{P}\right) = \frac{\mathfrak{P}\cdot P}{\mathcal{L}_b \cdot P}\,\frac{12V}{\sqrt{\left(\mathfrak{P}\cdot \boxed{\mathcal{M}_4} \cdot {}^t\mathfrak{P}\right)}}$$

Proof. This formula is homogeneous in P, in $\mathfrak{P}$ and in a,b,c,A,B,C. Moreover, it obeys to the general model $\mathrm{dist}\left(x,\ker\phi\right) = \phi(x)/\|\phi\|$. □

Proposition 8.4.5. *There are 8 points that are at the same distance from the faces of the standard tetrahedron. They are the centers of eight spheres tangents to theses faces. The coordinates of the centers and the radiuses are:*

$$I_j \simeq \pm S_A : \pm S_B : \pm S_C : \pm S_D\ ;\ \frac{1}{\rho_j} = \frac{1}{h_A} + \frac{1}{h_B} + \frac{1}{h_C} + \frac{1}{h_0}$$

where $h_j = 3V/S_j$ are the altitudes of the tetrahedron.

Proof. The distances from point $x:y:z:t$ to the four faces of the reference tetrahedron are:

$$\left[\frac{x}{S_A},\frac{y}{S_B},\frac{z}{S_C},\frac{t}{S_0}\right] \times \frac{3V}{x+y+z+t}$$

□

Proposition 8.4.6. *The distance d_j from the in/excenter I_j to the circumcenter is given by:*

$$d_j^2 - R^2 = -\left(\sum_6 |AB|\,S_C S_D\right) \div \left(\sum_4 S_j\right)^2$$

Proof. Compute ${}^{t}nor\,(\Omega)\cdot \boxed{\mathrm{Pyth}_4}\cdot nor\,(\Omega)$ and obtain $-R^2$, proving that $\boxed{\mathrm{Pyth}_4}$ gives the power of a point wrt the sphere (and not some multiple). And then use the barycentrics of I_j. Thus $\sum_6$ sums the product of each sidelength by the areas of the two adjacent faces (taken with the relevant sign). □

Example 8.4.7. The tetrahedron used in Hecquet (1980) is characterized by

$$a^2 = 19,\ b^2 = 13,\ c^2 = 7,\ A^2 = 21,\ B^2 = 28,\ C^2 = 37$$

It's existence granted by the reality of the areas:

$$\frac{S_A}{26} = \frac{S_B}{19} = \frac{S_C}{14} = \frac{S_D}{11} = \frac{\sqrt{3}}{4}$$

One has $V^2 = 105/2$, $R^2 = 973/90$, $r^2 = 18/35$, $d^2 = 2851/630$.

8.5 HH: hyperbolic hyperboloids

Remark 8.5.1. Caveat 1: the $\mathfrak{H}$ used here means "hyperbolic hyperboiloid" and therefore is not the $\mathfrak{H}$ used at Section 28.9, in the Sister Marie Cordia Karl section.
Caveat 2: HH are also called one-sheet hyperboloids.

Definition 8.5.2. Let L_1, L_2, L_3 be three lines in the 3D space, no two of them being incident. For each point M_3 on L_3, it exists a line Δ going through M_3 and incident to L_1 and L_2. The union of these lines Δ is called the **hyperbolic hyperboloid** defined by the three L_j.

Proposition 8.5.3. *It exist a 4×4 matrix $\boxed{\mathfrak{H}}$ such that ${}^{t}\boxed{\Delta^*}\cdot\boxed{\mathfrak{H}}\cdot\boxed{\Delta^*} = \boxed{0_4}$ is verified by the (planar) matrix of Δ when M_3 moves on L_3. Moreover, any point M on such a Δ verifies: ${}^{t}M\cdot\boxed{\mathfrak{H}}\cdot M = 0$.*

Proof. Use the four points $A, B \in L_1$ and $C, D \in L_2$ as barycentric basis, and note L_3 as $\left(\overrightarrow{B}, \overrightarrow{E}\right)$. Let Δ be a line incident to L_1 at $M_1 \doteq t\,A + (1-t)B$ and to L_2 at $M_2 \doteq s\,C + (1-s)D$. This gives:

$$\Delta \simeq \left[\begin{array}{cccc} 0 & 0 & (1-s)\,(1-t) & s\,(t-1) \\ 0 & 0 & (s-1)\,t & st \\ (s-1)\,(1-t) & (1-s)\,t & 0 & 0 \\ s\,(1-t) & -st & 0 & 0 \end{array}\right]$$

and now write that Δ and L_3 are incident. This leads to an homomorphic relation between s and t. Eliminate t, k in $M = k\,M_1 + (1-k)\,M_2$ and obtain ${}^{t}M\cdot\boxed{\mathfrak{H}}\cdot M = 0$ where

$$M \simeq \left(\begin{array}{c} \left(\left(E_x - E_y + B_x + B_y\right)kt + \left(E_y - B_x\right)k\right)\,t \\ \left(\left(E_x - E_y + B_x + B_y\right)kt + \left(E_y - B_x\right)k\right)(1-t) \\ \left(\left(E_x - E_y\right)t + E_y\right)(1-k) \\ \left(\left(B_x + B_y\right)t - B_x\right)(1-k) \end{array}\right) \;;\; \boxed{\mathfrak{H}} \simeq \left[\begin{array}{cccc} 0 & 0 & -B_y & E_x \\ 0 & 0 & B_x & E_y \\ -B_y & B_x & 0 & 0 \\ E_x & E_y & 0 & 0 \end{array}\right]$$ □

Proposition 8.5.4. *Suppose that M, N and $M+N$ belong all three to $\mathfrak{H}$. Then any point of line $\Delta = MN$ belongs to $\mathfrak{H}$, while ${}^{t}\boxed{\Delta^*}\cdot\boxed{\mathfrak{H}}\cdot\boxed{\Delta^*} = \boxed{0_4}$ is satisfied.*

Proof. Use $\mathfrak{H}_0$ and consider $M \simeq \overrightarrow{V} : v$; $N \simeq \overrightarrow{W} : w$, write the three equations and eliminate. This leads to

$$v = \frac{\left(V_xB_y - V_yB_x\right)V_z}{E_xV_x + E_yV_y},\ w = \frac{\left(B_yW_x - B_xW_y\right)V_z}{E_xV_x + E_yV_y},\ W_z = \frac{\left(E_xW_x + E_yW_y\right)V_z}{E_xV_x + E_yV_y}$$

and the conclusion follows. □

Proposition 8.5.5. *By any point M of $\mathfrak{H}$, two straight lines can be drawn which belong to $\mathfrak{H}$. One of them, say Δ_M, is incident to L_1, L_2, L_3 while the other one, say δ_M, is incident to all the lines Δ_N where $N \in \mathfrak{H}$.*

Proof. Use $\mathfrak{H}_0$ and consider $M \simeq \overrightarrow{U} : u$; $N \simeq \overrightarrow{V} : v$, write that ${}^t(M + k\,N) \cdot \boxed{\mathfrak{H}} . (M + k\,N)$ is the null polynomial in k and solve in $u, v, \overrightarrow{V}$. This gives two families of solutions for N (depending on two parameters, e.g. v, V_x). Then compute the lines MN, check that parameters are cancelling and get:

$$\boxed{\Delta} \simeq \left[\begin{array}{cccc} 0 & 0 & +uU_y & -U_yU_z \\ 0 & 0 & -uU_x & +U_xU_z \\ -uU_y & +uU_x & 0 & 0 \\ U_yU_z & -U_xU_z & 0 & 0 \end{array}\right] ; \boxed{\delta} \simeq \left[\begin{array}{ccccc} 0 & \dfrac{-E_zB_zU_z^2}{E_xU_x+E_yU_y} & -B_yU_z & E_xU_z \\ * & 0 & +B_xU_z & E_yU_z \\ * & * & 0 & -E_xU_x-E_yU_y \\ * & * & * & 0 \end{array}\right]$$

And then Klein $(\Delta_M, \delta_N) = 0$ proves the incidences. One can check that $L_1 = \delta(A)$, $L_2 = \delta(C)$ together with $L_3 = \delta(-E_yU_y - E_zU_z : E_xU_y : E_xU_z : E_xu)$ □

Remark 8.5.6. When dealing with a circumscribed QH, i.e. when $\boxed{\mathfrak{H}} = \left[\begin{array}{cccc} 0 & H_z & H_y & K_x \\ H_z & 0 & H_x & K_y \\ H_y & H_x & 0 & K_z \\ K_x & K_y & K_z & 0 \end{array}\right]$, both series of lines Δ, δ are involving the same radical W, where

$$W^2 = H_x^2K_x^2 + K_y^2H_y^2 + H_z^2K_z^2 - 2\,H_xK_xH_yK_y - 2\,K_yH_yH_zK_z - 2\,H_zK_zH_xK_x$$

Proposition 8.5.7. *The four altitudes of a tetrahedron $ABCD$ belong to a same HH. When using barycentrics wrt $ABCD$, we have*

$$\boxed{\mathfrak{H}} \simeq \left[\begin{array}{cccc} 0 & u(a,b,c,A,B,C) & u(c,a,b,C,A,B) & u(b,C,A,B,c,a) \\ * & 0 & u(b,c,a,B,C,A) & u(c,A,B,C,a,b) \\ * & * & 0 & u(a,B,C,A,b,c) \\ * & * & * & 0 \end{array}\right]$$

where $u(a,b,c,A,B,C) =$

$$\left(B^2 - A^2 + b^2 - a^2\right)\left(\left(A^2 - B^2\right)\left(a^2 - b^2\right) + \left(a^2 + b^2 + A^2 + B^2 - 2\,C^2\right)c^2 - c^4\right)$$

$$\text{so that : } \det \boxed{\mathfrak{H}} = 82944\,V^2 \times \prod_3 \left(B^2 - C^2 + b^2 - c^2\right)^2$$

Proof. Read the direction δ_D of the fourth altitude Δ_D at (8.3). Compute $\Delta_D \simeq \left(D \underset{6}{\wedge} \delta_D\right)$, etc. And check that the four sets of equations: ${}^t\boxed{\Delta_D^*} \cdot \boxed{\mathfrak{H}} \cdot \boxed{\Delta_D^*} = \boxed{0_4}$, etc are compatible. □

Proposition 8.5.8. *Orthocentric tetrahedron. When $A^2 + a^2 = B^2 + b^2$, then $A_0B_0 \perp C_0D_0$ and conversely. When two sets of opposite edges are orthogonal, so is the third set and the HH is totally degenerate: the four altitudes are concurrent, defining an orthocenter H. One has:*

$$H \simeq \left(\begin{array}{c} S_b\,S_c\,\left(A^2 - S_a\right) \\ S_c\,S_a\,\left(A^2 - S_a\right) \\ S_a\,S_b\,\left(A^2 - S_a\right) \\ S_a\,S_b\,S_c \end{array}\right) ; \; V_H^2 = \frac{1}{9}\,S^2A^2 - \frac{1}{36}\,a^2S_a^2$$

Proof. Direct computation. □

Exercise 8.5.9. What happens when there is only one pair of orthogonal opposite edges ?

8.6 HH: some examples

Proposition 8.6.1. *Consider a tetrahedron A, B, C, D. Affect a coefficient to each edge, e.g. $BC = a$,* etc *and $AD = d$,* etc. *On each face, take the barycenter of the three vertices using the weight of the opposite side, and obtain*

$$\begin{array}{rcl} D^{+} & = & (a\ A + b\ B + c\ C) \,/\, (a + b + c) \\ A^{+} & = & (f\ B + e\ C + a\ D) \,/\, (f + e + a) \\ B^{+} & = & (f\ A + d\ C + b\ D) \,/\, (f + d + b) \\ C^{+} & = & (e\,A + d\,B + c\_D) \,/\, (e + d + c) \end{array}$$

Then lines AA^{+}, BB^{+}, CC^{+}, DD^{+} are imbedded in a same HH.

Proof. A simple computation gives:

$$\boxed{\mathfrak{H}} = \left(\begin{array}{cccc} 0 & (ad - be)\, c & (cf - ad)\, b & (be - cf)\, d \\ (ad - be)\, c & 0 & (be - cf)\, a & (cf - ad)\, e \\ (cf - ad)\, b & (be - cf)\, a & 0 & (ad - be)\, f \\ (be - cf)\, d & (cf - ad)\, e & (ad - be)\, f & 0 \end{array} \right)$$

together with $W = (ad - be)\,(be - cf)\,(cf - ad)$. □

Corollary 8.6.2. *If we take $a = |BC|$ and so on, we obtain a property relative to the four lines joining a vertex to the incenter of the opposite face in a tetrahedron. And the same occurs for the Lemoine centers. And for their isotomic images X(75) and X(76). When all coefficients are equal, the four lines are simply concurrent at the gravity center.*

Exercise 8.6.3. Consider the four points:

$$A, B, C, D \simeq \left(\begin{array}{c} -13 \\ -43 \\ 43 \\ 1 \end{array} \right), \left(\begin{array}{c} -21 \\ -11 \\ 41 \\ 1 \end{array} \right), \left(\begin{array}{c} -13 \\ 21 \\ -91 \\ 1 \end{array} \right), \left(\begin{array}{c} -1 \\ 22 \\ 53 \\ 1 \end{array} \right)$$

and compute the orthocenters, by minimizing $\delta^2\,(A, xB + yC + zD)$ or otherwise. Consider the lines AH_a, etc. Show they aren't independent. Compute the QH containing these four lines. One obtains

$$\boxed{\mathfrak{H}} = \left[\begin{array}{rrrr} -6048 & 40004 & -96004 & 3786528 \\ 40004 & 94944 & 64701 & -288685 \\ -96004 & 64701 & -88896 & -4296565 \\ 3786528 & -288685 & -4296565 & 520174122 \end{array} \right]$$

Exercise 8.6.4. (Follow-up). Take two points P, Q on AH_a and two points R, S on BH_b. Consider the line Δ joining $eP + fQ$ with $gR + hS$. Solve in h so that Δ becomes incident to CH_c. And now check that the generic point of Δ belongs to the HH.

Chapter 9

Pedal stuff

In a previous life, this Section was intended as foreword to Chapter 7 (orthogonality). Now, this Section is rather the symmetric aisle of Chapter 3 (cevian stuff).

9.1 Pedal triangle

Definition 9.1.1. The **pedal triangle** of point P is the triangle whose vertices are the orthogonal projections of P on the sides of the triangle.

Remark 9.1.2. Crossover the Channel, the pedal triangle is called "triangle podaire", while "pédal triangle" is used to denote the Cevian triangle. *Plaisante vérité qu'une rivière borne* (Pascal, 1670, p. 46). For the *anti-pedal* triangle, see Proposition 26.4.8.

Proposition 9.1.3. *The pedal triangle of point P has the following barycentrics (each point is a column) :*

$$pedal\left(\begin{array}{c} p\\ q\\ r \end{array}\right) \underset{b}{\simeq} \left[\begin{array}{ccc} 0 & S_c\,q+b^2p & S_b\,r+c^2p\\ S_c\,p+a^2q & 0 & S_a\,r+c^2q\\ S_b\,p+a^2r & S_a\,q+b^2r & 0 \end{array}\right] \simeq \left[\begin{array}{ccc} 0 & \dfrac{q\,S_c}{b^2}+p & \dfrac{r\,S_b}{c^2}+p\\ \dfrac{p\,S_c}{a^2}+q & 0 & \dfrac{r\,S_a}{c^2}+q\\ \dfrac{p\,S_b}{a^2}+r & \dfrac{q\,S_a}{b^2}+r & 0 \end{array}\right] \tag{9.1}$$

Proof. As usual, $S_a = \left(b^2+c^2-a^2\right)/2,\quad$ *etc.* Use (7.27) and obtain directly the result. □

Proposition 9.1.4. *Condition for an inscribed triangle $P_1P_2P_3$ to be the pedal triangle of some P is :*

$$\frac{q_1-r_1}{q_1+r_1}\,a^2+\frac{r_2-p_2}{p_2+r_2}\,b^2+\frac{p_3-q_3}{p_3+q_3}\,c^2 \tag{9.2}$$

In such a case, point P is the perspector between $P_1P_2P_3$ and triangle $\boxed{\mathcal{M}_b}$ and is given by either following expressions :

$$\left[\begin{array}{c} b^2c^2p_2p_3-\left(S_cr_2-S_ap_2\right)\left(S_bq_3-S_ap_3\right)\\ b^2c^2p_2q_3+\left(S_bq_3-S_ap_3\right)b^2r_2\\ b^2c^2p_3r_2+\left(S_cr_2-S_ap_2\right)c^2q_3 \end{array}\right]$$

$$\left[\begin{array}{c} a^2c^2p_3q_1+\left(S_ap_3-S_bq_3\right)a^2r_1\\ a^2c^2q_1q_3-\left(S_ap_3-S_bq_3\right)\left(S_cr_1-S_bq_1\right)\\ a^2c^2q_3r_1+\left(S_cr_1-S_bq_1\right)c^2p_3 \end{array}\right]$$

$$\left[\begin{array}{c} a^2b^2p_2r_1+\left(S_ap_2-S_cr_2\right)a^2q_1\\ a^2b^2q_1r_2+\left(S_bq_1-S_cr_1\right)b^2p_2\\ a^2b^2r_1r_2-\left(S_bq_1-S_cr_1\right)\left(S_ap_2-S_cr_2\right) \end{array}\right]$$

Proof. P is on the line through P_1 and orthopoint of BC etc. The required condition is the determinant of these three lines. The various ways of writing P are the wedge product of two rows at a time. A more symmetric formula would be great... □

Proposition 9.1.5. *It exists exactly one pedal triangle of a given shape. When the shape is given by the tangents* $t_A \doteq \tan\left(\overrightarrow{A_P B_P}, \overrightarrow{A_P C_P}\right)$, etc *(bound by* $t_A + t_B + t_C = t_A t_B t_C$*), the central point is given by*

$$P \simeq \left[\begin{array}{c} a^2 t_B t_C \left(S_a\, t_A + 2\, S\right) \\ b^2 t_C t_A \left(S_b\, t_B + 2\, S\right) \\ c^2 t_A t_B \left(S_c\, t_C + 2\, S\right) \end{array}\right]$$

Proof. Substitute 9.1 in the tan formula (7.22) and obtain three equations. And then eliminate. □

Exercise 9.1.6. Spoiler: using Morley affixes, the pedal triangle of can be written

$$pedal \left(\begin{array}{c} z \\ t \\ \zeta \end{array}\right) \underset{z}{\simeq} \left(\begin{array}{ccc} \dfrac{\beta+\gamma+\dfrac{z-\beta\gamma\zeta}{t}}{2} & \dfrac{\alpha+\gamma+\dfrac{z-\alpha\gamma\zeta}{t}}{2} & \dfrac{\alpha+\beta+\dfrac{z-\alpha\beta\zeta}{t}}{2} \\ \dfrac{\beta+\gamma}{\beta\gamma}+\dfrac{\beta\gamma\zeta-z}{\beta\gamma t} & \dfrac{\alpha+\gamma}{\alpha\gamma}+\dfrac{\gamma\alpha\zeta-z}{\gamma\alpha t} & \dfrac{\alpha+\beta}{\alpha\beta}+\dfrac{\alpha\beta\zeta-z}{\alpha\beta t} \end{array}\right)$$

Exercise 9.1.7. The area of the P-pedal triangle is $\dfrac{S}{4}\left(1-\dfrac{|OP|^2}{R^2}\right)$. Spoiler: a proof is given at Proposition 28.8.14.

Exercise 9.1.8. From Proposition 9.1.5, there is exactly one pedal triangle of each shape. What to say about the only pedal triangle who is skew similar to a given one ?

Construction 9.1.9. *Construct the* ABC*-pedal triangle similar to a given triangle* $T_a T_b T_c$*. Through point* A*, draw the parallel* $B_p C_p$ *to* $T_b T_c$, etc *and obtain triangle* $A_p B_p C_p$*. Draw circles* BCA_p, etc. *Call* gE *their common (Miquel) point. Then* $\mathcal{E} \doteq \operatorname{isogon}(gE)$ *is the center of the sought pedal triangle.*

9.2 Isogonal conjugacy and Steiner triangle

Definition 9.2.1. Use barycentrics wrt ABC and suppose that the point $P = p : q : r$ is not on a sideline. Then the point Q defined by:

$$Q \simeq \operatorname{isogon}(p : q : r) \doteq \frac{a^2}{p} : \frac{b^2}{q} : \frac{c^2}{r} \tag{9.3}$$

is not a vertex and is called the isogonal conjugate of P.

Remark 9.2.2. The isogonal conjugate of point P can be introduced as the circumcenter of the P-Steiner triangle. This will be done at Section 10.1. Another introduction comes from the following eponymous property.

Proposition 9.2.3. *The* ***isogonal conjugate*** Q *of the point* P *(not on a sideline) is characterized by the three relations:*

$$(AB,\, AP) + (AC, AQ) = 0, \text{ etc}$$

In other words, lines AP *and* AQ *are equally inclined over lines* AB, AC*. This property is obviously symmetric between* P, Q *and both points are said to form a pair of isogonal conjugates.*

Proof. One has $\tan(A, B, P) + \tan(A, C, Q) = 0$, etc. Mind the order ! When line AP cuts BC between B and C, then line AQ does the same. □

Remark 9.2.4. The isogonal conjugate of a point P is often noted P^{-1} in ETC since its trilinears are $1/p : 1/q : 1/r$ when those of P are $p : q : r$.

9.3 Cyclopedal conjugate

Definition 9.3.1. The pedal circle of a given point is the circle circumscribed to the pedal triangle of this point. (spoiler) It's representative is:

$$\mathcal{V}_{\mathcal{P}} \simeq \left[\begin{array}{c} p\left(c^2q+S_a\,r\right)\left(b^2r+S_a\,q\right) \\ q\left(a^2r+S_b\,p\right)\left(c^2p+S_b\,r\right) \\ r\left(b^2p+S_c\,q\right)\left(a^2q+S_c\,p\right) \\ \left(a^2qr+b^2pr+c^2pq\right)(p+q+r) \end{array}\right]$$

Proposition 9.3.2 (Matthieu)**.** *When P and Q are isogonal conjugates, they share the same pedal circle. The center of this circle is the middle of P and Q (cf Figure 9.1)*

Proof. Straightforward computation (using $\mathcal{V}$ or not !). □

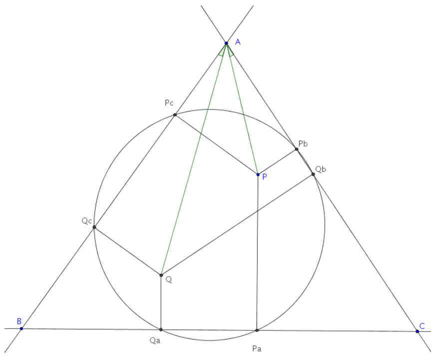


Figure 9.1: Cyclopedal congugates are isogonal conjugates

Remark 9.3.3. The definition of "cyclopedal conjugacy" has been coined to enforce symmetry with the cyclocevian conjugacy, cf. Section 13.23. Some examples are :

point	code	bary	cycp	circumcenter
incenter	$X(1)$	a	$X(1)$	$X(1)$
centroid	$X(2)$	1	$X(6)$	$X(597)$
Lemoine	$X(6)$	a^2	$X(2)$	$X(597)$
circumcenter	$X(3)$	$a^2\left(-a^2+b^2+c^2\right)$	$X(4)$	$X(5)$
orthocenter	$X(4)$	$1/\left(-a^2+b^2+c^2\right)$	$X(3)$	$X(5)$

Proposition 9.3.4. *(Spoiler). The center K_P of the RH through A, B, C, H, P* belongs to the pedal circle of P. As a result, K_P and K_Q are the common points of the (P, Q) pedal circle and the Euler circle.

Proof. This can be checked by using:

$$K_P \simeq \left[\begin{array}{c} \left(r\,(p+q)\,b^2-q\,(r+p)\,c^2\right)(S_b\,q-S_c\,r)\,p \\ \left(p\,(q+r)\,c^2-r\,(p+q)\,a^2\right)(S_c\,r-S_a\,p)\,q \\ \left(q\,(r+p)\,a^2-p\,(q+r)\,b^2\right)(S_a\,p-S_b\,q)\,r \end{array}\right]$$ □

Chapter 10

Orthogonal stuff

10.1 Steiner triangle, definition

Remark 10.1.1. More details are given at Section 18.5.2.

Definition 10.1.2. The **Steiner triangle** of point P is the triangle whose vertices are the orthogonal reflections of P on the sides of the triangle.

Proposition 10.1.3. *The Steiner triangle of point $P \doteq p : q : r$ has the following barycentrics (each point is a column) :*

$$\text{Steiner}\begin{pmatrix} p \\ q \\ r \end{pmatrix} \underset{b}{\simeq} \begin{bmatrix} -p & \dfrac{2q\,S_c}{b^2}+p & \dfrac{2r\,S_b}{c^2}+p \\ \dfrac{2p\,S_c}{a^2}+q & -q & \dfrac{2r\,S_a}{c^2}+q \\ \dfrac{2p\,S_b}{a^2}+r & \dfrac{2q\,S_a}{b^2}+r & -q \end{bmatrix} \tag{10.1}$$

Proof. Use (7.28) and obtain directly the result. □

Proposition 10.1.4. *The circumcenter of the Steiner triangle has the following center and radius:*

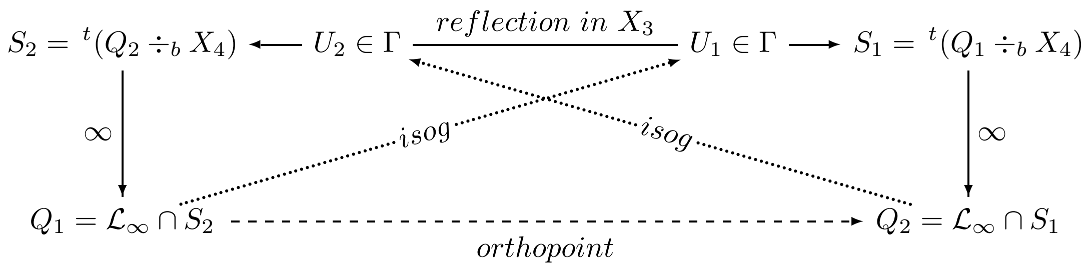
$$Q \simeq \begin{pmatrix} a^2\,qr \\ b^2\,rp \\ c^2\,pq \end{pmatrix} \;;\; \rho = \frac{\prod \sqrt{b^2r^2 + 2\,qrS_a + c^2q^2}}{(a^2qr + b^2rp + c^2qb)\,(p+q+r)}$$

Proof. Determine Q using the perpendicular bissectors $|QM_a|^2 - |QM_b|^2 = 0$ and then compute $|QM_a|$. (Spoiler) Another method will be described at Section 14.3: compute $\underset{b}{\mathcal{V}} = \bigwedge_3 \underset{b}{Ver}(M_j)$, and then use (14.14) together with (14.15) □

10.2 Steiner line

Proposition 10.2.1 (Steiner line)**.** *Let P be a point in the barycentric plane. When its Steiner triangle $A'B'C'$ is flat, the corresponding line is called the Steiner line of point P. This occurs when either:*
(1) P is on the line at infinity. Then $A' = B' = C' = P$ and $\text{Steiner}(P) = \mathcal{L}_b$.
(2) P is on the circumcircle. Then $Q =$ $\text{isogon}(P) \in \mathcal{L}_b$ *while* $\text{Steiner}(U)$ *goes through the orthocenter X_4 and has the following equation :*

$$\begin{aligned} \text{Steiner}(P) &\simeq {}^t\text{isogon}\left(P \underset{b}{*} X_4\right) \simeq {}^t\left(Q \underset{b}{\div} X_4\right) \\ &\simeq \left[S_a(\tau-\sigma), S_b(\rho-\tau), S_c(\sigma-\rho)\right] \\ &\simeq \left[\frac{a^2\,S_a}{p}, \frac{b^2\,S_b}{q}, \frac{c^2\,S_c}{r}\right] \end{aligned} \tag{10.2}$$



$$\begin{aligned} \mathit{orthopoint}\,(Q) &= \mathcal{L}_b \wedge {}^t\!\left(Q \underset{b}{\div} X_4\right) && (10.3)\\ \mathit{Steiner}\,(P) &= X_4 \wedge \mathit{orthopoint}\,(\mathit{isogon}\,(P)) && (10.4) \end{aligned}$$

Figure 10.1: The orthopoint transform

Proof. The Steiner triangle is depicted at (10.1). Its determinant factors into :

$$(p+q+r)\left(a^2qr + b^2rp + c^2pq\right)/R^2$$

The case $p+q+r=0$ is the line at infinity and is not to be discarded, since the orthogonal projection of a point at infinity onto a line at finite distance is the point at infinity of this line. When $P \in \Gamma$, computing (10.2)is straightforward. □

Remark 10.2.2. The study of the properties of the so called Simson lines is delayed until Section 28.6. Indeed, their envelope is a third degree curve, and not a simple point as for the Steiner lines.

Proposition 10.2.3. *When U_1 and U_2 are on the circumcircle, then :*

$$\left(\overbrace{\mathit{Steiner}\,(U_1)\,,\,\mathit{Steiner}\,(U_2)}\right) = -\frac{1}{2}\left(\overrightarrow{X_3U_1},\,\overrightarrow{X_3U_2}\right)$$

where the "overbrace" denotes the oriented angle between two straight lines. Therefore, it exists a one-to-one correspondence between lines through X_4 and points U on the circumcircle. Moreover, Steiner lines relative to diametrically opposed points on the circumcircle are orthogonal to each other.

Proof. From elementary euclidian geometry... or using $\tan(2\vartheta) = 2\tan\vartheta/\left(1-\tan^2\vartheta\right)$ and (7.22). □

Corollary 10.2.4. *For each point on the circumcircle, the isogonal conjugate is the orthopoint of the Steiner line (cf. Figure 10.1, and also –far below– Figure 22.7).*

Proof. Let Q_1 be a point at infinity. Take the isogonal conjugate of Q_1 and obtain $U_1 \in \Gamma$. Take the Steiner line of U_1 and obtain S_1. Take the point at infinity of S_1 and obtain Q_2. Take the isogonal conjugate of Q_2 and obtain $U_2 \in \Gamma$. Take the Steiner line of U_2 and obtain S_2. Now Q_1 is the point at infinity of S_2 while Q_1, Q_2 are orthopoints of each other and U_1, U_2 are antipodes of each other on the circumcircle. □

Proposition 10.2.5. *When Q_1, Q_2 are orthopoints, their barycentric product lies on the orthic axis, i.e. the tripolar of X_4.*

Proof. Use parametrization (7.16), eliminate k, σ in $k\,X = Q_1 \underset{b}{*} Q_2$ and obtain an equation of first degree. □

Example 10.2.6. The following list gives the triples (I, J, K) where $X(I)$ and $X(J)$ are named orthopoints of each other and $X(K)$ is their (named) barycentric product :

30	523	1637
511	512	2491
513	517	3310
514	516	676
2574	2575	647
3307	3308	3310

10.3 Parallelogy

This section has moved to Section 26.3

10.4 Orthology

This section has moved to Section 26.4

10.5 Orthopole

The section has moved to "Pedal LFIT and orthopole" (Section 28.8), while "orthojoin" has disappeared.

10.6 Spoiler: Moebius-Steiner-Cremona transform

This section should be skipped by a reader not familiar with Morley spaces (Chapter 15) and Cremona transforms (Chapter 18).

Definition 10.6.1. Let $\alpha : 1 : 1/\alpha$, etc be the reference triangle ABC, and $z : t : \zeta$ the Morley affix of a point P.

Proposition 10.6.2. *Let* $\boxed{\mathcal{T}}$ *be the Steiner triangle of point* P. *The affine transform* ψ *characterized by* $\boxed{ABC} \mapsto \boxed{\mathcal{T}}$ *sends* O *on* H *and satisfies:*

$$z\left(\psi\left(M\right)\right) = +s_1 - z_M - \frac{\zeta\, s_3}{t}\,\zeta_M$$

Proof. The Steiner triangle of P is described by:

$$\boxed{\mathcal{T}} \simeq \frac{1}{t}\left(\begin{array}{ccc} \dfrac{t\,(\beta+\gamma)-\beta\gamma\,\zeta}{t} & \dfrac{t\,(\gamma+\alpha)-\gamma\alpha\,\zeta}{t} & \dfrac{t\,(\alpha+\beta)-\alpha\beta\,\zeta}{t} \\ \dfrac{t\,(\beta+\gamma)-z}{\beta\,\gamma} & \dfrac{t\,(\gamma+\alpha)-z}{\gamma\,\alpha} & \dfrac{t\,(\alpha+\beta)-z}{\alpha\,\beta} \end{array}\right)$$

and, therefore, the matrix of ψ is:

$$\boxed{\mathcal{T}}\cdot\boxed{Lu^{-1}} \simeq \left[\begin{array}{ccc} -1 & s_1 & -\dfrac{\zeta\, s_3}{t} \\ 0 & 1 & 0 \\ -\dfrac{z}{s_3\, t} & \dfrac{s_2}{s_3} & -1 \end{array}\right]$$

□

Corollary 10.6.3. *Applied to* $Q \doteq \operatorname{isogon}(P)$ *and using Fact 18.5.9, this gives* (18.7).

Proposition 10.6.4. *Let us consider two points and their Steiner triangles. Define the collineation ϕ by $\mathcal{T}_1 \mapsto \mathcal{T}_2$; $\mathcal{L}_z \mapsto \mathcal{L}_z$. Its matrix is :*

$$\boxed{\phi} \simeq \boxed{\psi_2} \cdot \boxed{\psi_1}^{-1} \simeq \boxed{\mathcal{T}_2} \cdot \boxed{\mathcal{T}_1}^{-1}$$
$$\simeq \left(\begin{array}{ccc} t_1\,t_2-\zeta_2\,z_1 & \dfrac{(\zeta_2\,t_1-\zeta_1\,t_2)\,(z_1\,\sigma_1-\sigma_2 t_1)}{t_1} & \sigma_3\,(\zeta_2\,t_1-\zeta_1\,t_2) \\ 0 & \left(t_1^2-z_1\zeta_1\right)\,t_2/t_1 & 0 \\ \dfrac{1}{\sigma_3}\,(z_2\,t_1-z_1\,t_2) & \dfrac{(z_2\,t_1-z_1\,t_2)\,(\zeta_1\,\sigma_2-\sigma_1 t_1)}{t_1\,\sigma_3} & t_1\,t_2-z_2\,\zeta_1 \end{array}\right)$$

its characteristic polynomial is

$$\chi(\mu)=(\mu-1)\left(\mu^2-\frac{t_1\,(2\,t_1\,t_2-\zeta_2\,z_1-z_2\,\zeta_1)}{t_2\,(t_1^2-z_1\,\zeta_1)}\,\mu+\frac{\left(t_2^2-z_2\zeta_2\right)t_1^2}{\left(t_1^2-z_1\zeta_1\right)t_2^2}\right)$$

and we have $\phi(H)$=H.

Remark 10.6.5. Considering some special cases:

1. For ϕ to be parallelogic, condition is $\phi_{11}+\phi_{33}=0$, i.e: $z_2\,\zeta_1+\zeta_2\,z_1-2\,t_1\,t_2=0$. Each point is on the circle-polar of the other.
2. For ϕ to be orthologic, condition is $\phi_{11}-\phi_{33}=0$, i.e. $z_2\,\zeta_1-\zeta_2\,z_1=0$. Both points are aligned with the circumcenter X(3).
3. For ϕ to be an skew similarity, condition is $\phi_{11}=\phi_{33}=0$. Both points are inverse in the circumcircle.
4. For triangles $\boxed{\mathcal{T}_1}$, $\boxed{\mathcal{T}_2}$ to be in perspective (homologic), condition is

   $$\sigma_4\,(z_2\,t_1-t_2\,z_1)\,(\zeta_2\,t_1-\zeta_1\,t_2)\,((z_1\,t_2-t_1 z_2)\,\sigma_2+(\zeta_1 z_2-z_1\zeta_2)\,\sigma_3+(t_1\,\zeta_2-\zeta_1\,t_2)\,\sigma_3\sigma_1)=0$$

   In other words, either the points are equal in one of the spherical maps, or they are aligned with X(4). In this case, the perspector is a point S on the circumcircle, and line U_1HU_2 is nothing but the Steiner line of S.

Proof. Compute the perspector of $\mathcal{T}(M_1)$ and $\mathcal{T}(k\,M_1+(1-k)\,H)$. □

Proposition 10.6.6. *In the map $\mathbf{Z}:\mathbf{T}$, the homography μ defined by $A\mapsto A'$, $B\mapsto B'$, $C\mapsto C'$ is called the Moebius-Steiner transform related to U. This transform is involutive and we have:*

$$\mu(\mathbf{Z}:\mathbf{T})=\frac{t^2\mathbf{Z}-\left(t^2\sigma_1-t\zeta\,\sigma_2+\zeta^2\sigma_3\right)\mathbf{T}}{\zeta\,t\mathbf{Z}-t^2\mathbf{T}}$$

Image $\mu(U)$ is isogon(U), i.e. the other focus of the inscribed conic whose U is the first focus. Moreover, $\mu(\infty)=t/\zeta$, i.e. the symmetric of U wrt the circumcircle. Finally, A',B',C',H are concyclic when either U is on the circumcircle (the Steiner property) or U is on the orthoptic of the polar circle (centered at H).

Proof. Write the reality of the cross-ratio and obtain $\left(t^2-z\zeta\right)\left(s_1\,s_3\,t\zeta+s_2\,tz-s_3\,z\zeta-s_3\,t^2\right)=0$. □

10.7 Orthocorrespondents

Definition 10.7.1. Suppose P is a point in the plane of triangle ABC. The perpendiculars through P to the lines AP, BP, CP meet the lines BC, CA, AB, respectively, in collinear points. Let L denote their line. The trilinear pole of L is $P^\perp$, the *orthocorrespondent of* P. This definition is introduced in Gibert (2003). If $P=p:q:r$ is given in barycentrics, then $P^\perp=u:v:w$ is given by :

$$-\left(b^2+c^2-a^2\right)p^2+\left(a^2-b^2+c^2\right)pq+\left(a^2+b^2-c^2\right)pr+2qra^2$$

Remark 10.7.2. If follows that if $P = x : y : z$ is given in trilinears, then $P^{\perp}$ has trilinears given cyclically by :

$$yz + (-x\cos A + y\cos B + z\cos C)x$$

Example 10.7.3. Pairs (I,J) for which the orthocorrespondent of X(I) is X(J) include the following:

1	57	11	651	62	2005	109	1813	125	648	1566	677
2	1992	13	13	80	2006	111	895	132	287	1785	57
3	1993	14	14	98	287	112	**110**	186	1994	1845	2006
4	2	15	62	100	1332	113	2986	242	1999	1878	1997
5	1994	16	61	101	1331	114	2987	403	1993	3563	2987
6	1995	19	2000	103	1815	115	**110**	468	1992		
7	1996	32	2001	105	1814	117	2988	915	2990		
8	1997	33	2002	106	1797	118	2989	917	2989		
9	1998	36	2003	107	648	119	2990	1300	2986		
10	1999	61	2004	108	651	120	2991	1560	895		

Proposition 10.7.4. *The orthocorrespondent of every point on the line at infinity is the centroid. Conversely, given a finite point U, different from the centroid, it exists exactly two* orthoassociate points *P_1 and P_2 (real or not, distinct or not) that share the same orthocorrespondent U. When P_1 is given, then :*

$$p_2 \simeq (q_1 + r_1)p_1 + \frac{a^2 - b^2 + c^2}{a^2 - b^2 - c^2}\, q_1^2 + \frac{a^2 + b^2 - c^2}{a^2 - b^2 - c^2}\, r_1^2$$

When U is given, the condition of reality is $\Delta \geq 0$ where :

$$\Delta \doteq S^2(u + v + w)^2 - u(w + v)S_c\, S_b - v(u + w)S_a\, S_c - w(u + v)S_b\, S_a$$

$S = area$ and $S_a = \left(b^2 + c^2 - a^2\right)/2$. Then, cyclically, we have :

$$p_1, p_2 \simeq \begin{pmatrix} S\left((u - w)(u + v - w)S_b + (u - v)(u - v + w)S_c\right) \\ \pm\left((u - w)S_b + (u - v)S_c\right)\sqrt{\Delta} \end{pmatrix}$$

Proof. Write *orthocorr* $(P) = kU$ and eliminate (rationally) k, p. Obtain a second degree equation, whose discriminant is $S^2\,(v - w)^2\,\Delta$. In order to obtain a symmetric form for the barycentrics $p : q : r$, all these expressions must be simplified using $\left(\sqrt{\Delta}\right)^2 = \Delta$, then rationalized and simplified again, and finally normalized using $p + q + r = 1$. □

10.8 Isoscelizer

An isoscelizer is a line perpendicular to an angle bisector. If P is a point, then the A-isoscelizer of P is the line $L(P, A)$ through P perpendicular to the line that bisects vertex angle A; the B- and C- isoscelizers are defined cyclically. Let D and E be the points where $L(P, A)$ meets sidelines AB and AC. Unless $D = E = A$, the triangle ADE is isosceles.

In ETC, there are several triangle centers defined in terms of isoscelizers. These were discovered or invented by Peter Yff, in whose notebooks the word *isoscelizer* dates back to 1963.

Chapter 11

Circumcevian stuff

11.1 Circum-cevians, circum-anticevians

Definition 11.1.1. Circumcevian. Let P be a point, not on Γ (the circumcircle of ABC). Let A' be the other intersection of line AP with Γ and define B', C' cyclically. Then $A'B'C'$ is the circumcevian triangle of P. Using barycentric columns,

$$circumcevian(P) = \begin{pmatrix} \dfrac{-qra^2}{c^2q+b^2r} & p & p \\ q & \dfrac{-rpb^2}{a^2r+c^2p} & q \\ r & r & \dfrac{-pqc^2}{b^2p+a^2q} \end{pmatrix} \tag{11.1}$$

Proposition 11.1.2. *A circumcevian triangle is a central triangle. When P is on Γ, the corresponding triangle is totally degenerate. Three points on the circumcircle form a circumcevian triangle when the triangle obtained by killing the diagonal of the matrix is a cevian triangle.*

Proof. Centrality follows directly from (11.1) (that's a reason to keep denominators), while killing the diagonal gives the intersections with sidelines. Determinant is $\Gamma(P)^3$ over $\prod a^2q+b^2p$. □

Definition 11.1.3. Circum-anticevian. Consider the anticevian triangle $P_AP_BP_C$ of a point P that is not a vertex of triangle ABC. Line P_BP_C cuts circumcircle Γ at A. Let A' be the other intersection and define B', C' cyclically. Then $A'B'C'$ is the circum-anticevian triangle of P. Using barycentric columns,

$$circumanticevian(P) = \begin{pmatrix} \dfrac{qra^2}{c^2q-b^2r} & -p & p \\ q & \dfrac{rpb^2}{a^2r-c^2p} & -q \\ -r & r & \dfrac{pqc^2}{b^2p-a^2q} \end{pmatrix} \tag{11.2}$$

Proposition 11.1.4. *This triangle should be a central triangle (does the definition allows that ?). When one of the three other points $\pm p : \pm q : \pm r$ is on Γ, the circum-anticevian triangle degenerates.*

11.2 Steinbart transform

Definition 11.2.1. Exceter point. The circumcevian triangle of the centroid, X_2, is perspective to the tangential triangle $\mathcal{A}_6$. The perspector, X_{22}, is named Exceter point, for Phillips Exceter Academy in Exceter, New Hampshire, USA, where X_{22} was detected in 1986 using a computer.

Definition 11.2.2. Steinbart transform. The circumcevian triangle of a point P is ever perspective to the tangential triangle $\mathcal{A}_6$. The corresponding perspector has been called Steinbart point by Funck (2003) and was called in TCCT (p. 201). This transformation carries triangle centers to triangle centers. Using barycentrics :

$$Steinbart\,(P) = a^2\left(\frac{b^4}{q^2}+\frac{c^4}{r^2}-\frac{a^4}{p^2}\right) : b^2\left(\frac{a^4}{p^2}-\frac{b^4}{q^2}+\frac{c^4}{r^2}\right) : c^2\left(\frac{a^4}{p^2}+\frac{b^4}{q^2}-\frac{c^4}{r^2}\right)$$

Example 11.2.3. On the circumcircle, Steinbart transform is the identity. Here is a list of other (I,J) such that Steinbart(X(I)) = X(J) :

1	3	14	1606	56	1616	162	1624	365	55
2	22	17	1607	57	1617	163	1625	366	1631
3	1498	18	1608	58	595	174	1626	509	1486
4	24	19	1609	59	1618	188	2933	648	1632
5	1601	21	1610	63	1619	251	1627	651	1633
6	6	25	1611	64	1620	254	1628	662	1634
7	1602	28	1612	81	1621	259	198		
8	1603	31	1613	83	1078	266	56		
9	1604	54	1614	84	1622	275	1629		
13	1605	55	1615	88	1623	284	1630		

Points X(1601)-X(1634) have been contributed in ETC by Jean-Pierre Ehrmann (August 2003).

Remark 11.2.4. See Grinberg (2003e) and his Extended Steinbart Theorem in Hyacinthos #7984, 2003/09/23.

11.3 Circum-eigentransform

Definition 11.3.1. The circum-eigentransform of point $U = u : v : w$, different from X_6, is the eigencenter of the circumcevian triangle of point U and is denoted by $CET(U)$. In trilinears, we have :

$$\frac{avw}{av^2+aw^2-buv-cuw} : \frac{bwu}{bw^2+bu^2-cvw-avu} : \frac{cuv}{cu^2+cv^2-awu-bwv}$$

and in barycentrics (cyclically) :

$$\frac{a^2vw}{a^2c^2v^2+a^2b^2w^2-b^2c^2uv-bc^2uw}$$

Proposition 11.3.2. *Point $CET(U)$ lies on the circumcircle, and we have $CET(U) = isog\,(U)$ if and only if $U \in \mathcal{L}_b$.*

Remark 11.3.3. My own computations are leading to :

$$\frac{a^2vw}{-a^2c^2v^2+a^2b^2w^2-b^2c^2uv+bc^2uw}$$

This point is also on the circumcircle, but property $CET(U) = isog\,(U)$ if and only if $U \in \mathcal{L}_b$ is lost. Some signs have changed : why ?

Example 11.3.4. Apart from points on the infinity line, pairs (I, J) such that $X(J) = CET(X(I))$ include :

1	106	41	767	74	1294	238	741
2	729	42	2368	75	701	265	1300
3	1300	43	106	81	2375	670	3222
4	1294	44	106	110	99	694	98
9	1477	55	2369	125	827	895	2374
19	2365	56	2370	184	2367	1084	689
25	2366	57	2371	194	729	1279	1477
31	767	58	2372	213	2368	1634	689
32	2367	67	2373	219	2376		
37	741	69	2374	220	2377		

Exercise 11.3.5. For a given point P on the circumcircle, which points U satisfy $CET(U) = P$? For example, CET carries each of the points X(1), X(43), X(44), X(519) to X(106).

11.4 Dual triangles, DC and CD Points

Definition 11.4.1. Dual triangle. Suppose DEF is a triangle (at finite distance) in the plane of triangle ABC. Let D' be the isogonal conjugate of the point at infinity of line EF. Define E' and F' cyclically. The triangle $D'E'F'$ is called the *dual* of DEF. Its vertices lie on the circumcircle.

Proposition 11.4.2. *The dual triangle $D'E'F'$ characterizes the class of all triangles homothetic to DEF. Moreover, this triangle is similar to the original one.*

Proof. Vertices of $D'E'F'$ depends only on direction of sidelines DE, EF, FD and conversely. Similarity between DEF and $D'E'F'$ can be proved in many ways. Brute force method : Pythagoras Theorem 7.4.4 applied to both triangles leads to proportional sidelengths. □

Remark 11.4.3. (Proof of similarity follows from Theorem 6E in TCCT, as the "gamma triangle" there is the dual of a triangle whose sidelines are respectively perpendicular to those of DEF.)

Definition 11.4.4. DC point. Suppose $U = u : v : w$ is a point having cevian triangle DEF and dual triangle $D'E'F'$. It happens that the later triangle is also the circum-anticevian triangle of some point. This point will be described as $DC\,(U)$. Using barycentrics :

$$DC\,(U) = \frac{a^2}{u\,(v+w)} : \frac{b^2}{v\,(w+u)} : \frac{c^2}{w\,(u+v)}$$

Remark 11.4.5. The barycentrics of triangle $D'E'F'$ are :

$$\begin{array}{ccc} \dfrac{a^2}{wu-vu} & \dfrac{-a^2}{wu+vu} & \dfrac{a^2}{wu+vu} \\ \dfrac{b^2}{vu+wv} & \dfrac{b^2}{vu-wv} & \dfrac{-b^2}{vu+wv} \\ \dfrac{-c^2}{wv+wu} & \dfrac{c^2}{wv+wu} & \dfrac{c^2}{wv-wu} \end{array}$$

Proposition 11.4.6. *To construct $DC(U)$ from U and $D'E'F'$, let $A' = AD' \cap BC$ and let A" be the harmonic conjugate of A' with respect to B and C. Define B" and C" cyclically. The lines AA", BB" and CC" concur in $DC(U)$. We have also the formula :*

$$DC\,(U) = cevamul\,(isog\,(U)\,,\,X\,(6))$$

Example 11.4.7. There are 115 pairs (I,J) such that I<2980 and X(J) = DC(X(I)). Among them (1,81), (2,6), (3,275), (4,2), (5,288), (6,83), (7,1), (8,57), (9,1170), (10,1171). The longest chain for this relation is : 69 $\mapsto$ 4 $\mapsto$ 2 $\mapsto$ 6 $\mapsto$ 83 $\mapsto$ 3108.

Proposition 11.4.8. *Inversely, the circum-anticevian triangle of a point P is the dual of the cevian triangle of a point CD(P), given for $P = p : q : r$ by the inverse of the DC-mapping; that is:*

$$\frac{1}{-a^2qr+b^2pr+c^2pq} : \frac{1}{a^2qr-b^2pr+c^2pq} : \frac{1}{a^2qr+b^2pr-c^2pq}$$

In other words, we have :

$$CD\,(P) = isog\,(cevadiv(P,\,X(6)))$$

11.5 Saragossa points

Definition 11.5.1. Saragossa points. Let P be a point not on the circumcircle of ABC. Let $\mathcal{T}' = A'B'C'$ be the cevian triangle of P and $\mathcal{T}'' = A'', B'', C''$ the circumcevian triangle of P. Consider triangle $\mathcal{T}$ that is the crosstriangle of $\mathcal{T}'$ and $\mathcal{T}''$, i.e. the triangle whose vertices are $U = B'C'' \cap B''C'$, $V = C'A'' \cap C''A'$ and $W = A'B'' \cap A''B'$. Then (Figure 11.1) triangles ABC, $\mathcal{T}'$, $\mathcal{T}''$ and $\mathcal{T}$ are pairwise perspective ((Grinberg, 2003d). The first, second and third Saragossa points of P are the perspectors of $\mathcal{T}$ with, respectively, ABC, $\mathcal{T}'$ and $\mathcal{T}''$. The name *Saragossa* refers to the king who proved Ceva's theorem before Ceva did (Hogendijk, 1995).

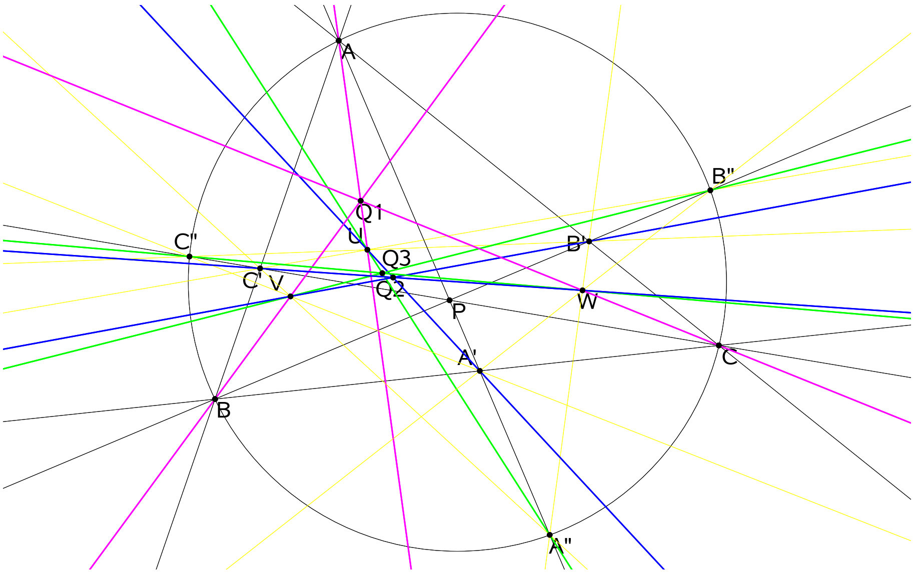


Figure 11.1: Saragossa points of point P

Proposition 11.5.2. *The barycentrics of the Saragossa points of $P = p : q : r$ are (cyclically) :*

$$\begin{aligned} g_1(a,b,c) &= \frac{a^2}{a^2qr - (a^2qr + b^2pr + c^2pq)} \\ g_2(a,b,c) &= p - \left(\frac{1}{c^2q} + \frac{1}{b^2r}\right)\left(a^2qr + b^2pr + c^2pq\right) \\ g_3(a,b,c) &= 2\,p - \left(\frac{1}{c^2q} + \frac{1}{b^2r}\right)\left(a^2qr + b^2pr + c^2pq\right) \end{aligned} \tag{11.3}$$

Points P, Q_2, Q_3 are clearly collinear. When one of the Saragossa points is equal to P then P is $X(6)$ or lies on the circumcircle.

Proof. Computations are straightforward. □

Example 11.5.3. The following table give the Saragossa points of the $X\,(I)$ whose number is given in the first line.

1	2	3	4	5	6	19	21	24	25	28	31
58	251	4	54	1166	6	284	961	847	2	943	81
386	1180	1181	?	?	6	1182	1183	?	1184	?	1185
1193	1194	185	389	?	6	1195	?	?	1196	?	1197

11.6 Vertex associates

Definition 11.6.1. Vertex associate. Consider the circumcevian triangles $A_pB_pC_p$, $A_uB_uC_u$ of points P, U (not both on the circumcircle) and draw their may be degenerate vertex triangle $\mathcal{T}$ i.e. the triangle whose sidelines are A_pA_u, B_pB_u, C_pC_u. It happens that $\mathcal{T}$ is perspective to ABC : the corresponding perspector X is called the vertex associate of P and U.

Proposition 11.6.2. *When $P \in \Gamma$ but $U \notin \Gamma$, $A_pB_pC_p$ and $\mathcal{T}$ are totally degenerate at P, so that $X = P$ (regardless of U). Otherwise, the barycentrics of the vertex associate of $p : q : r$ and u:v:w are (cyclically) :*

$$\mathit{vertexthird}\,(P, U) = \frac{a^2}{wra^4qv - p\,(wb^2 + vc^2)\,u\,(rb^2 + c^2q)}$$

Proof. When both P, U are on Γ, both circumcevians are totally degenerate and X is not defined. □

Remark 11.6.3. The definition of vertex conjugate allows X = U. To extend the geometric interpretation to the case that X = U, as X approaches U, the vertex triangle approaches a limiting triangle which we call the tangential triangle of U, a triangle perspective to ABC with perspector U-vertex conjugate of U.

Proposition 11.6.4. *When P is not on Γ, but U is on P_Γ^* (the Γ-polar of P), then $\mathcal{T}$ is totally degenerate to a point X that is the Γ-pole of line PU. Finally, triangle PUX is autopolar wrt Γ.*

Proof. Concurrence of A_pA_u, B_pB_u, C_pC_u in a point X is straightforward, and $X \in P_\Gamma^*$ too. □

Proposition 11.6.5. *Operation vertexthird is commutative and "formally involutory" i.e. :*

$$vertexthird\,(P,\,vertexthird\,(P,U)) \simeq U$$

unless P lies on the circumcircle (where $vertexthird\,(P,U) = P$, regardless of U).

Proof. Commutativity is from the very definition, and the formal involutory property is from straightforward computations. It remains only to track degeneracies. Determinant of the vertex triangle $\mathcal{T}$ is the square of determinant of the corresponding trigone ($\mathcal{T}$ is either a genuine triangle, or totally degenerate). The factors are the conditions for $U \in \Gamma$, $P \in \Gamma$ and the condition for one point to be on the Γ-polar of the other. □

Exercise 11.6.6. In the general case, what can be said about the way the three circumcevian triangles are lying on Γ ?

Example 11.6.7. Here are some vertex conjugates X(I), X(J), X(K) :

	1	2	3	4	5	6	7	8	9
1	56	3415	84	3417		2163	3418		3420
2	3415	25	3424	3425		1383			
3	84	3424	64	4		3426	3427		
4	3417	3425	4	3		3431			
5					3432				
6	2163	1383	3426	3431		6			
7	3418		3427				3433		
8								3435	
9	3420								1436

Proposition 11.6.8. *For a given U, not on the circumcircle, the associated first Saragossa point (11.3) is the sole and only point X such that $vertexthird\,(U,X) = X$.*

Proof. We want $U = VT\,(X, VT\,(X,U)) = VT\,(X,X)$ while $sarag1\,(VT\,(P,P)) = P$ is straightforward. □

Proposition 11.6.9. *Vertex association wrt X_3 maps the Darboux cubic to the Darboux cubic (X_3 is the reflection center of this cubic, whose pole is X_6 and pivot X_{20}). The appearance of (I,J) in the following list means that X(I),X(J) are on the Darboux cubic and that X(3),X(I),X(J) are vertex associates :*

1	3	4	20	40	1490	1498	2131	3182
84	64	4	3346	3345	3347	3348	3183	3354

Chapter 12

About conics

Notation 12.0.1. $\mathcal{C}$ is a conic, $\boxed{\mathcal{C}}$ is the matrix of the punctual equation while $\boxed{\mathcal{C}^*}$ is the matrix of the tangential equation. Point P is (often) the *perspector*, $U \simeq u : v : w$ is (often) the *center*, $Q \simeq f : g : h$ the auxiliary point (of an inconic), $F_j \simeq f_j : g_j : h_j$ the focuses i.e. F_0 for a parabola and F_1, F_2 otherwise.

12.1 Tangent to a curve

Definition 12.1.1. An algebraic curve $\mathcal{C}$ is the set of all the points $x : y : z$ that satisfy a polynomial equation $\mathcal{C}(x, y, z) = 0$. In order to be a projective property, the polynomial $\mathcal{C}(x, y, z)$ is required to be homogeneous (this is ever assumed in what follows).

Proposition 12.1.2. *Consider an algebraic curve $\mathcal{C}$ (not necessarily a conic). The line tangent to $\mathcal{C}$ at point $P = p : q : r$ is given by :*

$$\overrightarrow{grad}(\mathcal{C})_{p:q:r} = \left[\left(\frac{\partial \mathcal{C}}{\partial x}\right)_{X=P}, \left(\frac{\partial \mathcal{C}}{\partial y}\right)_{X=P}, \left(\frac{\partial \mathcal{C}}{\partial z}\right)_{X=P}\right] \tag{12.1}$$

Proof. Let $P \in \mathcal{C}$ be the contact point and $P + kQ$ be a point in the vicinity. If we require $P + kQ \in \mathcal{C}$, we must have $\mathcal{C}(P + kQ) - \mathcal{C}(P) = \mathrm{O}\left(k^2\right)$ and this is $\overrightarrow{grad}(\mathcal{C})_{p:q:r} \cdot U = 0$. But polynomial $\mathcal{C}$ is homogeneous and we have $\overrightarrow{grad}(\mathcal{C})_{p:q:r} \cdot P = dg(\mathcal{C})\,\mathcal{C}$. The result follows. □

Exercise 12.1.3. Use parametrization (7.17) to describe the points P of the circumcircle. Obtain the tangent at P. Take the orthodir and obtain the normal. Differentiate and wedge to catch the contact point of the envelope of all the normals... and obtain X(3).

Definition 12.1.4. Pole and polar. The polar line of point X with respect to an algebraic (homogeneous) curve $\mathcal{C}$ is the line whose affix is the gradient of $\mathcal{C}$ evaluated at point X. Point X is called a pole of its polar.

Remark 12.1.5. When point X is a simple point on an algebraic curve, its polar is nothing but the line tangent at X to the curve. Finding all the points whose polar is a given line is not an easy task in the general case.

Proof. Well-known result. In fact, this is the rationale for the concept of polarity. □

Definition 12.1.6. To avoid misunderstandings, it is often useful to specify the curve used to polarize. So we will use **circumpolar** to describe polarity wrt the circumcircle, and conipolar to describe polarity wrt a given specified conic.

Remark 12.1.7. Triangle ABC can be seen as the curve xyz. Gradient evaluated at point $P \simeq p : q : r$ is $[qr, rp, pq]$ i.e. the already defined tripolar. And we can see why the tripolar transform is not "as nice as" the usual conipolar transform: the degree of the underlying curve is not 2: there is no more "reciprocity".

12.2 Folium of Descartes

The curve know as the "folium of Descartes" is surely not a conic ! But, in our opinion, it could be useful to see how some general methods are working in the general case, before using them in the rather specific situation of the algebraic curves of degree two.

Definition 12.2.1. The **folium** $\mathcal{F}$ is the curve which Descartes used to check his methods regarding the coordinate system. The Cartesian equation of this curve is $x^3 + y^3 - 6xy = 0$, and its homogeneous equation is

$$X^3 + Y^3 - 6\,X\,Y\,T = 0$$

Proposition 12.2.2. *The folium presents a double point at* $0:0:1$*. If we cut by the line* $Y = pX$*, we obtain the parametrization :* $M \simeq 6p : 6p^2 : 1 + p^3$*. A better parametrization is :*

$$M \simeq 3\,(1+q)\,(1-q)^2 : 3\,(1+q)^2\,(1-q) : 3\,q^2 + 1 \tag{12.2}$$

Then the tangent Δ_M *at* $M \in \mathcal{F}$ *is given by*

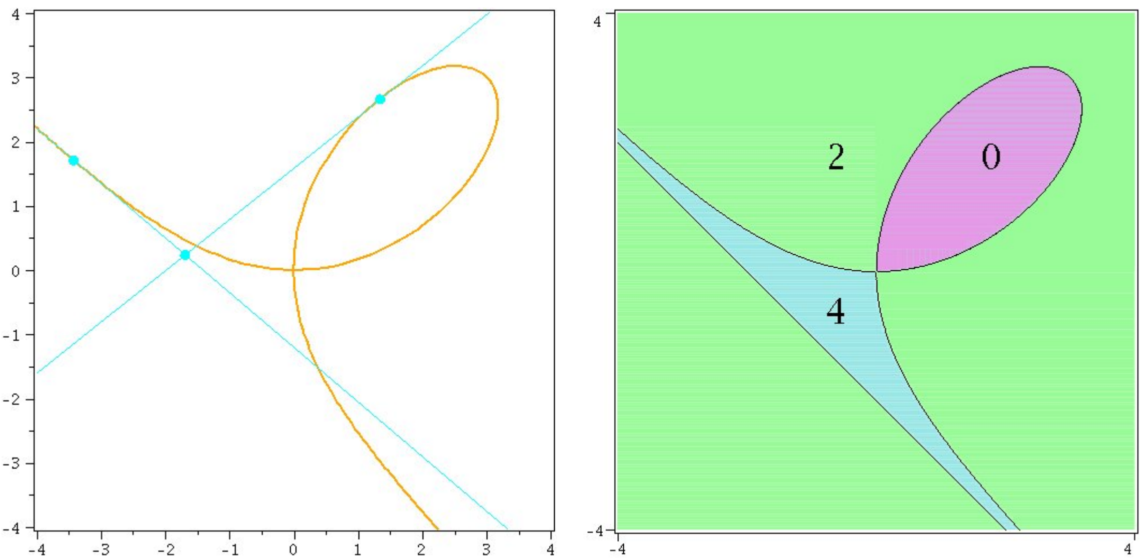
$$N = x : y : z \in \Delta_M \quad when \quad \left[3X^2 - 6T,\, 3Y^2 - 6T,\, -6XY\right] \cdot \left(\begin{array}{c} x \\ y \\ z \end{array}\right) = 0$$

Proof. Homography $q = (p-1)\,/\,(p+1)$ has been used to move point $p = -1$ at $q = \infty$ in order to have a "one piece" curve. Tangency condition is $\overrightarrow{grad\phi} \cdot \overrightarrow{MN} = 0$, while, due to the Darboux property, we already have $\overrightarrow{grad\phi} \cdot M = 0$. □

Example 12.2.3. When $q_1 = 1/3$ and $q_2 = -3$, we obtain points $M_1 = 4 : -5 : 8$ and $M_2 = 24 : -12 : 7$. Tangents are $\left[\begin{array}{ccc} 4 & -5 & 8 \end{array}\right]$, $\left[\begin{array}{ccc} 17 & 20 & 24 \end{array}\right]$ while their common point is (see Figure 12.1) given by :

$$\left[\begin{array}{ccc} 4 & -5 & 8 \end{array}\right] \wedge \left[\begin{array}{ccc} 17 & 20 & 24 \end{array}\right] = 56 : -8 : -33 \approx -1.70 : +0.24 : 1$$

Visible asymptote is the tangent at the visible $T = 0$ point, i.e. at $+1 : -1 : 0$. Using the gradient at that point, we see that asymptote is $[3, 3, -6] \simeq [1, 1, -2]$.



(a) Two intersecting tangents (b) Dividing the plane in three regions

Figure 12.1: Folium of Descartes

Proposition 12.2.4. *Tangential equation of the folium (i.e. the condition for a line* $\Delta \simeq [u, v, w]$ *to be tangent to the curve is :*

$$\mathcal{F}^*\,(u, v, w) \doteq 48\,u^2 v^2 - 32\,w\left(u^3 + v^3\right) + 24\,uvw^2 - w^4 = 0$$

Proof. Tangency requires a contact point, so that :

$$\left\{(q-1)\left(3\,q^3+9\,q+3\,q^2+1\right)=Kv,\ (q+1)\left(3\,q^3+9\,q-3\,q^2-1\right)=Ku,\ 6\,(q+1)^2\,(q-1)^2=Kw\right\}$$

is required. Apart from $w=0$ or $q=\pm1$, last equation gives a K value, that can be substituted into the other equations. Writing that the remaining two polynomials have the same roots, we obtain the resolvant :

$$\begin{vmatrix} -w-6\,v & -9\,w-6\,v & -3\,w+6\,v & -3\,w+6\,v & 0 & 0 \\ 0 & -w-6\,v & -9\,w-6\,v & -3\,w+6\,v & -3\,w+6\,v & 0 \\ 0 & 0 & -w-6\,v & -9\,w-6\,v & -3\,w+6\,v & -3\,w+6\,v \\ w+6\,u & -9\,w-6\,u & 3\,w-6\,u & -3\,w+6\,u & 0 & 0 \\ 0 & w+6\,u & -9\,w-6\,u & 3\,w-6\,u & -3\,w+6\,u & 0 \\ 0 & 0 & w+6\,u & -9\,w-6\,u & 3\,w-6\,u & -3\,w+6\,u \end{vmatrix}$$

Suppressing the non vanishing factors leads to the given result. □

Example 12.2.5. Start from a point $N\simeq x:y:z$ and search the u,v,w such that

$$\{ux+vy+wz=0,\ \Psi=0\}$$

We have three different possibilities, that are exemplified by :

$$N\simeq 1:1:1,\quad \begin{array}{ccc} u & v & w \\ 1.0 & -0.51903-1.16372\,i & -0.48097+1.16372\,i \\ 1.0 & -0.31967+0.71674\,i & -0.68033-0.71674\,i \\ 1.0 & -0.51903+1.16372\,i & -0.48097-1.16372\,i \\ 1.0 & -0.31967-0.71674\,i & -0.68033+0.71674\,i \end{array}$$

$$N\simeq 4:2:1,\quad \begin{array}{ccc} u & v & w \\ 1.0 & 1.0 & -6.0 \\ 1.0 & -1.52334 & -0.95333 \\ 1.0 & -0.73833-1.09791\,i & -2.52334+2.19582\,i \\ 1.0 & -0.73833+1.09791\,i & -2.52334-2.19582\,i \end{array}$$

$$N=-\frac{1}{2}:-\frac{1}{2}:1,\quad \begin{array}{ccc} u & v & w \\ 1.0 & 1.35307 & 1.17653 \\ 1.0 & 0.73906 & 0.86953 \\ 1.0 & -0.43584 & 0.28208 \\ 1.0 & -2.29442 & -0.64721 \end{array}$$

In other words, the curve and its asymptote are dividing the plane into three zones. From a magenta point (see Figure 12.1b) no tangents can be drawn to the curve, but two from a green point and four from a cyan point.

Proposition 12.2.6 (Plucker formulas)**.** *Let d, δ, κ be respectively the degree of a curve $\mathcal{C}$, its number of nodes (double points with two tangents), its number of cups (double point with a single tangent) and d', δ', κ' be the corresponding numbers for the dual curve $\mathcal{C}'$ then we have the following relations :*

$$\begin{aligned} d' &= d\,(d-1)-2\delta-3\kappa \\ \kappa' &= 3d\,(d-2)-6\delta-8\kappa \\ d &= d'\,(d'-1)-2\delta'-3\kappa' \\ \kappa &= 3d'\,(d'-2)-6\delta'-8\kappa' \\ g &\doteq \frac{1}{2}\,(d-1)\,(d-2)-\delta-\kappa=\frac{1}{2}\,(d'-1)\,(d'-2)-\delta'-\kappa' \end{aligned}$$

Proof. Proof is not obvious since the formula turns weird when points with multiplicity greater than two are occurring. There is no simpler formula giving δ than $g=g'$. A remark : we have $3d-\kappa=3d'-\kappa'$. Application : see Figure 12.2. Cups are occurring at $x=y=1/2$ and at $x=j/2$, $y=j^2/2$ and conjugate. □

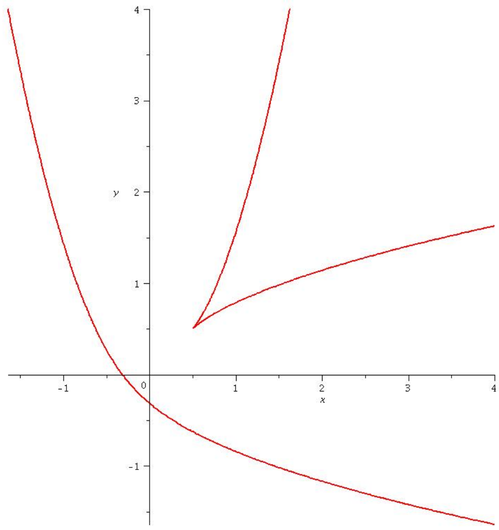


Figure 12.2: Dual curve of the folium

12.3 General facts about conics

Definition 12.3.1. A conic $\mathcal{C}$ is a curve whose barycentric equation is an homogeneous polynomial of second degree. This can be written using the usual matrix apparatus :

$$X \in \mathcal{C} \Longleftrightarrow {}^tX \cdot \boxed{\mathcal{C}} \cdot X = (x, y, z) \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Lemma 12.3.2. ***Comatrix***. *Let* $\boxed{M}$ *be a* $n \times n$ *matrix and* $\boxed{M^*}$ *the matrix of the cofactors (at the right place, such that cofactors of a row form a column). Then :*

$$\boxed{M} \cdot \boxed{M^*} = \boxed{M^*} \cdot \boxed{M} = \det\left(\boxed{M}\right) \mathbf{1}_n$$

Moreover rank $\boxed{M^*} = n$ *when* rank $\boxed{M} = n$, rank $\boxed{M^*} = 1$ *when* rank $\boxed{M} = n-1$, *and* otherwise $\boxed{M^*}$ is the 0 matrix.

Proof. When $\det \boxed{M} \neq 0$, both matrices can be inverted. In the second case a row of $\boxed{M^*}$ describes the hyperplane obtained by any non-zero wedge of two columns of $\boxed{M}$. In the last one, all minors are 0 since rank is less than $n - 1$. ☐

Remark 12.3.3. Comatrix $\boxed{M^*}$ is also called the **adjoint** matrix of M. Frenchies are usually proud to dispose wrongly the cofactors... and then are reduced to use the transpose of what they are wrongly calling "comatrices".

Definition 12.3.4. Proper conic. A quadratic form can be written as the sum of as many squares of independent linear forms as its rank, leading to the following classification :

1. When rank is one, $\mathcal{C}$ is a straight line, whose points are each counted twice (strange object).
2. When rank is 2, $\mathcal{C}$ is the union of two intersecting lines. When these lines are complex conjugate of each other (not real), only their intersection is real and this point appears as an isolated point. When one of these lines is the line at infinity, $\mathcal{C}$ is considered as some kind of extended circle (see Chapter 13).
3. When $\det \boxed{\mathcal{C}} \neq 0$, intersection of $\mathcal{C}$ and any straight line contains exactly two points (real or not, may be a double point). A such conic is called a proper conic.

Proposition 12.3.5. ***Conic by five points***. *Let $P_j \simeq p_j : q_j : r_j$ be five fixed points and $P = P_6 \simeq x : y : z$ be a generic sixth point. In this section we define the (conic) Veronese map as*

$$\underset{b}{Ver} : [p : q : r] \mapsto \left[p^2 : q^2 : r^2 : pq : qr : rp\right]$$

Let $\boxed{\widehat{Q}}$ be the 6×6 matrix $\left[\underset{b}{Ver}\left(P_j\right)\right]$. Then the six points $P_1 ... P_5, P$ are coconic when the six Veronese are co-hyperplanar, i.e. when $Q(x, y, z) \doteq \det \boxed{\widehat{Q}} = 0$. Let $\boxed{Q}$ be the matrix of Q so that $Q = {}^tP \cdot \boxed{Q} \cdot P$. Then :

1. When none of the 10 P_j-triples are collinear, then $\boxed{Q}$ defines a proper conic.
2. When four of the five points P_j are collinear, then $Q \equiv 0$ and no conic is defined.
3. Otherwise, the conic degenerates into the reunion of two lines.

Proof. In all of the cases, a simple computation leads to the "ten determinants formula"

$$\det \boxed{Q} = \prod_{j,k,l}^{10} \det \left(P_j P_k P_l\right)$$

while the relation $P_3 = a\, P_1 + b\, P_2$ leads to the factorization:

$$\det \boxed{\widehat{Q}} = ab \det \left(P_1 P_2 P_4\right) \det \left(P_1 P_2 P_5\right) \times \det \left(P_1 P_2 P\right) \det \left(P_4 P_5 P\right)$$ □

Proposition 12.3.6. ***Proper parametrization***. *Given a proper conic $\mathcal{C}$, a basis can be found where the equation of $\mathcal{C}$ becomes $xz - y^2 = 0$. And then, a parametrization is $u^2 : uv : v^2$.*

Proof. Rewrite the equation as a sum of three squares. And then use $x^2 + z^2 = (x - i\, z)(x + i\, z)$. More precisely, defining

$$K^2 \doteq m_{22}\, m_{33} - m_{23}^2 \;;\; \chi \doteq \begin{bmatrix} m_{12} & m_{22} & m_{23} \\ 1 & 0 & 0 \\ m_{22}\, m_{13} - m_{23}\, m_{12} & 0 & K^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -iK & 0 & iK \end{bmatrix}$$

$$\text{leads to } {}^t\chi \cdot \boxed{\mathcal{C}} \cdot \chi \simeq \begin{bmatrix} 0 & 0 & K^2 \\ 0 & 2\, m_{22} \det \mathcal{C} & 0 \\ K^2 & 0 & 0 \end{bmatrix}$$ □

Definition 12.3.7. Pole and polar. According to the general definitions, the polar of a point X wrt a proper conic $\mathcal{C}$ is the line ${}^tX \boxed{\mathcal{C}}$, i.e. the locus of the points Y such that ${}^tX \boxed{\mathcal{C}} Y = 0$, while the pole of line $\{Y \mid \Delta\, Y = 0\}$ is the point $X = \text{Adjoint}\left(\boxed{\mathcal{C}}\right) {}^t\Delta$.

Remark 12.3.8. This polarity is not to be confused with polarity wrt the main triangle. Therefore, it can be useful to describe line ${}^tX \boxed{\mathcal{C}}$ as the conipolar of X and line ${}^t isotom\,(X)$ as the tripolar of X.

Remark 12.3.9. The relation "point Q belongs to the polar of P" is symmetric. When point P belongs to $\mathcal{C}$, its polar wrt the conic is nothing but the line tangent at X to the conic.

Proposition 12.3.10. *For two distinct points A, B, and for two distinct lines Δ_1, Δ_2, we have*

$$\begin{aligned} \text{polar}\,(A \wedge B) &= (\text{polar}\, A) \wedge (\text{polar}\, B) \\ \text{polar}\,(\Delta_1 \wedge \Delta_2) &= (\text{polar}\, \Delta_1) \wedge (\text{polar}\, \Delta_2) \end{aligned}$$

Proof. When M is a 3×3 invertible matrix, then, for any columns A, B, we have:

$$\left({}^tA \cdot M\right) \wedge \left({}^tB \cdot M\right) = M^* \cdot {}^t(A \wedge B)$$

This can be seen by taking coordinates, and expanding $\left({}^{t}A\cdot M\right)\wedge\left({}^{t}B\cdot M\right)-M^{*}\cdot{}^{t}(A\wedge B)$ to 0. Another proof is using a generic row, called X in what follows, and check that:

$$\begin{aligned}X\cdot\left({}^{t}A\cdot M\right)\wedge\left({}^{t}B\cdot M\right) &= \det\left[X\cdot M^{-1}\cdot M,\ {}^{t}A\cdot M,\ {}^{t}B\cdot M\right]\\ &= \det\left[X\cdot M^{-1},\ {}^{t}A,\ {}^{t}B\right]\times\det M\\ &= \det\left[X\cdot M^{*},\ {}^{t}A,\ {}^{t}B\right]\\ &= X\cdot M^{*}\cdot\left({}^{t}A\wedge{}^{t}B\right)=X\cdot M^{*}\cdot{}^{t}(A\wedge B)\end{aligned}$$

□

Definition 12.3.11. Taking the polar lines of the vertices of a triangle $\mathcal{T}$ gives a trigone. The associate triangle is called the polar triangle of $\mathcal{T}$ and noted polar $\mathcal{T}$. When both triangles are equal, we say that $\mathcal{T}$ is autopolar wrt $\mathcal{C}$.

Proposition 12.3.12. *When quadrangle A,B,C,D is inscribed in a proper conic $\mathcal{C}$, its diagonal triangle, i.e. $AB\cap CD$, $AC\cap BD$, $AD\cap BC$ is autopolar wrt $\mathcal{C}$.*

Computed Proof. From 12.3.6, we can describe the problem by:

$$\boxed{\mathcal{C}}\simeq\left[\begin{array}{ccc}0&0&-1\\0&2&0\\-1&0&0\end{array}\right]\ ;\ A,B,C,D\simeq\left(\begin{array}{c}u_1^2\\u_1v_1\\v_1^2\end{array}\right),\left(\begin{array}{c}u_2^2\\u_2v_2\\v_2^2\end{array}\right),\left(\begin{array}{c}u_3^2\\u_3v_3\\v_3^2\end{array}\right),\left(\begin{array}{c}u_4^2\\u_4v_4\\v_4^2\end{array}\right)$$

and ${}^{t}(AB\wedge CD)\cdot\boxed{\mathcal{C}}\cdot(AC\wedge BD)=0$ is easy to verify. □

Construction 12.3.13. *The polar line of a point P wrt a conic $\mathcal{C}$ is the locus of the points $AC\cap BD$ where AB and CD are chords of $\mathcal{C}$ that meet in P. As a result, the required conipolar is the line $AC\cap BD$; $AD\cap BC$.*

Proof. Obvious from the previous proposition. □

Proposition 12.3.14. *Triangle $\mathcal{T}$ is autopolar wrt a proper conic $\mathcal{C}$ if and only if the matrix of $\mathcal{C}$ wrt triangle $\mathcal{T}$ is diagonal.*

Proof. Use $\mathcal{T}$ as barycentric basis. Then eliminate and see that either $\mathcal{C}$ is diagonal, or contains a null column □

Proposition 12.3.15. *Perspector of a conic wrt a triangle*. *When the polar triangle of $\mathcal{T}$ is not $\mathcal{T}$ itself, then both triangles are in perspective. This defines a perspector and a perspectrix (related to the triangle). When $\mathcal{T}$ is the reference triangle itself, the polar triangle is $\boxed{\mathcal{C}^{*}}$ so that:*

$$P\simeq\left(\begin{array}{c}(m_{22}\,m_{13}-m_{23}\,m_{12})\,(m_{33}\,m_{12}\,m_{13}\,m_{23})\\(m_{33}\,m_{12}-m_{13}\,m_{23})\,(m_{11}\,m_{23}-m_{13}\,m_{12})\\(m_{11}\,m_{23}-m_{13}\,m_{12})\,(m_{22}\,m_{13}-m_{23}\,m_{12})\end{array}\right)\ ;\ \Delta\simeq[m_{13}\,m_{12},m_{23}\,m_{12},m_{13}\,m_{23}]$$

When nothing vanishes, the following shorter formulas can be used:

$$\begin{aligned}\text{isotom}\,P &\simeq m_{11}\,m_{23}-m_{13}\,m_{12}\ :\ m_{22}\,m_{13}-m_{23}\,m_{12}\ :\ m_{33}\,m_{12}\,m_{13}\,m_{23}\\ \text{tripolar}\,\Delta &\simeq m_{23}\ :\ m_{13}\ :\ m_{12}\end{aligned}$$

Proof. The $\mathcal{T}=ABC$ case is easy to compute. And suffices to prove the general case. □

Definition 12.3.16. Dual of a conic. The dual of a given conic $\mathcal{C}_1$ is the conic $\mathcal{C}_2$ such that point $x:y:z$ belongs to $\mathcal{C}_2$ when line (x,y,z) is tangent to $\mathcal{C}_1$. When dealing with proper conics, we have $\boxed{\mathcal{C}_2}=\text{Adjoint}\left(\boxed{\mathcal{C}_1}\right)$ and conversely. When rank is 2, the dual is rank 1 : all tangents have to pass through the common point.

Definition 12.3.17. The **center** U of a conic $\mathcal{C}$ is the pole of the line at infinity $\mathcal{L}_b$ with respect to the conic. Its barycentrics are :

$$\begin{aligned}&-m_{23}^2+(m_{13}+m_{12})\,m_{23}+m_{22}m_{33}-m_{13}m_{22}-m_{12}m_{33}\\&-m_{13}^2+(m_{12}+m_{23})\,m_{13}+m_{33}m_{11}-m_{12}m_{33}-m_{23}m_{11}\\&-m_{12}^2+(m_{23}+m_{13})\,m_{12}+m_{11}m_{22}-m_{23}m_{11}-m_{13}m_{22}\end{aligned}$$

Definition 12.3.18. A **parabola** is a conic whose center is at infinity (more about parabolas in Section 12.21). Two parallel lines make a non proper parabola. The union of the line at infinity and another line is ... some kind of circle rather than a "special special" parabola.

Fact 12.3.19. *When a conic goes through its center and this center is not at infinity, the conic is the union of two different straight lines. When a conic is a single line whose points are counted twice,* $\det \mathcal{C}$ *vanishes and center has no meaning.*

Proposition 12.3.20. *When* $\mathcal{C}$ *is not a parabola, its center is the symmetry center of the conic.*

Computed Proof. Substitute (3.1) into the equation and obtain $\mathcal{C}(x, y, z)$ times the square of the condition to be a parabola. □

Proposition 12.3.21. *Let* P *be a point not on the sidelines of* ABC*. Six points are obtained by intersecting a sideline with a parallel through* P *to another sideline. These points are on the same conic* $\mathcal{C}$*. Equation, perspector* T *and center* U *are :*

$$\begin{aligned} \mathcal{C} &= \sum (q+r)qrx^2 - \sum (p^2+pq+pr+2qr)pyz \\ T &= \frac{p}{2\,pr+2\,pq+qr} : \frac{q}{2\,pq+pr+2\,qr} : \frac{r}{pq+2\,pr+2\,qr} \\ U &= p\left(2\,qr+pr+pq-p^2\right) : q\left(2\,pr+pq+qr-q^2\right) : r\left(2\,pq+pr+qr-r^2\right) \end{aligned}$$

Center U *is at infinity (and* $\mathcal{C}$ *is a parabola) when* P *is at infinity or on the Steiner inconic. Points* P *and* $Q = p\,(q+r-p) : q\,(r+p-q)\,, r\,(p+q-r)$ *are leading to the same center* U*. Point* Q *is at infinity when* P *is on the Steiner inconic.*

Proof. Equation in Q is of third degree. The discriminant factors into minus a product of squares. Other computations are straightforward. Examples are [P,U], [115, 523], [1015, 513], [1084, 512], [1086, 514], [1146, 522], [2482, 524], [3163, 30] where P is on the Steiner inconic and [P,U], [2, 2], [6, 182], [3, 182], [9, 1001], [1, 1001], [190, 1016], [664, 1275] for other points. □

Exercise 12.3.22. Show that points X(13),X(14),X(15),X(16),X(17),X(18) belong to a same conic (Evans, 2002). Identify 50 ETC points on its perimeter. Center is X(3054).

12.4 Tangential conics

Definition 12.4.1. A point-conic is a 'conic as usual', i.e. a locus of points, whose equation is ${}^tM \cdot \boxed{\mathcal{C}} \cdot M = 0$. In the general case, a line is tangent to $\mathcal{C}$ when $\Delta \cdot \boxed{\mathcal{C}^*} \cdot {}^t\Delta = 0$. A line-conic or **tangential conic** is what is obtained when seeing $\mathfrak{C}$ in equation $\Delta \cdot \boxed{\mathfrak{C}} \cdot {}^t\Delta = 0$ as the primitive object, and seeing the punctual conic ${}^tM \cdot \boxed{\mathfrak{C}^*} \cdot M = 0$ as a derivative object.

Proposition 12.4.2. *Degenerate line-conic. When* $\det \boxed{\mathfrak{C}} = 0$*, the equation splits in two first degree factors, and a line belongs to the conic when it goes through either of the fixed points (the centers) defined by each factor (supposed distinct).*

Example 12.4.3. The degenerate line-conic of the isotropic lines (see Proposition 7.6.7).

Proposition 12.4.4. *The degenerate conic formed by the two tangents drawn from a point* X_0 *to a given conic* $\boxed{\mathcal{C}}$ *is given by*

$$\begin{aligned} {}^tX \cdot \boxed{\mathcal{D}_0} \cdot X &\simeq \left({}^tX_0 \cdot \boxed{\mathcal{C}} \cdot X_0\right) * \left({}^tX \cdot \boxed{\mathcal{C}} \cdot X\right) + (-1) * \left({}^tX_0 \cdot \boxed{\mathcal{C}} \cdot X\right)^2 \\ \boxed{\mathcal{D}_0} &\simeq \left({}^tX_0 \cdot \boxed{\mathcal{C}} \cdot X_0\right) \boxed{\mathcal{C}} - \boxed{\mathcal{C}} \cdot X_0 \cdot {}^tX_0 \cdot \boxed{\mathcal{C}} \end{aligned} \tag{12.3}$$

Computed Proof. From 12.3.6, we can describe the problem by:

$$\boxed{\mathcal{C}} \simeq \left[\begin{array}{ccc} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{array}\right] \; ; \; X_1, X_2 \simeq \left(\begin{array}{c} u_1^2 \\ u_1 v_1 \\ v_1^2 \end{array}\right), \left(\begin{array}{c} u_2^2 \\ u_2 v_2 \\ v_2^2 \end{array}\right)$$

$$tan_1, tan_2 \simeq [v_1^2, -2\,u_1v_1, u_1^2],\ [v_2^2, -2\,u_2v_2, u_2^2] \; ; \; X_0 \doteq tan_1 \wedge tan_2 \simeq 2\,u_1u_2 : u_1v_2 + u_2v_1 : 2\,v_1v_2$$

Then compute $(tan_1 \cdot X) * (tan_2 \cdot X)$ and check the formula. □

more geometrico. Consider the pencil of all the conics that are bi-tangent to conic $\mathcal{C}$. Among them, we have $\mathcal{C}$ itself, the required $\mathcal{D}_0$ and the polar line of X_0... when counted twice. Thus:

$$\boxed{\mathcal{D}_0} \simeq \alpha \left(\boxed{\mathcal{C}}\right) + \beta \left(\boxed{\mathcal{C}} \cdot X_0 \cdot {}^t X_0 \cdot \boxed{\mathcal{C}}\right)$$

It remains to choose α, β so that $\mathcal{D}_0$ goes through X_0. □

12.5 Locusconi

Theorem 12.5.1. ***locusconi***. *Suppose that the projective coordinates of a point $P(s)$ are second degree polynomials in a given parameter s. Then the locus of $P(s)$ is a conic. Moreover, we have the following algorithm.*

```
Require: tmp is the column of the coordinates, and s is the parameter to eliminate
  LOCUSCONI := proc tmp, s, conitan :: uneval ; local tmp1, tmp0, coni2
  tmp1 := Matrix([seq](map(coeff, reduce(tmp), s, j), j = [2, 1, 0]))
  tmp0 := Matrix([[0, 0, 2], [0, −1, 0], [2, 0, 0]])
  coni2 := reduce(tmp1.tmp0.Tr(tmp1))
  if nargs = 3 then assign(conitan, coni2) end if
  reduce(Adjoint(coni2))
```

Listing 12.1: The locusconi procedure

Proof. When using $\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}$ as the algebraic basis, the equation of the fundamental circle is $\mathbf{Z}\overline{\mathcal{Z}} - \mathbf{T}^2$. We are doing the same here, using $1 : t : t^2$ as algebraic basis. This leads to the tangential equation of the conic... and we save it immediately. Don't compute it afterwards by taking the adjoint of the main result ! See also Kimberling (2001, p. 2). □

Exercise 12.5.2. Use the example $P(s) = \begin{pmatrix} 7\,s^2 - 12 \\ 3\,s^2 + 2\,s \\ 3\,s^2 - 2\,s - 6 \end{pmatrix}$. Show that the brute force identification method, i.e. solve $\left({}^t P \cdot \boxed{m_{jk}} \cdot P = 0, m_{jk}\right)$ amounts to write

$$[s^4, s^3, s^2, s, 1] \cdot \begin{bmatrix} 49 & 9 & 9 & 42 & 18 & 42 \\ 0 & 12 & -12 & 28 & 0 & -28 \\ -168 & 4 & -32 & -72 & -44 & -156 \\ 0 & 0 & 24 & -48 & -24 & 48 \\ 144 & 0 & 36 & 0 & 0 & 144 \end{bmatrix} \cdot \begin{bmatrix} m_{11} \\ m_{22} \\ m_{33} \\ m_{12} \\ m_{23} \\ m_{13} \end{bmatrix} = 0$$

and then to compute the 6 cofactors of this 5×6 matrix, while locusconi amounts to write

$${}^t\begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} \cdot \boxed{\Gamma} \cdot \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} = 0 \qquad \text{where } \boxed{\Gamma} \doteq {}^t Q \cdot \boxed{m_{jk}} \cdot Q \;;\; Q \doteq \begin{bmatrix} 7 & 0 & -12 \\ 3 & 2 & 0 \\ 3 & -2 & -6 \end{bmatrix}$$

and then to solve this system as: Adjoint $\left(\boxed{m_{jk}}\right) \doteq {}^t Q \cdot$ Adjoint $\left(\boxed{\Gamma}\right) \cdot Q$.

12.6 Founding configuration

Definition 12.6.1. Let us start from triangle ABC and perspector $p : q : r$. Its tripolar $\Delta \simeq [qr : rp : pq]$ goes through points $Q_a \simeq 0 : +q : -r$, $Q_b \simeq -p : 0 : +r$, $Q_c \simeq +p : -q : 0$. From

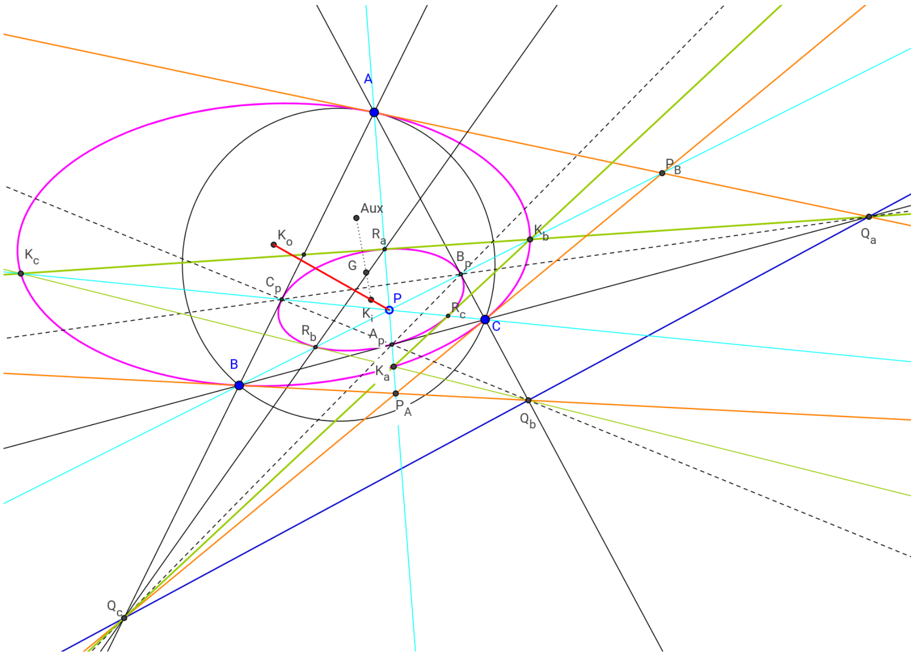


Figure 12.3: Starting from perspector P

Proposition 3.8.12, all the triangles that can be written as $\mathcal{T}_t \simeq \begin{pmatrix} tp & p & p \\ q & tq & q \\ r & r & tr \end{pmatrix}$ share P and Δ as perspector and perspectrix. And their vertices are on the P-cevian lines (dotted cyan).

Proposition 12.6.2. *We have* cross_ratio $(A, A_P, P, A_t) =$ cross_ratio $(\infty, 0, 1, t) = t$, etc.

Proof. Obvious since the cevian vertex A_p is nothing but A_0. □

Proposition 12.6.3. *Vertices of* $\boxed{\mathcal{T}_t}$ *and* $\boxed{\mathcal{T}_s}$ *are on the same conic (noted $\mathcal{C}(s,t)$) if and only if*

$$2ts + t + s - 4 = 0 \;;\; s = \sigma(t) \doteq \frac{-t+4}{2t+1}$$

Fixed points of σ are $t = 1$, $t = -2$. And we have the decomposition:

$$\boxed{\mathcal{C}_t} = \left(t^2+2\right) \begin{bmatrix} 2\,q^2r^2 & -pqr^2 & -pq^2r \\ -pqr^2 & 2\,p^2r^2 & -p^2qr \\ -pq^2r & -p^2qr & 2\,p^2q^2 \end{bmatrix} - 2\,(t-1)^2 \begin{bmatrix} q^2r^2 & 0 & 0 \\ 0 & p^2r^2 & 0 \\ 0 & 0 & p^2q^2 \end{bmatrix}$$

Proof. The corresponding 6×6 determinant factors into $(t-1)^2\,(s-1)^2\,(t-s)^3\,q^4r^4p^4\,(2\,st+s+t-4)$. □

Proposition 12.6.4. *The center K_t of conic $\mathcal{C}_t$ belongs to line PP^2. More precisely, we have*

$$\left(t^2+2\right)(p+q+r)\begin{pmatrix} p \\ q \\ r \end{pmatrix} - 2\,(t-1)^2 \begin{pmatrix} p^2 \\ q^2 \\ r^2 \end{pmatrix}$$

And we have the formula:

$$\begin{aligned} \text{cross\_ratio}\,(K(t_j)) &= \text{cross\_ratio}\,(t_1,t_2,t_3,t_4) \times \text{cross\_ratio}\,(\sigma(t_1),\sigma(t_2),t_3,t_4) \\ &= \text{cross\_ratio}\,(t_1,t_2,t_3,t_4) \times \text{cross\_ratio}\,(t_1,t_2,\sigma(t_3),\sigma(t_4)) \end{aligned}$$

Proof. Mind the fact that cross_ratio $(\sigma(t_1), \sigma(t_2), \sigma(t_3), \sigma(t_4)) =$ cross_ratio (t_1, t_2, t_3, t_4) ! □

Exercise 12.6.5. Prove that cross_ratio $\left(K_{in}, P^2, P, K_{out}\right) = -1$. that involves the conic $t = 1 = \sigma(1)$ which is centered at P and degenerates into $3\left(jqr\,x + j^2 rp\,y + pq\,z\right)\left(j^2 qr\,x + jrp\,y + qp\,z\right)$ and the conic $t^2 + 2 = 0$ which is centered at P^2.

Example 12.6.6. We have the following cases of interest:

$\mathcal{C}(-2,-2)$ degenerates into $-6\,(qr\,x + rp\,y + pq, z)^2$, i.e. the tripolar line $Q_a Q_b Q_c$.

$\mathcal{C}_1 \doteq \mathcal{C}(1,1)$ is centered at P, and degenerates into

$$3\left(j\,qr\,x + j^2 rp\,y + pq\,z\right)\left(j^2 qr\,x + j\,rp\,y + pq\,z\right) = 0$$

$\mathcal{C}_{diag} = \mathcal{C}\left(\pm i\sqrt{2}\right)$ is centered at P^2 and is diagonal.

$\mathcal{C}_{out} \doteq \mathcal{C}(\infty, -1/2)$ is the so called P-circumconic (see Section 12.7). It goes through the vertices ($t = \infty$) and the extra points $K_a = -p : 2q : 2r$, etc ($t = -1/2$).

$\mathcal{C}_{in} \doteq \mathcal{C}(0,4)$ is the so called P-inconic (see Section 12.8). It goes through the cevian points $A_P B_P C_P$ ($t = 0$) and the extra points $R_a = 4p : q : r$, etc ($t = +4$).

$\mathcal{T}(-1)$ is the anticevian triangle $P_A P_B P_C$, obtained from the orange trigone AQ_a, BQ_b, CQ_c.

Proposition 12.6.7. *The six sidelines of the following triangles:*

$$\mathcal{T}(\kappa(t)), \mathcal{T}(\sigma(\kappa(t))) \qquad \text{where } \kappa(t) \doteq \frac{2-t}{t}$$

are tangent to the conic $\mathcal{C}(t, \sigma(t))$ at the vertices of the defining triangles. Note that $(\sigma\kappa\sigma) = \kappa$, enforcing the symmetry, while Q_a, X_b, X_c are aligned for any triangle $\mathcal{T}$.

Proof. Obvious computations □

Example 12.6.8. Three tangents to $\mathcal{C}_{in} \doteq \mathcal{C}(0,4)$ are provided by $\mathcal{T}(-1/2)$, see the green lines $Q_a R_a K_b K_c$, etc . As it should be, the other triangle is $\mathcal{T}(\infty)$. i.e. the ABC triangle itself.

Six tangents to $\mathcal{C}_{out} \doteq \mathcal{C}(\infty, -1/2)$ are provided by $\mathcal{T}(-5)$ and, as it should be, by the anticevian triangle $\mathcal{T}(-1)$, see the orange lines $Q_a A P_B P_C$, etc.

Moreover, we have the alignments $Q_a R_b R_c$ not drawn), $Q_a B_P C_P$ (black dotted).

12.7 Circumconics

Definition 12.7.1. A **circumconic** is a conic that contains the vertices A, B, C of the reference triangle. Its equation can be written as :

$$CC(P) = {}^t X \boxed{\mathcal{C}_c} X = 0 \qquad \text{where } \boxed{\mathcal{C}_c} = \begin{pmatrix} 0 & r & q \\ r & 0 & p \\ q & p & 0 \end{pmatrix} \tag{12.4}$$

Construction 12.7.2. *Graphical tools can construct any conic from five points. Given the perspector $P \simeq p : q : r$, the other points on the cevian lines are $-p : 2q : 2r$, etc i.e points $2qB + 2rC - p\,A$.*

Proposition 12.7.3. *When $p : q : r$ is the perspector, a **parametrization** of CC(P) is*

$$M(t) \simeq \frac{p}{1} : \frac{q}{t} : \frac{-r}{1+t} \tag{12.5}$$

Proof. Direct inspection. □

Theorem 12.7.4 (circumconics)**.** *We have the following four properties :*
(i) Point P is the perspector of the conic, and the polar triangle of ABC wrt $CC(P)$ is the anticevian triangle of P wrt ABC (in other words, $CC(P)$ is tangent at A to $P_B P_C$ etc.
(ii) When U is the center of $CC(P)$ then P is the center of $CC(U)$. Both are related by :

$$U = cevadiv(X_2, P) = P \underset{b}{*} anticomplem(P)$$

(iii) Circumconic $CC(P)$ is the P isoconjugate of $\mathcal{L}_b$. Inter alia,

$$X \in circumcircle \Longleftrightarrow isog(X) \in \mathcal{L}_b$$

(iv) The polar line of $M \simeq x : y : z$ wrt $CC(P)$ is $\simeq[qz + ry,\, rx + pz,\, py + qx]$ ... aka the polarmul of (P, M).

Proposition 12.7.5. *Points at infinity*. *A circumscribed conic is an ellipse, a parabola or an hyperbola when its perspector is inside, on or outside the Steiner in-ellipse. Moreover, its points at infinity, expressed from the perspector $P = p : q : r$ have the following barycentrics :*

$$M_\infty \simeq \begin{pmatrix} -2\,p \\ p+q-r-W \\ p+r-q+W \end{pmatrix} \qquad \text{where } W^2 = p^2+q^2+r^2-2\,pq-2\,qr-2\,rp$$

When the point at infinity of a circumparabola is $u : v : w$, its perspector is $P = u^2 : v^2 : w^2$.

Proof. Immediate computation. Mind the fact that $W^2 = -3$ when P is at X(2). For the second part, start from $T = u : v : -u-v$, and compute the circumconic relative to $u^2 : v^2 : (u+v)^2$. □

Remark 12.7.6. Properties of the CircumRH, aka the circumscribed rectangular hyperbola, are collected at Subsection 12.22.2

Proposition 12.7.7. *The four common points of two (non equal) circumconics $CC(P)$ and $CC(Q)$ are the three vertices and the tripole of line PQ.*

Proof. Conics that share five distinct points are equal. The value of X follows by direct inspection. □

Exercise 12.7.8. A stupid person would rewrite this property as "the fourth point is isotom $(P \wedge Q)$. But the clever reader wouldn't. Explain why !

Proposition 12.7.9. *The perspector P of the circumconic through point Q lies on the tripolar of Q. In other words,*

$$P = Q \underset{b}{*} T \qquad \text{where } T \in \mathcal{L}_b$$

Therefore, the perspector of the circumconic which goes through additional points Q_1, Q_2 is the intersection of the tripolars of Q_1 and Q_2 (caveat: this is not the tripole of line Q_1Q_2). In other words, :

$$P = Q_1 \underset{b}{*} Q_2 \underset{b}{*} {}^t(Q_1 \wedge Q_2)$$

Proof. Direct inspection. □

Remark 12.7.10. Collineations can be used to transform any circumconic into the circumcircle or the Steiner out-ellipse, so that many proofs can be done assuming such special cases. More details in Proposition 16.4.3.

Proposition 12.7.11. *Let be given Δ and K where Δ is a line (tripole G) that cuts the sidelines BC, CA, AB in A', B', C' and K is a circumconic (perspector P) . Let $M = x : y : z$ be a point on Δ, and A" the other intersection of MA with K and cyclically $B" \in MB \cap K$, $C" \in MC \cap K$. Then lines $A'A"$, $B'B"$, $C'C"$ are concurrent on a point $Q \in K$. Moreover, the conjugacy that exchanges P and G exchanges also M and $Q(M)$.*

Proof. Use $\Delta \simeq [\rho, \sigma, \tau]$, $P = p : q : r$, $M = 1/\rho : t/\sigma : -(1+t)/\tau$ (where t is a parameter) and obtain

$$Q = \frac{p}{\rho x} : \frac{q}{\sigma y} : \frac{r}{\tau z}$$ □

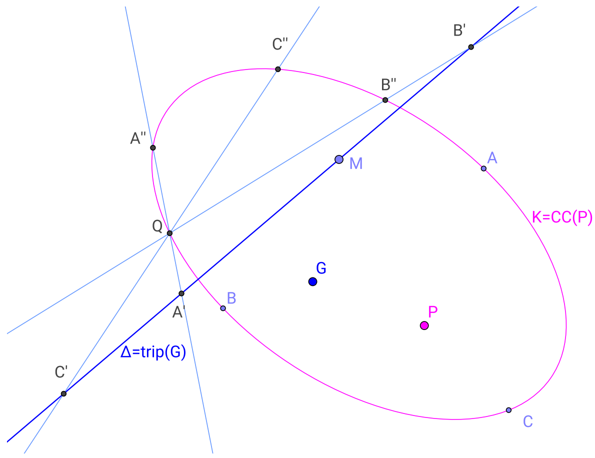


Figure 12.4: Conjugacy between a line and a circumconic

12.8 Inconics

Definition 12.8.1. An inconic is a conic that is tangent to the three sides of the reference triangle.

Exercise 12.8.2. Compute the determinant wich asserts that $pB+(1-p+K)\,C$, etc and $pB+(1-p-K)\,C$, etc are con-conics where K is evanescent. Conclude.

Theorem 12.8.3 (inconics)**.** *The punctual and tangential equations of an inconic $\mathcal{C}_i$ can be written as :*

$$^{t}X\cdot\boxed{\mathcal{C}_i}\cdot X,\ \Delta\cdot\boxed{\mathcal{C}_i^{*}}\cdot{}^{t}\Delta \qquad \text{where}\ \boxed{\mathcal{C}_i}\simeq\left[\begin{array}{ccc} f^2 & -fg & -fh\\ -fg & g^2 & -hg\\ -fh & -hg & h^2\end{array}\right],\ \boxed{\mathcal{C}_i^{*}}\simeq\left[\begin{array}{ccc} 0 & h & g\\ h & 0 & f\\ g & f & 0\end{array}\right] \tag{12.6}$$

*where $\Delta=[f,g,h]$ is the so-called **auxiliary line** of the conic. Let us note $P\doteq\operatorname{tripolar}(\Delta)\simeq 1/f:1/g:1/h$ and $Q=\operatorname{isotom}P\simeq f:g:h$. The dual conic of $\mathcal{C}_i$ is precisely $CC\,(Q)$ (that acts over lines). The contact points of $\boxed{\mathcal{C}_i}$ are $0:h:g$, etc (mind the order !). They are the cevians of point P. This point P is the perspector between triangle ABC and its polar triangle with respect to the conic, while Δ is its perspectrix. Finally, the center of $\mathcal{C}_i$ is the complement of $\operatorname{isotom}(P)$.*

Proof. By definition, $\boxed{\mathcal{C}_i^{*}}$ must have a zero diagonal. Then $\boxed{\mathcal{C}_i}$ is its adjoint. Perspectivity is obvious, while center is the pole of the line at infinity. □

Corollary 12.8.4. *Direct relations between center and perspector are as follows :*

$$\begin{array}{ccccccc} U &=& complem\,(isot\,(P)) &=& crossmul(X_2,P) &=& P\underset{b}{*}complem\,(P)\\ P &=& isot\,(anticomplem\,(U)) &=& crossdiv\,(U,X_2) \end{array}$$

Construction 12.8.5. *(inconics) A graphical tool can construct any conic from five points. Given the perspector $P\simeq p:q:r$, the points on the cevian lines are $A_1=qB+rC$, etc and $A_2=qB+rC+4pA$, etc. And therefore,*

$$\text{cross\_ratio}\ (A,A_1,P,A_2)=4$$

Proposition 12.8.6. *When $p:q:r$ is the perspector, a **parametrization** of IC(P) is*

$$M\,(t)\simeq p:q\,t^2:r\,(1+t)^2 \tag{12.7}$$

Proof. Direct inspection. □

Proposition 12.8.7. *The 'complem formula' is as follows:*

$$1\,\frac{M_{cc}\,(t)}{\operatorname{tripolar}(P)\cdot M_{cc}\,(t)}+2\,\frac{M_{ic}\,(t)}{\operatorname{tripolar}(P)\cdot M_{ic}\,(t)}=3\,\frac{P}{\operatorname{tripolar}(P)\cdot P}=P$$

where $M_{cc}(t)$ is the standard parametrization of $CC(P)$ and $M_{ic}(t)$ the standard parametrization of $IC(P)$. This amounts to use $\text{tripolar}(P)$ *as 'line at infinity' to normalize the columns in the assertion $M_{cc} + 2M_{ic} \simeq P$.*

Proof. From (12.5), (12.7) and the obvious :

$$1 \times \left(\frac{1}{t^2+t+1} \begin{bmatrix} t(1+t)\,p \\ (1+t)\,q \\ -tr \end{bmatrix} \right) + 2 \times \left(\frac{1/2}{t^2+t+1} \begin{bmatrix} p \\ qt^2 \\ r(1+t)^2 \end{bmatrix} \right) = 3 \times \left(\frac{1}{3} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \right)$$

□

Construction 12.8.8. *An inscribed conic can be generated as follows. Given the perspector $P \simeq p : q : r$, draw the cevians, obtain $A_P B_P C_P$ and draw the cevian triangle. Draw an arbitrary line Δ through B. Define $B_a = (B_P A_P) \cap \Delta$ and $B_c = (B_P C_P) \cap \Delta$. Then $M = B_a C_P \cap B_c A_P$ is on the conic. Thereafter, you can define Δ by a point X on the circumcircle, and generate $\mathcal{C}_i$ as the locus of M (parametrization by a turn).*

Proof. Use $X = x : y : z$ so that $\Delta \simeq [-z, 0, x]$. Thereafter :

$$B_a \simeq \begin{pmatrix} prx \\ pqz - qrx \\ zpr \end{pmatrix},\ B_c \simeq \begin{pmatrix} prx \\ qrx - pqz \\ zpr \end{pmatrix},\ M \simeq \begin{pmatrix} pr^2x^2 \\ q\,(pz - rx)^2 \\ z^2p^2r \end{pmatrix}$$

One can check that M is on the conic and that $t = \dfrac{pz}{rx} - 1$. □

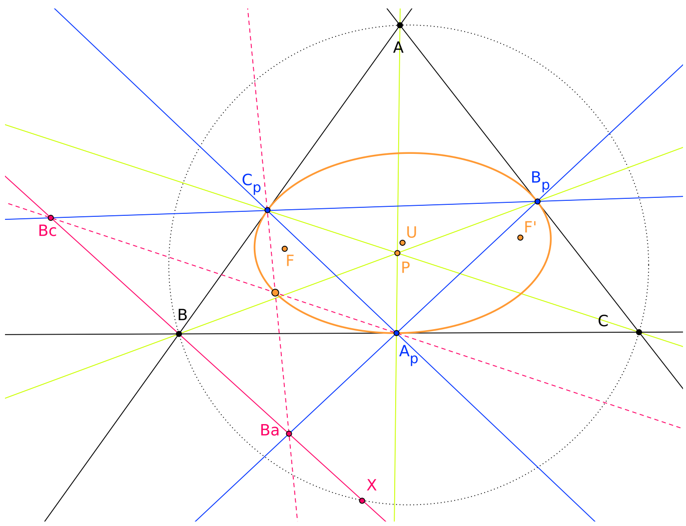


Figure 12.5: How to generate an inscribed conic from its perspector.

Proposition 12.8.9. ***Points at infinity****. An inscribed conic, with auxiliary point $Q \simeq f : g : h$, perspector $P \simeq p : q : r$ and center $U \simeq u : v : w$ is an ellipse, a parabola or an hyperbola when quantity*

$$W^2 = fgh\,(f+g+h) = \frac{pq+pr+rq}{p^2q^2r^2} = (u+v+w)\,(v+w-u)\,(u+v-w)\,(w+u-v)$$

is, respectively, positive, null or negative. The boundaries are the line at infinity and the sidelines of ABC for Q, the Steiner circum-ellipse for P and the line at infinity and the sidelines of the medial triangle for U. Moreover, its points at infinity have the following barycentrics :

$$M_\infty^{\pm} \simeq \begin{pmatrix} (h+g)^2 \\ fg - (f+g+h)\,h \pm 2\sqrt{-W^2} \\ fh - (f+g+h)\,g \mp 2\sqrt{-W^2} \end{pmatrix}$$

Proof. Immediate computation for the M_∞, followed by $P = \text{isotom}(Q)$, $U = \text{anticomplem}(Q)$. □

Proposition 12.8.10. *When center U is given, the perspectors P_i, P_c of the corresponding in- and circum-conics are related by :*

$$P_i \underset{b}{*} P_c = U \qquad ; \qquad P_i = anticomplem\,(P_c)$$

When perspector P is given, the centers U_i, U_c are aligned with $P = p : q : r$ together with $p^2 : q^2 : r^2$.

Proof. Direct inspection. □

Proposition 12.8.11. *Let P be a fixed point, not on the sidelines, and Q be a point moving point on* tripolar(P). *The envelope of all the lines $\Delta \doteq$* tripolar(Q) *is the inconic $IC(P)$. Moreover, the contact point of Δ is $T = Q \underset{b}{*} Q \underset{b}{\div} P$.*

Proof. This result has already be given. But now, this is the right place to prove it. Write :

$$\begin{array}{rcl} P & = & p : q : r \\ Q & = & p\,(\sigma-\tau) : q\,(\tau-\rho) : r\,(\rho-\sigma) \\ \Delta & \simeq & (1 \div p(\sigma-\tau) : 1 \div q(\tau-\rho) : 1 \div r(\rho-\sigma)) \\ T & = & p\,(\sigma-\tau)^2 : q\,(\tau-\rho)^2 : r\,(\rho-\sigma)^2 \end{array}$$

and check that : $\Delta \cdot \text{Adjoint}\left(\boxed{\mathcal{C}_i}\right) \cdot {}^t\Delta = 0$, $\Delta \cdot T = 0$ and ${}^tT \cdot \boxed{\mathcal{C}_i} \cdot T = 0$ where $\boxed{\mathcal{C}_i}$ is given in (12.6) □

12.9 Poncelet porism

Definition 12.9.1. We will say that two conics $\mathcal{C}_{\text{in}}, \mathcal{C}_{\text{out}}$ form a Poncelet configuration when it exists a triangle A, B, C inscribed in $\mathcal{C}_{\text{out}}$ and circumscribed to $\mathcal{C}_{\text{in}}$.

Proposition 12.9.2. ***Poncelet porism****. Let A, B, C be the existing triangle, taken as reference, and $P \simeq p : q : r$, $U \simeq u : v : w$ the perspectors of the circumconic $\mathcal{C}_{\text{out}}$ and inconic $\mathcal{C}_{\text{in}}$. Then any point $M \in \mathcal{C}_{\text{out}}$ can be taken as the initial vertex of a poristic triangle.*

The vertices and the contact points are given by

$$\begin{array}{rcl} M_t \in \mathcal{C}_{\text{out}} & \simeq & p : \dfrac{q}{t} : \dfrac{-r}{1+t} \\ N_t \in \mathcal{C}_{\text{in}} & \simeq & u : v\,t^2 : w\,(1+t)^2 \end{array}$$

where parameters $t = t_1, t_2, t_3$ are bound by the relations

$$\begin{array}{rcl} t_1 + t_2 + t_3 & = & \mu \\ t_1 t_2 + t_2 t_3 + t_3 t_1 & = & -\mu - 1 - (qu \div pv) - (ru \div pw) \\ t_1\, t_2\, t_3 & = & (qu) \div (pv) \end{array}$$

In other words, parameter μ describes the poristic triangle as a whole, while the t_j describe the individual vertices and contact points.

Proof. Start from $M_t, M_s \in \mathcal{C}_{\text{out}}$ and write that $M_t M_s$ is tangent to $\mathcal{C}_{\text{in}}$. This gives an equation in s where

$$s_1 + s_2 = -\frac{p\,vw\,t\,(1+t) + q\,wu\,(1+t) + r\,uv\,t}{t\,(1+t)\,pvw} \;;\; s_1 s_2 = \frac{q\,u}{t\,p\,v}$$

And now, compute the line

$$M_{s;1} \wedge M_{s;2} \simeq [qr,\ pr\,s_1 s_2,\ pq\,(1 + s_1 + s_2 + s_1 s_2)]$$

This gives $M_2 M_3 \simeq \left[\dfrac{1}{u}, \dfrac{1}{v\,t}, \dfrac{-1}{w\,(1+t)}\right]$, which is indeed tangent to $\mathcal{C}_{\text{in}}$, while the contact point is $N_1 \simeq u : v\,t^2 : w\,(1+t)^2$.

From now on, using $t = t_1$ as main parameter would be an error: due to the symmetry, each object would be described three times, increasing the degree of the expressions by a factor 3. □

Proposition 12.9.3. *Synchronization:* $M' \doteq M \underset{b}{\div} P$ *and* $N \underset{b}{\div} U$ *belong, respectively, to the out-Steiner and in-Steiner conics. And then* $N' \underset{b}{*} M' \underset{b}{*} M' \simeq G = X(2)$.

Proposition 12.9.4. *When both conics are circles, we have* $R(R-2r) = d^2$ *where* R, r *are both radiuses and* d *the distance between the centers. When they are the circum- and the in-circle of* ABC *then we have the so called* Brisse *Transform (2001):*

$$N = \text{isogon}\,(M) \underset{b}{*} \text{isogon}\,(M) \underset{b}{*} X(7)$$

Proof. First part is the Euler's formula, second part is the previous proposition. □

12.10 Conic cross-ratios

Proposition 12.10.1. *All the lines* λ *through a given point* P *form a linear projective family* F. *Consider a transversal line* D *(i.e. a line that doesn't go through* P*). Then cross ratio remains unchanged by application* $F \hookrightarrow D$, $\lambda \mapsto \lambda \cap D$.

Proof. Obvious since the wedge operator is a linear transform $\lambda \mapsto \lambda \wedge D$: the parametrization is preserved. □

Proposition 12.10.2. *Consider four fixed points* A, B, C, U *with no alignments and define the moving cross-ratio of a point* M *in the plane as the cross-ratio of lines* MA, MB, MC, MU. *The level lines of this function are the conics passing through* A, B, C, U. *Using barycentrics with respect to* ABC, *we have the more precise statement: the level line of a given* μ *is the circumscribed conic whose perspector is* $P \simeq -u : v\mu : w\,(1-\mu)$, *on the tripolar of* $U = u : v : w$.

Definition 12.10.3. Conic-cross-ratio. Consider a fixed proper conic $\mathcal{C}$, and four points A, B, C, U lying on this conic. Then the cross-ratio of lines MA, MB, MC, MU does not depends upon the choice of the auxiliary point M as long as this point M remains on the conic. If we consider $\mathcal{C}$ as a circumconic with perspector $P = p : q : r$ wrt triangle ABC, we have :

$$\text{cross\_ratio}\;(A, B, C, U) = \mu \quad \text{when} \quad U \simeq \begin{pmatrix} (\mu^2 - \mu)\,p \\ (1-\mu)\,q \\ \mu\,r \end{pmatrix}$$

Remark 12.10.4. In the complex plane, the quantity defined in Theorem 3.2.10 is nothing but the usual cross-ratio, as computed from complex affixes. In order to be sure that, given four points on a circle, the conic cross-ratio and the usual $\mathbb{C}$ cross-ratio are the same quantity, let us consider the stereographic projection.

Definition 12.10.5. Stereographic projection. See Figure 12.6. Use cartesian coordinates. The North and South points are $(+1, 0)$ and $(-1, 0)$.

Start from $P\,(c, s)$. Define $M\,(C, S)$ by doubling the rotation and $T\,(0, t)$ by intersection of SM with the y-axis. Apply Thales to similar triangles SOT and SKM, and Pythagoras to OHP. This gives :

$$c^2 + s^2 = 1,\; (C,\, S) = \left(c^2 - s^2,\, 2cs\right),\; \frac{t}{1} = \frac{S}{1+C}$$

and leads to $t = s/c$, proving that $(SO, ST) = (ON, OP)$ and therefore that

$$\cos 2\vartheta = \frac{1-t^2}{1+t^2},\; \sin 2\vartheta = \frac{2t}{1+t^2}$$

Moreover, circular cross-ratio between M_j points is equal to linear cross-ratio between T_j points (this is the definition), while complex cross-ratio between M_j points is equal to complex cross-ratio between T_j points due to the homography :

$$z = (1 + \boldsymbol{i}\,t)/(1 - \boldsymbol{i}\,t)\,;\; t = \boldsymbol{i}\,(1-z)/(1+z)$$

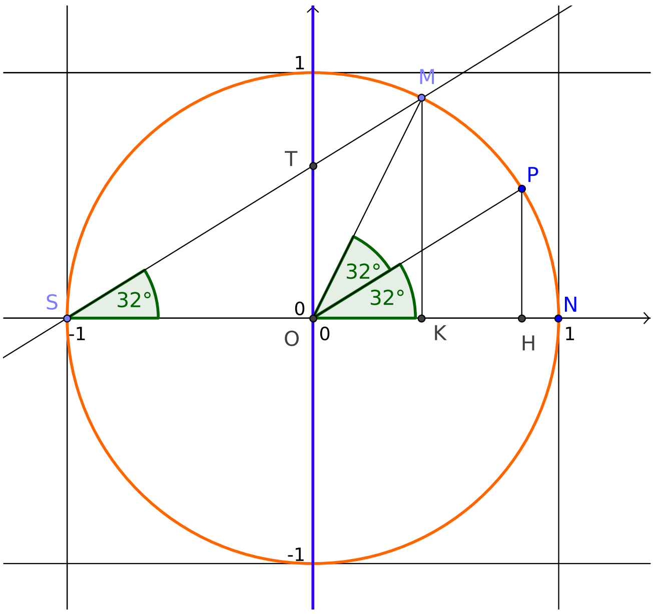


Figure 12.6: Stereographic projection

P	IC	center	CC	center
X_1		X_{37}		X_9
X_2	inSteiner 12.19	X_2	circumSteiner	X_2
X_3		X_{216}	outMacBeath 12.11.3	X_6
X_4	orthic	X_6		X_{1249}
X_5		X_{233}		X_{216}
X_6	Brocard 12.20	X_{39}	circumcircle 13.4	X_3
X_7	incircle 13.5	X_1	Soddy conic 12.14.1	X_{3160}
X_8	Mandart	X_9		X_{3161}
X_{99}	Kiepert parabola 15.5.3	IX_{523}		???
X_{190}	Yff parabola	IX_{514}		???
X_{264}	inMacBeath 12.11.2	X_5		???
∞X_{523}		X_{115}	Kiepert RH 13.22.2	X_{115}
X_{598}	inLemoine12.2	X_{597}		???
X_{647}		???	Jerabek RH 12.22.12	X_{125}
X_{650}		???	Feuerbach RH 12.22.13	X_{11}

RH is rectangular hyperbola

Table 12.1: Some Inconics and Circumconics

12.11 Some in- and circum- conics

A list of specific in- and circum- conics is given in Table 12.1. Kiepert RH is studied at Brocard Section.

Example 12.11.1. The **Steiner** ellipses (centers=perspector=X_2) are what happen to both the circum- and in-circle of an equilateral triangle when this triangle is transformed into an ordinary triangle by an affinity. The Steiner circumellipse (S) is the isotomic conjugate of $\mathcal{L}_b$ and the isogonal conjugate of the Lemoine axis. Since $isog\,(\mathcal{L}_b)$ is the circumcircle (C), the mapping $X \mapsto X \underset{b}{*} X_6$ sends (S) onto (C). Steiner in-ellipse is the envelope of the line whose tripole is at infinity (more about Steiner in-ellipse in Section 12.19).

Example 12.11.2. The **MacBeath-inconic** was introduced as follows:

> Lemma 1: Let O, H be the common points of a coaxal system of circles. Let a variable circle of the system cut the line of centers at C. Let T be a point on the circumference such that $TC = k * OC$, where k is a fixed ratio. Then the locus of T is a conic with foci at O, H (Macbeath, 1949)

Its perspector is X(264), center X(5) and foci X(3) and X(4) . Its barycentric equation is :

$$\sum \frac{a^4\left(-a^2+b^2+c^2\right)x^2}{\left(c^2+a^2-b^2\right)\left(a^2+b^2-c^2\right)} - 2\sum \frac{b^2c^2yz}{b^2+c^2-a^2}$$

and this conic goes through X(I) for I =

$$339, 1312, 1313, 2967, 2968, 2969, 2970, 2971, 2972, 2973, 2974$$

(spoiler) imaginary foci are $X(5) \pm 4\boldsymbol{i}S\,X(523)$, on "the shortest cubic" Proposition 22.4.28.

Example 12.11.3. The **MacBeath-circumconic** , is the dual to the MacBeath-inconic. Its perspector is X(3) and its center X(6). Its barycentric equation is :

$$a^2(b^2+c^2-a^2)yz + b^2(c^2+a^2-b^2)zx + c^2(a^2+b^2-c^2)xy = 0$$

and it goes through X(I) for I =

$$110, 287, 648, 651, 677, 895, 1331, 1332, 1797, 1813, 1814, 1815$$

12.12 Cevian conics

Proposition 12.12.1. ***Cevian conic.*** *For any points $P = p : q : r$ and $Q = u : v : w$, not on a sideline of ABC, the cevians of P and Q are on a same conic, whose equation in $x : y : z$ is conicev (P, Q) given by :*

$$\frac{1}{up}x^2 + \frac{1}{qv}y^2 + \frac{1}{rw}z^2 - \left(\frac{1}{qu}+\frac{1}{vp}\right)yx - \left(\frac{1}{ru}+\frac{1}{wp}\right)zx - \left(\frac{1}{rv}+\frac{1}{qw}\right)zy = 0$$

$$conicev\,(P,Q) \simeq \begin{bmatrix} 2\,qr\,vw & -\left(pv+uq\right)rw & -\left(pw+ru\right)qv \\ -\left(pv+uq\right)rw & 2\,rp\,uw & -\left(qw+rv\right)pu \\ -\left(pw+ru\right)qv & -\left(qw+rv\right)pu & 2\,pq\,uv \end{bmatrix}$$

Proof. Apply $(x,y,z) \to \left[x^2, xy, y^2, yz, z^2, zx\right]$ to the six points and check that rank isn't 6. □

Example 12.12.2. Any inconic is a cevian conic : $IC\,(P) = conicev\,(P,P)$. For example, the incircle is $IC\,(X_7) = conicev\,(X_7, X_7)$.
A non trivial example is the nine-points circle, aka $conicev\,(X_2,\ X_4)$.

Proposition 12.12.3. *Assume that a conic $\mathcal{C}$ encounters the sidelines of ABC in six (real) points, none of them being a vertex A, B, C. Three of these points are the cevians of some point P if and only if :*

$$m_{33}m_{22}m_{11} - m_{11}\,m_{23}^2 - m_{22}\,m_{13}^2 - m_{33}\,m_{12}^2 - 2\,m_{13}m_{23}m_{12}$$
$$= \det M - 4\,m_{13}m_{23}m_{12} = 0$$

In this case, the remaining three intersections are the cevians of some point Q and both P, Q are given by :

$$P, Q \simeq \begin{bmatrix} \left(+m_{13}m_{12} + m_{11}m_{23} + m_{13}\sqrt{m_{12}^2 - m_{22}m_{11}}\right)/m_{11} \\ \left(-m_{22}m_{13} - m_{23}m_{12} + m_{23}\sqrt{m_{12}^2 - m_{22}m_{11}}\right)/m_{22} \\ -\sqrt{m_{12}^2 - m_{22}m_{11}} \end{bmatrix}$$

Proof. Hypotheses are implying $m_{11} \neq 0$ ($A \notin \mathcal{C}$) and $m_{12}^2 - m_{22}m_{11} \geq 0$ (existence of intersections). □

Exercise 12.12.4. The fourth common point F between cevian conics $conicev\left(P, Q_1\right)$ and $conicev\left(P, Q_2\right)$ can be obtained from the tripolars of Q_1, Q_2. We have :

$$\begin{aligned} F &\simeq anticomplem\left(X \underset{b}{\div} P\right) \underset{b}{*} X \qquad \text{where} \\ X &\doteq tripolar\left(Q_1\right) \cap tripolar\left(Q_2\right) = \left(Q_1 \wedge Q_2\right) \underset{b}{*} Q_1 \underset{b}{*} Q_2 \end{aligned}$$

Exercise 12.12.5. When Q =X(2), then $conicev\left(P, Q\right)$ contains also points $\left(P + A\right)/2$, etc (de Villiers, 2006).

Proposition 12.12.6. *For any points $P = p : q : r$ and $Q = u : v : w$, not on a sideline of ABC, the eight points $\pm p : \pm q : \pm r$ and $\pm u : \pm v : \pm w$ (i.e. P, Q and their anticevians) are on a same conic, whose equation in $x : y : z$ is $conacev\left(P, Q\right)$ given by :*

$$conacev\left(P, Q\right) \simeq \begin{bmatrix} g^2 r^2 - h^2 q^2 & 0 & 0 \\ 0 & h^2 p^2 - f^2 r^2 & 0 \\ 0 & 0 & f^2 q^2 - g^2 p^2 \end{bmatrix}$$

Proof. Straightforward computation. Formally, the coefficients are $\left[f^2 : g^2 : h^2\right] \wedge \left[p^2 : q^2 : r^2\right]$. □

12.13 Direction of axes

Proposition 12.13.1. *Let $\mathcal{C}$ be a conic, but not a circle, and γ an auxiliary circle. Consider, in any order, the common points X_1, X_2, X_3, X_4 of $\mathcal{C}$ and γ. Then axes of $\mathcal{C}$ have the same directions as bisectors of angle $\left(\overbrace{X_1X_2,\ X_3X_4}\right)$.*

Proof. Use rectangular Cartesian coordinates. Then $\mathcal{C}$ is $y^2 = 2px + qx^2$ and γ is $\left(x - a\right)^2 + \left(y - b\right)^2 = r^2$. Substitution $y^2 = Y$ gives :

$$2\,by = \left(1 + q\right)x^2 + \left(2\,p - 2\,a\right)x + b^2 + a^2 - r^2 \quad ; \quad Y = 2px + qx^2$$

By substitution and reorganization :

$$2b\left(\frac{y_2 - y_1}{x_2 - x_1} + \frac{y_4 - y_3}{x_4 - x_3}\right) = 4\left(p - a\right) + \left(1 + q\right)\left(x_1 + x_2 + x_3 + x_4\right)$$

But the fourth degree equation $0 = Y - y^2 = \left(1 + q\right)^2 x^4 - 4\,\left(1 + q\right)\left(a - p\right)x^3 \cdots$ leads to $\sum x_i = 4\left(a - p\right)/\left(1 + q\right)$. This proves that lines X_1X_2 and X_3X_4 are symmetric wrt the axes and the conclusion follows. By the way, it has been proven that points X_i can be sorted in any order without changing the result. □

Definition 12.13.2. Gudulic point. The gudulic point of a circumconic is defined as its fourth intersection with the circumcircle. Its barycentrics are rational wrt the barycentrics of the perspector.

$$\begin{aligned} Gu &= \text{isotom}\left(b^2 r - c^2 q\ :\ c^2 p - a^2 r\ :\ a^2 q - b^2 p\right) \\ &= \text{tripole}\left(P \wedge X\left(6\right)\right) \end{aligned} \tag{12.8}$$

Definition 12.13.3. Gudulic point (general method). The former proposition shows that any pair of orthogonal directions can be specified by giving a point M on the circumcircle Γ of ABC such that the bisectors of $\left(\overbrace{BC,\ AM}\right)$ have the required directions. This method was firstly used by Lemoine (1900) who called it "the point M method". In order to have a more specific name, the expression "gudulic point" was coined in a discussion at www.les-mathematiques.net. May be in honor of St Gudula of Brussels.

Proposition 12.13.4. *When $\mathcal{C}$ is a circumconic, but not the circumcircle itself, directions of axes are given by the bisectors of $\left(\widehat{BC,\, AG_u}\right)$ where G_u is the fourth common point of $\mathcal{C}$ and Γ. When $\mathcal{C}$ is not a circumconic, it exists nevertheless an unique point $G_u \in \Gamma$ so that axes of $\mathcal{C}$ have the same directions as the bisectors of $\left(\widehat{BC,\, AG_u}\right)$. We have the formulas :*

$$\boxed{\mathcal{C}} \simeq \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} \mapsto G_u \simeq \begin{bmatrix} \dfrac{1}{(m_{11}-2\,m_{12}+m_{22})\,b^2-(m_{11}-2\,m_{13}+m_{33})\,c^2} \\ \dfrac{1}{(m_{22}-2\,m_{23}+m_{33})\,c^2-(m_{11}-2\,m_{12}+m_{22})\,a^2} \\ \dfrac{1}{(m_{11}-2\,m_{13}+m_{33})\,a^2-(m_{22}-2\,m_{23}+m_{33})\,b^2} \end{bmatrix}$$

$$CC\,(P) \mapsto \begin{pmatrix} 1/\left(rb^2-qc^2\right) \\ 1/\left(pc^2-ra^2\right) \\ 1/\left(qa^2-pb^2\right) \end{pmatrix} \quad ; \quad IC\,(U) \mapsto \begin{pmatrix} 1/\left(w^2b^2-v^2c^2\right) \\ 1/\left(u^2c^2-w^2a^2\right) \\ 1/\left(v^2a^2-u^2b^2\right) \end{pmatrix}$$

Proof. First part is the preceding proposition. For the second part, we have :

$$\begin{aligned} \tan\left(\widehat{BC,\, BU}\right) + \tan\left(\widehat{AG_u,\, BU}\right) &= 0 \\ \tan\left(\widehat{BU_1,\, BU}\right) + \tan\left(\widehat{BU_2,\, BU}\right) &= 0 \end{aligned}$$

where U_1, U_2 are the points at infinity of the conic, U is the unknown direction of either axis and G_u is the required gudulic point. We use the usual parametrization of points at infinity (using t, t_1, t_2, s) to describe $U, U_1, U_2, \mathrm{isogon}\,(G_u)$. We extract $t_1 + t_2$ and $t_1 t_2$ from the very equation of the conic, and substitute. This gives a quadratic equation in t alone, and a linear equation in s, with t as parameter. Eliminating leads to s and thus to G_u. □

Proposition 12.13.5. *(Spoiler) Let $\alpha, \beta, \gamma, \delta$ be four turns on the unit circle. Then bisectors of the two lines obtained by pairing these four points have clinant τ^2 where $\tau^4 = \alpha\beta\gamma\delta$.*

Proof. Write that $\tan(AB, OT) + \tan(CD, OT) = 0$, and factor the numerator. □

12.14 Focuses of a conic

Definition 12.14.1. A point F is a **focus** for a curve when the isotropic lines from this point are tangent to the curve.

Proposition 12.14.2. *The geometric focuses of a conic are examples of the preceding definition. But a conic has, in the general case, four analytical focuses : the two geometrical ones, and two extra focuses which stay on the other axis and are not visible*

Proof. Let us consider ellipse $x^2/a^2 + y^2/b^2 = 1$ and isotropic line $(x - x_0) + i\,(y - y_0) = 0$. Their intersection is given by a second degree equation whose discriminant is $(x_0 + i\,y_0)^2 - \left(a^2 - b^2\right) = 0$. It vanishes when $y_0 = 0$, $x_0 = \pm f$ (as usual) but also when $x_0 = 0$, $y_0 = \pm i\,f$. □

Proposition 12.14.3. *(spoiler) When using the Morley space to compute the focuses, one equation applies to map $\mathbf{Z} : \mathbf{T}$ and the other one applies to map $\overline{\mathcal{Z}} : \mathbf{T}$. When going back to the Morley space, there are four ways of pairing the two "focuses" of the first map with the two "focuses" of the second map.*

Proof. When, for a visible conic, the first map says $z = u \pm v\sqrt{w}$, the second map says $\zeta = \overline{u} \pm \overline{v}\sqrt{w}$ (with $w \in \mathbb{R}$). But the conjugate of $\sqrt{w}$ is either $\sqrt{w}$ or $-\sqrt{w}$, depending of the sign of the real number w. □

Proposition 12.14.4. *Let be given two points $F_1 \simeq f_1 : g_1 : h_1$ and $F_2 \simeq f_2 : g_2 : h_2$. All the conics that admit F_1, F_2 as foci form a tangential pencil. This pencil is generated by (1) the tangential conic $\{F_1, F_2\}$ of all the lines $\Delta \simeq [u, v, w]$ through F_1 or F_2, whose equation is :*

$$(f_1 u + g_1 v + h_1 w)(f_2 u + g_2 v + h_2 w)$$

and (2) the tangential conic $\{\Omega^+, \Omega^-\}$ of all the isotropic lines (through one or another umbilic), whose equation is : $\Delta \cdot \boxed{\mathcal{M}_b} \cdot {}^t\Delta$.

Proof. Giving the focuses gives four lines that are tangent to the conic. □

Remark 12.14.5. The "Joint Orthoptic Circle" of two confocal conics is now described at 12.25.

Remark 12.14.6. Non parabolic inconics are considered at 12.18, while inscribed parabola and circumscribed parabola are considered at Section 12.21

12.14.1 Soddy Conic

Definition 12.14.7. The **Soddy line** is the line that goes through X(1) [incircle], X(7) [Gergonne], X(20) [Longchamps], 77, 170, X(175) and X(176) [Soddy focuses], 269, 279, 347, 390, 481, 482, 516 [infinity] ... 360 ETC points (in 2024).

Definition 12.14.8. The **Soddy conic** is the circumconic whose perspector is X(7).

Proposition 12.14.9. *Center is X(3160). Gudulic point is X(927). Direction of axes are X(516) and X(514). The foci are X(175) and X(176). This curve is bi-tangent with the polar circle (along the $[a, b, c]$ line).*

Proof. Foci can be obtained by the Plucker method. This leads to a fourth degree equation, that factors easily. For the contact, one can check that:

$$\text{Soddy} = \text{polar\_circle} + (ax + by + cz)^2$$

□

12.15 Heptagonal triangle

Remark 12.15.1. I am not sure about the right place of this section in this Glossary. For the moment, this section rather deals with conics and focuses.

Definition 12.15.2. Symbols δ, τ are intended to be unevaluated, inert variables, while quantities $\widehat{\delta}, \widehat{\tau}$ are defined as "the first root after $z = 1$" of polynomials $\delta^7 - 1$ and $\tau^{28} - 1$. Introducing the algebraic number $\widehat{\delta}$ to study the heptagon is self-explanatory. And then $\widehat{\tau}$ is introduced in order to deal with surds of surds. As a result

$$\delta = \widehat{\delta} \doteq \text{RootOf}\left(\mathbf{Z}^6 + \mathbf{Z}^5 + \mathbf{Z}^4 + \mathbf{Z}^3 + \mathbf{Z}^2 + \mathbf{Z} + 1\right) \qquad \approx 0.6235 + 0.7818\boldsymbol{i}$$
$$\tau = \widehat{\tau} \doteq \text{RootOf}\left(\mathbf{Z}^{12} - \mathbf{Z}^{10} + \mathbf{Z}^8 - \mathbf{Z}^6 + \mathbf{Z}^4 - \mathbf{Z}^2 + 1\right) \qquad \approx 0.9749 + 0.2225\boldsymbol{i}$$

Moreover, companion matrices of these polynomials will be noted $\boxed{\delta}$ and $\boxed{\tau}$.

Theorem 12.15.3. *Surd $\sqrt{7}$ belongs to $\mathbb{Q}(\delta, \boldsymbol{i})$ and, therefore, belongs to $\mathbb{Q}(\tau)$. Moreover,*

$$\text{rac7}_\delta \doteq \quad -\boldsymbol{i}\left(2\delta^4 + 2\delta^2 + 2\delta + 1\right) = \quad \sqrt{7}$$
$$\text{rac7}_\tau \doteq \quad -2\tau^{11} + 2\tau^9 - \tau^7 + 2\tau = \quad \sqrt{7}$$

Proof. Let $\boxed{\mathcal{C}}$ be the conic through the points $1, \delta, \delta^2, \delta^4$ and $\mathbf{Z}$. Then $\left\langle \boxed{\mathcal{M}_z} \mid \boxed{\mathcal{C}} \right\rangle = 0$ is the equation of the RH through $1, \delta, \delta^2, \delta^4$. Its matrix is

$$\mathcal{H} \simeq \left[\begin{array}{ccc} 2 & -1-K & 0 \\ -1-K & -2 & K \\ 0 & K & 2 \end{array}\right]$$

where $K \doteq \delta + \delta^2 + \delta^4$. It happens that $K = \left(-1 + \boldsymbol{i}\sqrt{7}\right)/2$. This can be seen by obtaining $7I_{12}$ when substituting $\delta = \boxed{\delta}$ into $\left(-\boldsymbol{i}(2K + 1)\right)^2$... and obtaining a non negative value when substituting δ by an approximate value before squaring. □

Proposition 12.15.4. *Algebraic number δ is constructible by circles and conics.*

Proof. This result is attributed to Archimedes (Hogendijk, 1984). Quantity $\sqrt{7}$ is constructible. Vertices of $\mathcal{H}$ are $\left(1 \pm \sqrt{2} + \boldsymbol{i}\sqrt{7}\right)/4$, while the center of $\mathcal{H}$ is $\left(1 + \boldsymbol{i}\sqrt{7}\right)/4$ and can be used to construct the replicas of $z_0 = 1$ (axes are directed along X, Y axes). □

Proposition 12.15.5. *Sidelength a, b, c don't belong to $\mathbb{Q}(\delta)$. Nevertheless, one has:*

$$a^2 = \delta^5 + \delta^4 + \delta^3 + \delta^2 + 3 \qquad b^2 = -\delta^4 - \delta^3 + 2 \qquad c^2 = -\delta^5 - \delta^2 + 2$$
$$a = -\tau^9 + \tau^5 \qquad b = -\tau^{11} + \tau^9 - \tau^7 + \tau^5 - \tau^3 + 2\tau \qquad c = -\tau^{11} + \tau^3$$

together with $S = \sqrt{7}/4$.

Proof. Direct computation. □

Exercise 12.15.6. Consider the Kiepert RH, which goes through $ABCGH$ (see Definition 13.22.1). Compute the focuses –they belong to $\mathbb{Q}(\tau)$. One of them is on the circumcirle, the other on the Euler line.

Exercise 12.15.7. Consider the conic which goes through A, B, C, G, O. Compute the focuses –they belong to $\mathbb{Q}(\tau)$. What can be said about the circle $[F_1, F_2]$?

12.16 PPPP, the four points pencil

Definition 12.16.1. The conics $\mathcal{C}$ that are going through four fixed points $M_1 = A, M_2 = B, M_3 = C, M_0 = D$ form a linear pencil, called a PPPP, i.e. a "four points pencil".

Notation 12.16.2. Using triangle ABC as barycentric basis, the fourth point is described as $1/p : 1/q : 1/r$ while the pencil will be parametrized by $E^* \simeq 1 : t : -1 - t \in \mathcal{L}_b$, i.e. by $E \simeq a^2 : b^2/t : -c^2/(1+t)$ – the gudulic point of $\mathcal{C}$ (see (12.8)).

Proposition 12.16.3. *The matrix of $\mathcal{C}$, the $ABCDE$ conic, is*

$$\left[\begin{array}{ccc} 0 & c^2\left(b^2q - a^2pt\right) & -b^2\left(a^2p(1+t) + c^2r\right) \\ c^2\left(b^2q - a^2pt\right) & 0 & a^2\left(b^2q(1+t) + c^2rt\right) \\ -b^2\left(a^2p(1+t) + c^2r\right) & a^2\left(b^2q(1+t) + c^2rt\right) & 0 \end{array}\right]$$

The tangents at ABC determine a trigone. Let $\mathcal{T}_P \doteq P_aP_bP_c$ be the corresponding triangle. Then

$$P_a \simeq a^2\left(b^2q(1+t) + c^2rt\right) : b^2\left(a^2p(1+t) + c^2r\right) : c^2\left(a^2pt - b^2q\right)$$

Triangle $\mathcal{T}_P$ is perspective with $\mathcal{T}_0 = ABC$, perspector P, and with $\mathcal{T}_G$(the midpoints triangle), perspector U. Point P is the usual perspector of $\mathcal{C}$ when seen as an ABC circumconic, while U is the center of $\mathcal{C}$. The locus of P when $E \in \Gamma$ is the tripolar of D, while the locus of U is the conic G through the six midpoints of quadrangle $ABCD$.

Proof. One has $\mathrm{loc}(P) = [p, q, r]$ and $\mathrm{loc}(U) = \left[\begin{array}{ccc} -2p & p+q & p+r \\ p+q & -2q & q+r \\ p+r & q+r & -2r \end{array}\right]$. □

Proposition 12.16.4. *Let $N_a = BC \cap DA$,* etc *be the diagonal points of quadrangle $ABCD$. All three are on G. The locus of the foci is a sixth degree algebraic curve, say $\mathcal{K}_6$, with six double points: the three N_j and the three R_j where*

$$N_a \simeq \left[\begin{array}{c} 0 \\ r \\ q \end{array}\right] \;;\; R_a \simeq \left[\begin{array}{c} 2\,rq\left(a^2p + S_b\,r + S_c\,q\right) \\ r\left(p^2a^2 - q^2b^2 + c^2r^2 + 2\,S_b\,rp\right) \\ q\left(p^2a^2 + q^2b^2 - c^2r^2 + 2\,S_c\,qp\right) \end{array}\right]$$

Triangles $\mathcal{T}_0$ and $\mathcal{T}_N$ are perspective (at D), while $\mathcal{T}_0$ and $\mathcal{T}_R$ are perspective at

$$K_R \simeq \left[\begin{array}{c} qr\left(2\,S_a\,rq - p^2a^2 + q^2b^2 + c^2r^2\right) \\ rp\left(2\,S_b\,rp + p^2a^2 - q^2b^2 + c^2r^2\right) \\ pq\left(2\,S_c\,qp + p^2a^2 + q^2b^2 - c^2r^2\right) \end{array}\right]$$

Moreover, R_a belongs to line N_bN_c, so that triangles $\mathcal{T}_N$ and $\mathcal{T}_R$ are perspective at:

$$\begin{aligned} K &\simeq \operatorname{crossmul}(P, K_R) \\ &\simeq \left[\begin{array}{c} \left(a^2p+S_b\,r+S_c\,q\right)\left(2\,S_a\,rq-p^2a^2+q^2b^2+c^2r^2\right) \\ \left(b^2q+S_c\,p+S_a\,r\right)\left(2\,S_b\,rp+p^2a^2-q^2b^2+c^2r^2\right) \\ \left(c^2r+S_a\,q+S_b\,p\right)\left(2\,S_c\,qp+p^2a^2+q^2b^2-c^2r^2\right) \end{array}\right] \end{aligned}$$

Proof. Direct computation. Moreover, one has $R_a \underset{b}{*} R_b \underset{b}{*} R_c \simeq K \underset{b}{*} K_R \underset{b}{*} D$. □

12.17 FF, the focal tangential pencil

Definition 12.17.1. All of the tangential conics that share the same foci (homofocal conics) form a linear pencil. We will use B, C to note the foci, and A for a specific point (on the punctual conic).

Proposition 12.17.2. *The FF pencil is generated by* $\mathrm{hyp}A^*$ *and* $\mathrm{lip}A^*$ *which are, respectively, the hyperbola and the ellipse through A. One has:*

$$\mathrm{hyp}A \simeq \left[\begin{array}{ccc} 0 & 2\,c\,(b-c) & 2\,b\,(c-b) \\ 2\,c\,(b-c) & a^2-(b-c)^2 & -a^2-(b-c)^2 \\ 2\,b\,(c-b) & -a^2-(b-c)^2 & a^2-(b-c)^2 \end{array}\right] \tag{12.9}$$

$$\mathrm{lip}A \simeq \left[\begin{array}{ccc} 0 & 2\,c\,(b+c) & 2\,b\,(b+c) \\ 2\,c\,(b+c) & (b+c)^2-a^2 & a^2+(b+c)^2 \\ 2\,b\,(b+c) & a^2+(b+c)^2 & (b+c)^2-a^2 \end{array}\right]$$

Proof. The four isotropic tangents provide four equations implying the matrix $\boxed{\mathcal{C}^*}$. Taking the adjoint matrix of the result and saying that $A \in \mathcal{C}$ leads to a second degree equation, which provides two solutions (as a reminder of the fact that the set of the $\mathcal{C}$ is **not** a linear pencil). □

Proposition 12.17.3. *Tangent at A to* $\mathrm{hyp}A$ *is the internal bisector (through I_0, I_a), while the tangent to* $\mathrm{lip}A$ *is the external one (through I_b, I_c). Moreover (spoiler !) the (12.16) formulas are giving:*

$$\sigma, \pi, f^4 = \frac{(b-c)^2}{2} - \frac{a^2}{4}, \; \frac{-S^2\,(b-c)^2}{(b+c-a)\,(b+c+a)}, \; \frac{a^4}{16}$$

$$\sigma, \pi, f^4 = \frac{(b+c)^2}{2} - \frac{a^2}{4}, \; \frac{+S^2\,(b+c)^2}{(a+b-c)\,(a+c-b)}, \; \frac{a^4}{16}$$

Proof. Obvious computations. Moreover, the value of f^4 shouldn't be a surprise, while the signum of π characterizes beyond any doubt which one is the hyperbola and which one is the ellipse among this pair of conics. □

12.18 Focuses of an inconic

Proposition 12.18.1. *Let $F = p : q : r$ be a point not lying on the sidelines. As in Figure 12.8, we note F_a, F_b, F_c and F'_a, F'_b, F'_c the projections and the reflections of F about the sidelines ; G the isogonal conjugate of F, $\omega = (F+G)/2$ and $P_a = FG'_a \cap F'_aG$,* etc. *Then points P_a, P_b, P_c are the cevians of $P = (\mathrm{isotom} \circ \mathrm{anticomplem})(\omega)$, and are the contact points of $\mathcal{C} \doteq \mathrm{inconic}(P)$. This conic admits ω as center and F, G as geometrical focuses. Moreover circle $F'_aF'_bF'_c$, centered at G is the circular directrix of this conic wrt focus F while circle $F_aF_bF_cG_aG_bG_c$ (the common pedal circle of F and G) is the principal circle of $\mathcal{C}$ (tangent at major axis).*

When either F or G is at infinity, the other is on the circumcircle, P is on the Steiner circumconic, and $\mathcal{C}$ is a parabola.

P	F_1	F_2	P	F_1	F_2	P	F_1	F_2	P	F_1	F_2	P	F_1	F_2
2	2239162	39163	4555	900	901	18811	8	56	34393	102	515	35172	9111	9055
7	1	1	4569	934	3900	18812	10	58	34410	20	64	35174	2222	3738
13	13	15	4577	826	827	18813	17	61	34413	40	84	35179	1296	1499
14	14	16	4586	824	825	18814	18	62	35136	3565	3566	35181	4160	8691
69	46357	46358	4597	4588	4777	18815	36	80	35137	7927	7953	35510	42411	42412
80	39150	39151	5641	542	842	18816	104	517	35138	3906	11636	39626	39624	39625
99	110	523	6189	1380	3414	18817	186	265	35139	476	526	41072	30664	30665
190	101	514	6190	1379	3413	18818	187	671	35140	1297	1503	42371	688	689
264	3	4	6528	107	520	18819	371	485	35141	17768	28471	43091	530	2378
290	98	511	6540	4977	8701	18820	372	486	35142	3563	3564	43092	531	2379
598	2	6	6606	6362	?	18821	528	840	35143	35101	35105	43093	674	675
648	112	525	6613	42337	?	18822	537	2382	35145	2249	8680	43094	702	703
664	109	522	6635	6550	6551	18823	543	843	35147	2703	2787	43095	716	717
666	918	919	6648	3910	8687	18824	696	697	35148	2702	2786	43097	752	753
670	99	512	9487	9136	?	18825	712	713	35149	2708	2792	43099	760	761
671	111	524	10512	23	67	18827	740	741	35150	2700	2784	46132	788	789
886	888	9150	10604	25	69	18830	932	4083	35151	2699	2783	46133	912	915
889	891	898	11117	532	2380	18831	933	6368	35152	2711	2795	46134	924	925
892	690	691	11118	533	2381	23895	5995	23870	35156	1290	8674	46135	926	927
903	106	519	14727	42341	?	23896	5994	23871	35157	6366	14733	46136	952	953
1494	30	74	14970	732	733	32036	16806	23872	35159	35104	35108	46137	971	972
2481	105	518	15164	1113	2574	32037	16807	23873	35162	17770	28482	46138	1141	1154
2966	2715	2799	15165	1114	2575	32038	23880	32693	35164	2717	2801	46139	930	1510
3225	698	699	18026	108	521	32040	26716	?	35168	545	2384	46142	2698	2782
3227	536	739	18810	7	55	32041	4762	8693	35171	1308	3887	46143	2705	2789

Figure 12.7: Inconics: some P, F_1, F_2

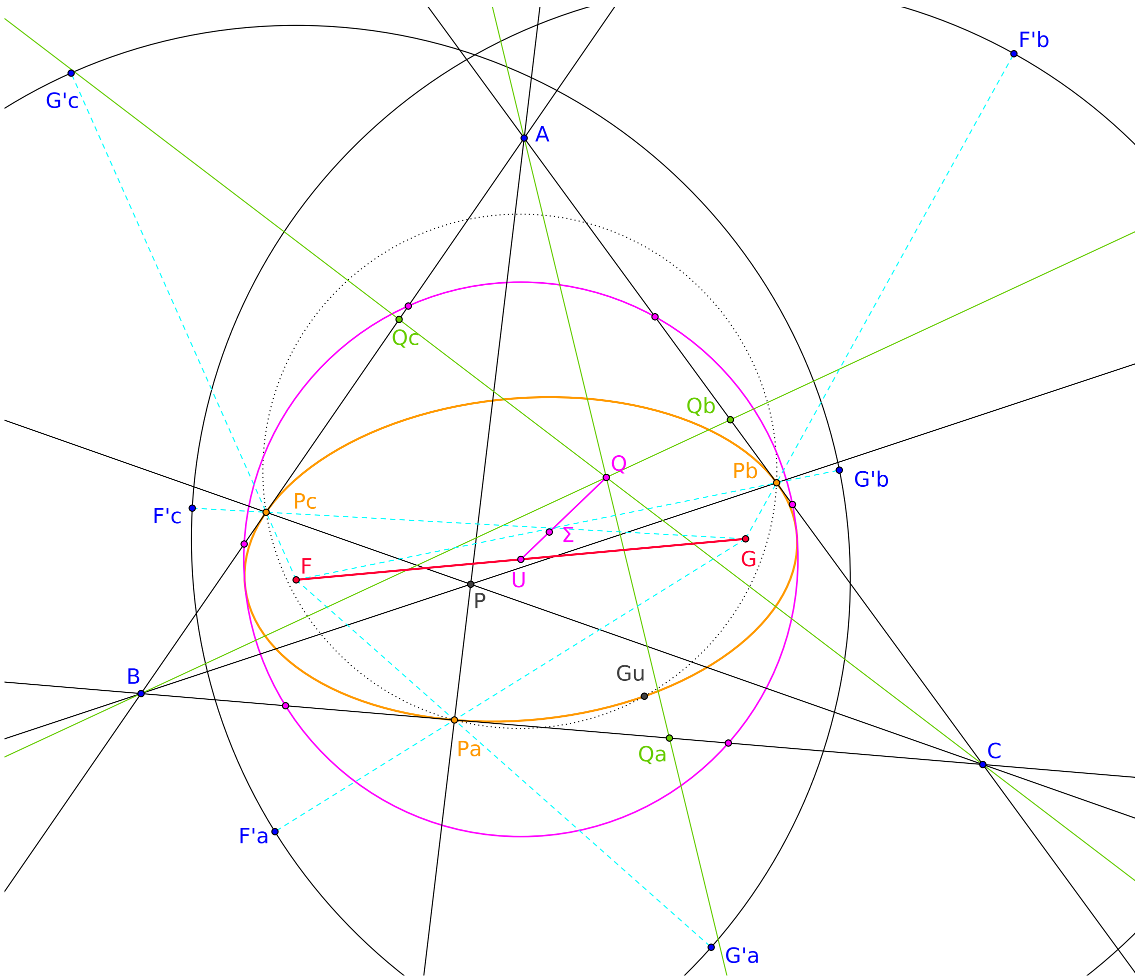


Figure 12.8: Focus of an inconic

Proof. One obtains easily :

$$G'_a = -\frac{a^4}{p} : \frac{S_c a^2}{p} + \frac{a^2 b^2}{q} : \frac{S_b a^2}{p} + \frac{a^2 c^2}{r}$$

leading to the symmetric expression :

$$|F'_a G|^2 = \frac{\left(c^2q^2 + 2S_a qr + b^2r^2\right)\left(a^2r^2 + 2S_b rp + c^2p^2\right)\left(b^2p^2 + 2S_c pq + a^2q^2\right)}{(r+q+p)^2\left(a^2qr + b^2pr + c^2pq\right)^2}$$

proving that $\gamma \doteq F'_a F'_b F'_c$ is centered at G. When P is inside ABC, we can define an ellipse $\mathcal{C}$ by focus F and circular directrix γ. We have $P_a \in \mathcal{C}$ since $|FP_a| = |F'_a P_a|$. Moreover BC is the bisector of $(P_a F,\, P_a F'_a)$, again by symmetry. Therefore $\mathcal{C}$ is the inscribed conic tangent at P_a, P_b, P_c and we have :

$$\omega \simeq \begin{pmatrix} \left(rb^2 + qc^2\right)p^2 + (2\,p + q + r)\,qra^2 \\ \left(pc^2 + ra^2\right)q^2 + (p + 2\,q + r)\,rpb^2 \\ \left(qa^2 + pb^2\right)r^2 + (p + q + 2\,r)\,pqc^2 \end{pmatrix} \;;\; P \simeq \begin{pmatrix} \dfrac{rq}{q^2c^2 + 2\,S_a rq + r^2b^2} \\ \dfrac{rp}{c^2p^2 + 2\,S_b rp + a^2r^2} \\ \dfrac{qp}{b^2p^2 + 2\,S_c qp + q^2a^2} \end{pmatrix}$$

When P is outside ABC, the simplest method is to revert the process and define P, and therefore $\mathcal{C}$, using the given formula and, thereafter, check that lines $F\Omega^{\pm}$ are tangents to $\mathcal{C}$. □

Remark 12.18.2. When substituting $F = p : q : r$ by an umbilic $\Omega^{\pm}$, the ω formula gives $0 : 0 : 0$. Therefore, umbilics are expected to appear as artifacts when trying to revert the ω formula to obtain the foci.

P	ω	F	G	FG	HK	*nom*
2	2			see 12.19		Steiner
7	1	1	1	$X1$	$X1$	inscrit
598	597	2	6	$X597 \pm X524$	$X597 \pm \dfrac{i}{4S}\, X1499$	Lemoine
264	5	4	3	$X5 \pm X30$	$X5 \pm i4S\, X523$	MacBeath
6	39	ω^+	ω^-	$X39 \pm X512$	$X39 \pm \dfrac{i}{4S}\, X511$	Brocard
80	44			$X44 \pm \dfrac{\sqrt{3}}{4S}\, X517$	$X44 \pm i\sqrt{3}\, X513$	
673	3008			$X3008 \pm W\, X514$	$X3008 \pm \dfrac{i}{4S}\, W\, X516$	
694	3229			$X3229 \pm W\, X512$	$X3229 \pm \dfrac{i}{4S}\, W\, X511$	

Table 12.2: Perspector and focuses of some in-conics

Remark 12.18.3. The Moebius-Steiner-Cremona transform (Section 10.6) provides another point of view... but computations aren't easier (nor worse).

Proposition 12.18.4. *The focus of an inconic can be obtained from the perspector by two successive second degree equations (ruler and compass construction). Let $P = p : q : r$ be the perspector. Then the four focuses can be written as :*

$$F_i \simeq \left(\begin{array}{c} p\,(r+q) + \sqrt{K} \\ q\,(r+p) + t\sqrt{K} \\ r\,(q+p) - (1+t)\,\sqrt{K} \end{array} \right)$$

Then t is homographic in K, while K is solution of a second degree equation. The converse is true : K is homographic in t, while t is solution of a second degree equation (with same discriminant). Obviously, the two solutions in t lead to orthogonal directions (orthopoints at infinity)

Proof. Straightforward elimination. □

Example 12.18.5. Table 12.2 gives some examples of perspectors and focuses. In this table, expressions like $X5 \pm X30$ are not "up to a proportionality factor", but are addressing the usual simplified values. All expressions are "centered" at the center, the $\pm$ term being at infinity. Imaginary focuses associated with perspector X(80) have a very simple expression, namely $a : bj : cj^2$ and $a : bj^2 : cj$ where j is the third of a full turn.

$$\begin{aligned} W_{673} &= \sqrt{a^2 + c^2 + b^2 - 2\,bc - 2\,ac - 2\,ba} \\ W_{694} &= a^2b^2c^2 \sqrt{\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right)\left(\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right)\left(-\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)} \end{aligned}$$

Proposition 12.18.6. *Figure 12.7 gives some triples (P, F_1, F_2) where P is the perspector of an inscribed conic while F_1, F_2 are the corresponding focuses (isogonal conjugate of each other).*

Proof. For each point of the Kimberling's database taken as F_1, compute $F_2 = isogon(F_1)$ and see if the corresponding P is also on the database. □

12.19 Focuses of the Steiner inconic

Remark 12.19.1. While having the simplest equation, this inconic rather illustrates that going from perspector to focuses is a hard road to follow !

Definition 12.19.2. The **Steiner** in-ellipse $\mathcal{S}$ is what happen to the incircle of an equilateral triangle when this triangle is transformed into an ordinary triangle by an affinity. One has center =perspector = X(2). Moreover, the Steiner inconic is the envelope of the lines whose tripoles are at infinity.

Lemma 12.19.3. *Conic* $\mathcal{S}$ *goes through* $(B+C)/2$, etc *and therefore goes also throught* $(4A+B+C)/6$.

12.19.1 Using barycentrics

Notation 12.19.4. In this section, the following radicals will be used :

$$W = \sqrt{a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2}$$
$$W_a = \sqrt{\left(a^2+W\right)^2-b^2c^2}\,;\, W_b = \sqrt{\left(b^2+W\right)^2-c^2a^2}\,;\, W_c = \sqrt{\left(c^2+W\right)^2-a^2b^2}$$

Proposition 12.19.5. *Equation of the Steiner inconic* $\mathcal{S}$ *is :*

$$x^2+y^2+z^2-2xy-2yz-2zx = (x,y,z)\begin{pmatrix} +1 & -1 & -1 \\ -1 & +1 & -1 \\ -1 & -1 & +1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

When point $P \simeq p:q:r$ *is inside the medial triangle, then* $Q \simeq p^2:q^2:r^2$ *is inside* $\mathcal{S}$*. Moreover point* $M_a \doteq (q+r)^2:q^2:r^2$ *is the intersection of* $\mathcal{S}$ *and segment* $[AQ]$.

Proof. Direct computation. □

Exercise 12.19.6. Applying the mksigpi formulas, we have:

$$\widehat{a} = \sqrt{\frac{S_\omega+W}{18}}, \widehat{b} = \sqrt{\frac{S_\omega-W}{18}}, f = \sqrt{\frac{W}{9}}$$

Proposition 12.19.7. *The four radicals* W, W_j *are real. Moreover, assuming* $c>a$, $c>b$, *we have :*

$$\begin{array}{rcl} \left(c^2-b^2\right)W_b &=& WW_a+\left(b^2-a^2\right)W_a \\ W_c &=& W_a+W_b \end{array}$$

Proof. For each of the W_j: let $\widehat{W_a} = \sqrt{\left(a^2-W\right)^2-b^2c^2}$. Then (1) W_a^2, $\widehat{W_a}^2$ are real ; (2) $W_a^2\times\widehat{W_a}^2 = -16S^2\left(b^2-c^2\right)^2<0$; (3) $W_a^2>\widehat{W_a}^2$. Therefore $W_a^2>0$ and, since W_a is real, we can assume that $W_a\geq 0$.

Consider now the expression $Q_1 \doteq$

$$\left(W_a+W_b+W_c\right)\left(-W_a+W_b+W_c\right)\left(W_a-W_b+W_c\right)\left(W_a+W_b-W_c\right)$$

Depending only on the W_i^2, quantity Q is intended to be rational in W. Substituting the value of W^2, one obtains $Q=0$ so that one of the W_i is the sum of the other two. If c is the greatest side, this leads to $W_c = W_a+W_b$.

In the same manner, the product $Q_2 \doteq$

$$\left(\left(c^2-b^2\right)W_c+\left(c^2-a^2+W\right)W_a\right)\left(\left(c^2-b^2\right)W_c-\left(c^2-a^2+W\right)W_a\right)$$

simplifies to 0. If c is the greatest side, the first factor cannot vanish. And the W_b formula results using $W_c = W_a+W_b$. □

Lemma 12.19.8. *Points X(3413) and and X(3414), given by*

$$\begin{pmatrix} (b^2-c^2)(a^4-b^2c^2-a^2\,W) \\ (c^2-a^2)(b^4-a^2c^2-b^2\,W) \\ (a^2-b^2)(c^4-a^2b^2-c^2\,W) \end{pmatrix}, = \begin{pmatrix} (b^2-c^2)(a^4-b^2c^2+a^2\,W) \\ (c^2-a^2)(b^4-a^2c^2+b^2\,W) \\ (a^2-b^2)(c^4-a^2b^2+c^2\,W) \end{pmatrix}$$

belong to $\mathcal{L}_b$ and are orthopoints of each other.

Proposition 12.19.9. *The foci of the Steiner inconic are X(39162)...X(39165). The visible one's are given by $F_\pm = X_2 \pm X_{3413}/W_Q$ where* X(2)=1 : 1 : 1, X(3413) is as given just above while

$$W_Q \doteq \sqrt{2a^2b^2c^2\,W^3 - 16\left(a^4b^4+b^4c^4+c^4a^4\right)S^2 + a^2b^2c^2\left(a^2b^2+b^2c^2+c^2a^2\right)\left(a^2+b^2+c^2\right)}$$

Proof. Isotropic lines through a focus are tangent to the curve. Write that $F\Omega^+$ is tangent to $\mathcal{C}$ and separate real and imaginary parts. Eliminate one of the coordinates of F from this system. It remains a fourth degree equation (E) giving the two real and two imaginary foci. The discriminant of this equation contains W^4 in factor. Using this indication, we factorize (E) over $\mathbb{R}(W)$ and obtain :

$$\left(c^2v^2 - \left(2\,wa^2+2\,wW\right)v + b^2w^2\right)\left(c^2v^2 - \left(2\,wa^2-2\,wW\right)v + b^2w^2\right) = 0$$

The discriminants of these second degree factors are W_a^2 and $\left(\widehat{W_a}\right)^2$. And we obtain the non symmetric expression :

$$F_+ \simeq \begin{pmatrix} \left(W+b^2\right)\left(b^2-c^2\right) + \left(W+b^2-a^2\right)W_a \\ \left(b^2-c^2\right)\left(W_a+W+a^2\right) \\ \left(b^2-c^2\right)c^2 \end{pmatrix}$$

In order to obtain a more symmetric expression, one can compute $U = \mathcal{L}_b \wedge (F_+ \wedge F_-)$, i.e. the point at infinity of the focal line. This point happens to be X_{3413}, the first Kiepert infinity point. The existence of W_Q is obvious since X_2 is the middle of the foci. A straightforward computation leads to the given formula.

□

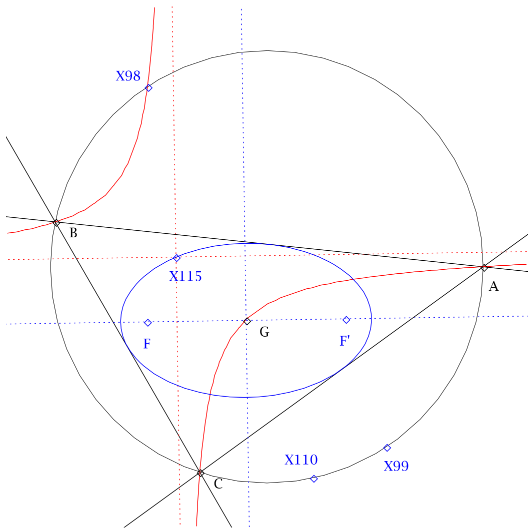


Figure 12.9: The Steiner in ellipse

Remark 12.19.10. Figure 12.9 summarizes these properties. The hyperbola is Kiepert RH, the Tarry point X(98) is the gudulic point of the KRH axes, while it's circumcircle antipode, the Steiner point X(99), is the gudulic point of both the KRH asymptotes and the Steiner axes. Moreover W_Q is the Vassillia's radical in LMN, while the ETC radical at X(39162) is $4S\left(2S_\omega - W\right)\prod_3\left(b^2-c^2\right) \div W_Q$

12.19.2 Using Morley affixes

Lemma 12.19.11. *Using Morley affixes, the tangential equation of the Steiner inconic is :*

$$\boxed{\mathcal{C}_z^*} \simeq \begin{pmatrix} 2\,\sigma_2\sigma_3 & 2\,\sigma_1\sigma_3 & \sigma_2\sigma_1 - 3\,\sigma_3 \\ 2\,\sigma_1\sigma_3 & 6\,\sigma_3 & 2\,\sigma_2 \\ \sigma_2\sigma_1 - 3\,\sigma_3 & 2\,\sigma_2 & 2\,\sigma_1 \end{pmatrix}$$

Proof. Start from barycentric equation and transmute. The ponctual equation is not so handy, and we know that the adjoint matrix will look better. □

Proposition 12.19.12 (Marden's theorem)**.** *(1945). When the vertices aren't aligned, the foci of the Steiner in-ellipse relative to the triangle ABC are the roots of the derivative polynomial, i.e. the roots of* $\frac{\partial}{\partial \mathbf{Z}}(\mathbf{Z}-\alpha)(\mathbf{Z}-\beta)(\mathbf{Z}-\gamma)$. *Therefore, the four foci are given by :*

$$F_j \simeq \begin{pmatrix} \sigma_1 \pm W_0\, W_f \\ 3 \\ \dfrac{\sigma_2}{\sigma_3} \pm \dfrac{W_g}{W_0} \end{pmatrix} \qquad \text{where } W_0 = \sqrt{\sigma_3},\ W_f = \sqrt{\frac{\sigma_1^2 - 3\,\sigma_2}{\sigma_3}},\ W_g = \sqrt{\frac{\sigma_2^2 - 3\,\sigma_1\sigma_3}{\sigma_3}}$$

Proof. Write that isotropic lines $\Omega_\pm F_j$ are tangent to the ellipse. The only difficulty is a sound management of the conjugacies: the conjugate of W_0 is $1/W_0$, while W_f, W_g are the conjugate of each other. From all the four possibilities for the $\pm$, two of them lead to visible points (the real focuses), the other two lead to non visible points (the analytical focuses). □

12.19.3 Using one of the focuses

Remark 12.19.13. The main result at 12.19.9 is mostly that going from a, b, c to focuses is a hard road. In this subsection, the road from focuses to a, b, c will be examined.

Proposition 12.19.14. *Let $F_1 \simeq p : q : r$ be the first focus of the Steiner inconic. Then*

$$\begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix} \simeq \begin{pmatrix} p\,(-p+2q+2r) \\ q\,(+2p-q+2r) \\ r\,(+2p+2q-r) \end{pmatrix}$$

As it should be, the other focus is $F_2 \simeq a^2/p : b^2/q : c^2/r$, so that $F_1 + F_2 = 2X(2)$.

Proof. As said at 12.19.9, isotropic lines through the focus $f : g : h$ are tangent to the curve. Write that $F\Omega^+$ is tangent to $\mathcal{C}$ and separate real and imaginary parts. And now, solve in a, b, c instead of solving in $f : g : h$. □

12.20 The Brocard ellipse, aka the K-ellipse

Remark 12.20.1. Since the K-circumconic, i.e. CC(X(6)), is nothing but the circumcircle, the name "K-ellipse" applies only to the K-inconic.

1. Equation of the K-ellipse is :

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} - 2\,\frac{xy}{a^2b^2} - 2\,\frac{yz}{b^2c^2} - 2\,\frac{zx}{a^2c^2} = 0$$

   perspector is X(6)= $a^2 : b^2 : c^2$, center U is X(39) $= a^2\left(b^2+c^2\right)$, etc.

2. Draw the circumcircle of the contact points $A_K B_K C_K$ and obtain :

$$a^2yz + b^2xz + xyc^2 - (x+y+z)\ \frac{\sum xb^2c^2\left(b^4+c^4+a^2b^2+b^2c^2+a^2c^2-a^4\right)}{2\left(b^2+c^2\right)\left(a^2+c^2\right)\left(a^2+b^2\right)} = 0$$

3. Compute the fourth intersection of this circle with the conic and obtain :

$$Q = a^2\left(b^2-c^2\right)^2\left(a^4+a^2b^2+a^2c^2-b^4-b^2c^2-c^4\right)^2, \text{ etc}$$

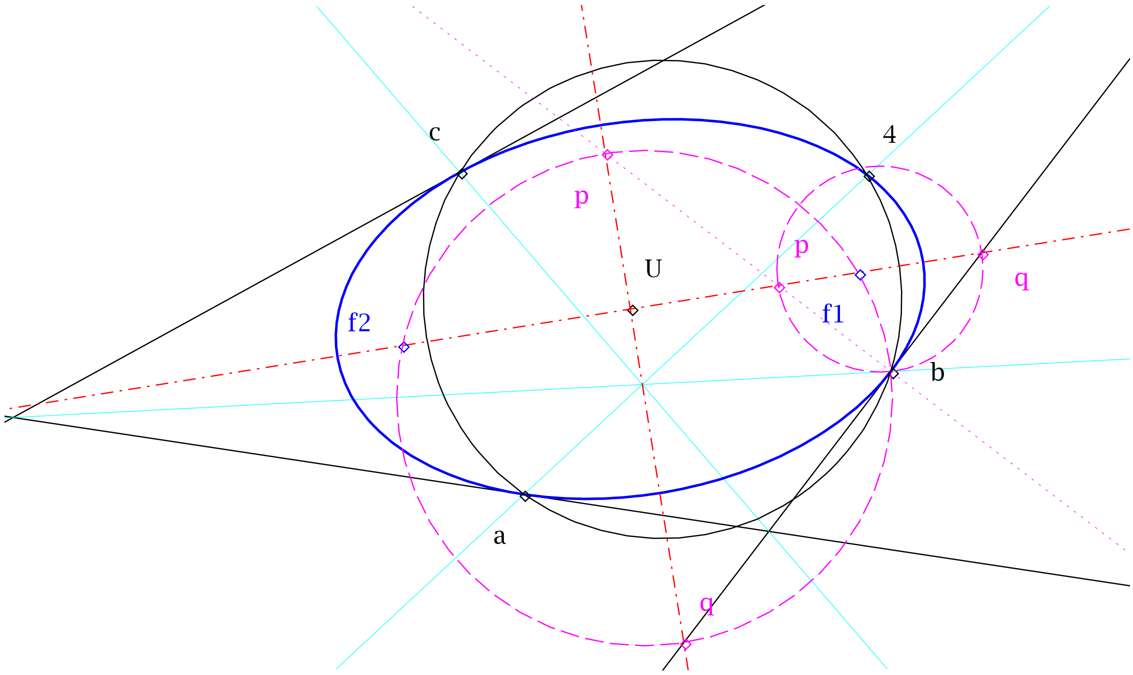


Figure 12.10: The K-ellipse

4. The axes are the lines through the center that are parallel to the bisectors of $\left(\overbrace{A_K C_K,\, B_K Q}\right)$. Therefore, compute :
$$T = \tan\left(\overbrace{A_K C_K,\, A_K B_K}\right)$$
$$t = \tan\left(\overbrace{A_K C_K,\, UV}\right)$$
where $V \simeq \rho : 1 : -1-\rho$ is an unknown point at infinity, substitute into $T = 2t/\left(1-t^2\right)$ and solve. Solutions are rational, leading to $V_1 = a^2\left(b^2-c^2\right)$, etc =X(512) and $V_2 = a^2\left(a^2b^2+a^2c^2-b^4-c^4\right)$, etc =X(511).

5. Compute the axes as $U \wedge V_1 = \left(a^4-b^2c^2\right) \div a^2$, etc and $U \wedge V_2 = \left(b^2-c^2\right) \div a^2$, etc : the Brocard axis X(3)X(6).
Having the perspector on an axis is special.

6. The sideline AC and the perpendicular to AC through B_K cut the first axis in P_1, Q_1 and the second in P_2, Q_2. The idea is to draw circle having diameter $[P_1, Q_1]$, then the circle centered at U orthogonal to the former and obtain the focuses by intersection with the axis.

7. More simpler, write $F_i = \mu P_1 + (1-\mu)\, Q_1$ and find μ such that $\left(\overline{UF_i}/\overline{UP_1}\right) \div \left(\overline{UF_i}/\overline{UQ_1}\right) = 1$. These ratios involve vectors that all have the same direction, and no radicals are appearing. In our special case, the equation factors, leading to a well-known result (the Brocard points) :
$$F_1 = a^2b^2 : b^2c^2 : c^2a^2 \quad ; \quad F_2 = c^2a^2 : a^2b^2 : b^2c^2$$

8. Proceed the same way with the other axis. Obtain an equation that doesn't factors directly, but whose discriminant splits nevertheless when using the Heron formula (7.8). Finally,
$$F_3,\, F_4 = \left(\begin{array}{c} 4\,\left(a^2b^2+a^2c^2\right) S + i\left(b^4+c^4-a^2c^2-a^2b^2\right) a^2 \\ 4\,\left(b^2c^2+b^2a^2\right) S + i\left(a^4+c^4-a^2b^2-b^2c^2\right) b^2 \\ 4\,\left(c^2a^2+c^2b^2\right) S + i\left(a^4+b^4-a^2c^2-b^2c^2\right) c^2 \end{array}\right)$$

9. To summarize, F_1, F_2 =X(39)±X(512), F_3, F_4 =X(39)±i X(511)/4S. As it should be, the focal distance (from center to a focus) is the same since X(512) and X(511)/4S are obtained by a rotation (in space $\overrightarrow{\mathcal{V}}$).

12.21 Parabola

For the sake of completeness, let us recall the definition.

Definition 12.21.1. A **parabola** is a conic tangent to the infinity line. Two parallel lines make a non proper parabola. The union of line at infinity and another line is ... some kind of circle rather than a "special special" parabola.

Corollary 12.21.2. *The conic defined by matrix* $\boxed{\mathcal{C}}$ *is a parabola when*

$$\mathcal{L}_b \cdot \text{Adjoint}\left(\boxed{\mathcal{C}}\right) \cdot {}^t\mathcal{L}_b = 0$$

Definition 12.21.3. The **directrix** of a parabola is the polar line of the focus.

Remark 12.21.4. This directrix is also the orthoptic cycle of the parabola. See Section 12.25.

12.21.1 Inscribed parabola

Proposition 12.21.5. *The focus F of an inscribed parabola is the isogonal conjugate of its point at infinity U (and is therefore on the circumcircle), while the perspector P is the isotomic conjugate of U (and is therefore on the Steiner circumconic). Moreover line FP goes through X(99), the fourth intersection of the circumcircle and the outSteiner ellipse. Finally, the directrix is the Steiner line of F and therefore goes through the orthocenter H =X(4).*

Proof. The first part is from Proposition 12.18.1, the last one is detailed at Section 12.25. And the X(99) part is easy to compute. □

12.21.2 Circumscribed parabola

Remark 12.21.6. Let $T_0 = u : v : w$ be the barycentrics of the point at infinity of a circumparabola. Then, from Proposition 12.22.9, its perspector is $P = u^2 : v^2 : w^2$ and lies on the inSteiner ellipse.

Remark 12.21.7. The perspectors of the two circumparabolas through the four points A, B, C, D are the intersection of tripolar D and the inSteiner conic.

Proposition 12.21.8. *Using Morley affixes, let $\kappa : 0 : 1$ be the point at infinity of a given circumparabola. Then equation, perspector and focus are:*

$$\boxed{\mathcal{C}_z} \simeq \begin{pmatrix} 2\,\sigma_3 & -\kappa^2 - \sigma_1\sigma_3 & -2\,\sigma_3\kappa \\ -\kappa^2 - \sigma_1\sigma_3 & 2\,\sigma_1\kappa^2 + 4\,\sigma_3\kappa + 2\,\sigma_2\sigma_3 & -\sigma_2\kappa^2 - \sigma_3^2 \\ -2\,\sigma_3\kappa & -\sigma_2\kappa^2 - \sigma_3^2 & 2\,\sigma_3\kappa^2 \end{pmatrix}$$

$$P \simeq \begin{pmatrix} \left(\dfrac{3\,\sigma_2 - 4\,\sigma_1^2}{\sigma_3} + \dfrac{\sigma_1\sigma_2^2}{\sigma_3^2}\right)\kappa + \dfrac{4\,\sigma_2^2}{\sigma_3} - 12\,\sigma_1 + (\sigma_2\sigma_1 - 9\,\sigma_3)\,\dfrac{1}{\kappa} \\ \left(\dfrac{2\,\sigma_2^2}{\sigma_3^2} - 6\,\dfrac{\sigma_1}{\sigma_3}\right)\kappa + 2\,\dfrac{\sigma_2\sigma_1}{\sigma_3} - 18 + \left(2\,\sigma_1^2 - 6\,\sigma_2\right)\dfrac{1}{\kappa} \\ \left(\dfrac{\sigma_2\sigma_1}{\sigma_3^2} - \dfrac{9}{\sigma_3}\right)\kappa + \dfrac{4\,\sigma_1^2 - 12\,\sigma_2}{\sigma_3} + \left(3\,\sigma_1 + \dfrac{\sigma_1^2\sigma_2 - 4\,\sigma_2^2}{\sigma_3}\right)\dfrac{1}{\kappa} \end{pmatrix}$$

$$F \simeq \begin{pmatrix} \left(\dfrac{4\,\sigma_1}{\sigma_3} - \dfrac{\sigma_2^2}{\sigma_3^2}\right)\kappa^2 + 8\,\kappa + 2\,\sigma_2 - \dfrac{\sigma_3^2}{\kappa^2} \\ 4\left(\dfrac{1}{\sigma_3}\kappa^2 + \dfrac{\sigma_2}{\sigma_3}\kappa + \sigma_1 + \sigma_3\,\dfrac{1}{\kappa}\right) \\ \dfrac{-1}{\sigma_3^2}\kappa^3 + \dfrac{2\,\sigma_1}{\sigma_3}\kappa + 8 + \left(4\,\sigma_2 - \sigma_1^2\right)\dfrac{1}{\kappa} \end{pmatrix}$$

Proof. Use point $\kappa + i\,\kappa\,h$ with $h \to 0$ as the fifth point of the conic and compute the determinant. Thereafter, compute the polar triangle and its perspector. Finalize by writing that $\Omega \pm F$ are tangent to the conic. □

Proposition 12.21.9. *The locus of the foci of all the circumscribed parabola is a circular quintic. Singular focus (not on the curve) is X(143), the nine points center of the orthic triangle. Other asymptotes are through points whose barycentrics are respectively,* $2:1:1,\ 1:2:1,\ 1:1:2$. *Its equation is :*

$$\begin{array}{l}
1024\,\sigma_3^3\,\mathbf{Z}\overline{\mathcal{Z}}\,\left(\mathbf{Z}+\beta\gamma\,\overline{\mathcal{Z}}\right)\left(\mathbf{Z}+\gamma\alpha\,\overline{\mathcal{Z}}\right)\left(\mathbf{Z}+\alpha\beta\,\overline{\mathcal{Z}}\right) \\
-256\left(\begin{array}{c}
\left(4\,\sigma_2-\sigma_1^2\right)\sigma_3^2\mathbf{Z}^4+\left(4\,\sigma_2^2+9\,\sigma_1\sigma_3-\sigma_1^2\sigma_2\right)\sigma_3^2\mathbf{Z}^3\overline{\mathcal{Z}} \\
+\left(3\,\sigma_3^2+13\,\sigma_1\sigma_2\sigma_3-\sigma_1^3\sigma_3-\sigma_2^3\right)\sigma_3^2\mathbf{Z}^2\overline{\mathcal{Z}}^2 \\
+\left(4\,\sigma_1^2\sigma_3+9\,\sigma_2\sigma_3-\sigma_1\sigma_2^2\right)\sigma_3^3\mathbf{Z}\overline{\mathcal{Z}}^3+\sigma_3^4\left(4\,\sigma_1\sigma_3-\sigma_2^2\right)\overline{\mathcal{Z}}^4
\end{array}\right)\mathbf{T} \\
+64\left(\begin{array}{c}
512\,\sigma_3^2\,\left(\left(\sigma_3+4\,\sigma_2\sigma_1-\sigma_1^3\right)\mathbf{Z}^3+\left(\sigma_3^3+4\,\sigma_1\sigma_2\sigma_3^2-\sigma_2^3\sigma_3\right)\overline{\mathcal{Z}}^3\right) \\
+\left(\sigma_1^3\sigma_2^2-4\,\sigma_1^4\sigma_3-8\,\sigma_1\sigma_2^3+30\,\sigma_1^2\sigma_2\sigma_3+28\,\sigma_2^2\sigma_3+33\,\sigma_1\sigma_3^2\right)\sigma_3^2\mathbf{Z}\overline{\mathcal{Z}}^2 \\
+\left(\sigma_1^2\sigma_2^3-4\,\sigma_2^4-8\,\sigma_1^3\sigma_2\sigma_3+30\,\sigma_1\sigma_2^2\sigma_3+28\,\sigma_1^2\sigma_3^2+33\,\sigma_2\sigma_3^2\right)\sigma_3\mathbf{Z}^2\overline{\mathcal{Z}}
\end{array}\right)\mathbf{T}^2 \\
+16\,\sigma_3\left(\begin{array}{c}
2\,\left(-\sigma_1^3\sigma_2^2+4\,\sigma_1\sigma_2^3+12\,\sigma_1^4\sigma_3-42\,\sigma_1^2\sigma_2\sigma_3-24\,\sigma_2^2\sigma_3-29\,\sigma_1\sigma_3^2\right)\mathbf{Z}^2 \\
\left(\begin{array}{c}
12\,\sigma_1\sigma_2^4-3\,\sigma_1^3\sigma_2^3+\left(12\,\sigma_1^4\sigma_2-31\,\sigma_1^2\sigma_2^2-28\,\sigma_2^3\right)\sigma_3 \\
-\left(28\,\sigma_1^3+177\,\sigma_1\sigma_2\right)\sigma_3^2-77\,\sigma_3^3
\end{array}\right)\mathbf{Z}\overline{\mathcal{Z}} \\
2\,\left(-\sigma_1^2\sigma_2^3+4\,\sigma_1^3\sigma_2\sigma_3+12\,\sigma_2^4-42\,\sigma_1\sigma_2^2\sigma_3-24\,\sigma_1^2\sigma_3^2-29\,\sigma_2\sigma_3^2\right)\sigma_3\overline{\mathcal{Z}}^2
\end{array}\right)\mathbf{T}^3 \\
+32\,\sigma_3\left(\begin{array}{c}
\left(\sigma_1^2\sigma_2^4-5\,\sigma_1^3\sigma_2^2\sigma_3+4\,\sigma_1^4\sigma_3^2-4\,\sigma_2^5+14\,\sigma_1\sigma_2^3\sigma_3+6\,\sigma_1^2\sigma_2\sigma_3^2+13\,\sigma_2^2\sigma_3^2+35\,\sigma_1\sigma_3^3\right)\overline{\mathcal{Z}} \\
+\left(\sigma_1^4\sigma_2^2-5\,\sigma_1^2\sigma_2^3+4\,\sigma_2^4-4\,\sigma_1^5\sigma_3+14\,\sigma_1^3\sigma_2\sigma_3+6\,\sigma_1\sigma_2^2\sigma_3+13\,\sigma_1^2\sigma_3^2+35\,\sigma_2\sigma_3^2\right)\mathbf{Z}
\end{array}\right)\mathbf{T}^4 \\
+\left(\begin{array}{l}
\sigma_1^4\sigma_2^4-8\,\sigma_1^2\sigma_2^2\left(\sigma_2^3+\sigma_1^3\sigma_3\right)+16\left(\sigma_2^6+\sigma_1^6\sigma_3^2\right)-80\left(\sigma_1^4\sigma_2\sigma_3+\sigma_1\sigma_2^4\right)\sigma_3 \\
+52\,\left(\sigma_1^3\sigma_2^3+2\,\sigma_1^2\sigma_2^2\sigma_3-4\,\sigma_1\sigma_2\,\sigma_3^2\right)\sigma_3-104\,\left(\sigma_2^3+\sigma_1^3\sigma_3\right)\sigma_3^2-343\,\sigma_3^4
\end{array}\right)\mathbf{T}^5
\end{array}$$

Proof. Elimination is straightforward. The real asymptotes are parallel to the sidelines. □

12.22 Hyperbola

Definition 12.22.1. An **hyperbola** is a conic that intersects the line at infinity in two different points. An ellipse is a special hyperbola (the intersection points are not visible) and a parabola is not an hyperbola.

Proposition 12.22.2. *Let* $\Delta_1 \simeq (\rho,\sigma,\tau)$ *and* $\Delta_2 \simeq (u,v,w)$ *be the asymptotes of an hyperbola* $\mathcal{C}$. *Then equation of* $\mathcal{C}$ *can be written as :*

$$(\rho x+\sigma y+\tau z)\,(ux+vy+wz)-k\,(x+y+z)^2=0 \tag{12.10}$$

Proof. Consider the line $\Delta_1 \simeq (\rho,\sigma,\tau)$ and its point at infinity $T_1=\sigma-\tau:\tau-\rho:\rho-\sigma$. The matrix of the quadratic form is :

$$\boxed{\mathcal{C}}=\frac{1}{2}\left({}^t\Delta_1\cdot\Delta_2+{}^t\Delta_2\cdot\Delta_1\right)-k\left({}^t\mathcal{L}_b\cdot\mathcal{L}_b\right) \tag{12.11}$$

It can be seen that ${}^tT_1\cdot\boxed{\mathcal{C}}\cdot T_1=0$ (T_1 belongs to conic) while ${}^tT_1.\boxed{\mathcal{C}}=\Delta_1$ (the tangent to the conic at T_1 is line Δ_1). Another method is $\Delta_1\cdot\operatorname{Adjoint}\left(\boxed{\mathcal{C}}\right)\cdot{}^t\Delta_1=0$ (line Δ_1 is tangent to the conic) while $\boxed{\mathcal{C}}\cdot{}^t\Delta_1=T_1$ (the contact point of Δ is T). □

Corollary 12.22.3. *Equation* (12.10) *is the parametrization in* k *of the pencil of hyperbola that share a given pair of asymptotes.*

Proposition 12.22.4. *The angle between the asymptotes* $\Delta_1,\ \Delta_2$ *of a conic is characterized by*

$$\tan^2(\Delta_1,\Delta_2)=\left(\frac{\Delta_1\cdot\boxed{W_b}\cdot{}^t\Delta_2}{\Delta_1\cdot\boxed{\mathcal{M}_b}\cdot{}^t\Delta_2}\right)^2=(-4)\,\frac{\mathcal{L}_b\cdot\operatorname{Adjoint}\boxed{\mathcal{C}}\cdot{}^t\mathcal{L}_b}{\left\langle\boxed{\mathcal{M}_b}\mid\boxed{\mathcal{C}}\right\rangle^2} \tag{12.12}$$

See some additional comments at Corollary 12.23.6.

Proof. The first equality is the general formula for the tangent, while the second part is easily checked using (12.11). As it should be, $\tan V$ itself is not accessible, since it depends on the order chosen for the asymptotes.

Caveat: everything must be used "as is", without any reduction by a proportionality factor. □

Remark 12.22.5. This amounts to restate (12.10) as searching k so that

$$\det\left(\boxed{\mathcal{C}} - k\,{}^t\mathcal{L}_b \cdot \mathcal{L}_b\right) = \det\boxed{\mathcal{C}} - k\left(\mathcal{L}_b \cdot \text{Adjoint}\,\boxed{\mathcal{C}} \cdot {}^t\mathcal{L}_b\right)$$

vanishes. Solution is unique... except when $\mathcal{C}$ is a parabola.

Proposition 12.22.6. *A rectangular hyperbola is an hyperbola with orthogonal asymptotes. Such an RH is characterized among all the conics by :*

$$\left\langle \boxed{\mathcal{C}} \mid \boxed{\mathcal{M}_b} \right\rangle \doteq \text{trace}\left(\boxed{\mathcal{C}} \cdot \boxed{\mathcal{M}_b}\right) = 0 \tag{12.13}$$

Spoiler: when using Morley coordinates, this reduces to $m_{13} = 0$.

Proof. Obvious from the previous proposition. Another method: for any matrix Q, we have $\text{trace}\left({}^t\Delta_1 \cdot \Delta_2 \cdot Q\right) = \Delta_2 \cdot {}^tQ \cdot {}^t\Delta_1$. Since matrix $\boxed{\mathcal{M}_b}$ is symmetric, $\text{trace}\left(\boxed{\mathcal{C}} \cdot \boxed{\mathcal{M}_b}\right)$ equals $\Delta_2 \cdot \boxed{\mathcal{M}_b} \cdot {}^t\Delta_1$ and the result follows. □

12.22.1 Circum-hyperbolas

Proposition 12.22.7. *A circumconic $\mathcal{C}$ can be characterized by one asymptote $\Delta_1 \simeq (\rho, \sigma, \tau)$. Then the second asymptote Δ_2 is $k/\rho : k/\sigma : k/\tau$ where k is the constant appearing in (12.10). Perspector P, center U, points at infinity T_1, T_2 are given by :*

$$\begin{array}{ccccccc} P & = & \rho\left(\sigma-\tau\right)^2 & : & \sigma\left(\tau-\rho\right)^2 & : & \tau\left(\rho-\sigma\right)^2 \\ U & = & \rho\left(\sigma^2-\tau^2\right) & : & \sigma\left(\tau^2-\rho^2\right) & : & \tau\left(\rho^2-\sigma^2\right) \\ T_1 & = & \sigma-\tau & : & \tau-\rho & : & \rho-\sigma \\ T_2 & = & \rho\left(\sigma-\tau\right) & : & \sigma\left(\tau-\rho\right) & : & \tau\left(\rho-\sigma\right) \end{array}$$

while the equation of the conic can be rewritten into $T_1 \underset{b}{*} T_2 \underset{b}{\div} X \in \mathcal{L}_b$, and asymptotes as $T_1 \underset{b}{*} X \underset{b}{\div} T_2 \in \mathcal{L}_b$ and $T_2 \underset{b}{*} X \underset{b}{\div} T_1 \in \mathcal{L}_b$. Moreover, $P = T_1 \underset{b}{*} T_2$ and $U = T_1 \underset{b}{*} T_2 \underset{b}{*} \text{polarmul}\left(T_1, T_2\right)$.

Proof. Direct examination. □

Proposition 12.22.8. *Consider a circumconic and its perspector P. The points at infinity are given by :*

$$\begin{pmatrix} \left(q^2+r^2-pq-pr\right)p + p\left(q-r\right)\,IST \\ \left(r^2+p^2-qr-qp\right)q + q\left(r-p\right)\,IST \\ \left(p^2+q^2-rp-qr\right)r + r\left(p-q\right)\,IST \end{pmatrix}$$

where $IST^2 = p^2+q^2+r^2-2pq-2qr-2rp$ is the equation of the Steiner in-ellipse.

Proof. Direct inspection. □

12.22.2 Circum-rectangular-hyperbolas

Proposition 12.22.9. *When a circumscribed conic is a rectangular hyperbola, its perspector is on the tripolar of $H = X(4)$ –the so-called orthic axis–, while its center $C = \text{cevadiv}\left(X_2, P\right)$ is on the Euler circle (see also Corollary 12.23.8).*

Proof. Write that $\text{trace}\left(\boxed{\mathcal{M}_b}\,\boxed{\mathcal{C}}\right) = 0$ and obtain a first degree equation for the perspector. Then substitute $P = \textit{cevadiv}(X_2, U)$. Even better: write $U = 2G - V$ and see that $V \in \Gamma$. □

Proposition 12.22.10. *Let M_1, M_2, M_3 be three distinct points on a RH. Then the Euler circle of $M_1M_2M_3$ goes through the center of the RH.*

Proof. Obvious from preceding proposition. □

Fact 12.22.11. *The perspector of a circumRH can be written as:*

$$\frac{1}{S_a} : \frac{-\mu}{S_b} : \frac{\mu-1}{S_c}$$

on the tripolar of X(4), while the RH itself is :

$$-S_a x\,(S_b y - z S_c)\,\mu + S_b y\,(S_a x - z S_c) = 0$$

Kiepert RH is $\mu = \left(a^2-c^2\right) S_b \div \left(b^2-c^2\right) S_a$.

Proof. A direct proof is that $\mathcal{H}$ belongs to the pencil generated by $BC \cup AH$ and $AC \cup BH$. □

Example 12.22.12. JRH, the Jerabek hyperbola, is the circumscribed RH through O and also the isogonal conjugate of the Euler line. Its perspector is X(647), on the H tripolar, its center is X(125), on the NPC. Points at infinity are X(2574) and X(2575), characterized by $\omega^4 \underset{1}{=} \sigma_3\sigma_2/\sigma_1$.

Example 12.22.13. FRH, the Feuerbach hyperbola, is the circumscribed RH through I and also the isogonal conjugate of the line (circumcenter,incenter). Its perspector is X(650), on the H tripolar, its center is X(11), on the NPC, Gu=X(104). Points at infinity are X(3307) and X(3308), characterized by $\omega^4 \underset{2}{=} s_3^3 s_1/s_2$.

12.22.3 Inscribed hyperbolas

Proposition 12.22.14. *An inconic $\mathcal{C}$ can be characterized by one asymptote $\Delta_1 \simeq (\rho,\sigma,\tau)$. Then :*

$$\begin{array}{ccccccc}
\Delta_1 & \simeq & \rho & , & \sigma & , & \tau \\
T_1 & = & \sigma-\tau & : & \tau-\rho & : & \rho-\sigma \\
N_1 & \simeq & \rho\sigma+\rho\tau-\sigma\tau & : & \rho\sigma+\sigma\tau-\rho\tau & : & \rho\tau+\sigma\tau-\rho\sigma \\
\Delta_2 & \simeq & \rho/f & ; & \sigma/g & ; & \tau/h \\
T_2 & = & (\sigma-\tau)\,f^2 & : & (\tau-\rho)\,g^2 & : & (\rho-\sigma)\,h^2 \\
N_2 & \simeq & (\rho\sigma+\rho\tau-\sigma\tau)^{-1} & ; & (\rho\sigma+\sigma\tau-\rho\tau)^{-1} & ; & (\sigma\tau+\rho\tau-\rho\sigma)^{-1} \\
P & = & 1\div\left((\sigma-\tau)\,\rho^2\right) & : & 1\div\left((\tau-\rho)\,\sigma^2\right) & : & 1\div\left((\rho-\sigma)\,\tau^2\right) \\
C & = & (\sigma-\tau)\,f & : & (\tau-\rho)\,g & : & (\rho-\sigma)\,h
\end{array}$$

where $(f,g,h) \simeq N_1 \simeq anticomplem\,(isot\,(\Delta))$ *is the Newton line associated with line* Δ_1 *(cf Proposition 12.27.6). Therefore :*

$$\Delta_2 = \Delta_1 \underset{b}{\div} N_1\ ;\ T_2 = T_1 \underset{b}{*} N_1 \underset{b}{*} N_1\ ;\ N_2 = G \underset{b}{\div} N_1$$

$$C = T_1 \underset{b}{*} N_1 = T_2 \underset{b}{*} N_2 = crossmul\,(G,P)\ ;\ P = crossdiv\,(G,C)$$

Proof. Let $\boxed{M}$ be the matrix $\boxed{IC}\,(p:q:r)$ of the general inconic with perspector $p:q:r$. Then formula giving P from Δ is *obtained* by elimination from $\Delta \cdot \text{Adjoint}\left(\boxed{M}\right) \cdot {}^t\Delta = 0$ (tangency) and $\mathcal{L}_b \cdot \text{Adjoint}\left(\boxed{M}\right) \cdot {}^t\Delta = 0$ ($T \in \mathcal{L}_b$). Thereafter, all formulas are *proven* by direct computing from matrix $\boxed{\mathcal{C}} = \boxed{IC}\,(P)$ where P is as given (cf. Stothers, 2003a). □

Proposition 12.22.15. *Consider an inconic and its center U. The points at infinity are given by :*

$$\left(\begin{array}{c} u^2\left(v^4+w^4-u^2v^2-u^2w^2\right) \\ v^2\left(u^4+w^4-v^2w^2-u^2v^2\right) \\ w^2\left(u^4+v^4-u^2w^2-v^2w^2\right)\end{array}\right) \pm OST\,(v+w-u)\,(w+u-v)\,(u+v-w)\left(\begin{array}{c}\left(v^2-w^2\right)u^2 \\ \left(w^2-u^2\right)v^2 \\ \left(u^2-v^2\right)w^2\end{array}\right)$$

where $OST^2 = -qr - rp - pq$ *is the equation of the Steiner out-ellipse.*

Proof. Direct inspection. □

Proposition 12.22.16. *Consider point* $U = v-w : w-u : u-v \in \mathcal{L}_b$ *and its tripolar* $\Delta_0 \simeq [1/(v-w), 1/(w-u), 1/(u-v)]$. *This line is tangent to Steiner in-ellipse and the contact point is* $T_0 = (v-w)^2 : (w-u)^2 : (u-v)^2$. *Define (index i=inscribed, c=circumscribed) lines* $\Delta_i = tripole(T_0)$, $\Delta_c \simeq [v-w,\ w-u,\ u-v]$ *and point*

$$C = TG(U) = (v-w)^2 (v+w-2u) : (w-u)^2 (u+w-2v) : (u-v)^2 (u+v-2w)$$

Then C *is the common center of a circum-hyperbola with asymptotes* Δ_0, Δ_c *and an in-hyperbola with asymptotes* Δ_0, Δ_c. *The locus of* C *(and also of the circum-perspector) is K219 :*

$$\sum_3 x^3 - \sum_6 x^2 y + 3xyz = 0$$

while the locus of the in-perspector is : $\sum_6 x^2 y - 6xyz = 0$. *All lines* Δ_c *contain* $G = X_2$ *while envelope of the* Δ_i *is cubic :* $\sum_3 x^3 + 3\sum_6 x^2 y - 21xyz = 0$.

Proof. In all these cubics, G is an isolated point (and don't belong to the locus). Otherwise, computing as usual. □

12.22.4 Inscribed-rectangular-hyperbolas

Proposition 12.22.17. *An inscribed conic is a rectangular hyperbola if, and only if, its auxiliary point is on the Longchamps circle. Equivalently, its center is on the polar circle. See Section 13.7 and Section 13.8. As a result, such conics are visible only for obtuse triangles.*

Proof. Use Proposition 12.22.6 and write that trace $\left(\boxed{\mathcal{M}_b}\boxed{\mathcal{C}}\right) = 0$. □

12.23 Metric elements

Proposition 12.23.1. *When a conic* $\mathcal{C}$ *degenerates in the reunion of two lines* Δ_j *then*

$$\cos^2(\Delta_1, \Delta_2) = \frac{\left\langle \boxed{\mathcal{M}_b} \mid \boxed{\mathcal{C}} \right\rangle^2}{2\left\langle \boxed{\mathcal{M}_b}\cdot\boxed{\mathcal{C}} \mid \boxed{\mathcal{M}_b}\cdot\boxed{\mathcal{C}} \right\rangle - \left\langle \boxed{\mathcal{M}_b} \mid \boxed{\mathcal{C}} \right\rangle^2} \tag{12.14}$$

Proof. Use $\Delta_1 = [f, g, h]$; $[u, v, w]$, obtain $\tan(\Delta_1, \Delta_2)$ from 7.22 and then use $\cos^2 = 1/\left(1+\tan^2\right)$. This gives a $4, 4$, rational fraction in u, v, w, f, g, h. What is given is the matrix (m_{jk}) of the bilinear form $(fx+gy+hz) \times (ux+vy+wz)$. An identification leads to a $2, 2$ rational fraction in the m_{jk}. And the $\cos^2$ formula follows. □

Definition 12.23.2. The metric elements of a proper conic are the a and b that are used to write the "standard equation relative to the standard axes"

$$P \in \mathcal{C} \Longleftrightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

when the only allowed transforms are isometries. Due to their symmetry properties, the following quantities will remain useful in every context:

$$\sigma \doteq a^2 + b^2 \ ; \ \pi \doteq a^2 b^2$$

Remark 12.23.3. In the elementary situation, were we have an ellipse and $b < a$ can be assumed, it is convenient to introduce f as the focal distance $|UF|$, δ as the distance center-directrix, e as the excentricity i.e. the constant ratio $dist(M, F)/dist(M, \Delta)$ and p, the parameter, as $e(\delta - f)$, all these quantities being ruled by:

$$f = \sqrt{a^2 - b^2} \ ; \ \delta = a^2/f \ ; \ e = f/a \ ; \ p = b^2/a = e(\delta - f) \tag{12.15}$$

In a context of dynamic geometry, the elementary quantities a, b, f, δ, e, p have to be replaced because we can only assume $a^2, b^2 \in \mathbb{R}$ while $|b| < |a|$ is not granted. On the contrary, using $\sigma \doteq a^2 + b^2$; $\pi \doteq a^2 b^2$ don't require to know who is the "main" axis and who is the "other" axis to be able to even define them.

Proposition 12.23.4. *Using $\langle | \rangle$ to denote the scalar product of two matrices (the trace of their product), we have:*

$$\begin{array}{rlrl} \sigma_b = & (-2S) \dfrac{\left\langle \boxed{\mathcal{M}_b} \mid \text{Adjoint} \boxed{\mathcal{C}_b^*} \right\rangle}{\left(\mathcal{L}_b \cdot \boxed{\mathcal{C}_b^*} \cdot {}^t\mathcal{L}_b\right)^2} & ; \; \pi_b = & (-2S)^2 \dfrac{\det \boxed{\mathcal{C}_b^*}}{\left(\mathcal{L}_b \cdot \boxed{\mathcal{C}_b^*} \cdot {}^t\mathcal{L}_b\right)^3} \\ = & (-2S) \dfrac{\left\langle \boxed{\mathcal{M}_b} \mid \boxed{\mathcal{C}_b} \right\rangle \det \boxed{\mathcal{C}_b}}{\left(\mathcal{L}_b \cdot \text{Adjoint} \boxed{\mathcal{C}_b} \cdot {}^t\mathcal{L}_b\right)^2} & = & (-2S)^2 \dfrac{\det^2 \boxed{\mathcal{C}_b}}{\left(\mathcal{L}_b \cdot \text{Adjoint} \boxed{\mathcal{C}_b} \cdot {}^t\mathcal{L}_b\right)^3} \\ \sigma_z = & \left(\dfrac{iR^2}{2}\right) \dfrac{\left\langle \boxed{\mathcal{M}_z} \mid \text{Adjoint} \boxed{\mathcal{C}_z^*} \right\rangle}{\left(\mathcal{L}_z \cdot \boxed{\mathcal{C}_z^*} \cdot {}^t\mathcal{L}_z\right)^2} & ; \; \pi_z = & \left(\dfrac{iR^2}{2}\right)^2 \dfrac{\det \boxed{\mathcal{C}_z^*}}{\left(\mathcal{L}_z \cdot \boxed{\mathcal{C}_z^*} \cdot {}^t\mathcal{L}_z\right)^3} \end{array} \tag{12.16}$$

Quantities f, e are only accessible through:

$$f^4 = \sigma^2 - 4\pi \; ; \; \frac{\pi}{\sigma^2} = \phi(e) \doteq \frac{1-e^2}{(2-e^2)^2}$$

so that we have ***four*** *focuses for a conic (not a parabola), while $\phi(e) = \phi(e')$ leads to $e' = \pm e$ but also to $e' = \pm e/\sqrt{e^2-1}$ for hyperbolas.*

Proof. Formulas for f^4 and $\phi(e)$ come from elimination between equations (12.15). Concerning (12.16), one can establish them for $x^2/a^2 + y^2/b^2 - 1$ in the Morley context and extend them to the barycentrics. Another method is the following algorithm. □

Algorithm 12.23.5. Consider a conic $\mathcal{C}^*$ not tangent to the infinity line (i.e. not a parabola). Its center is $U \simeq \boxed{\mathcal{C}^*_\cdot}\,{}^t\mathcal{L}_b$. Consider the point at infinity $\delta \simeq 1 : t : -1-t$, and draw line $\Delta = U\delta$. Then consider line $\Delta' \doteq U\delta + \mu\mathcal{L}_b$ and determine μ such that Δ' is tangent to the conic. This results in an equation that can be written as:

$$\left(\mathcal{L}_b \cdot \boxed{\mathcal{C}^*_\cdot}\,{}^t\mathcal{L}_b\right)\left(\mu^2 + poly_2(t)\right) = 0$$

The first factor is the condition for $U \in \mathcal{L}_b$ (parabola). The other factor gives μ^2. Then we consider $D^2(t) = dist^2(U, \Delta')$, that occurs to be a rational fraction of degrees $(+2, -2)$.

Condition $\partial\left(D^2(t)\right)/\partial t = 0$ is second degree in t. Let us call s, t its two roots. Then $s+t$ and st are known from $\boxed{\mathcal{C}^*}$. And therefore $\sigma = D^2(s) + D^2(t)$ and $\pi = D^2(s) \times D^2(t)$ are accessible.

Corollary 12.23.6. *The formula (12.12) that gives the angle between the asymptotes Δ_1, Δ_2 of a conic can be completed as:*

$$\tan^2(\Delta_1, \Delta_2) = (-4) \frac{\mathcal{L}_b \cdot \text{Adjoint}\, \mathcal{C} \cdot {}^t\mathcal{L}_b}{\left\langle \boxed{\mathcal{M}_b} \mid \mathcal{C} \right\rangle^2} = -4\phi(e) = \frac{-4\pi}{\sigma^2} = -\frac{4\left(1-e^2\right)}{(2-e^2)^2}$$

Proof. The first part is (12.12), the others are immediate from preceding results. □

Remark 12.23.7. The excentricity of the conic is controlled by this quantity $\phi(e)$, which is the quotient of $\mathcal{L}_b \cdot \text{Adjoint}\, \mathcal{C} \cdot {}^t\mathcal{L}_b$ which is null when $\mathcal{C}$ is a parabola, and of $\left\langle \boxed{\mathcal{M}_b} \mid \mathcal{C} \right\rangle^2$ which is null when $\mathcal{C}$ is a rectangular hyperbola RH.

Therefore, $\phi(e)$ can be perceived as the quotient of the measure of the parabolic character, by the measure of the RH character. The square at denominator allows for a formula that remains homogeneous in $\mathcal{C}$. The above formula doesn't depends on k since all hyperbolas that share the same asymptotes are similar to each other.

Corollary 12.23.8. *Applied to the circumscribed conic with perspector $P \simeq p : q : r$, we obtain:*

$$\phi(e) = \frac{\pi}{\sigma^2} = \frac{\left(1-e^2\right)}{(2-e^2)^2} = -\frac{S^2\left(p^2+q^2+r^2-2\,pq-2\,qr-2\,rp\right)}{\left(S_a p + S_b q + S_c r\right)^2}$$

We re-obtain that perspectors of the circumscribed parabolas stay on the in-Steiner conic, while those of the RH stay on the orthic axis.

Corollary 12.23.9. *Applied to the inscribed conic with perspector $P \simeq p:q:r$, auxiliary point* isotom $P \simeq f:g:h$ *and center $U \simeq u:v:w$ we obtain:*

$$\phi(e) = \frac{\pi}{\sigma^2} = \frac{(1-e^2)}{(2-e^2)^2} = \frac{16\,S^2p^2q^2r^2\,(pq+pr+rq)}{\left(a^2q^2r^2+b^2p^2r^2+c^2p^2q^2+2\,S_a\,p^2qr+2\,S_b\,pq^2r+2\,S_c\,pqr^2\right)^2}$$
$$= \frac{16\,S^2f\,g\,h\,(f+g+h)}{\left(a^2f^2+b^2g^2+c^2h^2+2\,S_a\,gh+2\,S_b\,hf+2\,S_c\,fg\right)^2}$$
$$= \frac{S^2\,(v+w-u)\,(w+u-v)\,(u+v-w)\,(w+v+u)}{\left(S_a\,u^2+S_b\,v^2+S_c\,w^2\right)^2}$$

We re-obtain that perspectors of the inscribed parabolas stay on the out-Steiner conic, auxiliary points and centers at infinity. On the contrary, auxiliary points of the inscribed RH are on the on Longchamps circle and centers on the polar circle.

12.24 Diagonal conics

Triangle ABC is autopolar wrt conic $\mathcal{C}$ if, and only if, the non-diagonal coefficients vanish. Such conic is called either autopolar or diagonal.

Remark 12.24.1. The only autopolar circle is the polar circle (see Section 13.7).

Remark 12.24.2. An autopolar parabola is tangent to $[1,1,1]$ (the line at infinity) and therefore to $[\pm1,\pm1,\pm1]$ the sidelines of the medial triangle $A'B'C'$. If U is its point at infinity, its focus F is the $A'B'C'$-isogonal of U (on the Euler circle) while the $A'B'C'$-perspector is the $A'B'C'$-isotomic of U.

12.24.1 Pencils of diagonal conics

Proposition 12.24.3. *Let $\gamma(\mu)$ be a pencil of diagonal conics :*

$$\gamma(\mu) \doteq (1-\mu)\left(\alpha_1x^2+\beta_1y^2+\gamma_1z^2\right)+\mu*\left(\alpha_2x^2+\beta_2y^2+\gamma_2z^2\right)=0$$

and $U = u:v:w$ a point, not a vertex of ABC. Then polar lines of U wrt all the conics of the pencil are concurring at a point U^* that will be called the isoconjugate of U wrt the pencil. In fact, $U^* = U_P^*$ –cf (18.4)– where :

$$P = \beta_1\gamma_2-\gamma_1\beta_2 : \gamma_1\alpha_2-\alpha_1\gamma_2 : \alpha_1\beta_2-\beta_1\alpha_2$$

The four fixed points (real or not) of the conjugacy, i.e. the points $\pm\sqrt{p} : \pm\sqrt{q} : \pm\sqrt{r}$, are the points common to all conics of the pencil.

When a pair of isoconjugates U_1 and U_2 is known, P is known and therefore the isoconjugacy. The pencil contains the conic γ_1 through U_1, *cevadiv* (U_2, U_1) and the vertices of their respective anti-cevian triangles. Conic γ_2 is defined cyclically. Both conics are tangent to U_1U_2.

Circumconic $CC(P)$ is together the P-isoconjugate of $\mathcal{L}_b$, the locus of centers of the conics of the pencil and the conic that contains the six midpoints of the quadrangle formed by the four fixed points.

12.25 Orthoptic cycle

Proposition 12.25.1. *The* ***orthoptic cycle*** $\mathfrak{O}$ *of a given conic $\mathcal{C}$ is the locus of points m where the tangents from m to $\mathcal{C}$ are orthogonal to each other. When $\mathcal{C}$ is a parabola, $\mathfrak{O}$ is nothing but its directrix. Otherwise, $\mathfrak{O}$ is concentric with $\mathcal{C}$ and its radius is given by:*

$$\rho^2 = \frac{(-2S)\,\det\boxed{\mathcal{C}}\,\left\langle\boxed{\mathcal{M}_b}\mid\boxed{\mathcal{C}}\right\rangle}{\left(\mathcal{L}_b\cdot\text{Adjoint}\boxed{\mathcal{C}}\cdot{}^t\mathcal{L}_b\right)^2} = \frac{(-2S)\,\left\langle\boxed{\mathcal{M}_b}\mid\text{Adjoint}\boxed{\mathcal{C}^*}\right\rangle}{\left(\mathcal{L}_b\cdot\boxed{\mathcal{C}^*}\cdot{}^t\mathcal{L}_b\right)^2} = a^2+b^2$$

Proof. This can be seen as an obvious corollary of 12.25.7. A direct proof is obtained by using (12.3) to describe the reunion $\mathcal{D}_0$ of the two tangents issued from m as a (degenerate) RH. We have:

$$\begin{aligned} {}^tM \cdot \boxed{\mathcal{D}_0} \cdot M &\doteq \left({}^tM \cdot \boxed{\mathcal{C}} \cdot M\right)\left({}^tm \cdot \boxed{\mathcal{C}} \cdot m\right) - \left({}^tm \cdot \boxed{\mathcal{C}} \cdot M\right)^2 \\ \boxed{\mathcal{D}_0} &= \boxed{\mathcal{C}}\left({}^tm \cdot \boxed{\mathcal{C}} \cdot m\right) - \boxed{\mathcal{C}} \cdot m \cdot {}^tm \cdot \boxed{\mathcal{C}} \end{aligned}$$

The required condition is $\left\langle \boxed{\mathcal{M}_b} \mid \boxed{\mathcal{D}_0} \right\rangle = 0$. One can check that the result is not only a conic, but actually a cycle. In the general case (not a parabola), center and ρ^2 are straightforward. The $\boxed{\mathcal{C}^*}$ part comes from Adjoint (Adjoint M) = (det M) M, while the $a^2 + b^2$ part comes from 12.16. □

Corollary 12.25.2. *The orthoptic cycle of an RH is the point-circle concentric with $\mathcal{C}$.*

Corollary 12.25.3. *The orthoptic cycle of a parabola is its directrix.*

Proof. From the previous proposition, this is a cycle with $\rho = \infty$, and therefore is a line. At Proposition 12.25.4, i.e. in the special case of an inscribed parabola, this line will be proven to be the directrix. But a parabola can ever be inscribed in some triangle. □

Proposition 12.25.4. *Orthoptic cycle of an inscribed conic.* *Let $P \simeq p : q : r$ be the perspector of an inscribed conic. Its auxiliary point is $Q \doteq$ isotom $P \doteq f : g : h$, while the center is $U \simeq p\,(q + r)$, etc. The orthoptic circle is described by:*

$$\boxed{\mathcal{C}^*} \simeq \begin{pmatrix} 0 & h & g \\ h & 0 & f \\ g & f & 0 \end{pmatrix} \; ; \; \mathfrak{O} \simeq \begin{pmatrix} f\,S_a \\ g\,S_b \\ h\,S_c \\ f+g+h \end{pmatrix} \simeq \begin{pmatrix} (v+w-u)\,S_a \\ (w+u-v)\,S_b \\ (u+v-w)\,S_c \\ w+v+u \end{pmatrix} \simeq \begin{pmatrix} qrS_a \\ prS_b \\ pqS_c \\ qr+rp+pq \end{pmatrix} \tag{12.17}$$

and is orthogonal to γ, the polar circle of ABC.

When $\mathcal{C}$ is not a parabola, circle $\mathfrak{O}$ and conic $\mathcal{C}$ are concentric at $u : v : w \simeq g+h : h+f : f+g$, while the radius of $\mathfrak{O}$ is given by:

$$\rho^2 = \frac{a^2 f^2 + b^2 g^2 + c^2 h^2 + 2\,S_a\,gh + 2\,S_b\,hf + 2\,S_c\,fg}{4\,(h+g+f)^2} = \frac{S_a\,u^2 + S_b\,v^2 + S_c\,w^2}{(w+v+u)^2}$$

(so that the center of any inscribed RH is on the polar circle).

When $\mathcal{C}$ is an inscribed parabola, $Q \in \mathcal{L}_b$ and the orthoptic circle becomes a line ($\rho = \infty$) and this line is the directrix of the conic.

Proof. Since $\gamma \simeq {}^t[S_a, S_b, S_c, 1]$, orthogonality is straightforward. For a RH, $\rho = 0$ implies $U \in \gamma$. For a parabola (see Proposition 12.21.5), one has $Q \in \mathcal{L}_b$, while the center is $U \simeq f^2 : g^2 : h^2$ (on in-Steiner), the focus is isogon Q (on the circumcircle) and its polar –the directrix, going through X(4)– is $[f\,S_a,\, g\,S_b,\, h\,S_c]$, as required. □

Proposition 12.25.5. *Circle $\mathfrak{O}$ is the locus of points M such that:*

$$f\overrightarrow{MB} \cdot \overrightarrow{MC} + g\overrightarrow{MC} \cdot \overrightarrow{MA} + h\overrightarrow{MA} \cdot \overrightarrow{MB} = 0$$

Construction 12.25.6. *Construct the orthoptic circle of an inconic.* *Let P be the perspector and $A_P B_P C_P$ its cevians. Circle $\delta \doteq [AA_P]$ cuts circle $\epsilon \doteq [BC]$ in two points, etc. These six points belong to the required circle.*

Proof. One has $\delta \simeq S_a : 0 : 0 : 1$; $\epsilon \simeq 0 : rS_b : qS_c : q+r$. And $x\delta + (1-x)\,\epsilon = \mathfrak{O}$ holds when $x = qr/\,(pq + pr + qr)$. Remark: all these circles are orthogonal to the polar circle. □

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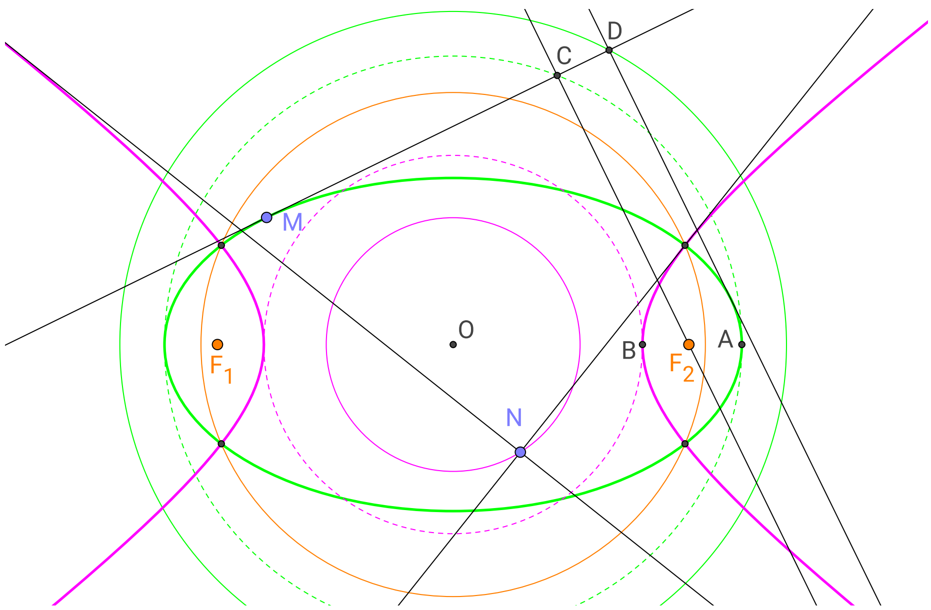


Figure 12.11: Confocal conics and orthoptic circles.

Theorem 12.25.7. *Joint-orthoptic circle. When $\mathcal{C}_t$ and $\mathcal{C}_s$ are two confocal conics, but not circles, the punctual pencil they generate contains exactly one circle Ω. Then $\Omega, \mathcal{C}_s, \mathcal{C}_t$ are concentric while Ω is the locus of points from which one can issue a tangent Δ_t to $\mathcal{C}_t$ and a tangent Δ_s to $\mathcal{C}_s$ so that $\Delta_t \perp \Delta_s$. As special cases, using $\mathcal{C}_s = \mathcal{C}_t$ leads to previously described orthoptic cycle, while using $\mathcal{C}_F \doteq \{F_1, F_2\}$ as conic $\mathcal{C}_s$ leads to the auxiliary circle of $\mathcal{C}_t$. Moreover, using obvious notations, we have:*

$$2\rho^2(\mathcal{C}_s, \mathcal{C}_t) = \rho^2(\mathcal{C}_s) + \rho^2(\mathcal{C}_t) \ ; \ \rho^2(\mathcal{C}_t) = a^2 + b^2 \ ; \ \rho^2(\mathcal{C}_F) = f^2$$

Proof. The conics having their foci at $F_1\,(z=-1)$ and $F_2\,(z=+1)$ are:

$$\boxed{\mathcal{C}_t^*} \simeq (1-t)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -t \\ 0 & 1 & 0 \\ -t & 0 & -1 \end{pmatrix} \ ; \ \boxed{\mathcal{C}_t} \simeq \begin{pmatrix} -1 & 0 & t \\ 0 & 1-t^2 & 0 \\ t & 0 & -1 \end{pmatrix}$$

Thus the circle that belongs to the punctual pencil generated by $\boxed{\mathcal{C}_t}$ and $\boxed{\mathcal{C}_s}$ is $\boxed{\mathcal{C}_t} - \boxed{\mathcal{C}_s}$, whose center is $z = 0$ while $2\rho^2 = s + t$. Consider now a generic point $m \simeq z_0 : t_0 : \zeta_0$. Using (12.3) to describe the tangents issued from m to $\mathcal{C}_t$, we obtain:

$$\left({}^t M \cdot \boxed{\mathcal{C}_t} \cdot M\right)\left({}^t m \cdot \boxed{\mathcal{C}_t} \cdot m\right) - \left({}^t m \cdot \boxed{\mathcal{C}_t} \cdot M\right)^2 = 0$$

where $M \simeq \mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}$ is the generic point. The ω^2 of the tangents are obtained by substituting $T = 0$, leading to

$$\left(\frac{\mathbf{Z}}{\overline{\mathcal{Z}}}\right)^2 - 2\,\frac{\left(t \times t_0^2 - z_0\,\zeta_0\right)}{\left(t_0^2 - \zeta_0^2\right)}\left(\frac{\mathbf{Z}}{\overline{\mathcal{Z}}}\right) + \frac{\left(t_0^2 - z_0^2\right)}{\left(t_0^2 - \zeta_0^2\right)} = 0$$

Their product is constant, well known property: the tangents are reflected by the bisectors of (mF_1, mF_2). Thus the orthogonality involves also the other pair of tangents, so that the sums of the ω^2 are opposite. But this gives $(t+s)\,t_0^2 - z_0\zeta_0 = 0$, i.e. the joint circle. □

12.26 PTPT, the bitangent pencil

PTPT means point,tangent,point,tangent.

Remark 12.26.1. Many properties are better stated when using complex projective coordinates (the Morley frame) and therefore, it could be better to read the corresponding chapter before the present section.

12.26.1 The focal cubic

Definition 12.26.2. All of the conics that are tangent to two fixed lines at two given points form a linear pencil, called the bitangent pencil $\mathcal{F}$. We define B, C as the contact points and A as the intersection of the tangents.

Definition 12.26.3. A', R, F_s. Midpoint A' is defined by $A' = (B+C)/2$. Then $A' \simeq_b 0:1:1$. Gudulic point R is the second intersection of the ABC circumcircle and the A-symmedian of this triangle. We have:

$$R \simeq_b a^2 : -2\,b^2 : -2\,c^2 \;;\; R \simeq_z \frac{\alpha\,\beta+\alpha\,\gamma-2\,\gamma\,\beta}{2\,\alpha-\gamma-\beta} : 1 : \frac{2\,\alpha-\gamma-\beta}{\alpha\,\beta+\alpha\,\gamma-2\,\gamma\,\beta}$$

Lastly, we define F_s as $(A+R)/2$.

Proof. One can verify that:

$$\omega_{AR} \times \omega_{AM} = \frac{\alpha\,(\gamma\,\alpha+\alpha\,\beta-2\,\beta\,\gamma)}{\beta+\gamma-2\,\alpha} \times \frac{\alpha\,\beta\,\gamma\,(\beta+\gamma-2\,\alpha)}{\gamma\,\alpha+\alpha\,\beta-2\,\beta\,\gamma} = (-\alpha\beta) \times (-\alpha\gamma) = \omega_{AB} \times \omega_{AC}$$

□

Theorem 12.26.4. *The punctual and tangential equation of the conics $\mathcal{C}_\lambda$ of the tangential pencil $\mathcal{F}$ are :*

$$\boxed{\mathcal{C}_b} \simeq \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \;;\; \boxed{\mathcal{C}_b^*} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & \lambda & 0 \end{pmatrix}$$

Their centers are $O_\lambda \simeq 1:\lambda:\lambda$ (on the median line AA'). Their two focuses F_λ lie on the cubic $\mathcal{K}$, whose barycentric equation is :

$$\begin{aligned} \mathcal{K}_b\,(x,y,z) &= \left(c^2y^2-b^2z^2\right)x+2\,S_b\,y^2z-2\,S_c\,z^2y \\ &= (z-y)(a^2yz+b^2zx+c^2xy)+(b^2-c^2)yz(x+y+z) \end{aligned} \qquad (12.18)$$

This curve goes through the umbilics, while the singular focus $F_s = (A+R)/2 \simeq 2S_a : b^2 : c^2$ belongs to the curve. The curve goes also through points B, C and twice the point A. The asymptote Δ_∞ is the parallel to the median AM through point $\Omega' = (3A-R)/2$.

As a consequence, the curve is unicursal, i.e. has rational parametrizations.

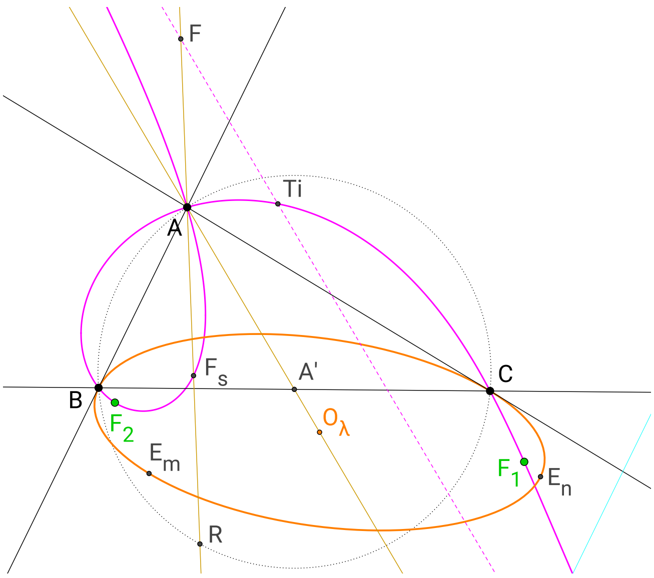


Figure 12.12: The tangential pencil

Proof. Usual computations, using the Plucker's method, i.e. $(F \wedge \Omega_x) \cdot \boxed{\mathcal{C}^*} \cdot {}^t(F \wedge \Omega_x) = 0$ and then taking real and imaginary parts. Other properties are straightforward from the gradient. □

Proposition 12.26.5. *Line FA is a bisector of angle FB, FC. This comes from the following relations between parameter λ and focuses F_λ described as $\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}$ in the Morley frame:*

$$\frac{-1}{2\lambda} = \frac{(\mathbf{Z} - \beta\mathbf{T})(\mathbf{Z} - \gamma\mathbf{T})}{(\mathbf{Z} - \alpha\mathbf{T})^2} = \frac{\left(\overline{\mathcal{Z}} - \mathbf{T}/\beta\right)\left(\overline{\mathcal{Z}} - \mathbf{T}/\gamma\right)}{\left(\overline{\mathcal{Z}} - \mathbf{T}/\alpha\right)^2} \tag{12.19}$$

Moreover, the Morley equation of the "focal cubic" is $\mathcal{K}_z$:

$$\left(\frac{2}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma}\right)\mathbf{Z}^2\overline{\mathcal{Z}} + (\gamma + \beta - 2\,\alpha)\,\mathbf{Z}\overline{\mathcal{Z}}^2 + \left(\frac{1}{\beta\,\gamma} - \frac{1}{\alpha^2}\right)\mathbf{Z}^2\mathbf{T} + \left(\alpha^2 - \beta\,\gamma\right)\overline{\mathcal{Z}}^2\mathbf{T} +$$
$$2\left(\frac{\alpha}{\gamma} + \frac{\alpha}{\beta} - \frac{\beta+\gamma}{\alpha}\right)\mathbf{Z}\overline{\mathcal{Z}}\mathbf{T} + \left(\frac{\beta+\gamma}{\alpha^2} - \frac{2\alpha}{\beta\,\gamma}\right)\mathbf{Z}\mathbf{T}^2 + \left(\frac{2\,\beta\gamma}{\alpha} - \frac{\alpha^2}{\gamma} - \frac{\alpha^2}{\beta}\right)\overline{\mathcal{Z}}\mathbf{T}^2 + \left(\frac{\alpha^2}{\beta\,\gamma} - \frac{\beta\,\gamma}{\alpha^2}\right)\mathbf{T}^3$$

Proof. Use $\boxed{\mathcal{C}_z} = {}^t\boxed{Lu}^{-1} \cdot \boxed{\mathcal{C}_b} \cdot \boxed{Lu}^{-1}$; $\boxed{\mathcal{C}_z^*} = \boxed{Lu} \cdot \boxed{\mathcal{C}_b^*} \cdot {}^t\boxed{Lu}$ to obtain the matrices, and then use the Plucker's equations. A separation of the variables occurs, giving one equation in the upper view $\mathbf{Z} : \mathbf{T}$ and another in the lower view $\overline{\mathcal{Z}} : \mathbf{T}$. Equation of $\mathcal{K}_z$ is easily obtained from $\mathcal{K}_b$ and even more easily by subtracting both sides of (12.19). □

Proposition 12.26.6. *In the Morley frame, the focal cubic can be parametrized by a turn τ:*

$$F_\tau \quad \simeq \quad \left[\begin{array}{c} \dfrac{(\alpha-\gamma)(\alpha-\beta)\,\tau^2 + \alpha\,(2\,\beta\,\gamma - \alpha\,\beta - \alpha\,\gamma)\,\tau + \beta\,\gamma\,\left(\alpha^2 - \beta\,\gamma\right)}{(2\,\beta\,\gamma - \alpha\,\beta - \alpha\,\gamma)\,\tau + \beta\,\gamma\,(2\,\alpha - \beta - \gamma)} \\ 1 \\ \dfrac{1}{\alpha\,\tau}\,\dfrac{\left(\beta\,\gamma - \alpha^2\right)\tau^2 + \beta\,\gamma\,(2\,\alpha - \beta - \gamma)\,\tau + \beta\,\gamma\,(\alpha-\gamma)(\alpha-\beta)}{(2\,\beta\,\gamma - \alpha\,\beta - \alpha\,\gamma)\,\tau + \beta\,\gamma\,(2\,\alpha - \beta - \gamma)} \end{array}\right]$$

Proof. Let K be the point $\tau : 1 : 1/\tau$. Cut the cubic by line AK, and obtain A (twice) and F. □

Proposition 12.26.7. *Three points $F(\tau), F(\kappa), F(\delta)$ on the focal cubic are aligned when:*

$$(\beta\,\gamma + \delta\,\kappa + \delta\,\tau + \kappa\,\tau)(2\,\beta\,\gamma - \alpha\,\beta - \alpha\,\gamma) + (\beta\,\gamma\,(\delta + \kappa + \tau) + \tau\,\kappa\,\delta)(2\,\alpha - \beta - \gamma) = 0 \tag{12.20}$$

Therefore, the tangential of $F(\tau)$, i.e. the point where the tangent of $F(\tau)$ cuts again the curve, is $F(\delta)$ where:

$$\delta = -\frac{2\,\beta\,\gamma\,(\beta + \gamma - 2\,\alpha)\,\tau + \left(\beta\,\gamma + \tau^2\right)(\alpha\,\beta + \alpha\,\gamma - 2\,\beta\,\gamma)}{2\,(\alpha\,\beta + \alpha\,\gamma - 2\,\beta\,\gamma)\,\tau + \left(\beta\,\gamma + \tau^2\right)(\beta + \gamma - 2\,\alpha)}$$

while points $F(\tau)$ and $F(\tau')$ have the same tangential when $\tau\tau' = \beta\gamma$.

point	A	B	C	Ω_x	Ω_y	∞	F_s						
	$\pm\sqrt{\beta\gamma}$	β	γ	∞	0	$\dfrac{\beta\,\gamma\,(2\,\alpha - \beta - \gamma)}{\alpha\,\beta + \alpha\,\gamma - 2\,\beta\,\gamma}$	z_R						
	$\mp\sqrt{\beta\gamma}$	$-\alpha$		$z_R = \dfrac{\alpha\,\beta + \alpha\,\gamma - 2\,\beta\,\gamma}{2\,\alpha - \beta - \gamma}$		$-z_R$							
tang.	A	T_{BC}		$F_s = (A + R)/2$		T_∞							

Proof. Compute the determinant of the three points. □

12.26.2 More constructions of the focal cubic

The focal cubic can be constructed in many ways (apart from the parametrization given at Proposition 12.26.6).

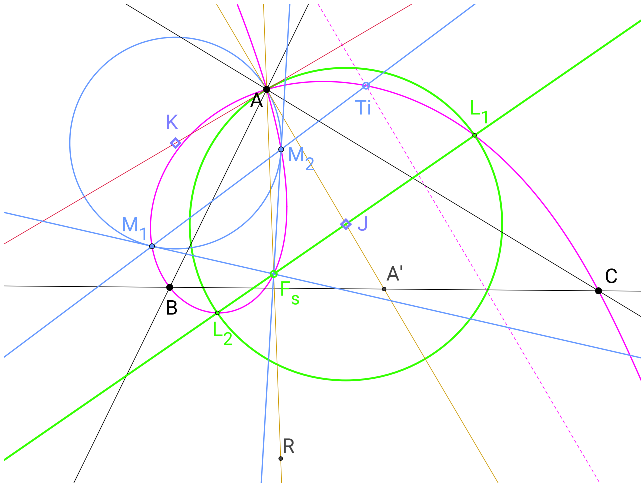


Figure 12.13: Two constructions of the focal cubic

Proposition 12.26.8. ***Using the tau+kappa property.*** *When points $F(\tau)$and $F(\kappa)$ are aligned with F_s then $\tau+\kappa=0$ and circle with diameter $[F(\tau), F(\kappa)]$ goes through A. This gives a construction of the focal cubic (see Figure 12.13): choose a point $J \in AA'$. The circle (J, A) cuts line JF_s in two points L_j that belongs to the cubic.*

Proof. Use the (12.20) formula and then $J = (F(\tau) + F(-\tau))/2$. □

Proposition 12.26.9. ***Using the median pencil.*** *Draw a circle γ_k tangent at A to the median AM (and center K, see Figure 12.13). Draw the tangents from F_s to this circle. The contact points M_j are on the focal cubic, while M_1M_2 goes through T_i, the common tangential of F_s and F_∞.*

Proof. The equation of the circles γ_k are parametrized by:

$$\left(\begin{array}{c} -1 \\ 2\,\alpha \\ -\alpha^2 \\ \alpha \end{array}\right) + k \left(\begin{array}{c} (\alpha\,\beta+\alpha\,\gamma-2\,\beta\,\gamma) \\ (\beta+\gamma)\left(\beta\,\gamma-\alpha^2\right) \\ \alpha\,\beta\,\gamma\,(2\,\alpha-\beta-\gamma) \\ 0 \end{array}\right)$$

Multiply by the Veronese of $F(\tau)$ and obtain the condition $F(\tau) \in \gamma_k$: first degree in k, second degree in τ. And conclude, since this condition divides the condition ensuring that $F_sF(\tau)$ is tangent to γ_k. Then substitute the k values of $\tau+\kappa$ and $\tau\kappa$ into (12.20). □

Proposition 12.26.10. ***Using the circumcircle.*** *Start from the variable point $K = \tau : 1 : 1/\tau$ on the circumcircle It defines a variable line AK. Reflect B, C into AK and obtain B_k, C_k. Then point $F = BC_k \cap CB_k$ is on the cubic.*

Proof. The idea comes from the bisector property of the last proposition. □

Proposition 12.26.11. ***Using the A-Apollonian circle.*** *The isogonal $\mathcal{K}^*$ of the focal cubic is the circle:*

$$\mathcal{K}^* \simeq a^2\left(yc^2-zb^2\right)(x+y+z)+\left(b^2-c^2\right)\left(a^2yz+b^2xz+c^2xy\right)$$

i.e. the A-Apollonian circle, centered at $0 : b^2 : -c^2$ and going through A (a diameter is given by the feet of the A-bisectors).

Proof. Obvious from (12.18). Using the Morley isogonal formula (18.5), one obtains:

$$F^* \simeq \left[\begin{array}{c} \dfrac{\alpha\,(\alpha\,\beta+\alpha\,\gamma-2\,\beta\,\gamma)}{\alpha^2-\beta\,\gamma} - \dfrac{(\alpha-\gamma)\,(\alpha-\beta)}{(\alpha^2-\beta\,\gamma)}\,\dfrac{\beta\,\gamma}{\tau} \\ 1 \\ \dfrac{2\,\alpha-\beta-\gamma}{\alpha^2-\beta\,\gamma} + \dfrac{(\alpha-\gamma)\,(\alpha-\beta)}{(\alpha^2-\beta\,\gamma)}\,\dfrac{\tau}{\beta\gamma} \end{array}\right] \quad \square$$

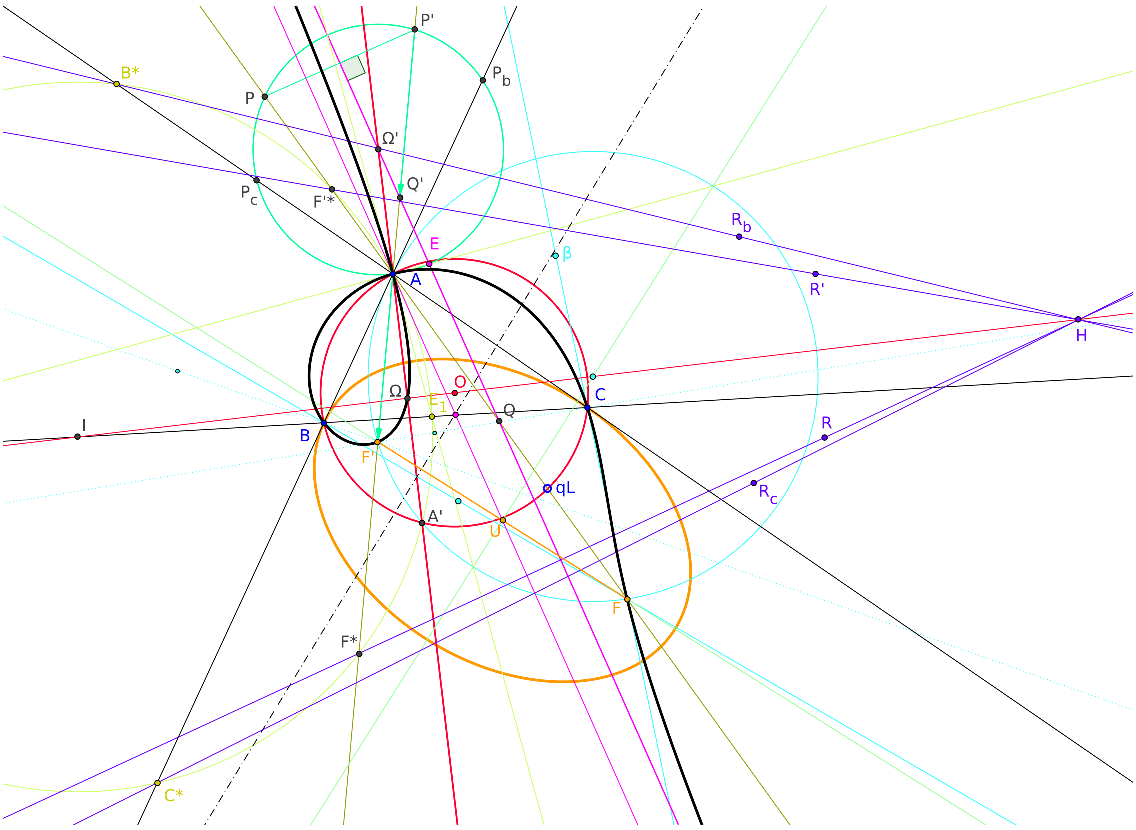


Figure 12.14: Two transformations of the focal cubic.

Proposition 12.26.12. ***Knowing the center.*** *When center O_λ is given on the median AA', the focuses F, F' can be constructed as follows (see Figure 12.15). Draw the bisectors Δ_1, Δ_2 of $O_\lambda A$, $O_\lambda R$. Cut them at H_1, H_2 by the perpendicular bisector of $[A, R]$. Draw circle $\gamma_1\,(H_1,\, A)$ and cut Δ_2. Additionally, draw circle $\gamma_2\,(H_2,\, R)$ and cut Δ_1. This gives the four focuses.*

Proof. This comes from the involutory homography ψ. □

Proposition 12.26.13. ***Using the cissoidal property.*** *Consider a point P on the circle γ through A and centered at $\Omega' = (3A - R)\,/2$. Define Q as the intersection of AP with Δ_∞. Then $F \doteq A + Q - P$ belongs to the cubic. The cissoidal property is the relation : $\overrightarrow{AF} = \overrightarrow{PQ}$.*

Proof. A possible parametrization of the cissoidal circle is :

$$P_\mu \quad \simeq \quad \left[\begin{array}{c} \dfrac{3\,\alpha^2 + \beta\,\gamma - 2\,\alpha\,(\beta + \gamma)}{2\,\alpha - \beta - \gamma} + \dfrac{(\alpha - \gamma)\,(\alpha - \beta)}{\alpha\,\beta + \alpha\,\gamma - 2\,\beta\,\gamma}\,\tau \\ 1 \\ \dfrac{2\,\alpha\,(\beta + \gamma) - \alpha^2 - 3\,\beta\,\gamma}{\alpha\,(\alpha\,\beta + \alpha\,\gamma - 2\,\beta\,\gamma)} - \dfrac{(\alpha - \gamma)\,(\alpha - \beta)}{(2\,\alpha - \beta - \gamma)\,\alpha}\,\dfrac{1}{\tau} \end{array}\right]$$

Here A is $\mu = (\alpha\,\beta + \alpha\,\gamma - 2\,\beta\,\gamma) \div (\beta + \gamma - 2\,\alpha)$. Everything else is straightforward. □

Proposition 12.26.14. *The two visible focuses F, F' of a given conic $\mathcal{C}_\lambda$ are exchanged :*

1. *in the parametrization of Proposition 12.26.6 by $\tau\,\tau' = \beta\gamma$.*
2. *in the construction of Proposition 12.26.10 by using lines AK and AK' that are equally inclined on lines AB, AC.*
3. *in the upper Riemann sphere by the involutory homography*

$$\psi : \left(\begin{array}{c} \mathbf{Z}' \\ \mathbf{T}' \end{array}\right) \simeq \left(\begin{array}{cc} \beta\,\gamma - \alpha^2 & \alpha^2\beta + \gamma\,\alpha^2 - 2\,\alpha\,\beta\,\gamma \\ \beta + \gamma - 2\,\alpha & \alpha^2 - \beta\,\gamma \end{array}\right) \left(\begin{array}{c} \mathbf{Z} \\ \mathbf{T} \end{array}\right)$$

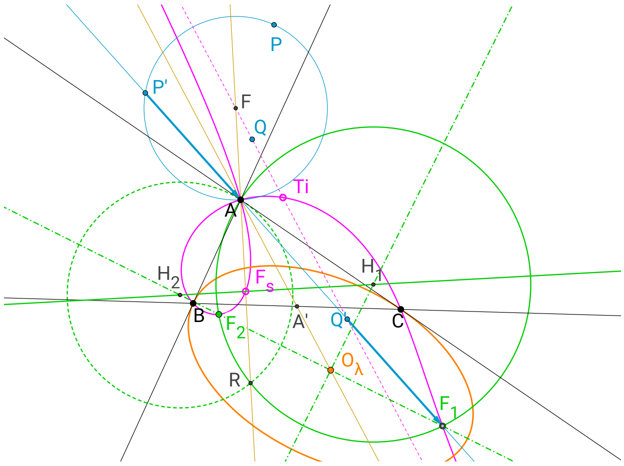


Figure 12.15: Constructions of paired focuses

The fixed points of ψ are A and R (the second intersection of the A-symmedian with the circumcircle).

4. *in the isogonal parametrization of Proposition 12.26.11, points F^{*}, F'^{*} are symmetric with respect to BC.*

5. *in the construction Proposition 12.26.13, P' is the reflection of P into the asymptote Δ_∞ while Q' is the reflection of Q' in the cissoidal circle.*

Proof. (1) One has $\dfrac{-1}{2\lambda}=\dfrac{(\tau-\gamma)(\tau-\beta)\left(\tau-\gamma\,\dfrac{\alpha-\beta}{\alpha-\gamma}\right)\left(\tau-\beta\,\dfrac{\alpha-\gamma}{\alpha-\beta}\right)}{\left(\tau^2-\beta\,\gamma\right)^2}$. Then $\lambda(\tau)-\lambda(\tau')=0$ gives four values for τ, since there are four focuses. But, apart τ itself, only $\tau'=\beta\gamma/\tau$ is unimodular. The other two are not turns (and lead to both non visible focuses). (2) $\kappa\tau=\beta\gamma$.

(3) Write and factor $\lambda(\mathbf{Z}:\mathbf{T})-\lambda(z:t)$ from (12.19). This gives $(z\mathbf{T}-t\mathbf{Z})$ together with another first degree factor with respect to $\mathbf{Z},\mathbf{T}$ and also with respect to z,t: this is our homography. The fixed points of ψ are α and $z_R=(\alpha\beta+\alpha\gamma-2\,\beta\,\gamma)\,/\,(2\alpha-\beta-\gamma)$.

(4) Reflection into the asymptote is obvious from parametrization. Reflection into the circle comes from $\overrightarrow{\Omega' Q}\cdot\overrightarrow{\Omega' Q'}=\left|\overrightarrow{\Omega' A}\right|^2$. □

12.27 LLLL, the Miquel pencil

Definition 12.27.1. All of the conics that are tangent to four fixed lines form a pencil, called the Miquel pencil $\mathcal{F}$ of these four lines.

Lemma 12.27.2. *Let us use describe our four lines as the three sidelines of ABC and the transversal whose tripole is $p:q:r$. In other words, $\mathcal{L}_0\underset{b}{\simeq}[qr,rp,pq]$. See Theorem 3.5.5 for more details. In the Lubin frame, the transversal points, the Newton line, the Miquel point M_q of the quadrilateral and the Clawson-Schmidt homography Ψ are :*

$$A',\,B',\,C'\underset{z}{\simeq}\left(\begin{array}{c}\beta\,q-\gamma\,r\\ q-r\\ \dfrac{q}{\beta}-\dfrac{r}{\gamma}\end{array}\right),\left(\begin{array}{c}\gamma\,r-\alpha\,p\\ r-p\\ \dfrac{r}{\gamma}-\dfrac{p}{\alpha}\end{array}\right),\left(\begin{array}{c}\alpha\,p-\beta\,q\\ p-q\\ \dfrac{p}{\alpha}-\dfrac{q}{\beta}\end{array}\right)$$

$$\text{Newton}\underset{z}{\simeq}\left[2\,c_2,\ -s_2\,c_1-c_3,\ 2\,s_3\,c_1\right]$$

$$M_q \underset{z}{\simeq} \left(\begin{array}{c} c_2/c_1 \\ 1 \\ c_1/c_2 \end{array}\right) \; ; \; \Psi \left(\begin{array}{c} \mathbf{Z} \\ \mathbf{T} \\ \overline{\mathcal{Z}} \end{array}\right) \underset{z}{\simeq} \left[\begin{array}{c} \dfrac{\mathbf{Z}c_2 - \mathbf{T}c_3}{\mathbf{Z}c_1 - \mathbf{T}c_2} \\ 1 \\ \dfrac{c_1\overline{\mathcal{Z}} - c_0\mathbf{T}}{c_2\overline{\mathcal{Z}} - c_1\mathbf{T}} \end{array}\right]$$

$$\text{where} \begin{cases} c_3 & = \alpha^2 (\gamma-\beta)\, p + \beta^2 (\alpha-\gamma)\, q + \gamma^2 (\beta-\alpha)\, r \\ c_2 & = \alpha\, (\gamma-\beta)\, p + \beta\, (\alpha-\gamma)\, q + \gamma\, (\beta-\alpha)\, r \\ c_1 & = (\gamma-\beta)\, p + (\alpha-\gamma)\, q + (\beta-\alpha)\, r \\ c_0 & = \frac{1}{\alpha}\, (\gamma-\beta)\, p + \frac{1}{\beta}\, (\alpha-\gamma)\, q + \frac{1}{\gamma}\, (\beta-\alpha)\, r \end{cases}$$

Remark 12.27.3. The Miquel point is the pole of the Clawson-Schmidt homography.

Remark 12.27.4. The four quantities c_j are bound by relations :

$$\begin{aligned} c_2 \frac{\sigma_1}{\sigma_3} + c_0 &= c_3 \frac{1}{\sigma_3} + c_1 \frac{\sigma_2}{\sigma_3} \in i\mathbb{R} \\ \overline{c_k} &= -\frac{c_{3-k}}{\sigma_3} \end{aligned}$$

Proposition 12.27.5. *The four fixed points of the isoconjugacy Ψ are given by :*

$$\Phi \simeq \left(\begin{array}{c} c_2/c_1 + W_u/c_1 \\ 1 \\ c_1/c_2 + W_d/c_2 \end{array}\right)$$

where the up and down radicals are given by :

$$\begin{aligned} W_u^2 &= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha) \times d_1 \\ W_d^2 &= \frac{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}{\alpha\beta\gamma} \times d_2 \end{aligned}$$

$$\text{where} \begin{cases} d_2 & = \alpha\, (\gamma-\beta)\, qr + \beta\, (\alpha-\gamma)\, rp + \gamma\, (\beta-\alpha)\, pq \\ d_1 & = \quad (\gamma-\beta)\, qr + \quad (\alpha-\gamma)\, rp + \quad (\beta-\alpha)\, pq \end{cases}$$

They can be constructed as follows. Lines Δ_1, Δ_2 are the common bisectors of $\Omega A, \Omega A'$, $\Omega B, \Omega B'$, $\Omega C, \Omega C'$ and δ_A is the perpendicular bisector of $[A, A']$. Then $H_1 = \Delta_1 \cap \delta_A$ (resp. $H_2 = \Delta_2 \cap \delta_A$) is the center of a circle by A, A' that cuts Δ_2 (resp Δ_1) at the four Φ_j points.

Proof. These points are characterized by

$$c_1\mathbf{Z}^2 - 2\, c_2\, \mathbf{ZT} + c_3\mathbf{T}^2 = 0 \quad ; \quad c_2\overline{\mathcal{Z}}^2 - 2\, c_1\overline{\mathcal{Z}}\mathbf{T} + c_0\, \mathbf{T}^2 = 0$$

□

Proposition 12.27.6 (Newton)**.** *All the conics that are tangent to four given lines have their centers on a line, that goes through the midpoints M_j of the diagonal pairs AA', BB', CC' (called the Newton axis of the quadrilateral). When this center $U \in$* Newton *is defined as $K\, M_b + (1-K)\, M_c$ the conic can be written as :*

$$\boxed{\mathcal{C}_b^*} \simeq (r-p) \left(\begin{array}{ccc} 0 & 0 & -p \\ 0 & 0 & q \\ -p & q & 0 \end{array}\right) + K\, p \left(\begin{array}{ccc} 0 & p-q & r-p \\ p-q & 0 & q-r \\ r-p & q-r & 0 \end{array}\right)$$

Proof. Let $\rho : \sigma : \tau$ be the "service point" of a conic $\mathcal{C} \in \mathcal{F}$. We have $\boxed{\mathcal{C}_b^*} = [0, \tau, \sigma; \tau, 0, \rho; \sigma, \rho, 0]$ and therefore $p\rho + q\sigma + r\tau = 0$. Since $U = \sigma + \tau : \tau + \rho : \rho + \sigma$, the Newton line is :

$$[q + r - p,\, r + p - q,\, p + q - r]$$

and the conclusions follow (remember that $p : q : r$ is the tripole of the transversal). □

Theorem 12.27.7. *The Morley affixes* $\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}$ *of a focus of the conic* $\mathcal{C}(K)\in\mathcal{F}$ *are bound to the parameter* K *by :*

$$K = \frac{(p-r)(\mathbf{Z}-\gamma\mathbf{T})((q-p)\mathbf{Z}+(\alpha\,p-\beta\,q)\mathbf{T})}{p\mathbf{T}(c_1\mathbf{Z}-c_2\mathbf{T})} \tag{12.21}$$

$$= \frac{(p-r)\left(\mathbf{T}-\overline{\mathcal{Z}}\gamma\right)\left(\alpha\beta(p-q)\overline{\mathcal{Z}}+(\alpha q-p\beta)\mathbf{T}\right)}{p\mathbf{T}\left(c_1\mathbf{T}-c_2\overline{\mathcal{Z}}\right)} \tag{12.22}$$

and therefore the focuses are located on the "focal cubic" $\mathcal{K}$ *:*

$$\frac{c_2}{\sigma_3}\mathbf{Z}^2\overline{\mathcal{Z}}+c_1\,\mathbf{Z}\overline{\mathcal{Z}}^2-\frac{c_1}{\sigma_3}\mathbf{Z}^2\mathbf{T}-\frac{c_3+c_1\sigma_2}{\sigma_3}\mathbf{Z}\overline{\mathcal{Z}}\mathbf{T}-c_2\overline{\mathcal{Z}}^2\mathbf{T}+\frac{c_1\sigma_1}{\sigma_3}\mathbf{Z}\mathbf{T}^2+\frac{c_2\sigma_2}{\sigma_3}\overline{\mathcal{Z}}\mathbf{T}^2+\frac{c_3-c_2\sigma_1}{\sigma_3}\mathbf{T}^3$$

Proof. Matrix $\boxed{\mathcal{C}_z^*}$ is obtained as $\boxed{Lu}\cdot\boxed{\mathcal{C}_b^*}\cdot{}^t\boxed{Lu}$ and then Plucker method is used. Some factors $(q-r)$ are appearing during the elimination process, but not all the $(p-q)(q-r)(r-p)$. Nothing special occurs when P is on a median (but not at the centroid). One can check that the cubic $\mathcal{K}$ is turned into its opposite when taking the (complex) conjugate. □

Theorem 12.27.8. *This focal cubic is nothing else than the vanRees cubic studied at Section 28.11.*

Proof. Check the six points A,B,C,A',B',C' together with the six points $n_A,n_B,n_C,n_{A'},n_{B'},n_{C'}$ More than nine is used to avoid a Cayley-Bacharach phenomenon. □

Theorem 12.27.9. *The focuses* F_j *of a given conic* $\mathcal{C}(K)\in\mathcal{F}$ *are exchanged by homographies* ψ, $\overline{\psi}$. *They can be constructed as follows. Call* $New^{\perp}$ *the perpendicular to the Newton line at* Ω. *Draw the bisectors* Δ_1, Δ_2 *of* $U\Phi_1$, $U\Phi_2$. *Cut them at* H_1, H_2 *by* $New^{\perp}$. *Draw circle* $\gamma_1(H_1,\Phi_1)$ *and cut* Δ_2. *Additionally, draw circle* $\gamma_2(H_2,\Phi_2)$ *and cut* Δ_1.

Proof. Write and factor $K\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)-K(z:t:\zeta)$ from (12.21). This gives $(p-q)(p-r)$ but not $(q-r)$, being smooth except at the centroid (i.e. when the fourth line at infinity). Otherwise, this gives $(z\mathbf{T}-t\mathbf{Z})$ together with another first degree factor with respect to $\mathbf{Z},\mathbf{T}$ and also with respect to z,t. In the upper view $\mathbf{Z}:\mathbf{T}$, this induces the identity together with another homography. Since the later has to provide $A\longleftrightarrow A'$, $B\longleftrightarrow B'$, $C\longleftrightarrow C'$, it has to be ψ. The same occurs in the lower view $\overline{\mathcal{Z}}:\mathbf{T}$, and the conclusion follows. □

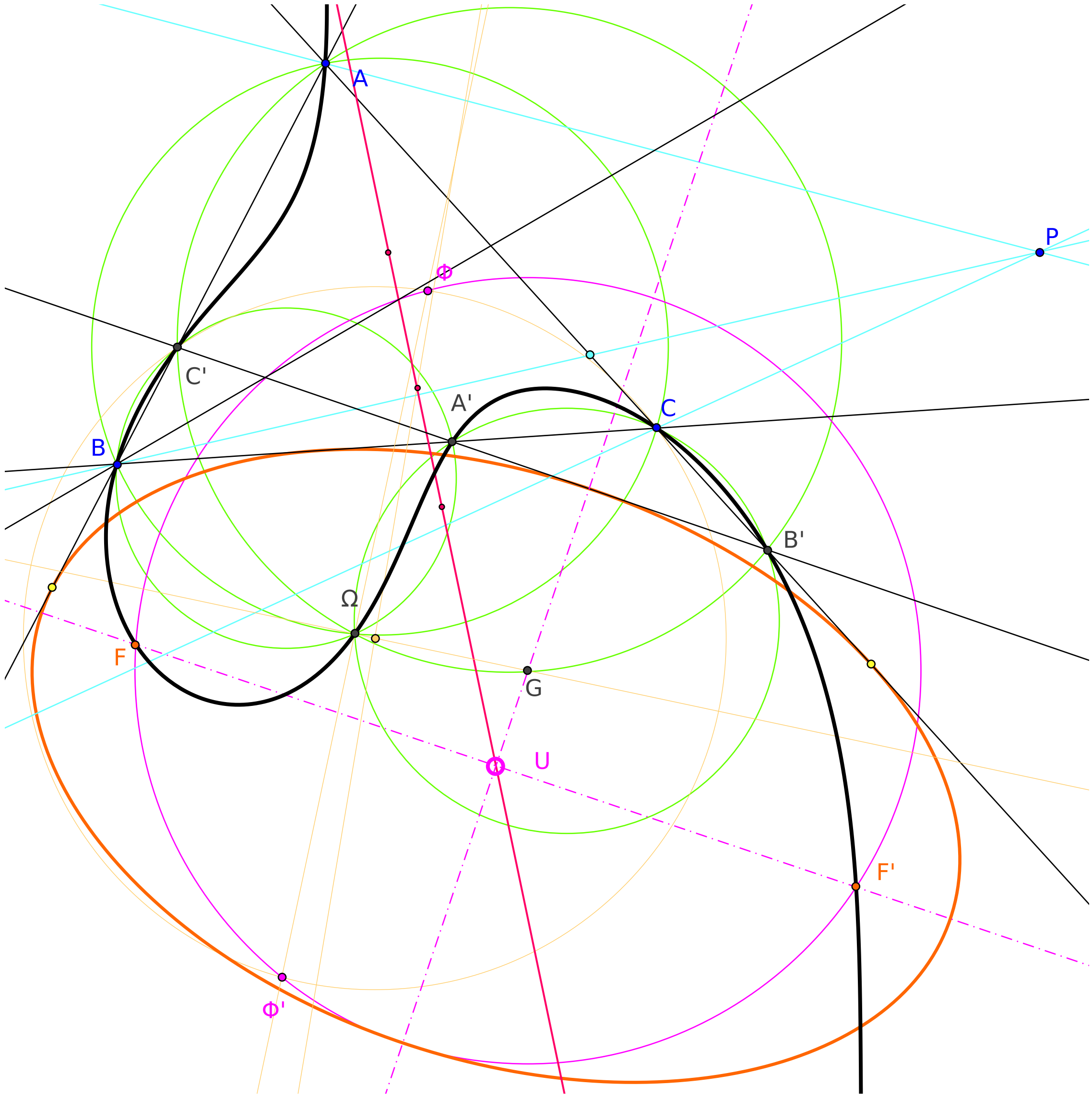
Remark 12.27.10. In the Geogebra Figure 12.16, the orange conic is drawn as follows. Reflect focuses F,F' into sideline AC and obtain F_b,F'_b. Then point $E_b=FF'_b\cap F'F_b$ on sideline AC belongs to the conic. In the same way, obtain the point E_c on AB. We draw both conics, an ellipse and an hyperbola, with focuses F,F' that go through E_b, but only the one that goes also through E_c is displayed.

Proposition 12.27.11. *When the transversal is tangent to* one *of the inexcircles of triangle* ABC, *the pencil contains* one *circle and the focal conic has a double point. This point is the center of the circle (see Figure 12.16b). When the transversal touches* two *of the inexcircles, the focal cubic degenerates into the Newton line and a circle having the corresponding inexcenters as antipodal points.*

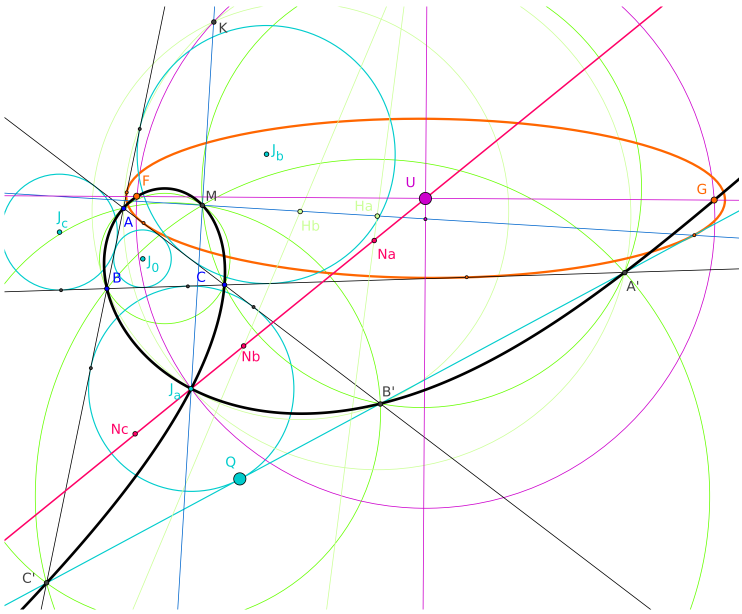
Proof. Only circles have equal focuses. □

12.28 Tg and Gt mappings

Definition 12.28.1. Tg and Gt mappings. Suppose U is a point not on a sideline of ABC. Let :
$gU=$ isogonal conjugate of U, $tU=$ isotomic conjugate of U
$tgU=$ isotomic conjugate of gX, $gtU=$ isogonal conjugate of tU
$GtU=$ intersection of lines $U-tU$ and $gU-gtU$,



(a) The general case



(b) The unicursal case

Figure 12.16: The Miguel pencil and its focal cubic

TgU= intersection of lines $U - gU$ and $tU - tgU$.
If $U = u : v : w$ (barycentrics), then :

$$GtU = \frac{a^2\left(b^2 - c^2\right)}{\left(v^2 - w^2\right)u} : \frac{b^2\left(c^2 - a^2\right)}{\left(w^2 - u^2\right)v} : \frac{c^2\left(a^2 - b^2\right)}{\left(u^2 - v^2\right)w}$$

$$TgU = \frac{b^2 - c^2}{\left(w^2b^2 - v^2c^2\right)u} : \frac{c^2 - a^2}{\left(u^2c^2 - w^2a^2\right)v} : \frac{a^2 - b^2}{\left(v^2a^2 - u^2b^2\right)w}$$

Proposition 12.28.2. *For any point U, not on a sideline of ABC, points A, B, C, gU, tU, TgU, GtU are on a same conic (Tuan, 2006). The perspector of this conic is $X_{512} \underset{b}{\div} U$. This conic is the isogonal image of line $U - gtU$ and also the isotomic image of line $U - tgU$.*

Proof. Straightforward computation. □

Example 12.28.3. Points X(3112) to X(3118) are related to Gt and Tg functions.
The X(31)-conic passes through X(I) for
I = 75, 92, 313, 321, 561, 1441, 1821, 1934, 2995, 2997, 3112, 3113
The X(32)-conic passes through X(I) for
I = 76, 264, 276, 290, 300, 301, 308, 313, 327, 349, 1502, 2367, 3114, 3115
The X(76)-conic passes through X(I) for
I = 6, 32, 83, 213, 729, 981, 1918, 1974, 2207, 2281, 2422, 3114, 3224, 3225

12.29 Polar coordinates

Definition 12.29.1. The polar equation of a curve is $z_\vartheta = \exp\left(\boldsymbol{i}\vartheta\right) f\left(\vartheta\right)$. Quantity $\rho \doteq f\left(\vartheta\right)$ is the so called algebraic radius and equals $+\mathrm{dist}\left(O,\, M\right)$ ... or $-\mathrm{dist}\left(O,\, M\right)$.

Proposition 12.29.2. *The polar equation of the circle through O and centered at $a + \boldsymbol{i}b$ is*

$$f\left(\vartheta\right) = 2a\cos\vartheta + 2b\sin\vartheta$$

Proof. Parametrize using $t = \tan\left(\vartheta/2\right)$. □

Proposition 12.29.3. *Equation $\rho = \dfrac{p}{1 + e\cos(\theta)}$ describes the conic whose focuses are O and $2pe/\left(e^2 - 1\right)$, with excentricity $e = f/a$ and parameter $p = b^2/a$.*

Proof. Parametrize as above and obtain the matrix $\begin{bmatrix} e^2 & -2pe & e^2 - 2 \\ -2pe & 4p^2 & -2pe \\ e^2 - 2 & -2pe & e^2 \end{bmatrix} \simeq \begin{bmatrix} f^2 & -2b^2f & -a^2 - b^2 \\ -2b^2f & 4b^4 & -2b^2f \\ -a^2 - b^2 & -2b^2f & f^2 \end{bmatrix}$

Exercise 12.29.4. When ρ ranges over the four intersections of the conic and the circle of the former two propositions, then $\sum 1/\rho$ doesn't depends on the circle (Koehler, 1886-1888, n°32, p.37).

Chapter 13

More about circles

13.1 General results

Let us start by recalling two key results.

Theorem 13.1.1 (Already stated in Section 7.5 as Theorem 7.5.2)**.** *Let Ω be the circle centered at P with radius ω. The* ***power formula*** *giving the Ω-power of any point $X = x : y : z$ from the power at the three vertices of the reference triangle is :*

$$\begin{aligned} power\,(\Omega,\, X) &\doteq |PX|^2 - \omega^2 = \frac{ux+vy+wz}{x+y+z} - \frac{a^2yz+b^2xz+c^2xy}{(x+y+z)^2} \\ \text{where } u &= power\,(\Omega,\, A)\,,\ \text{etc} \end{aligned} \tag{13.1}$$

Definition 13.1.2 (Already stated in Section 7.5 as Definition 7.5.3)**.** From $power\,(\Gamma,\, A) = 0$, etc, we have defined the standard equation of the circumcircle as :

$$\Gamma_{std}\,(x,y,z) \doteq -\frac{a^2yz+b^2xz+c^2xy}{x+y+z} = 0 \tag{13.2}$$

Proposition 13.1.3. *Let $\mathcal{C}$ be a conic, with matrix $\boxed{\mathcal{C}} = (m_{jk})$ (notations of Definition 12.3.1) . Then $\mathcal{C}$ is a cycle if and only if, for a suitable factor k, we have :*

$$\boxed{\mathcal{C}} - \frac{1}{2}\begin{pmatrix} 2\,m_{11} & m_{11}+m_{22} & m_{33}+m_{11} \\ m_{11}+m_{22} & 2\,m_{22} & m_{22}+m_{33} \\ m_{33}+m_{11} & m_{22}+m_{33} & 2\,m_{33} \end{pmatrix} = k\begin{pmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{pmatrix}$$

Proof. Obvious from (13.1). □

Proposition 13.1.4. *Four points at finite distance belong to the same cycle (aka circle or straight line) when their barycentrics $p_i : q_i : r_i$ are such that :*

$$\det_{i=1}^{i=4}\left[p_i,\, q_i,\, r_i,\, \Gamma_{std}\,(p_i,\, q_i,\, r_i)\right] = 0 \tag{13.3}$$

Proof. Obvious from (13.1). Don't forget how Γ_{std} was defined in (13.2) ! □

Computed Proof. Denominators are a reminder of the fact that circles don't escape to infinity. Write the Cartesian equation of the circle as :

$$\Delta_{cart} \doteq \det_{i=1}^{i=4}\left[\xi_i^2+\eta_i^2,\, \xi_i,\, \eta_i, 1\right] = 0$$

where ξ, η are the Cartesian coordinates of the points. Substitute these coordinates by :

$$\xi = \frac{x\xi_a+y\xi_b+z\xi_c}{x+y+z},\ \eta = \frac{x\eta_a+y\eta_b+z\eta_c}{x+y+z}$$

and obtain another determinant $\Delta'(x,y,z)$. Then compute $F\cdot\Delta'\cdot T^{-1}\cdot G$ where F is the diagonal matrix $diag\,(p_i+q_i+r_i)$ and

$$T=\left[\begin{array}{cccc}1&0&0&0\\0&\xi_a&\eta_a&1\\0&\xi_b&\eta_b&1\\0&\xi_c&\eta_c&1\end{array}\right],\qquad G=\left[\begin{array}{cccc}-1&0&0&0\\\xi_a^2+\eta_a^2&1&0&0\\\xi_b^2+\eta_b^2&0&1&0\\\xi_c^2+\eta_c^2&0&0&1\end{array}\right]$$

Matrix F acts on rows and kills quite all denominators, T acts on the last three columns and goes back to barycentrics while G acts on the first column to kill all square terms. After what everything simplifies nicely and leads to (13.3) □

Proposition 13.1.5. *The barycentric equation of circle with center $P=p:q:r$ and radius ω is :*

$$\left(u_0x+v_{.0}y+w_0z\right)(x+y+z)-\omega^2\,(x+y+z)^2-\left(a^2\,yz+b^2\,zx+c^2\,xy\right)=0 \tag{13.4}$$

where quantities u_0,v_0,w_0 are defined as :

$$\begin{array}{lll}u_0&\doteq|PA|^2=&\left(c^2q^2+b^2r^2+\left(b^2+c^2-a^2\right)qr\right)\div(p+q+r)^2\\v_0&\doteq|PB|^2=&\left(a^2r^2+c^2p^2+\left(c^2+a^2-b^2\right)rp\right)\div(p+q+r)^2\\w_0&\doteq|PC|^2=&\left(b^2p^2+a^2q^2+\left(b^2+a^2-c^2\right)pq\right)\div(p+q+r)^2\end{array}\tag{13.5}$$

Proof. Obvious from (13.1). The added value here is the emphasis on center and ω^2. It must be noticed that $u:v:w$ is not a point nor a line. Quantities u, v, w are strongly defined objects and are not defined up to a proportionality factor. They are to be considered exactly as ω^2, i.e. are of the same nature as a surface. It can be observed that u (or v or w) is zero-homogeneous wrt the barycentrics of point $P=p:q:r$. More details are given in Chapter 14 □

Proposition 13.1.6. ***Center.*** *The center of a circle defined by its equation (13.1) is given by :*

$$center\;\simeq\;\frac{1}{2}vX(3)-\boxed{\mathcal{K}}\cdot{}^tU=\left(\begin{array}{c}a^2\,S_a\\b^2\,S_b\\c^2\,S_c\end{array}\right)-\left(\begin{array}{ccc}a^2&-S_c&-S_b\\-S_c&b^2&-S_a\\-S_b&-S_a&c^2\end{array}\right)\cdot\left(\begin{array}{c}u\\v\\w\end{array}\right)\tag{13.6}$$

$$\simeq\;\frac{1}{2S}\left(a^2S_a:b^2S_b:c^2S_c\right)-\boxed{\mathcal{M}_b}\cdot{}^tU \tag{13.7}$$

Proof. As stated in Definition 12.3.17, the center of a conic is the pole of the line at infinity wrt the conic. Computations are straightforward. It can be noticed that product $\boxed{\mathcal{K}}\cdot{}^tU$ gives the orthodir of line U (i.e. the point at infinity in the orthogonal direction). Since the line of centers is orthogonal to the radical axis of Γ and Ω, this formula describes the coefficients to be used when the circumcenter X(3) is described as in ETC by $vX\,(3)=a^2\left(b^2+c^2-a^2\right)$ ::. Let us recall the Al-Kashi formula $\boxed{\mathcal{K}}=2S\boxed{\mathcal{M}_b}$. □

Remark 13.1.7. In both the center and the radius formulas, everything must be used exactly as written, and not to a proportionality factor. A more efficient formulation will be given later, with formula (14.15)

Proposition 13.1.8. ***Radius.*** *The radius of a circle defined by its equation (13.1) is given by :*

$$\omega^2=\frac{1}{16\,S^2}\left(U\cdot\boxed{\mathcal{K}}\cdot^t U-U.vX\,(3)+a^2b^2c^2\right)\tag{13.8}$$

Proof. Subtract formula (13.1) from $|PX|^2$ obtained from (13.6) and Pythagoras formula. □

Definition 13.1.9. ***kitW.*** Some usual square roots are given in Table 13.1, and some other notations in Table 13.2.

Example 13.1.10. Table 13.3 describes some of the usual circles in triangle geometry. For further information on many circles, refer to
http://mathworld.wolfram.com/Circle.html

name	#	value	where	
Lemoine	W_1	$\sqrt{a^2b^2+a^2c^2+b^2c^2}$	$e^{i\omega}=\dfrac{a^2+b^2+c^2+4\,iS}{2W_1}$	(13.11)
Brocard	W_2	$\sqrt{a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2}$	$\lvert OK\rvert=\dfrac{2\,W_2\,R}{a^2+b^2+c^2}$	(13.12)
Euler	W_3	$\sqrt{\sum_3 a^6-\sum_6 a^4b^2+3\,a^2b^2c^2}$	$\lvert OH\rvert=\dfrac{W_3}{4\,S}=\dfrac{R}{abc}\,W_3$	(13.13)
Fuhrmann	W_4	$\sqrt{\sum_3 a^3-\sum_6 a^2b+3\,abc}$	$\lvert HN\rvert=2\,W_4R\sqrt{\dfrac{1}{abc}}$	(13.14)

Table 13.1: Some usual square roots (kitW)

#	value	name
s	$(a+b+c)\,/2$	half-perimeter
R	$\dfrac{abc}{\sqrt{s\,(s-a)\,(s-b)\,(s-c)}}$	circumradius
S	$\dfrac{abc}{4R}$	area of triangle
ω	$\exp\left(i\,\omega\right)=\dfrac{a^2+b^2+c^2}{2W_1}+i\,\dfrac{2\,S}{W_1}$	Brocard angle
e	$\sqrt{\frac{a^4+b^4+c^4}{b^2c^2+c^2a^2+a^2b^2}-1}$	$\sqrt{1-4\,\sin^2\omega}$
r	$\dfrac{S}{s}=\dfrac{a\,b\,c}{2R\,(a+b+c)}$	inradius

Table 13.2: Some usual notations (circle kit 2)

Definition 13.1.11. Pencil of cycles. When $\mathcal{C}_1$ and $\mathcal{C}_2$ are two circles, then $\lambda\mathcal{C}_1+\mu\mathcal{C}_2$ is also a cycle. The family generated from two given circles is called a pencil. Then all centers are on the same line, which is orthogonal to the only line contained in the pencil (the so called radical axis). More details in Chapter 14

Definition 13.1.12. The **radical trace** of two non-concentric circles is the point of intersection of the radical axis of the circles and the line of the centers of the circles. (For examples, see X(I) for I = 6, 187, 1570, 2021-2025, 2030-2032.)

13.2 Inversion in a circle

Remark 13.2.1. All results relative to the inversion in a cycle have moved to Section 14.8.

13.3 Antipodal Pairs on Circles

Remark 13.3.1. Since the previous section has moved, the present section is now orphaned... and probably should be moved.

Proposition 13.3.2. *Suppose (O_1) and (O_2) are circles and that P, P' are antipodes on (O_1). Let $U=insim(O_1,O_2)$ and $V=exsim\,(O_1,O_2)$ be the respective internal and external center of homothety of circles (O_1) and (O_2). Define $Q=P'U\cap PV$ and $Q'=PU\cap P'V$. Then Q, Q' are antipodes on (O_2). Moreover, the lines PP' and QQ' are parallel.*

Proof. The result is quite obvious, but giving a non-circular proof is not so obvious... except from using $z:t:\zeta$ coordinates. Write the antipodal points as $P=z_1/t_1+r_1\tau$ and $P'=z_1/t_1-r_1\tau$

Name	Center	Radius	
circumcircle	X(3)	R	13.4
incircle	X(1)	r	13.5
nine-point circle	X(5)	$\frac{1}{2}R$	13.6
polar circle	X(4)	$\sqrt{-S_aS_bS_c} \div 2S$	13.7
Longchamps circle	X(20)	$\sqrt{-S_aS_bS_c} \div S$	13.8
Bevan circle	X(40)	$2R$	13.9
Spieker circle	X(10)	$r/2$	13.10
Apollonius circle	X(970)	$\left(r^2+s^2\right) \div 4\,r$	13.13
1st Lemoine	X(182)	$R \div 2\cos\omega$	13.14
2nd Lemoine	X(6)	$abc/\left(a^2+b^2+c^2\right)$	13.15
Sin-triple-angle	X(49)	R_{sta}	13.16
Brocard circle	X(182)	$eR \div 2\cos\omega$	13.17
Brocard second	X(3)	eR	13.17
Orthocentroidal	X(381)	$\lvert OH\rvert/2$	13.19
Fuhrmann	X(355)	$\lvert HN\rvert/2$	13.20

Table 13.3: Some circles

where τ is a turn. Point U, V are obtained as

$$U = \left(r_2z_1/t_1 + r_1z_2/t_2\right) / \left(r_2 + r_1\right) \;;\; V = \left(r_2z_1/t_1 - r_1z_2/t_2\right) / \left(r_2 - r_1\right)$$

Two wedges later, we have:

$$Q = z_2/t_2 + r_2\tau \;;\; Q' = z_2/t_2 - r_2\tau$$

and we are done. □

In the following examples, suppose $P = p : q : r$ on the first circle.

13.4 Circumcircle

Definition 13.4.1. The circumcircle is the circle through A, B, C. Perspector is X_6 and center X_3. Equation, matrix, column are :

$$a^2yz + b^2zx + c^2xy = 0 \;;\; \boxed{\mathrm{Pyth}_b} \;;\; \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Its standard parametrization is (7.17), i.e. :

$$\frac{a^2}{\sigma-\tau} : \frac{b^2}{\tau-\rho} : \frac{c^2}{\rho-\sigma} \mid \rho : \sigma : \tau \neq \mathcal{L}_b$$

Lemma 13.4.2. *The distance $|PX_3|$ to center from any point of the plane is given by :*

$$\left|PX_3\right|^2 = R^2 - \frac{a^2qr + b^2rp + c^2pq}{(p+q+r)^2}$$

Proof. Direct inspection using Theorem 7.4.4. As it should be, $|X_3X_3| = 0$ while the equation of the circumcircle is $\left|PX_3\right|^2 = R^2$. □

Proposition 13.4.3. *For any finite point, other than the circumcenter X_3, the inverse-in-circumcircle of $P = p : q : r$ has barycentrics $u : v : w$ obtained cyclically from :*

$$u = -p^2 + \frac{c^2 - a^2}{b^2}\, pq + \frac{b^2 - a^2}{c^2}\, pr + \frac{a^2\left(b^2 + c^2 - a^2\right)}{b^2 c^2}\, qr$$

1	36	24	403	54	1157	352	353	1692	3053
2	23	25	468	55	1155	371	2459	2482	2930
4	186	26	2072	56	1319	372	2460	2935	3184
5	2070	27	2073	57	2078	399	1511	3110	3286
6	187	28	2074	58	1326	667	1083	3438	3480
10	1324	29	2075	67	3455	859	3109	3439	3479
15	16	32	1691	115	2079	1054	1283	3513	3514
20	2071	35	484	131	2931	1145	2932		
21	1325	39	2076	182	2080	1384	2030		
22	858	40	2077	237	1316	1687	1688		

Proof. Points X(3), P, U are on the same line, and distance from X(3) to U is $R^2/|PX_3|$. Therefore, in normalized barycentrics, we have : $u = x_3 + (p - x_3)\, R^2/|PX_3|^2$. □

Remark 13.4.4. On ETC $n \leq 3587$, there are :

- 258 named points that belongs to Γ
- 47 pairs of "true" inverses that both are named
- 220 named points of Γ that have a named isogonal conjugate (among the 229 points of $\mathcal{L}_b$)
- 62 pairs of named antipodal points

13.5 Incircle

Definition 13.5.1. The incircle is one of the four circles that are tangent to the sidelines. This circle is inside the triangle, and also inside the nine point circle (these circles are tangent). Center X(1), perspector X(7), radius $r = S/s = abc \div 2R\,(a + b + c)$, while equation and column are :

$$\Gamma_{std} + \frac{1}{4}\sum x\,(b - a + c)^2 \;\; ; \;\; \begin{pmatrix} (-a+b+c)^2 \\ (+a-b+c)^2 \\ (+a+b-c)^2 \\ 4 \end{pmatrix} \tag{13.9}$$

Proof. well-known properties. □

Remark 13.5.2. On ETC $n \leq 3587$, there are :

- 39 named points on the incircle
- 7 pairs of "true" inverses that both are named
- 10 pairs of named antipodal points

Proposition 13.5.3. *Centers of homothety with the circumcircle are in=X(55), and ex=X(56). When $p : q : r \in \Gamma$, then Q, Q' are antipodal points on the incircle :*

$$Q = \left((b-c)^2\, p + a^2 q + a^2 r\right)(b + c - a)\,,\ \text{etc}$$

$$Q' = \frac{(b+c)^2\, p + a^2 q + a^2 r}{b + c - a},\ \text{etc}$$

Proof. Proposition 13.3.2. □

Proposition 13.5.4. ***Incircle transform***. *Let $U = u : v : w$ be a point other than the symmedian point, X_6. Then reflection of $\mathcal{T}_7$ (the intouch triangle) in the line UX_1 is perspective with triangle ABC. The isogonal conjugate of the corresponding perspector is called the incircle transform of U. Its barycentrics are :*

$$IT\left(u:v:w\right) = \frac{a^2\left(bw-cv\right)^2}{b+c-a} : \frac{b^2\left(cu-aw\right)^2}{a+c-b} : \frac{c^2\left(av-bu\right)^2}{b+a-c}$$

and this point is on the incircle.

Proof. Line UX_1 is a diameter of the incircle and the reflected triangle $\mathcal{T}$ is also inscribed in the incircle. The barycentrics of UX_1 are $[bw-cv, cu-aw, av-ub]$. Reflection in this line is obtained using (7.28), and barycentrics of $\mathcal{T}$ are obtained. Perspectivity and perspector are easily computed and conclusion follows by substituting in the incircle equation. □

Remark 13.5.5. In ETC another formula is given... and the point is also on the incircle :

$$IT2\left(U\right) = \frac{a^2\left(b^2w-c^2v\right)^2}{b^2c^2\left(b+c-a\right)} : \frac{b^2\left(c^2u-a^2w\right)^2}{a^2c^2\left(c+a-b\right)} : \frac{c^2\left(a^2v-b^2u\right)^2}{a^2b^2\left(a+b-c\right)}$$

One has $IT\left(X\right) = IT\left(U\right)$ when U, X aligned with X_1 while $IT2\left(X\right) = IT\left(U\right)$ when U, X aligned with X_6.

13.6 Nine-points circle

Definition 13.6.1. The nine point circle is the circumcircle of the orthic triangle. It goes also through the six midpoints of the orthocentric quadrangle $ABCH$. Center X(5), radius $R/2$ (half the ABC circumradius), perspector X(3613) :

$$\frac{1}{b^2c^2+2\,S_a\,a^2} : \frac{1}{a^2c^2+2\,b^2S_b} : \frac{1}{b^2a^2+2\,c^2Sc}$$

while equation, matrix, column are :

$$\Gamma_{std} + \frac{1}{2}\left(xS_a+yS_b+zS_c\right)\ ;\ \begin{bmatrix} 2\,S_a & -c^2 & -b^2 \\ -c^2 & 2\,S_b & -a^2 \\ -b^2 & -a^2 & 2\,S_c \end{bmatrix}\ ;\ \begin{bmatrix} S_a \\ S_b \\ S_c \\ 2 \end{bmatrix} \qquad (13.10)$$

Proposition 13.6.2. *Centers of homothety with Γ are in=X(2), and ex=X(4). Ω intersects Γ when ABC is not acute. Radical trace X(468), direction of center axis (Euler line) X(30), direction of radical axis X(523). Standard parametrization (homothety from circumcircle) :*

$$U \simeq \begin{pmatrix} \left(\sigma-\tau\right)\left(c^2\tau-b^2\sigma+b^2\rho-c^2\rho\right) \\ \left(\tau-\rho\right)\left(a^2\rho-c^2\tau+c^2\sigma-a^2\sigma\right) \\ \left(\rho-\sigma\right)\left(b^2\sigma-a^2\rho+a^2\tau-b^2\tau\right) \end{pmatrix}$$

Remark 13.6.3. On ETC $n \leq 3587$, there are :

- 37 named points on the nine points circle
- 9 pairs of "true" inverses that both are named
- 12 pairs of named antipodal points

Proposition 13.6.4 (Feuerbach)**.** *The nine-point circle is tangent to the incircle and the three excircles. The contact with incircle is X(11), the so-called Feuerbach point.*

Proof. Use (13.9) and (13.10) to obtain the radical axis. □

13.7 Polar circle

Definition 13.7.1. The polar circle is the only circle whose matrix is diagonal (and triangle ABC is autopolar). Center X(4), the orthocenter, radius $\sqrt{-S_a S_b S_c} \div 2S$, equation and column are :

$$\frac{1}{x+y+z} \sum S_a x^2 = \Gamma_{std} + (x\,S_a + y\,S_b + z\,S_c) = 0 \; ; \; \begin{pmatrix} S_a \\ S_b \\ S_c \\ 1 \end{pmatrix}$$

Proposition 13.7.2. *This circle belongs to the same pencil as the circum- and the nine-points circles. This circle is real only when triangle ABC is not acute. Therefore, no named points can belong to this circle.*

Proposition 13.7.3. *The polar circle is the locus of the centers of the inscribed rectangular hyperbolas (cf Section 12.8).*

13.8 Longchamps circle

Definition 13.8.1. The Longchamps circle of ABC is the polar circle of the antimedial triangle. Center X(20), the Longchamps point, radius $\sqrt{-S_a S_b S_c}/S$, equation, matrix and column are :

$$\Gamma_{std} + \left(x\,a^2 + y\,b^2 + z\,c^2\right) = 0 \; ; \; \begin{bmatrix} a^2 & S_c & S_b \\ S_c & b^2 & S_a \\ S_b & S_a & c^2 \end{bmatrix} \; ; \; \begin{pmatrix} a^2 \\ b^2 \\ c^2 \\ 1 \end{pmatrix}$$

Proposition 13.8.2. *The Longchamps circle is the locus of the auxiliary points of the inscribed rectangular hyperbolas (cf Section 12.8).*

13.9 Bevan circle

Definition 13.9.1. The Bevan circle is the circumcircle of the excentral triangle. Perspector X(57), center X(40), radius $2R$, equation and column :

$$\Gamma_{std} + (-bcx - acy - abz) = 0 \; ; \; \begin{pmatrix} -bc \\ -ac \\ -ba \\ 1 \end{pmatrix}$$

Proposition 13.9.2. *Centers of homothety with Γ are in=X(165), and ex=X(1). Radical trace X(1155), center axis X(517), radical axis X(513). Moses parametrization leads to Q (bad looking) and $Q' = -(a + 2\,b + 2\,c)\,p + aq + ar$.*

Remark 13.9.3. On ETC $n \leq 3587$, there are :

- 9 named points on the Bevan circle, namely : 1054, 1282, 1768, 2100, 2101, 2448, 2449, 2948, 3464
- 5 pairs of "true" inverses that both are named
- 2 pairs of named antipodal points

13.10 Spieker circle

Definition 13.10.1. Spieker circle is the incircle of the medial triangle. Perspector X(2), center X(10), radius $r/2$ (half the ABC inradius), equation :

$$\Gamma_{std} + \frac{1}{16} \sum x \left(5\,c^2 + 5\,b^2 - 3\,a^2 + 2\,ac + 2\,ab - 6\,bc\right)$$

Proposition 13.10.2. *Centers of homothety with Γ are in=X(958), and ex=X(1376). Radical trace not named, center axis X(515), radical axis X(522). Moses parametrization leads to :*

$$Q = (b+c-a)\left(\left(ab^2+ac^2+b^3-b^2c-c^2b+c^3\right)p + a\,(q+r)\left(a^2+ab+ac+2\,bc\right)\right)$$

$$Q' = \left(ab^2+c^2a+b^2c+c^2b-c^3-b^3\right)p + a\,(q+r)\left(+a^2-ab-ac+2\,bc\right),\ \text{etc}$$

Centers of homothety with the incircle are in=X(8) and ex=X(2).

Remark 13.10.3. On ETC $n \leq 3587$, there are :

- 8 named points on the Spieker circle, namely : 3035, 3036, 3037, 3038, 3039, 3040, 3041, 3042
- no pairs of "true" inverses that both are named
- 2 pairs of named antipodal points [3035,3036], [3042,3042]

13.11 Alt-Spieker circle

Definition 13.11.1. The alt-Spieker circle is the common orthogonal cycle to the three excircles (see Subsection 14.11.4). Center X(10), radius $\sqrt{r_0^2+s^2} \div 2$, equation and column :

$$4\,\Gamma_{std} - \sum x\,(a-b+c)\,(a+b-c) = 0\ ;\ \begin{pmatrix} -(a-b+c)\,(a+b-c) \\ -(b-c+a)\,(b+c-a) \\ -(c-a+b)\,(c+a-b) \\ 4 \end{pmatrix}$$

13.12 Apollonian circles

Definition 13.12.1. Let J_a and P_a be the points where the interior and exterior bisectors of angle A meet the opposite sideline. In other words, the J_k and P_k are, respectively, the cevians and the cocevians of I_0 =X(1). The circle drawn using $[J_a, P_a]$ as diameter is called the ***A*-Apollonian circle**. It goes obviously through vertex A.

Proposition 13.12.2. *The equation of the A-circle is* $0 : -a^2c^2 : a^2b^2 : b^2-c^2$. *Its center is* $E_a \simeq 0 : -b^2 : c^2$, *the cocevian of X(6). The three* (E_k) *belong to a same pencil (the so-called Lemoine pencil) whose base points are the isodynamic points X(15), X(16), while the radical axis is the Brocard line.*

(Spoiler) Since J_a, P_a *are Lemoine-conjugates, the Apollonian circles can be described in the Lubin-1 frame. One obtains:*

$$(E_a) \underset{1}{\simeq} \beta+\gamma-2\,\alpha : \alpha^2-\beta\,\gamma : \alpha\,(2\,\beta\gamma-\alpha\beta-\alpha\gamma) : \alpha^2-\beta\,\gamma$$

Proof. Straightforward from $\bigwedge_3 (A, J_a, P_a)$. □

13.13 Apollonius circle

Definition 13.13.1. The **Apollonius circle** is tangent to the three excircles and encloses them (see Subsection 14.11.4). Center X(970), perspector not named, equation :

$$\Gamma_{std} - \frac{a+b+c}{4}\sum \frac{a^2+ab+ac+2\,bc}{a}\,x = 0\ ;\ \begin{pmatrix} a+b+c+2bc\div a \\ a+b+c+2ca\div b \\ a+b+c+2ab\div c \\ -4\div(a+b+c) \end{pmatrix}$$

$$\text{radius} = (abc + \sum_6 a^2b) \div 8S = \frac{r_0^2+s^2}{4\,r_0}$$

Proposition 13.13.2. *Centers of homothety with* Γ *are in=X(573), and ex=X(386). Radical trace not named, center axis X(511), radical axis X(512). Moses parametrization leads to bad looking Q and*

$$Q' = -\left(b+c\right)^{2}\left(a+c\right)\left(a+b\right)p + \left(ab+ac+bc+b^{2}+c^{2}\right)a^{2}q + r\left(ab+ac+bc+b^{2}+c^{2}\right)a^{2}, \text{ etc}$$

Centers of homothety with the nine-points circle are in=X(10) and ex=X(2051).

Remark 13.13.3. On ETC $n \leq 3587$, there are :

- 8 named points on the Apollonius circle, namely : 2037, 2038, 3029, 3030, 3031, 3032, 3033, 3034
- no pairs of "true" inverses that both are named
- 1 pairs of named antipodal points [2037, 2038].

13.14 First Lemoine circle

Definition 13.14.1. The **first Lemoine circle** of ABC is obtained as follows. Draw parallels to the sidelines of ABC through Lemoine point X_6. The six intersections of these lines with sidelines are concyclic on the required circle. The following surd is useful :

$$W_1 = \sqrt{a^2b^2 + a^2c^2 + b^2c^2} \tag{13.11}$$

Proposition 13.14.2. *Center is X(182) (i.e.* $[O, K]$ *midpoint), radius* $R \div 2\cos\omega = W_1\,R/\left(a^2+b^2+c^2\right)$*, perspector :*

$$\frac{a^2}{2\,a^2b^2 + 2\,a^2c^2 + b^2c^2}, \frac{b^2}{a^2c^2 + 2\,a^2b^2 + 2\,b^2c^2}, \frac{c^2}{a^2b^2 + 2\,a^2c^2 + 2\,b^2c^2}$$

is not named, equation :

$$\Gamma_{std} + \frac{1}{\left(a^2+b^2+c^2\right)^2}\sum x\,\left(b^2+c^2\right)b^2c^2$$

This circle is concentric with and external to the first Brocard circle.

Proof. Difference of squared radiuses factors into $\left(abc/\left(a^2+b^2+c^2\right)\right)^2$. □

Proposition 13.14.3. *Centers of homothety with* Γ *are in=X(1342), and ex=X(1343). Radical trace X(1691), center axis X(511), radical axis X(512). Moses parametrization leads to bad looking Q and* Q'*. Poncelet centers of the pencil : X(1687) (inside) and X(1688) outside.*

Proof. In order to see that X(1687) is inside, compute $\Omega\left(X(182)\right) \times \Omega\left(X\left(1687\right)\right)$ and obtain a quantity that is clearly positive. □

Remark 13.14.4. On ETC $n \leq 3587$, there are :

- 2 named points on the first Lemoine circle, namely : 1662, 1663 (intersection with the Brocard axis, X(3)X(6)).
- 7 pairs of "true" inverses that both are named :

$$\begin{bmatrix} 3 & 6 & 32 & 39 & 371 & 372 & 1687 \\ 2456 & 1691 & 1692 & 2458 & 2461 & 2462 & 1688 \end{bmatrix}$$

- 1 pairs of named antipodal points [1662, 1663].

13.15 Second Lemoine circle

Definition 13.15.1. The **second Lemoine circle** of ABC is obtained as follows. Draw parallels to the sidelines of orthic triangle through Lemoine point X_6. The six intersections of these lines with sidelines are concyclic on the required circle.

Proposition 13.15.2. *Center is X(6) itself, radius* $abc/\left(a^2+b^2+c^2\right)$, *perspector X(3527), equation :*

$$\Gamma_{std}+\frac{4}{\left(a^2+b^2+c^2\right)^2}\sum x\,S_a\,b^2c^2=0$$

Centers of homothety with Γ *are in=X(371), and ex=X(372). Radical trace X(1692), center axis X(511), radical axis X(512). Moses parametrization leads to bad looking* Q *and* Q'*. Poncelet centers of the pencil are involving radical* $\sqrt{6\sum a^2b^2-5\sum a^4}$ *and are not named. Moreover, the second Lemoine circle is bitangent to the Brocard ellipse.*

Remark 13.15.3. On ETC $n\leq 3587$, there are :

- 2 named points on the second Lemoine circle, namely : 1666, 1667 (intersection with the Brocard axis, X(3)X(6)).
- 5 pairs of "true" inverses that both are named :

$$\left[\begin{array}{ccccc} 3 & 576 & 1316 & 1351 & 2452 \\ 1570 & 1691 & 2451 & 1692 & 3049 \end{array}\right]$$

- 1 pairs of named antipodal points [1666, 1667].

13.16 Sine-triple-angle circle

Definition 13.16.1. Define inscribed triangles $\mathcal{T}_1$ and $\mathcal{T}_2$ by the properties :

$$\angle\left(AB,\,AC\right)=\angle\left(B_1A,\,B_1C_1\right)=\angle\left(C_2B_2,\,C_2A\right),\text{ etc}$$

the idea being to obtain isosceles "remainders". Then all the six vertices are on the same circle.

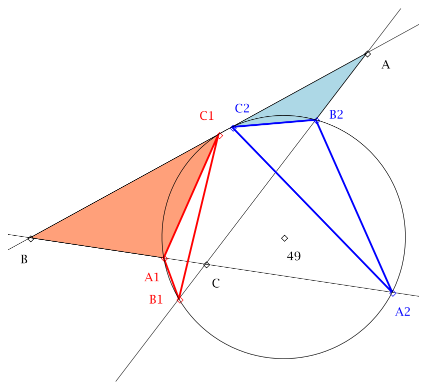


Figure 13.1: Sin triple angle circle

Proof. Using the tangent formula, the six points are easily obtained. $\mathcal{T}_1$ is a central triangle, and each vertex of $\mathcal{T}_2$ is obtained by a transposition.

$$A_1\simeq\left(\begin{array}{c} 0 \\ \left(a^2-ac-b^2\right)\left(a^2+ac-b^2\right)\left(a^2+b^2-c^2\right) \\ \left(a^2-bc-c^2\right)\left(a^2+bc-c^2\right)c^2 \end{array}\right)$$ □

Proposition 13.16.2. *Center X(49), perspector not named, equation horrific, radius* $R_{tsa} = R^3 / \left(|OH|^2 - 2\,R^2 \right)$. *Centers of homothety with* Γ *are in=X(1147), and ex=X(184), direction of radical axis X(924). Moses parametrization leads to bad looking Q and :*

$$Q' = a^2 \left(a^2 - c^2\right) \left(a^2 - b^2\right) p - a^4 \left(b^2 + c^2 - a^2\right) (q + r)\,, \text{ etc}$$

Remark 13.16.3. On ETC $n \leq 3587$, there are :

- 6 named points on the Sine Triple Angle circle, namely : 3043, 3044, 3045, 3046, 3047, 3048
- 0 pairs of "true" inverses that are both named
- 1 pairs of named antipodal points [3043, 3047].

13.17 Brocard 3-6 circle

Definition 13.17.1. The **Brocard 3-6 circle** has $[X_3,\, X_6]$ for diameter. Center X(182). Radius $e\,R \div 2\cos\omega = W_2\,R / \left(a^2 + b^2 + c^2\right)$ where :

$$W_2 = \sqrt{a^4 + b^4 + c^4 - (b^2c^2 + a^2b^2 + a^2c^2)} \tag{13.12}$$

while the perspector :

$$\frac{a^2}{2a^4 + b^2c^2} : \frac{b^2}{2b^4 + a^2c^2} : \frac{c^2}{2c^4 + a^2b^2}$$

is not named. Equation :

$$\Gamma_{std} + \frac{1}{(a^2 + b^2 + c^2)} \left(x\, b^2c^2 + y\, c^2a^2 + z\, a^2b^2\right) = 0$$

Proposition 13.17.2. *Centers of homothety with* Γ *are in=X(1340), and ex=X(1341). Radical trace X(187), center axis X(511), radical axis X(512). Moses parametrization leads to :*

$$Q = \begin{pmatrix} a^2 \left(a^4 - a^2c^2 - a^2b^2 - 2\,b^2c^2\right) (p + q + r) \\ \pm W \left(\left(a^2b^2 + a^2c^2 + 2\,b^2c^2 - b^4 - c^4\right) p - a^2 \left(b^2 + c^2 - a^2\right) (q + r) \right) \end{pmatrix}, \text{ etc}$$

Poncelet centers of the pencil are X(15) and X(16), the isodynamic points (see Section 14.10).

Remark 13.17.3. On ETC $n \leq 3587$, there are :

- 4 named points on the first Brocard circle, namely : 3, 6, 1083, 1316. Moreover, this circle goes through the Brocard points (cf 7.11.1).
- 47 pairs of "true" inverses that are both named
- 1 pairs of named antipodal points [3, 6].

13.18 Second Brocard circle

Definition 13.18.1. First anti-Brocard circle: circumcircle of the first anti-Brocard triangle. Equation:

$$\Gamma_{std} - \sum_{3} \frac{a^2 \left(b^2 + bc + c^2\right) \left(b^2 - bc + c^2\right)}{W_2^2}\, x = 0$$

Definition 13.18.2. The **Brocard second circle** is centered on X(3) and goes through the Brocard's centers. Radius $eR = W_2\,R / \sqrt{a^2b^2 + b^2c^2 + c^2a^2}$, while the perspector :

$$\frac{a^2}{2a^4 - a^2b^2 - a^2c^2 + b^2c^2}, \text{ etc}$$

is not named. Equation :

$$\Gamma_{std} + \frac{a^2b^2c^2}{a^2b^2 + a^2c^2 + b^2c^2} = 0$$

Remark 13.18.3. On ETC $n \leq 3587$, there are :

- 6 named points on the second Brocard circle, namely : 1670, 1671, 2554, 2555, 2556, 2557. Moreover, this circle goes through the Brocard points (cf 7.11.1).
- 17 pairs of "true" inverses that are both named

6	39	62	3106	1340	3558	2026	2561
15	3105	76	99	1341	3557	2027	2560
16	3104	182	3095	1666	2563		
32	3094	371	3103	1667	2562		
61	3107	372	3102	1689	1690		

- 3 pairs of named antipodal points [1670, 1671], [2554, 2555], [2556, 2557].

13.19 Orthocentroidal 2-4 circle

Definition 13.19.1. Orthocentroidal circle has $[X_2,\, X_4]$ for diameter. Center is X(381) and radius $R\,W_3 \div 3abc$ where :

$$W_3 = \sqrt{a^6+b^6+c^6-(a^4b^2+a^4c^2+a^2b^4+a^2c^4+b^4c^2+b^2c^4)+3a^2b^2c^2} \tag{13.13}$$

while the perspector :

$$\frac{1}{b^2c^2+2\,a^2\left(b^2+c^2-a^2\right)}$$

is not named. Equation and column are :

$$\Gamma_{std}+\frac{2}{3}\left(x\,S_a+y\,S_b+z\,S_c\right)=0\ ;\ \begin{pmatrix}2\,S_a\\2\,S_b\\2\,S_c\\3\end{pmatrix}$$

Proposition 13.19.2. *Centers of homothety with Γ are in=X(1344), and ex=X(1345). Radical trace X(468), center axis X(30), radical axis X(523). Moses parametrization leads to :*

$$Q=\begin{pmatrix}abc\left(a^4+a^2b^2+a^2c^2-2\,b^4+4\,b^2c^2-2\,c^4\right)(p+q+r)\\ \pm W\left(\left(a^2b^2+a^2c^2+2\,b^2c^2-b^4-c^4\right)p-a^2\left(b^2+c^2-a^2\right)(q+r)\right)\end{pmatrix},\ \text{etc}$$

Poncelet points are real when triangle is acute.

Remark 13.19.3. On ETC $n \leq 3587$, there are :

- 2 named points on the orthocentroidal circle, namely (antipodal) 2, 4
- 45 pairs of "true" inverses that are both named

Proposition 13.19.4. *The orthocentroidal circle goes through points* $A' = (A+2A_H)/3$, etc. *where $A_HB_HC_H$ are the feet of the altitudes.*

Proof. One has $A_H \simeq a^2 : 2\,S_c : 2\,S_b$. Then one has $Ver\,(A_H)\cdot \underset{b}{\mathcal{V}}=0$. ☐

Proposition 13.19.5. *Among the four in/ex-centers, the in-center is inside the orthocentroidal circle, and the other three are outside.*

Proof. One can check that

$$\begin{aligned} |O\,I_0|^2-4\,|N\,I_0|^2 &= +2r_0\,(R-2r_0) && = |O\,I_0|^2\times(+2r_0/R) && \geq 0\\ |O\,I_A|^2-4\,|N\,I_A|^2 &= -2r_A\,(R+2r_A) && = |O\,I_A|^2\times(-2r_A/R) && \leq 0 \end{aligned}$$

while the locus of $|O\,M|^2 = 4\,|N\,M|^2$ is precisely the said circle ($M=H$ and $M=G$ are solutions). ☐

Proposition 13.19.6. *Let O, N, I be three points on the plane and let $\psi(\Phi)$ be the image of the curve*

$$\Phi : 3\,\overline{\mathcal{Z}}^2 \mathbf{Z}^2 + 14\,\mathbf{Z}\overline{\mathcal{Z}}\,\left(\mathbf{Z}+\overline{\mathcal{Z}}\right)\mathbf{T} + 27\left(\mathbf{Z}^2 + \frac{4}{3}\,\mathbf{Z}\overline{\mathcal{Z}} + \overline{\mathcal{Z}}^2\right)\mathbf{T}^2 + 54\left(\mathbf{Z}+\overline{\mathcal{Z}}\right)\mathbf{T}^3 + 27\,\mathbf{T}^4 = 0$$

*by the similitude ψ which sends $z=-1$ to O and $z=0$ to N. When I is outside $\psi(\Phi)$, it exists a triangle such that O is the circum-center, N is the Euler-center and I is one of the in/ex-centers. When I is inside this **exclusion curve**, no such triangle can be found.*

***Sketch of the proof**, more details in Guinand, 1984.* Consider the polynomial:

$$P(X) \doteq X^3 - X^2\left(\frac{3}{2} - \frac{2\,\sigma}{\rho}\right) + X\left(\frac{2\,\sigma\,(\sigma-\kappa)}{\rho^2} - \frac{3\,\sigma}{\rho} + \frac{3}{4}\right) - \frac{1}{8} + \frac{2\,\sigma\,\kappa}{\rho^2}$$

When $\rho = |OI_0|^2$, $\sigma = |NI_0|^2$, $\kappa = |ON|^2$, the roots of P are the three $\cos A$, etc. When a is changed into $-a$, I_0 becomes I_a and the roots of P become $\cos A$, $-\cos B$, $-\cos C$. This is easily checked using the Al-Kashi formula. (How to obtain P is another story, see Guinand).

Since all roots have to be real, we must have

$$0 \geq \Delta(P) = 16\,\frac{\sigma^2\,S^2(\rho,\sigma,\kappa)}{\rho^6}\,\left(32\,\sigma\,\kappa - 27\,\rho^2 + 40\,\sigma\,\rho - 16\,\sigma^2\right)$$

And then Φ is obtained by substituting

$$\rho = \left(\overline{\mathcal{Z}} + \mathbf{T}\right)(\mathbf{Z}+\mathbf{T})/\mathbf{T}^2 \;;\; \sigma = \mathbf{Z}\overline{\mathcal{Z}}/\mathbf{T}^2 \;;\; \kappa = 1$$

in the last factor. When $\Delta = 0$, the roots are $X=-1$ (simple) and $X = 5/4 - \sigma/\rho$ (double). □

Remark 13.19.7. Due to the relation $A+B+C=\pi$, the three cosines are related by

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\,\cos A\,\cos B\,\cos C.$$

In other words, we have $s_1^2 - 2\,s_2 + 2\,s_3 - 1 = 0$. And this can be checked for polynomial P. The Lemoine's transforms changes two signs, and the relation remains.

Exercise 13.19.8. Generate "a lot" of α, β, γ on the unit circle, and draw the corresponding

$$\psi^{-1}(I) = -\frac{(\alpha+\beta+\gamma)^2}{\alpha^2+\beta^2+\gamma^2}$$

(the green points). Superpose the graph of Φ, and draw the circle $[-1/3; +1]$.

13.20 Fuhrmann 4-8 circle

Definition 13.20.1. Fuhrmann circle has $[X_4,\, X_8]$ for diameter (see also ???). Center X(355), radius $R\,W_4 \div \sqrt{abc}$ where :

$$W_4 = \sqrt{a^3+b^3+c^3-(a^2b+a^2c+ab^2+ac^2+b^2c+bc^2)+3\,abc} \tag{13.14}$$

perspector not named (and not handy). Equation and column are :

$$\Gamma_{std} + \frac{2}{a+b+c}\,(x\,aS_a + y\,bS_b + z\,cS_c) = 0 \;;\; \left(\begin{array}{c} 2\,S_a\,a \\ 2\,S_b\,b \\ 2\,S_c\,c \\ b+a+c \end{array}\right)$$

Remark 13.20.2. The only named points of this circle are X(2) and X(4). Five inverse pairs are :

1	11	72	2475	3434
80	1837	3419	3448	3436

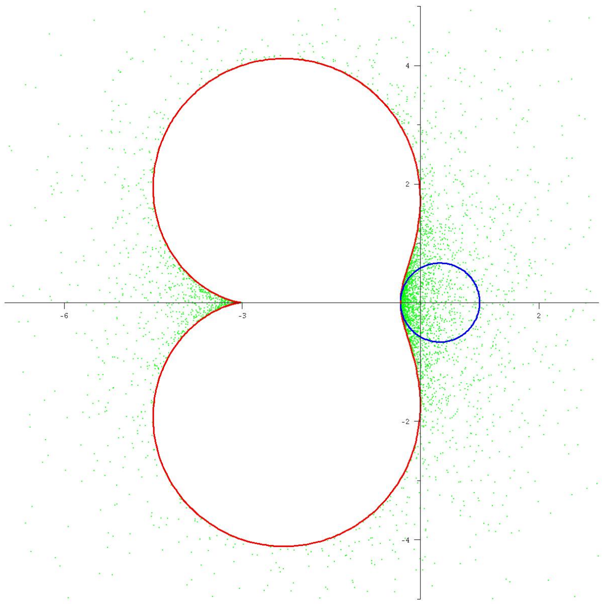


Figure 13.2: The exclusion curve

13.21 Taylor circle

Definition 13.21.1. Project the foot of any altitude onto the two other sidelines. The six points obtained are concyclic, defining the Taylor circle. Center X(389) with barycentrics

$$a^2\left(4\,S^2+S_b\,S_c\right)\left(2\,S_a^2-b^2c^2\right)-a^4b^2c^2S_a,\ \text{etc}$$

squared radius

$$\frac{4\,S^4}{a^2b^2c^2}+\frac{S_a^2S_b^2S_c^2}{16\,a^2b^2c^2S^2}=\frac{rad_H^4+4\,S^2}{16\,R^2}$$

where rad_H is the radius of the polar circle (so that $rad_H^4 \geq 0$), perspector not so simple (and not named). No named point on it. Equation and column are :

$$\Gamma_{std}+\frac{4\,S^2}{a^2b^2c^2}\left(x\,S_a^2+y\,S_b^2+z\,S_c^2\right)=0\ ;\ \begin{bmatrix}4S^2S_a^2\\4S^2S_b^2\\4S^2S_c^2\\a^2b^2c^2\end{bmatrix}\simeq\begin{bmatrix}S_a^2\\S_b^2\\S_c^2\\4R^2\end{bmatrix}$$

13.22 Kiepert RH and isosceles adjunctions

Definition 13.22.1. Kiepert RH is the rectangular hyperbola through A, B, C, G. It goes through X(4) (general RH property) ad also through 10,13,14,17,18... and circa 400 other ETC centers. Center is X(115), perspector X(523), points at infinity X(3413) and X(3414), i.e.

$$b^2-c^2:a^2-b^2\pm W_2:c^2-a^2\mp W_2$$

$$\text{where } W_2=\sqrt{a^4-a^2b^2-a^2c^2+b^4-b^2c^2+c^4}\qquad\text{is the Brocard radical}$$

Proposition 13.22.2. . *Chose angle ϕ and construct isosceles triangles $BA'C$, $CB'A$, $AC'B$ with basis angle $\angle(BC, BA')=\phi$ ($\phi<0$ when A' is outside). Then triangles ABC and $A'B'C'$ are perspective wrt a point $N(\phi)$:*

$$N(\phi)\quad\simeq\quad\frac{1}{2S\,\cot\phi-S_a},\ \text{etc}\simeq\frac{a}{\sin(A-\phi)},\ \text{etc}\tag{13.15}$$

and Kiepert RH is the locus of such points.

Proof. One has : $A'=a^2\tan\phi:2\,S-S_c\tan\phi:2\,S-S_b\tan\phi$. □

Remark 13.22.3. Triangle $A'B'C'$ is perspective from X(3) to the tangential triangle of ABC...

Proposition 13.22.4. *For a given K, the points $A'B'C'$are on the same cubic as the vertices, the inexcenters, orthocenter, circumcenter and the points $A'B'C'$ relative to the opposite value of K. This cubic can be written as $\left(K^2+1\right) K001 + \left(3-K^2\right) K003$ where $K001$ and $K003$ are the standardized equations of, respectively, the Neuberg and the McKay cubics, as given in Proposition 22.4.25 and Proposition 22.4.24. The pole is X(6), while pivot is $3\,vX(2) - K^2\, vX(20)$ i.e. : $\left(1+K^2\right) s_1\, s_3 : \left(3-K^2\right) s_3 : \left(1+K^2\right) s_2$.*

Proof. Details are given in Proposition 22.4.22. □

Remark 13.22.5. Points at infinity of Kiepert RH are parametrized by :

$$\cot\phi \quad = \quad \frac{-1}{3}\left(1+\frac{2\,W_2}{a^2+b^2+c^2}\right)\cot(\omega) = \frac{-1}{12S}\left(a^2+b^2+c^2+2\,W_2\right)$$

Remark 13.22.6. Fixed values of angle ϕ can be obtained by adding some regular shape to each side of the reference triangle. This created a race for the most inventive adjunction. Among them is the Pelle à Tarte, whose name was coined from a charade whose solution was "neon lamp, pie shovel" (i.e. lampe au néon, pelle à tarte, an approximation for Napoleon Bonaparte)

Example 13.22.7. In Figure 13.3, a Pelle à Tarte (pie shovel) like $AFUGC$ is made of a square $AFGC$ and an equilateral triangle FUG. It defines the angle $\phi = \left(\overbrace{AU,\, AC}\right) = -75°$. When the triangle is inside the square, we obtain a Pelle Pliée (folded shovel) like $AFWGC$ that defines angle $\phi = \left(\overbrace{AU,\, AC}\right) = -15°$.

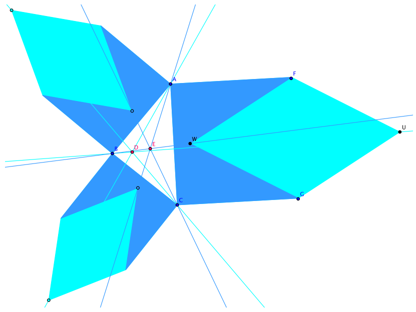


Figure 13.3: Cake server (Pelle a Tarte)

Example 13.22.8. Arbelos. Another idea to obtain some adding object is as follows. Divide $[AB]$ in a given ratio, obtaining D. Use the same ratio to obtain E, F. Draw the half circles and perpendicular DM, then the tangent circles. The triangle of the centers is perspective with ABC if and only if the ratio is $\left(\sqrt{5}-1\right)/2$. Perspector is X(2672). And belongs to Kiepert RH. Therefore, we have isosceles triangle. Value is $\pm\tan\phi = 3-\sqrt{5}$ (plus is inward, minus is outward).

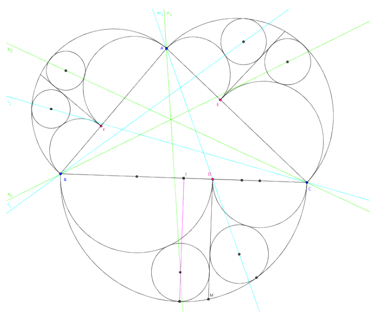


Figure 13.4: Arbelos configuration

-90°	-0°	-75°	-15°	-72°	-18°	-at3		-67.5°	-22.5°
4	2	3391	3366	1139	3370	1327	???	3387	3373
4	2	3367	3392	1140	3397	1328	???	3374	3388
+90°	+0°	+75°	+15°	+72°	+18°	+at3		+67.5°	+22.5°
H	G	Pelle à Tarte		pentagons		arctan 3			
-at2		-60°	-30°	-54°	-36°		-arb	-45°	
1131	3316	13	17	3381	???	???	2671	485	
1132	3317	14	18	???	3382	???	2672	486	
+at2		+60°	+30°	+54°	+36°		+arb	+45°	
		Fermat	Napol.			Arbelos		Vecten	
-2ω	$2\omega-\frac{\pi}{2}$	$-\omega$	$\omega-\frac{\pi}{2}$	$-\frac{1}{2}\omega$	$\frac{1}{2}\omega-\frac{\pi}{2}$	$-\frac{1}{2}\omega-\frac{\pi}{4}$			
3407	3399	83	262	1676	???	???			
1916	3406	76	98	???	1677	2010			
$+2\omega$	$\frac{\pi}{2}-2\omega$	$+\omega$	$\frac{\pi}{2}-\omega$	$+\frac{1}{2}\omega$	$\frac{\pi}{2}-\frac{1}{2}\omega$	$+\frac{1}{2}\omega+\frac{\pi}{4}$			
Gibert			Brocart	Lemoine circ		Galaty circ			

Table 13.4: Blocks related to Kiepert adjunctions

Example 13.22.9. When using the center of a regular triangle, square, pentagon, we have $\phi = 30°$, $\phi = 45°$, $\phi = 54°$; when using the farthest vertex (or the midpoint of the farthest side), $\phi = 60°$, $\phi = at2 \doteq \arctan 2$, $\phi = 72°$. We even have points relative to $at3 = \arctan 3$. Points are collected into 2×2 blocks corresponding to $-\phi$, $\phi - 90°$, $90° - \phi$, ϕ, see Table 13.4

Example 13.22.10. Brocard angle is often involved since :

$$P_\phi \simeq \frac{1}{(b^2+a^2+c^2)\cot(\phi) - S_a \cot(\omega)}, \text{ etc}$$

Proposition 13.22.11. *Let σ be a constant value. Then all lines $N(\phi)\,N(\sigma-\phi)$ are passing through point $T(\sigma)$ where :*

$$T(\sigma) \simeq \begin{pmatrix} a^2 S_a \\ b^2 S_b \\ c^2 S_c \end{pmatrix} + 2S\,\cot\sigma \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix}$$

Therefore all the $N(\phi)\,N(-\phi)$ lines (columns in a block) are passing through X(6), the Lemoine point. And all the $N(\phi)\,N\left(\frac{\pi}{2}-\phi\right)$ lines (rows in a block) are passing through X(3) the circumcenter. All the $T(\sigma)$ are on the Brocard axis X(3)X(6).

Proof. Obvious from (13.15). □

Proposition 13.22.12. *Let δ be a constant value. Then all lines $N(\phi)\,N(\phi+\delta)$ are tangent to a conic $D(\delta)$. When $\delta = 0$ this conic is the Kiepert RH itself. When $\delta = \pi/2$ (diagonals in a block), the conic reduces to a real point : the Euler center X(5). In the general case, $D(\delta) = 16S^2\left(\prod\left(a^2-b^2\right)\right)\cot^2(\delta)\;D(0) + D(\pi/2)$.*

Proof. This can be proved using the usual techniques : differentiate and wedge. When δ is rational wrt π, conics $D(\delta)$ and $D(0)$ are in a Poncelet configuration. □

Lemma 13.22.13. *Consider circle $\mathcal{C}_0 : x^2+y^2=1$ and points $U=(u,0)$, $V=(v,0)$ in the Cartesian plane. When $u+v \neq 0$, these points are the centers of homothety of circles $\mathcal{C}_0$ and $\mathcal{C}(P,\rho)$ where*

$$P = \left(\frac{2\,uv}{u+v}, 0\right),\ \rho = \left|\frac{u-v}{u+v}\right|$$

Proof. Consider $M=(0,+1)$ and $N=(0,-1)$. Their counterparts in circle $\mathcal{C}$ are obtained by the intersections $J = UM \cap VN$ and $K = UN \cap VM$. We have $P=(J+K)/2$ and $\rho = |J-K|/2$. □

Proposition 13.22.14. *A circle $\mathcal{C}(P,\rho)$ can be found such that points $U = N(\phi)$, $V = N(\phi + \pi/2)$ are the centers of homothety between $\mathcal{C}$ and the nine-points circle. We have :*

$$P \simeq \cot(2\phi) \begin{pmatrix} a^2 S_a \\ b^2 S_b \\ c^2 S_c \end{pmatrix} - 2S \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix} \cong \cot(2\phi)\ X_3 - 2S\,X_6$$

$$\rho = \frac{abc\sqrt{1+\cot^2(2\phi)}}{a^2+b^2+c^2-4S\cot(2\phi)}$$

Proof. These two points are aligned with E=X(5). Define $W = (U+V)/2$. From the above lemma, we have :

$$P = E + k(U-E)\ ;\ \rho = (1-k)\frac{R}{2} \qquad \text{where } k = \frac{V-E}{W-E}$$

Computations are greatly simplified when remarking that the results depends not really from $\cot\phi$ itself, but rather from $\cot\phi - 1/\cot\phi$. That is the reason why all these formulas are involving $\cot(2\phi)$. □

13.23 Cyclocevian conjugate

Definition 13.23.1. Two points are called cyclocevian conjugates when their cevian triangles share the same circumcircle. This definition has to be compared with cyclopedal conjugacy, see Section 9.3.

Proposition 13.23.2 (Terquem)**.** *Each point not on the sidelines has a cyclocevian conjugate. Its barycentrics are given by (Grinberg, 2003b) :*

$$U = cyclocevian(P) = (isot \circ anticomplement \circ isog \circ complement \circ isot)(P)$$

Proof. This proposition asserts that the other intersections of the circumcircle of $A_P B_P C_P$ with the sidelines are the vertices of another cevian triangle. As it should be, the formula is involutory. □

Example 13.23.3. Some examples :

point	code	bary	cycc	circumcenter
Gergonne	$X(7))$	$1/(-a+b+c)$	$X(7)$	$X(1)$
centroid	$X(2)$	1	$X(4)$	$X(5)$
orthocenter	$X(4)$	$1/\left(-a^2+b^2+c^2\right)$	$X(2)$	$X(5)$
Nagel	$X(8)$	$-a+b+c$	$X(189)$	$X(1158)$

Point X_7 is the only center that is invariant by *cyclocevian*. Three other points share this property, obtained by changing one of the a, b, c into its opposite in the barycentrics of X_7.

13.24 Mixtilinear circles

Definition 13.24.1. In triangle ABC, the circle γ tangent to the sidelines AB, AC and the circumcircle is called the A mixtilinear circle when γ_A is inside the circumcircle. The other circle tangent to these three lines is called the external mixtilinear circle and noted $\widehat{\gamma_A}$.

Proposition 13.24.2. *Using barycentrics or Lubin-2 lead to:*

$$\gamma_A \underset{b}{\simeq} \begin{bmatrix} 4b^2c^2 \\ c^2(a-b+c)^2 \\ b^2(a+b-c)^2 \\ (b+a+c)^2 \end{bmatrix} \ ;\ \gamma_A \underset{z}{\simeq} \left(\begin{bmatrix} (\alpha s_2 + s_3)^2 \\ \alpha^2(\beta-\gamma)^2 \\ (\alpha+s_1)^2 \end{bmatrix} \ ;\ \rho = \frac{2(s_1 s_2 - s_3)}{(\beta-\gamma)^2\alpha} \right)$$

The external mixtilinear circle is obtained using the A-Lemoine transform.

ϕ	U	V	P	ρ	*name*
0	2	4	3	R	*circum*
$\pi/12$	3392	3391	16	$\dfrac{2\sqrt{3}abc}{12\,S-\sqrt{3}\,(a^2+b^2+c^2)}$	
$-\pi/12$	3366	3367	15	$\dfrac{2\sqrt{3}abc}{12\,S+\sqrt{3}\,(a^2+b^2+c^2)}$	
$\pi/10$	3397	1139	3393		
$-\pi/10$	3370	1140	3379		
$\arctan(3)$	1328	?	?	$\dfrac{5\,abc}{16\,S+3\,(a^2+b^2+c^2)}$	
$-\arctan(3)$	1327	?	?	$\dfrac{5\,abc}{16\,S-3\,(a^2+b^2+c^2)}$	
$\pi/8$	3388	3387	372	$\dfrac{\sqrt{2}abc}{4\,S-(a^2+b^2+c^2)}$	
$-\pi/8$	3373	3374	371	$\dfrac{\sqrt{2}abc}{4\,S+(a^2+b^2+c^2)}$	
$\arctan(2)$	1132	3316	?	$\dfrac{5\,abc}{12\,S+4\,(a^2+b^2+c^2)}$	
$-\arctan(2)$	1131	3317	?	$\dfrac{5\,abc}{12\,S-4\,(a^2+b^2+c^2)}$	
$\pi/6$	18	13	62	$\dfrac{2\,abc}{4\,S-\sqrt{3}\,(a^2+b^2+c^2)}$	
$-\pi/6$	17	14	61	$\dfrac{2\,abc}{4\,S+\sqrt{3}\,(a^2+b^2+c^2)}$	
$\pi/5$	3382	3381	3395	$--$	
$-\pi/5$	?	?	3368	$--$	
$\arctan\left(3-\sqrt{5}\right)$	2672	?	?	$\dfrac{3\left(4\sqrt{5}+5\right)abc}{44\,S-2\left(9+5\sqrt{5}\right)(a^2+b^2+c^2)}$	
$-\arctan\left(3-\sqrt{5}\right)$	2671	?	?	$\dfrac{3\left(4\sqrt{5}+5\right)abc}{44\,S+2\left(9+5\sqrt{5}\right)(a^2+b^2+c^2)}$	
$\pi/4$	486	485	6	$\dfrac{abc}{a^2+b^2+c^2}$	2° *Lemoine*
$\operatorname{arccot}\dfrac{(a+b+c)^2}{4\,S}$	10	2051	970	$\dfrac{r^2+p^2}{4\,r}$	*Apollonius*
$-\operatorname{arccot}\dfrac{(a+b+c)^2}{4\,S}$	?	?	?		
$2\,\omega$	1916	3399	?	$\dfrac{\left(a^2b^2+a^2c^2+b^2c^2\right)R}{a^4+b^4+c^4+a^2c^2+a^2b^2+b^2c^2}$	
$-2\,\omega$	3407	3406	?		
ω	76	262	3095	R	
$-\omega$	83	98	3398	$\dfrac{\left(a^2b^2+a^2c^2+b^2c^2\right)R}{a^4+b^4+c^4+a^2c^2+a^2b^2+b^2c^2}$	
$\omega/2$	?	?	511	∞	*Linfty*
$-\omega/2$	1676	1677	182	$\dfrac{R}{2\,\cos(\omega)}$	1° *Lemoine*
$\pi/4+\omega/2$	2010	2009	39	$R\sin(\omega)$	*Gallatly*
$-\pi/4-\omega/2$	?	?	32	$\dfrac{abc\left(b^2+a^2+c^2\right)}{2\,\left(b^4+c^4+a^4\right)\cos(\omega)}$	

Table 13.5: Similicenters on Kiepert RH

Proof. Formulas are easy to verify... and then non-displayed ones are rather huge ! □

Construction 13.24.3. *The second intersection of line A-X(56) cuts again the circumcircle at* $A_1 \in \gamma_A$. *Line* OA_1 *cuts line* AI_0 *at* K_a,*the center of* γ_A. *Contacts* A_b, A_c *with the sidelines are aligned with* I. *Moreover* $A_b A_c \perp AI_0$.

Chapter 14

Pencils of Cycles in the Triangle Plane

14.1 Introductory remarks

14.1.1 How many points at infinity should be used ?

In the context of the (barycentric) Triangle Plane, points are described by projective columns living in $\mathbb{P}_{\mathbb{R}}(\mathbb{R}^3)$. In the same vein, the present chapter will describe the circles and their pencils by projective columns $\mathcal{V}$ living in $\mathbb{P}_{\mathbb{R}}(\mathbb{R}^4)$. Later, the Morley plane will be introduced, where points are described by columns living in $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^3)$ while, in Chapter 19, circles will be described by projective columns $\mathcal{V}$ living in $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^4)$.

Efficient notations are powerful, but poor notations can be confusing. Thus we will use $\underset{b}{\mathcal{V}}, \underset{c}{\mathcal{V}}, \underset{p}{\mathcal{V}}, \underset{s}{\mathcal{V}}$ to distinguish between barycentric, general Morley, Pedoe and spherical objects. When a formula is valid in all the four contexts, indices will be omitted.

Using such 4D spaces is not the most frequent method to describe the circles and their pencils. The tradition (Poncelet, 1822, 1865) is rather to use the Riemann sphere $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^2)$ where $(z_1, z_2) \simeq (\lambda z_1, \lambda z_2)$ for any non-zero $\lambda \in \mathbb{C}$. But Triangle Geometry deals with points and lines living in projective 3D-spaces, i.e. described as $x : y : z$ where $x : y : z \simeq kx : ky : kz$ for any non-zero multiplier k.

Obviously, the 2D and 4D points of view are reducing to the same elementary Cartesian coordinates when restricted to the finite domain. But they are conflicting *where they are the most useful*, i.e. where they are implementing the Poncelet's continuity principle for objects at infinity. This is even more true concerning the "circular points at infinity", the so-called umbilics of the plane.

An ordinary line must be completed in a way or another to become a "circle with infinite radius" and having a clear definition of this completion is required in order to unify the three concepts of circle ($0 < \rho < \infty$), point ($\rho = 0$) and line ($\rho = \infty$) into a single concept of cycle.

But the intuition of an "infinite sphere whose center is everywhere and its circumference nowhere"[1] (**?**, p. 3 [83)pascal:pensees has to be formulated in minute detail to become effective and fruitful.

In the Riemann sphere $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^2)$, there exists only one point at infinity (noted ∞). In this context, a "circle with infinite radius" is an ordinary line Δ completed with the point noted ∞, i.e. $\overline{\Delta} = \Delta \cup \{\infty\}$, while point-circles are either the set $\{\infty\}$ or a circle with radius 0 around an ordinary point.

In the Triangle Plane $\mathbb{P}_{\mathbb{R}}(\mathbb{R}^3)$, there exists a whole line $\mathcal{L}_b$ of points at infinity, verifying $x + y + z = 0$. In this document, barycentrics are used. Using trilinears would only change some formulas, but not the very nature of the underlying space. In this context, the barycentric equation of an ordinary circle leads to define a cycle as a second degree curve (a conic) that goes through the so-called umbilics $\Omega^{\pm}$. Therefore, the equation of a completed line becomes $(x + y + z)(ux + vy + wz) = 0$ so that we must define $\overline{\Delta}$ as $\Delta \cup \mathcal{L}_b$, while the role of $\{\infty\}$ in $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^2)$ is played now by the horizon circle $\mathcal{C}_\infty$ defined by $(x + y + z)^2 = 0$, i.e. defined as the object having the same points as $\mathcal{L}_b$ but each of them counted twice.

In Chapter 7, orthogonality of lines has been reduced to a polarity wrt operator $\boxed{\mathcal{M}_b}$. Here,

[1] une sphère infinie dont le centre est partout, la circonférence nulle part,

the same treatment will be applied to cycles, i.e. the family of all curves that are either a circle or a line. This leads to a fundamental quadric $\mathcal{Q}$ in a 4-D projective space $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$ [here] or $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^4\right)$ [some chapters later]. Finite points on the quadric can be interpreted as representatives of point-circles, quite the same thing as an ordinary point in the triangle plane. Points outside this quadric are representatives of cycles, while points inside are assigned to the later defined virtual circles.

When dealing with tangency of cycles, a better description would be secured by using oriented cycles, living in a Lie sphere, embedded into a 5D space obtained by a double coating of the ordinary 4D space of cycles. We haven't developped this concept here.

14.1.2 Umbilics

Lemma 14.1.1. *We have* $\boxed{\mathrm{OrtO}_b}\cdot\boxed{\mathrm{OrtO}_b}\cdot\boxed{W_b} = -\boxed{W_b}$. *Therefore, restricted to* $\overrightarrow{\mathcal{V}}$, $\boxed{\mathrm{OrtO}_b}^2$ *is nothing but an half turn. Multiplied by* $\boxed{\mathrm{Pyth}_b}$, *this leads to* $\boxed{\mathrm{OrtO}_b}^3 + \boxed{\mathrm{OrtO}_b} = 0$, *so that eigenvalues of* $\boxed{\mathrm{OrtO}_b}$ *are* $0, +\boldsymbol{i}, -\boldsymbol{i}$.

Notation 14.1.2. CAVEAT (2024-12-19). Among the two hands, the right hand is the hand where the thumb is on the left hand side. Notations $\Omega_x, \Omega_y, \Omega^+, \Omega^-$ have been used in this document. But a strict application of the rules may have been lacking. Here is the eternal rule ! $\Omega_y = \Omega^+ \underset{z}{\simeq} 1:0:0$. Let the heretics tremble with fear!

Definition 14.1.3. Umbilics. Let $U \in \mathcal{L}_b$ be a point at infinity, i.e. a point such that $u+v+w = 0$. The associated umbilics are the complex points :

$$\Omega^+ \simeq \left(\boldsymbol{i}\boxed{1} + \boxed{\mathrm{OrtO}}\right)\cdot U \quad ; \quad \Omega^- \simeq \left(\boldsymbol{i}\boxed{1} - \boxed{\mathrm{OrtO}}\right)\cdot U$$

$$\begin{aligned}\Omega^+ = \Omega_y &\simeq 4SX_{512} - \boldsymbol{i}\,X_{511} \\ &\simeq 4S\begin{pmatrix} a^2\left(b^2-c^2\right) \\ b^2\left(c^2-a^2\right) \\ c^2\left(a^2-b^2\right)\end{pmatrix} \pm \boldsymbol{i}\begin{pmatrix} a^2\left(a^2\left(b^2+c^2\right)-b^4-c^4\right) \\ b^2\left(b^2\left(c^2+a^2\right)-c^4-a^4\right) \\ c^2\left(c^2\left(a^2+b^2\right)-a^4-b^4\right)\end{pmatrix}\end{aligned} \tag{14.1}$$

where $\boldsymbol{i}$ is the imaginary unit and S the area of the triangle. Another choice is :

$$\Omega^+ = \Omega_y \simeq \begin{pmatrix} S_b + 2\,\boldsymbol{i}S \\ S_a - 2\,\boldsymbol{i}S \\ -c^2\end{pmatrix} \;;\; \Omega^- = \Omega_x \simeq \begin{pmatrix} S_b - 2\,\boldsymbol{i}S \\ S_a + 2\,\boldsymbol{i}S \\ -c^2\end{pmatrix} \tag{14.2}$$

These expressions are no more symmetric, but computations become easier.

Remark 14.1.4. Spoiler. Using Morley representation $z_A = \alpha$, etc, we obtain:

$$\Omega_y \underset{z}{\simeq} \begin{pmatrix} 1 \\ 0 \\ 0\end{pmatrix} \underset{b}{\simeq} \begin{pmatrix} \dfrac{1}{\gamma} - \dfrac{1}{\beta} \\ \dfrac{1}{\alpha} - \dfrac{1}{\gamma} \\ \dfrac{1}{\beta} - \dfrac{1}{\alpha}\end{pmatrix} \;;\; \Omega_x \underset{z}{\simeq} \begin{pmatrix} 0 \\ 0 \\ 1\end{pmatrix} \underset{b}{\simeq} \begin{pmatrix} \beta - \gamma \\ \gamma - \alpha \\ \alpha - \beta\end{pmatrix} \tag{14.3}$$

Proposition 14.1.5. *When seen as elements of* $\overrightarrow{\mathcal{V}}_{\mathbb{C}}$, *all* $\Omega^{\pm}$ *are eigenvectors of operator* $\boxed{\mathrm{OrtO}_b}$, *with eigenvalues (respectively)* $\pm\boldsymbol{i}$ *and therefore belong to the light cone. When seen as elements of* $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$, *points* Ω^+ *and* Ω^- *are now independent of the choice of* U, *and are the fixed points of the orthopoint transform. They both belong to the circumcircle, and are isogonal conjugates to each other.*

Proof. See Postnikov (1982, 1986) for better insights on real-complex spaces. Property $\mathcal{L}_b \cdot \Omega = 0$ is obvious. Umbilics are eigenvectors because of

$$\boxed{\mathrm{OrtO}}\cdot\Omega^+ = \boxed{\mathrm{OrtO}}\cdot\left(\boldsymbol{i}\boxed{1} + \boxed{\mathrm{OrtO}}\right)\cdot U$$

$$\boxed{\text{OrtO}} \cdot \Omega^{+} = \boxed{\text{OrtO}} \cdot \left(\boldsymbol{i}\boxed{1} + \boxed{\text{OrtO}}\right) \cdot U = \left(\boldsymbol{i}\boxed{\text{OrtO}} - \boxed{1}\right) \cdot U = +\boldsymbol{i} \times \left(\boldsymbol{i}\boxed{1} + \boxed{\text{OrtO}}\right) \cdot U$$

Since eigenvalues of $\boxed{\text{OrtO}}$ are simple, the $\mathbb{C}$-dimension of eigenspaces is one, and uniqueness in $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ follows. For this reason, points $\Omega^{\pm}$ are also called the "circular points at infinity". Finally, intersection of circumcircle and the infinity line must be invariant by isogonal conjugacy, while $\Omega^{+} \underset{b}{*} \Omega^{+}$ cannot be real, even up to a complex proportionality factor. □

Remark 14.1.6. And now, 2024-12-19, the "+" in Ω^{+} is linked to the "+" in the $\boxed{\text{OrtO}} \cdot \Omega^{+} = +\boldsymbol{i}\,\Omega^{+}$ formula. Whaow !

Proposition 14.1.7. *A circle is a conic that goes through the umbilics Samuel (1986, p. 53). Above all, choosing the umbilics is deciding which of the circum-ellipses is* ***the*** *circumcircle.*

Proof. By definition, umbilics are the (non real) points where the line at infinity intersects the circumcircle. By (13.3), these points belong to any circle. For the converse, consider the values taken by x^2, y^2, z^2, xy, yz, zx at A, B, C together with both umbilics. This gives a 5×6 matrix whose rank is 5 : the first three lines are $1, 0, 0, 0, 0, 0$ etc. and it remains only to show that rank of submatrix 4..5,4..6 is two. A direct inspection shows that critical factors are $a^2 + b^2 - c^2$ (straight angle, that can occur only once) and $a^4 + b^4 + c^4 - b^2c^2 - a^2b^2 - a^2c^2$ (condition of equilaterality). In such a case, the property remains when umbilics are written as $1 : j : j^2$ and $1 : j^2 : j$. □

Remark 14.1.8. When the umbilics are given, the euclidian structure of the Triangle Plane is known. From $\Omega^{+} \underset{b}{*} \Omega^{-} = a^2 : b^2 : c^2$, the $\boxed{\text{Pyth}_b}$ matrix is known (up to the value of R), while the orthopoint transform, and its matrix $\boxed{\text{OrtO}_b}$ is characterized by its diagonal shape, namely $(0, +\boldsymbol{i}, -\boldsymbol{i})$, when using the triple X(3), Ω^{+}, Ω^{-} as barycentric basis.

14.1.3 Notations

We have done our best effort to use unified notations. In this whole chapter,

Notation 14.1.9. Conventions about letters.

- P, Q — denote some flat true points in the Triangle Plane and X_n a Kimberling-named triangle center, all of them being 3-columns.
- Γ, Ω — Γ denotes the circumcircle of the fundamental triangle ABC (and nothing else) while Ω denotes a cycle, both of them being curves, i.e. a set of points. The 3×3-matrices describing their equations as a quadratic form are noted using a box.
- $U, \underset{b}{\mathcal{V}}$ — U denotes the representative of a circle-point, while $\underset{b}{\mathcal{V}}$ denotes the representative of any cycle, all of them being columns in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$.
- Y — denotes the representative of an oriented cycle in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^5\right)$
- G — denotes a Gram matrix. The elements of this matrix are noted w^2 along the diagonal and W for the non-diagonal elements.

Notation 14.1.10. To design a circle known by a pair center/radius, parentheses will be used, exemplified by $\Gamma = (X_3, R)$. Using parentheses around a single Roman letter –e.g. (P)– will be reserved to denote the circle $(P, 0)$ i.e. the circle whose unique real point is P itself.

14.2 Cycles and representatives

Definition 14.2.1. The (barycentric) Veronese map is the correspondence that maps a $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^3\right)$ column into a $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$ row proportional to:

$$^t\begin{pmatrix} x\,(x+y+z) \\ y\,(x+y+z) \\ z\,(x+y+z) \\ -a^2yz - b^2xz - c^2xy \end{pmatrix}$$

Proposition 14.2.2. *The Veronese map is obviously homogeneous. Both umbilics are send to* $[0,0,0,0]$ *(points of indeterminacy). The other points at infinity are send to* $[0,0,0,1]$. *Otherwise, the map is injective.*

Definition 14.2.3. Veronese map. For points at finite distance, we will use (7.14) and **define** $Ver\,(x,y,z)$ by the simpler formula:

$$Ver\,(x:y:z) \simeq [x,\ y,\ z,\ \Gamma_{std}\,(x,y,z)] \tag{14.4}$$

Remark 14.2.4. Requiring that four points are on the same circle leads to Proposition 13.1.4, i.e. to equation :

$$\det_{i=1}^{i=4} [p_i,\ q_i,\ r_i,\ \Gamma_{std}\,(p_i,\ q_i,\ r_i)] = 0$$

But, conversely, this equation only implies that our four points are on the same circle or on the same straight line. To summarize both situations under a single concept, we introduce :.

Definition 14.2.5. The **cycle** Ω associated with the **representative** $\underset{b}{\mathcal{V}} \simeq u:v:w:t \in \mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$ is Samuel (1988) the locus of the points $X \in \mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^3\right)$ that satisfy the equation :

$$\underset{b}{Ver}\,(X)\cdot(u:v:w:t) = 0 \tag{14.5}$$

For example, the representative of circumcircle Γ is $\underset{b}{\mathcal{V}}_{\Gamma} \simeq 0:0:0:1$.

Remark 14.2.6. When $t \neq 0$, cycle Ω is the (ordinary) circle whose standardized equation is :

$$\frac{u}{t}\,x + \frac{v}{t}\,y + \frac{w}{t}\,z + \Gamma_{std}\,(x,y,z) \tag{14.6}$$

Remark 14.2.7. The representative of a circle is seen as a 3D point (described by a column). The Veronese of a 2D point is an action over the 3D points: i.e. a plane, described by a row.

Definition 14.2.8. Cycle $\mathcal{C}_\infty$ represented by $\underset{b}{\mathcal{V}}_{\infty} \simeq 1:1:1:0$ has to be understood as the line at infinity $\mathcal{L}_b$ described twice, and will be called the **horizon circle**. This object has to be perceived as a circle "whose center is everywhere and circumference nowhere" (Empedocles).

The representative itself, i.e. the point $\underset{b}{\mathcal{V}}_{\infty} \simeq 1:1:1:0$, will be called Sirius, following Kimberling (1998-2024) in using stars to coin the name given to a point. While using that specific star for a very distant point is from (Voltaire, 1752).

Definition 14.2.9. Otherwise, the cycle represented by $u:v:w:0$ is the union of an ordinary line and $\mathcal{L}_b$, and will be called a **completed line**.

Theorem 14.2.10. *The* ***representative of the point-circle*** (P) *associated with a point at finite distance* $P \simeq p:q:r$ *is the column given by:*

$$U_P \simeq \boxed{\underset{b}{\mathcal{Q}}^{-1}}\cdot {}^tVer\,(P) \qquad \text{where}\ \boxed{\underset{b}{\mathcal{Q}}^{-1}} = \begin{pmatrix} 0 & c^2 & b^2 & 1 \\ c^2 & 0 & a^2 & 1 \\ b^2 & a^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \tag{14.7}$$

$$U_P = c^2q^2 + b^2r^2 + 2\,S_a qr : c^2p^2 + a^2r^2 + 2\,S_b pr : b^2p^2 + a^2q^2 + 2\,S_c pq : (p+q+r)^2 \tag{14.8}$$

Conversely, we have:

$$Ver\,(P) \simeq {}^tU_P\cdot\boxed{\underset{b}{\mathcal{Q}}} \qquad \text{where}\ \boxed{\mathcal{Q}} = \frac{-1}{8S^2}\begin{bmatrix} a^2 & -S_c & -S_b & -a^2S_a \\ -S_c & b^2 & -S_a & -b^2S_b \\ -S_b & -S_a & c^2 & -c^2S_c \\ -a^2S_a & -b^2S_b & -c^2S_c & a^2b^2c^2 \end{bmatrix} \tag{14.9}$$

Proof. Direct computation. Mind the fact that both formulas are hard equalities, and that, when both matrices are written exactly that way, the obvious result $\boxed{\underset{b}{\mathcal{Q}}}\cdot\boxed{\underset{b}{\mathcal{Q}}^{-1}} = +1$ is enforced ! □

Remark 14.2.11. It must be taken into account that radiuses that are not "up to a proportionality factor". The values of matrix $\boxed{\underset{b}{\mathcal{Q}}}$ and $\boxed{\underset{b}{\mathcal{Q}}^{-1}}$ were chosen to obtain the best looking formula at (14.10) and the normalized Minkowski formula (14.12)

$$^t\underset{b}{\mathcal{V}_1} \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot \underset{b}{\mathcal{V}_2} = d^2 - r_1^2 - r_2^2$$

The price to pay is the appearence of a -2 factor when computing radiuses, at the general formula (14.15), and at the orthogonal formula (14.18). Be prudent, don't over-simplify !

Remark 14.2.12. Sometimes, "the representative of the point-circle P" will be shortened into "the representative of P". This object (a column) is not to be confused with the Veronese image of P (a row) !

Example 14.2.13. Here are some point representatives :

$P \setminus x$	u	v	w	t
$1:0:0$	0	c^2	b^2	1
$0:1:1$	$2\,b^2+2\,c^2-a^2$	a^2	a^2	4
$X(1)$	$bc\,(b+c-a)$	$ac\,(c+a-b)$	$ab\,(b+a-c)$	$a+b+c$
$X(2)$	$2\,b^2+2\,c^2-a^2$	$2\,a^2+2\,c^2-b^2$	$2\,b^2+2\,a^2-c^2$	9
$X(3)$	R^2	R^2	R^2	1
$X(4)$	$R^2a^2\left(b^2+c^2-a^2\right)^2$	$R^2b^2\left(c^2+a^2-b^2\right)^2$	$R^2c^2\left(a^2+b^2-c^2\right)^2$	$a^2b^2c^2$
$X(6)$	$b^2c^2\left(2\,b^2+2\,c^2-a^2\right)$	$a^2c^2\left(2\,c^2+2\,a^2-b^2\right)$	$a^2b^2\left(2\,a^2+2\,b^2-c^2\right)$	$\left(a^2+b^2+c^2\right)^2$
umb	0	0	0	0
∞	1	1	1	0

The fact that formula (14.8) *would* give $Sirius \simeq 1:1:1:0$ for each point on $\mathcal{L}_b$ is the reason of their exclusion from the definition of the point representatives.

Corollary 14.2.14. *The **representative of the circle** (P, ω) where $P \simeq p:q:r$ is a point at finite distance is obtained by :*

$$\boxed{\underset{b}{\mathcal{Q}}^{-1}} \cdot {}^tVer\left(\frac{P}{p+q+r}\right) - \omega^2 Sirius \tag{14.10}$$

where Sirius is exactly $^t[1,1,1,0]$ *and* $\boxed{\underset{b}{\mathcal{Q}}^{-1}}$ *is exactly as in* (14.7).

Proof. Obvious from the above theorem. □

Example 14.2.15. Compute the representative of the incircle. Two equivalent methods are:

```
mQQI.(Tr@FActor@Ver@norb@vX)(1)-ri^2*Sirius:
method1:= (FActor@subs)(kitcircleS, valS6, %);
method2:= (nor4@wedge3)(seq(Ver(j), j=Column(matcev(vX(7)), 1..3)));
```

$$incircle = \frac{1}{4}\left((a-b-c)^2 : (b-c-a)^2 : (c-a-b)^2 : 4\right)$$

Remark 14.2.16. Assuming that representatives are living in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$ has many advantages. The top one could be to enforce the fact that a representative is not a point in the Triangle Plane. It is a key fact that the triple $[\widehat{u}, \widehat{v}, \widehat{w}]$ appearing in the standardized equation (14.6) is **not** defined up to a proportionality factor. The same remark applies to the so-called "circle function" $[\widehat{u} \div bc, \widehat{v} \div ca, \widehat{w} \div ab] \in \mathbb{R}^3$ that appears when using trilinears as in Weisstein (1999-2009).

14.3 Fundamental quadric and orthogonality

Theorem 14.3.1. *Any point representative $U = u : v : w : t$ belongs to the quadric $\mathcal{Q}$:*

$$ {}^{t}U \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot U = 0 \quad = \quad 0 \tag{14.11}$$

where $\boxed{\underset{b}{\mathcal{Q}}}$ is as given in (14.9).

Proof. One has $Ver\,(P)\,.\boxed{\underset{b}{\mathcal{Q}^{-1}}} \cdot {}^{t}Ver\,(P) = 0$ by the very definition of $Ver\,(P)$. □

Proposition 14.3.2. *Object Sirius $\simeq 1 : 1 : 1 : 0$ is the only (real) point at infinity of the quadric $\mathcal{Q}$. Therefore, $\mathcal{Q}$ is a paraboloid.*

Proof. Substitute $t = 0$, then compute the discriminant with respect to w and obtain $-(u - v)^2 a^2 b^2 c^2 / R^2$. This requires $u = v$, etc. □

Remark 14.3.3. In the usual $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^2\right)$ model, $\mathcal{L}_b$ is "in the South plane" while the horizon circle $\mathcal{C}_\infty$ is nothing but the point-circle $\{\infty\}$.

Proposition 14.3.4. *An element $\underset{b}{\mathcal{V}} = u : v : w : t$ of $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$ is the representative of a (real) cycle if only if $\underset{b}{\mathcal{V}}$ is outside of $\mathcal{Q}$ (i.e. on the same side as $0 : 0 : 0 : 1$ characterized by ${}^{t}(u : v : w : t) \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot (u : v : w : t) \geq 0$ when (14.11) is used.*

Proof. Obvious from (14.10), that states that representative of (P, ω) is "below" the representative of (P), while representatives of completed lines are at infinity in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$ and therefore outside of paraboloid $\mathcal{Q}$. □

Theorem 14.3.5. ***Orthogonal cycles.*** *Consider two cycles Ω_1, Ω_2 with representatives $\underset{b}{\mathcal{V}}_1, \underset{b}{\mathcal{V}}_2$. When $\underset{b}{\mathcal{V}}_2$ belongs to the polar plane of point $\underset{b}{\mathcal{V}}_1$ wrt the fundamental quadric then cycles Ω_1 and Ω_2 are orthogonal –and conversely.*

Computed Proof. Begin with two circles. Write representative $\underset{b}{\mathcal{V}}_j$ as in (14.10) from representative U_j of point-circle (P_j).This implies that $\underset{b}{\mathcal{V}}_j[4] = 1$. Compute ${}^{t}\underset{b}{\mathcal{V}}_1 \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot \underset{b}{\mathcal{V}}_2$ and –using (7.12)– obtain :

$$ {}^{t}\underset{b}{\mathcal{V}}_1 \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot \underset{b}{\mathcal{V}}_2 = \left(|P_1 P_2|^2 - \omega_1^2 - \omega_2^2 \right) \tag{14.12}$$

Compute now ${}^{t}\underset{b}{\mathcal{V}}_1 \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot \underset{b}{\mathcal{V}}_3$ where $\underset{b}{\mathcal{V}}_3 = u_3 : v_3 : w_3 : 0$ and obtain :

$$ {}^{t}\underset{b}{\mathcal{V}}_1 \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot \underset{b}{\mathcal{V}}_3 = (p_1 u_3 + q_1 v_3 + r_1 w_3) \div (p_1 + q_1 + r_1) \tag{14.13}$$

In both cases, the result is the orthogonality condition times a non vanishing factor. Finally, when the representatives of two lines are involved, the conclusion follows directly from the properties of the orthopoint transform. □

Remark 14.3.6. The elementary formula ${}^{t}\underset{b}{\mathcal{V}}_1 \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot \underset{b}{\mathcal{V}}_2 = d^2 - r_1^2 - r_2^2$ has to be enforced at any cost. But this implies that ${}^{t}\underset{b}{\mathcal{V}} \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot \underset{b}{\mathcal{V}} = -2r^2$, and we have to live with this disgracious -2 factor when computing the radius of a circle.

Corollary 14.3.7. *The locus of the representatives of the points of a given cycle Ω is the intersection between $\mathcal{Q}$ and the polar plane –wrt $\mathcal{Q}$– of the representative of Ω.*

Proof. By definition, point P belongs to cycle Ω if and only if Ω and (P) are orthogonal. □

Theorem 14.3.8. ***Back to barycentrics.*** *Let* $\underset{b}{\mathcal{V}} = u : v : w : t \in \mathbb{P}_{\mathbb{R}}(\mathbb{R}^4)$ *be a representative. Then either*
(1) $\underset{b}{\mathcal{V}} \simeq 1:1:1:0$. *Then* $\underset{b}{\mathcal{V}}$ *is* Sirius, *i.e. represents the horizon circle*
(2) $t = 0$, *but* $\underset{b}{\mathcal{V}} \neq$ Sirius. *Then* $\underset{b}{\mathcal{V}}$ *represents a line.*
(3) $t \neq 0$. *Then* $\underset{b}{\mathcal{V}}$ *represents a circle (may be reduced to a point). The associated center and squared radius are given by :*

$$(p:q:r) \simeq \left(\boxed{\mathcal{Q}} \cdot \underset{b}{\mathcal{V}}\right)_{1..3} \tag{14.14}$$

$$\omega^2 = \left(\frac{-1}{2}\right) \frac{{}^t\underset{b}{\mathcal{V}} \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot \underset{b}{\mathcal{V}}}{t^2} \tag{14.15}$$

Moreover, the representative of the center (as a point-circle) is $U = \underset{b}{\mathcal{V}}/t + \omega^2\,\text{Sirius} \in \mathcal{Q}$.

Proof. The radius formula is a corollary of the preceding theorem. Let $\underset{b}{\mathcal{V}} = x : y : z : \tau$ be any cycle representative and $U \in \mathcal{Q}$ be the representative of point $P = p : q : r$. Then ${}^tW \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot U = 0$ implies

$$xp + yq + zr + \tau\,\Gamma_{std}\,(p, q, r) = 0$$

where $\Gamma_{std}\,(p, q, r) = -\left(a^2qr + b^2rp + c^2pq\right) \div (p + q + r)$, so that equation (14.14) must hold for rank reason (and can be checked directly). Conversely, starting from any $\underset{b}{\mathcal{V}}$ and applying (14.14) and then (14.8) leads back to $\underset{b}{\mathcal{V}}$. □

Example 14.3.9. Re obtain the radius of the incircle.

```
factor(Tr(incircle).mQQ.incircle);  subs(kitcircleS, valS6, %/ri^2); ↦
1
```

Proposition 14.3.10. ***Power of a point wrt a circle.*** *When* $\mathcal{W}$ *is the Veronese of a point* M *and* $\underset{b}{\mathcal{V}}$ *is the representative of circle* $(P,\ \omega)$*, then*

$$power\,(M,\ (P,\omega)) \doteq |PM|^2 - \omega^2 = \frac{\mathcal{W}}{\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3} \cdot \frac{\underset{b}{\mathcal{V}}}{\underset{b}{\mathcal{V}}_4} \tag{14.16}$$

This requires that $\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 \neq 0$, *i.e.* $M \notin \mathcal{L}_b$ *and that* $\underset{b}{\mathcal{V}}_4 \neq 0$, *i.e. that* (P, ω) *is a true circle (and not a completed line).*

Proof. Obvious from definitions. Mind the normalizations ! □

Exercise 14.3.11. Compute the power of X(3) wrt the circumcircle (and obtain $-R^2$). Compute the power of X(4) wrt the same circle. This formula enforces the fact that X(4) is inside the circle when triangle is acute.

Proposition 14.3.12. *The angle of two circles is defined as:*

$$\cos\left(\Omega_1,\ \Omega_2\right) = \frac{-{}^t\underset{b}{\mathcal{V}}_1 \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot \underset{b}{\mathcal{V}}_2}{\sqrt{{}^t\underset{b}{\mathcal{V}}_1 \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot \underset{b}{\mathcal{V}}_1}\sqrt{{}^t\underset{b}{\mathcal{V}}_2 \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot \underset{b}{\mathcal{V}}_2}} \tag{14.17}$$

Proof. When the circles intersect at a visible point M, this value is nothing but the Al-Kashi formula applied to triangle $M\omega_1\omega_2$. In any case, this formula is homogeneous wrt each of $\boxed{\underset{b}{\mathcal{Q}}}$, $\underset{b}{\mathcal{V}}_1$ and $\underset{b}{\mathcal{V}}_2$. □

Theorem 14.3.13. ***Common orthogonal cycle.*** *Let be given three cycles $\Omega_1, \Omega_2, \Omega_3$. If they don't belong to the same pencil, the bundle they generate is exactly the set of all the cycles orthogonal to a fixed cycle $\Omega_\perp$. We have the formulas (see (14.7), (14.9) for the precise values of the matrices):*

$$W \doteq \bigwedge_3 \left(\underset{b}{\mathcal{V}}_1, \underset{b}{\mathcal{V}}_2, \underset{b}{\mathcal{V}}_3 \right) \quad \text{(a 4-sized row)} \tag{14.18}$$

$$\underset{b}{\mathcal{V}}_\perp = \boxed{\underset{b}{\mathcal{Q}}^{-1}} \cdot {}^t W$$

$$\text{center} = W_1 : W_2 : W_3 \tag{14.19}$$

$$\text{squared radius} = \left(\frac{-1}{2}\right) \frac{{}^t\underset{b}{\mathcal{V}}_\perp \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot \underset{b}{\mathcal{V}}_\perp}{\left(\underset{b}{\mathcal{V}}_\perp [4]\right)^2} = \left(\frac{-1}{2}\right) \frac{W \cdot \underset{b}{\mathcal{V}}_\perp}{\left(\underset{b}{\mathcal{V}}_\perp [4]\right)^2}$$

Proof. Let $j = 1, 2, 3$. By definition, $W \cdot \underset{b}{\mathcal{V}}_j = 0$, so that ${}^t\underset{b}{\mathcal{V}}_j \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot \left(\boxed{\underset{b}{\mathcal{Q}}}^{-1} \cdot {}^t W \right)$ vanishes. Then the center follows by (14.14) and the radius as well. □

Remark 14.3.14. Mind the normalizing factor in the formula just above (see Remark 14.2.11). Don't simplify anything in the formula giving the radius ! Remember that

$${}^t\underset{b}{\mathcal{V}}_1 \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot \underset{b}{\mathcal{V}}_2 = \left(d_{12}^2 - r_1^2 - r_2^2\right) \times \left(\underset{b}{\mathcal{V}}_1 [4] \ \underset{b}{\mathcal{V}}_2 [4] \right)$$

Example 14.3.15. Compute the radius of the Euler circle (useless factor K, means whatever).

```
seq(mQQI.Tr(Ver(j)), j= Column(matcev(vX(2)),1..3) ):;
   tmp1:= K*(wedge3)(%):;
   tmp2:= FActor(mQQI.Tr(tmp1)):  # (Tr@reduce)(tmp1[1..3]); ency(%);
   methode1:= (-1/2)* tmp1.tmp2/tmp2[4]^2:  subs(rapbpc, factor(%));
   methode2:= (-1/2)* Tr(tmp2).mQQ.tmp2/tmp2[4]^2:  subs(kitRcircle, factor(%));
```

And obtain $R^2/4$ by each method.

Definition 14.3.16. Radical center. The ever visible point $W_1 : W_2 : W_3$ in the former theorem is called the radical center of the three cycles. Having the same power wrt all the cycles of the bundle is a characteristic property of this point.

Remark 14.3.17. As emphasized later, the nature of the radius of Ω_4, i.e. real, zero or imaginary fixes the nature of the bundle defined by $\Omega_1, \Omega_2, \Omega_3$.

Example 14.3.18. The example of the three excircles is examined in Subsection 14.11.4. Center is X(10) and radius is

$$\omega_4 = \sqrt{\frac{b^2c + ab^2 + bc^2 + a^2b + ac^2 + a^2c + acb}{4\,(a+b+c)}}$$

14.4 Pencils of cycles

Definition 14.4.1. Pencil. When Ω_1, Ω_2 are distinct cycles (with non proportional representatives), all curves $\lambda_1\Omega_1 + \lambda_2\Omega_2 = 0$, where $(\lambda_1, \lambda_2) \neq (0, 0)$, are cycles and the set of all these cycles is called the pencil generated by Ω_1, Ω_2. It is clear that representatives of the cycles of a given pencil are on the same projective line in $\mathbb{P}_\mathbb{R}(\mathbb{R}^4)$ –called the representative of the pencil.

Example 14.4.2. Formula (14.6) describes circle Ω as a member of the pencil generated by the circumcircle and a completed line. Therefore, the ordinary line $ux+vy+wz=0$ is the radical axis Δ of both circles Ω and Γ. That's another way to see that knowing $u:v:w$ is not enough to determine a circle.

Example 14.4.3. Formula (14.10), i.e. $\Omega\left(P,\omega\right)=\boxed{\mathcal{Q}_{b}}^{-1}\cdot{}^{t}Ver\left(\dfrac{P}{p+q+r}\right)-\omega^{2}\mathcal{C}_{\infty}$, describes the circle (P,ω) as a member of the pencil generated by the point-circle (P) and the horizon circle, i.e. the pencil of all circles concentric with $(P,0)$.

Remark 14.4.4. Here again, the triple $u:v:w$ is not sufficient to specify P, and $[u;v;w;t]$ must be used. It can be checked that representative is well specified, i.e. doesn't depends on whichever triple (kp,kq,kr) is chosen as barycentrics of point P.

14.5 Classification of pencils

Theorem 14.5.1 (Classification)**.** *Pencils of cycles fall in three classes, depending on the way their representative line $\mathcal{P}$ intersects –in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^{4}\right)$– the fundamental quadric $\mathcal{Q}$.*

$\mathcal{Q},\mathcal{P}$ tangent : $\mathcal{P}$ is the **tangent pencil** *of all the cycles containing a given point ω_0 and tangent at ω_0 to a line Δ_1 containing ω_0. Archetype : $\omega_0=\infty$ and $\mathcal{P}$ is "all the lines parallel to a given line Δ_1".*

$\mathcal{Q},\mathcal{P}$ secant : $\mathcal{P}$ is the **isotomic pencil** *generated by two different point-circles $\{\omega_1\}$ and $\{\omega_2\}$ (ω_i are the **limit** points of $\mathcal{P}$). Archetype : $\omega_2=\infty$ and $\mathcal{P}$ is, apart from $\{\infty\}$, "all the circles centered at a finite point ω_1".*

$\mathcal{Q},\mathcal{P}$ disjoint : $\mathcal{P}$ is the **isoptic pencil** *of all the cycles going through two different points ω_1 and ω_2 (the **base** points). Archetype : $\omega_2=\infty$ and P is "all the lines through a finite point ω_1".*

When $\mathcal{P}$ is a tangent pencil, so is $\mathcal{P}^{\perp}$ (using ω_0 and $\Delta_1^{\perp}$ orthogonal to Δ_1 at ω_0). When $\mathcal{P}$ is isoptic$\left(\omega_1,\omega_2\right)$ then $\mathcal{P}^{\perp}$ is isotomic$\left(\omega_1,\omega_2\right)$ and conversely. In all cases, representative of $\mathcal{P}$ and $\mathcal{P}^{\perp}$ are conjugate lines wrt $\mathcal{Q}$.

Proof. Everything goes as in (Pedoe, 1970) –using another paraboloid– or (Douillet, 2009) –using a sphere. The only striking thing is that the usual point at infinity of the complex plane, namely $\infty\in\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^{2}\right)$, has to be replaced by the horizon circle $\mathcal{C}_{\infty}\ :\ \left(x+y+z\right)^{2}=0$. □

Proposition 14.5.2. *A pencil of cycles that contains two lines is a pencil of lines. A concentric pencil contains the horizon cycle. All other pencils (i.e. all the non archetypal pencils) contain exactly one straight line (the radical axis of the pencil).*

Proof. The representative of $\mathcal{P}$ ever intersects the plane at infinity of $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^{4}\right)$. □

Proposition 14.5.3. *Let $P=p:q:r$ be a point in the Triangle Plane. Define its shadow in the Triangle Plane as point $S=u:v:w$ where u,v,w are defined in* (14.8)*. Then S is not outside the inconic $IC(X_{76})$. Any point on the border of $IC\left(X_{76}\right)$ is the shadow of exactly one point on the circumcircle, while a point inside $IC\left(X_{76}\right)$ –except from X_2– is the shadow of exactly two points. Moreover, these points are inverse in the circumcircle.*

Remark 14.5.4. Figure 14.1 shows the shadows of all the named points in ETC (Kimberling, 1998-2024), using the standard values $a=6$, $b=9$, $c=13$. One can see two lines of points : $L\left(X_2,\,X_6\right)$ and $L\left(X_2,\,X_{39}\right)$ containing the shadows of points from $L\left(X_3,\,X_2\right)$ –Euler line– and $L\left(X_3,\,X_6\right)$ –Brocard axis– respectively.

Proof of Proposition 14.5.3 . The locus of representatives of the points P_0 that belongs to Γ is the intersection of quadric $\mathcal{Q}$ and the polar plane $\prod$ of $0:0:0:1$, namely :

$$ua^{2}\left(b^{2}+c^{2}-a^{2}\right)+vb^{2}\left(c^{2}+a^{2}-b^{2}\right)+wc^{2}\left(a^{2}+b^{2}-c^{2}\right)-2\,ta^{2}b^{2}c^{2}=0$$

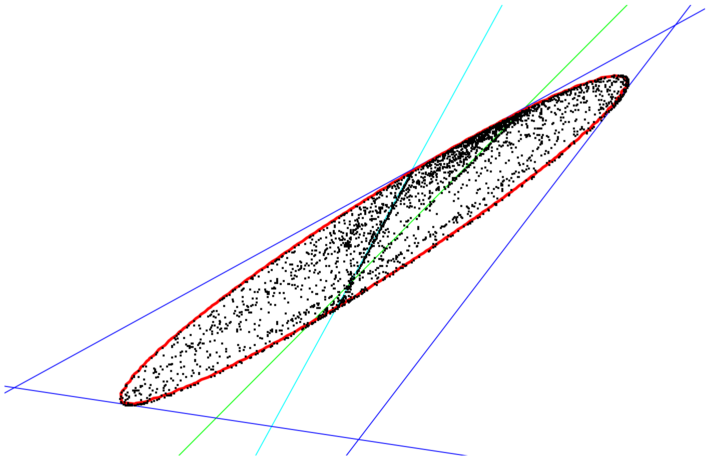


Figure 14.1: No point-shadow fall outside of the IC(X76) inconic

Extracting t and substituting in $\mathcal{Q}$ leads (apart from a constant non-zero factor) to :

$$u^2a^4 + b^4v^2 + c^4w^2 - 2\,uva^2b^2 - 2\,vwb^2c^2 - 2\,wuc^2a^2 = 0$$

i.e. the equation of $IC\,(X_{76})$.

When two points P_1, P_2 in the Triangle Plane share the same shadow, then points U_1, U_2 and $0:0:0:1$ are collinear in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$ so that cycles (P_1), (P_2) and Γ belongs to the same pencil. Therefore P_1, P_2 are inverse in the circumcircle. Moreover $U_0 \doteq U_1U_2 \cap \prod$ is inside $IC\,(X_{76})$ –otherwise U_0 would be the representative of a real circle belonging to pencil $(P_1)\,,(P_2)$ and orthogonal to Γ. □

Proposition 14.5.5. *Points inside of $\mathcal{Q}$ are representative of virtual circles (real center, imaginary radius). The reason to imagine such circles is that inversion in such a circle is a real transform. Moreover a real cycle Ω is orthogonal to $(X,\,i\omega)$ when Ω cuts $(X,\,\omega)$ along a diameter.*

Proof. Straightforward computation. □

Proposition 14.5.6. *Formally, the isoptic (ω_1, ω_2) pencil is also the isotomic (ω_3, ω_4) pencil where $\omega_3 + \omega_4 = \omega_1 + \omega_2$ and $\omega_3 - \omega_4 = i\ \ (\omega_1 - \omega_2)^{\perp}$ (ω_1, ω_2 are supposed to be at finite distance and normalized , while orthopoint is obtained using $\boxed{\text{OrtO}}$).*

Proof. Denote the common middle by ω_0 and use Pythagoras theorem to compute $|\omega_1\omega_3|^2$. One has $|\omega_1\omega_3|^2 = |\omega_0\omega_1|^2 + |\omega_0\omega_3|^2 = 0$ and thus both $\{\omega_3\}$,$\{\omega_4\}$ are orthogonal to $\{\omega_1\}$,$\{\omega_2\}$. Using $\boxed{\text{OrtO}}$, i.e. a rotation acting over $\overrightarrow{\mathcal{V}}$. □

14.6 Quadrimatrix of a pencil

Proposition 14.6.1. ***Orthogonality formula*** *(Spoiler). Let $\underset{b}{\mathcal{V}}_j$, $j = 1..4$ be the representatives of four cycles $\mathcal{C}_j$. Describe the pencil generated by $\mathcal{C}_1, \mathcal{C}_2$ using the matrix $\boxed{\Delta_{12}} = \left(\underset{b}{\mathcal{V}_1} \underset{6}{\wedge} \underset{b}{\mathcal{V}_2}\right)$, and the pencil generated by $\mathcal{C}_3, \mathcal{C}_4$ using the matrix $\boxed{\Delta_{34}} = \left(\underset{b}{\mathcal{V}_3} \underset{6}{\wedge} \underset{b}{\mathcal{V}_4}\right)$. When each pencil is orthogonal to the other, then*

$$\boxed{\Delta_{34}} \simeq \boxed{\underset{b}{\mathcal{Q}}} \cdot \boxed{\Delta_{12}^{*}} \cdot \boxed{\underset{b}{\mathcal{Q}}} \tag{14.20}$$

Proof. Cut Δ_{12} by the four base hyperplanes. Among the four expressions obtained, at most two are $0:0:0:0$. Do the same with $\boxed{\Delta_{34}}$, and assert that all the 16 orthogonality relations are fulfilled. Use `eliminate` to solve in C_x, C_y, C_z. Use `eliminate` onto the three shortest remaining equations and solve in F_x, F_y, F_z (up to a common multiplier). Use the Klein's relation ${}^t\Delta_{12} \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot \Delta_{12} = E_xB_x + E_yB_y + E_zB_z = 0$ to simplify the results, obtain first degree formulas and check the compliance with our claim. This is for the necessity. Sufficiency is easily checked by direct examination (if you don't trust the Maple's `eliminate`). One can also check that the process is involutive, as it should be. □

Exercise 14.6.2. Check this formula using the following circles

$$\mathcal{C}_1 \doteq \mathcal{C}\left(A\ ;\ 0:b:+c\ ;\ 0:b:-c\right)\ ;\ \mathcal{C}_2 \doteq \mathcal{C}\left(B\ ;\ a:0:+c\ ;\ a:0:-c\right)$$

The second pencil is supposed to contain both the circumcircle and the 3-6 Brocard circle. See Section 14.10 for more details.

14.7 Apexes

Definition 14.7.1. We define the (barycentric) **apex** of a point, or of a cycle, as the columns:

$$\begin{aligned} \mathcal{A}(M) &\doteq {}^{t}\underset{b}{Ver}(M) \\ \mathcal{A}(\mathcal{C}) &\doteq \boxed{\underset{b}{\mathcal{Q}}}\cdot \underset{b}{\mathcal{V}}(\mathcal{C}) \end{aligned}$$

With this definition, the apex of a point is the same as the apex of the null-radius circle centered at this point.

Maple 14.7.2. One can check these assertions by:

```
mQQ. bar2colu(vp,0), (Tr@Ver)(vp):  subs(sapbpc, zipd(%));
```

Proposition 14.7.3. *The apex of a line lies on the "South plane"* $[1:1:1:0]$.

Proof. Obvious from $\underset{b}{\mathcal{V}}([p,q,r]) = p:q:r:0$ and $[1,1,1,0]\cdot\boxed{\underset{b}{\mathcal{Q}}} = [0,0,0,1]$. □

14.8 Inversion

14.8.1 One cycle

Definition 14.8.1. Two points X_1, X_2 are inverses in a given circle with center P and radius ω when P, X_1, X_2 are in straight line and $\left\langle \overrightarrow{PU_1} \mid \overrightarrow{PU_2} \right\rangle = \omega^2$. Here $\omega \in \boldsymbol{i}\mathbb{R}$ is allowed.

Proposition 14.8.2. *The **inverse of a point** $X = x:y:z$ in the circle Ω having center $P = p:q:r$ and radius ω is given by :*

$$\operatorname{nor}(inv(X)) = nor(P) + (nor(X) - nor(P))\,\frac{\omega^2}{pytha(X,\,P)} \tag{14.21}$$

This can be rewritten as

$$inv(X) = \left(\frac{\Gamma_{std}(X)}{x+y+z} + \frac{\Gamma_{std}(P)}{p+q+r} - \omega^2\right)\operatorname{nor}(P) + \left(\omega^2 - \frac{2}{(p+q+r)^2}\,P\cdot{}^{t}P\cdot\boxed{\mathrm{Pyth}_b}\right)\cdot\operatorname{nor}(X)$$

This formula is to be compared with formula (14.22) given in Theorem 14.8.4.

Proof. The first formula is nothing but the definition. Multiplying by $pytha(X,\,P)$, we obtain $inv(X) \simeq \left(pytha(X,\,P) - \omega^2\right) nor(P) + \omega^2\, nor(X)$. Then $pytha$ is expanded using its definition, and we conclude using $\left({}^{t}P\cdot X\right)P = \left(P\cdot{}^{t}P\right)\cdot X$. □

Remark 14.8.3. When circle Ω is given by its equation (13.1) and $(P,\,\omega)$ are obtained from (13.6) and (13.8), the following identity can be useful :

$$\frac{\Gamma_{std}(P)}{(p+q+r)^2} - \omega^2 = \frac{U\cdot(vX(3)/2) - a^2b^2c^2}{8\,S^2}$$

(remember: U and $vX(3)$ are "as is" and not "up to a proportionality factor", while P is projective and ω is a number).

Theorem 14.8.4. ***Inversion of cycles in a cycle***. *Let Ω_0 be a fixed cycle with representative $\underset{b}{\mathcal{V}}_0$ and Ω_1 another cycle with representative $\underset{b}{\mathcal{V}}_1$. Assume that Ω_0 is not a point-circle and call $\underset{b}{\widehat{\mathcal{V}_1}}$ the intersection of line $\underset{b}{\mathcal{V}}_0\underset{b}{\mathcal{V}}_1$ with the polar plane of $\underset{b}{\mathcal{V}}_0$. Define σ as the transform $\Omega_1 \mapsto \Omega_3$ where $\underset{b}{\mathcal{V}}_3$, the representative of Ω_3, is such that division $\underset{b}{\mathcal{V}}_0, \underset{b}{\widehat{\mathcal{V}_1}}, \underset{b}{\mathcal{V}}_1, \underset{b}{\mathcal{V}}_3$ is harmonic. Then the cycle Ω_3 is the inverse of Ω_1 in cycle Ω_0 (inversion in a straight line is the ordinary reflection in this line) while the matrix of the transform $\underset{b}{\mathcal{V}}_1 \mapsto \underset{b}{\mathcal{V}}_3$ is given by :*

$$\boxed{\sigma} = \mathrm{Id} - 2\,\frac{\underset{b}{\mathcal{V}}_0 \cdot {}^t\underset{b}{\mathcal{V}}_0 \cdot \boxed{\underset{b}{\mathcal{Q}}}}{{}^t\underset{b}{\mathcal{V}}_0 \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot \underset{b}{\mathcal{V}}_0} \tag{14.22}$$

Moreover, when $\underset{b}{\mathcal{V}}_2$ is yet another cycle representative, we have the conservation law :

$${}^t\sigma\left(\underset{b}{\mathcal{V}}_1\right) \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot \sigma\left(\underset{b}{\mathcal{V}}_2\right) = {}^t\underset{b}{\mathcal{V}}_1 \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot \underset{b}{\mathcal{V}}_2 \tag{14.23}$$

Proof. Write $\underset{b}{\widehat{\mathcal{V}_1}}$ as $\alpha_1\underset{b}{\mathcal{V}}_0 + \underset{b}{\mathcal{V}}_1$ in ${}^t\underset{b}{\mathcal{V}}_0 \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot \underset{b}{\widehat{\mathcal{V}_1}} = 0$ and then obtain $\underset{b}{\mathcal{V}}_3$ as $2\alpha_1\underset{b}{\mathcal{V}}_0 + \underset{b}{\mathcal{V}}_1$ since division $(\infty, 1, 0, 2)$ is harmonic. Equation (14.23) is obvious from (14.21), and shows that σ preserves orthogonality. Moreover, (14.21) shows that cycles orthogonal to Ω_0 are invariant while cycles concentric with Ω_0 are transformed into cycles concentric with Ω_0 : all together, this proves that σ is the inversion in cycle Ω_0. □

14.8.2 Two cycles

Here, all barycentrics are supposed to be in their normalized form.

Proposition 14.8.5. Centers of homothety. *Let* $\mathcal{C}_j\left(O_j, r_j\right)$, etc *be two circles. Points U, V defined by :*

$$U = \frac{r_1 O_2 + r_2 O_1}{r_1 + r_2},\; V = \frac{r_1 O_2 - r_2 O_1}{r_1 - r_2}$$

are respectively the internal and the external centers of homothety of these two circles. At Kimberling, ETC, they are called insimilicenter and exsimilicenter. When $X_1 \in \mathcal{C}_1$ then :

$$\left(r_1 + r_2\right) U - r_2 X_1 \in \mathcal{C}_2 \quad \text{and} \quad \left(r_1 - r_2\right) V + r_2 X_1 \in \mathcal{C}_2$$

Proof. When $r_1 = r_2$, point V defines a translation, not an homothety. Otherwise, all steps are obvious. □

Proposition 14.8.6. *Let* $\mathcal{C}_j\left(O_j, r_j\right)$, etc *be two circles, X_1 the generic point of $\mathcal{C}_1$ and $U = \left(r_1 O_2 + r_2 O_1\right) / \left(r_1 + r_2\right)$ as above. Then the line UX_1 cuts $\mathcal{C}_2$ in two points. The first one is $X_2 = \left(\left(r_1 + r_2\right) U + r_2 X_1\right) / r_1$, obtained by homothety. The second one is Y_2, obtained by inversion into the circle centered at U with power :*

$$\rho^2 = \left(1 - \frac{\left|O_1 O_2\right|^2}{\left(r_1 + r_2\right)^2}\right) r_1 r_2$$

Changing one of the radiuses into its opposite give the results relative to V (assuming $r_1 \neq r_2$)

14.8.3 Three circles

Notation 14.8.7. We start with three generic circles $\mathcal{C}_j\left(z_j, r_j\right)$, i.e. the centers are not aligned and all the radiuses are different. And we note $\gamma_j(U_j, \rho_j)$ and $\gamma'_j\left(V_j, \rho'_j\right)$ the six circles

$$\gamma_1 \doteq \frac{r_3 \mathcal{C}_2 + r_2 \mathcal{C}_3}{r_3 + r_2}, \text{ etc } ;\; \gamma'_1 \doteq \frac{r_3 \mathcal{C}_2 - r_2 \mathcal{C}_3}{r_3 - r_2}, \text{ etc}$$

so that γ_j is the internal circle of homothety of $\mathcal{C}_i, \mathcal{C}_k$, and γ_j' is the external one. As stated in the previous subsection, we have:

$$U_1, V_1 \doteq \frac{r_3 O_2 \pm r_2 O_3}{r_3 \pm r_2} \; ; \; (\rho_1)^2 = r_2 r_3 \left(1 - \frac{|z_3 - z_2|^2}{(r_3 + r_2)^2}\right) \; ; \; (\rho_1')^2 = r_2 r_3 \left(\frac{|z_3 - z_2|^2}{(r_3 - r_2)^2} - 1\right)$$

Finally, $\mathcal{C}_4$ is the common orthogonal cycle to all these circles.

Lemma 14.8.8. *The product of three inversions α, β, γ wrt circles of a same pencil is another inversion wrt a circle of the pencil. Thus the chain $M_0 \underset{\alpha}{\mapsto} M_1 \underset{\beta}{\mapsto} M_2 \underset{\gamma}{\mapsto} M_3 \underset{\alpha}{\mapsto} M_4 \underset{\beta}{\mapsto} M_5 \underset{\gamma}{\mapsto} M_6$ closes with $M_6 = M_0$. Moreover, these 6 points are on a same circle, which belongs to the orthogonal pencil.*

Proof. (Spoiler) Use Morley affixes and consider the circles $-p : 0 : -p : 1$ (where $p, q, r, \in \mathbb{R}$). Then

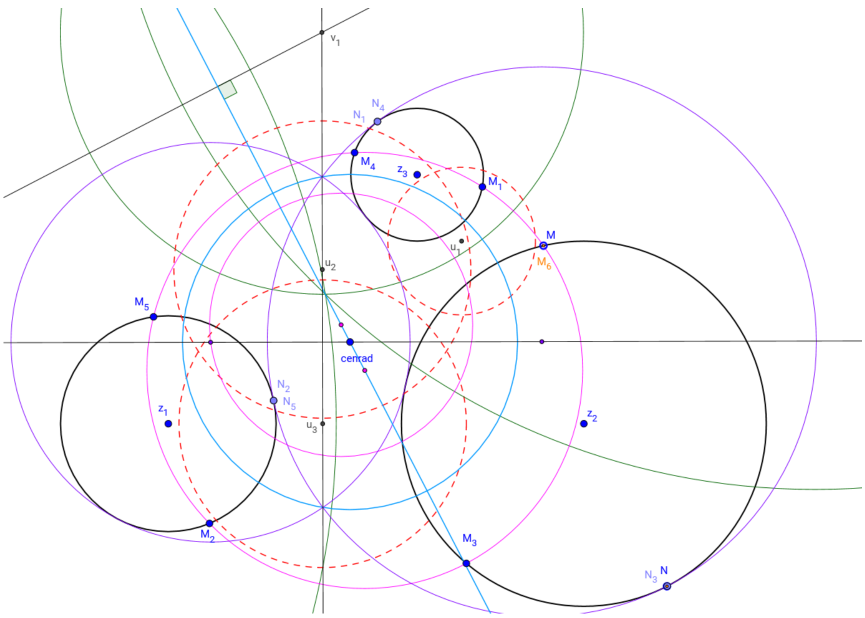
$$\alpha \begin{pmatrix} \mathbf{Z} \\ \mathbf{T} \\ \mathcal{Z} \end{pmatrix} \simeq \begin{pmatrix} \dfrac{p\overline{\mathcal{Z}} + \mathbf{T}}{\overline{\mathcal{Z}} - p\mathbf{T}} \\ 1 \\ \dfrac{p\mathbf{Z} + \mathbf{T}}{\mathbf{Z} - p\mathbf{T}} \end{pmatrix} \; ; \; \gamma\beta\alpha \begin{pmatrix} \mathbf{Z} \\ \mathbf{T} \\ \mathcal{Z} \end{pmatrix} \simeq \begin{pmatrix} \dfrac{(pqr + p - q + r)\,\overline{\mathcal{Z}} + (pq - pr + qr + 1)\,\mathbf{T}}{(pq - pr + qr + 1)\,\overline{\mathcal{Z}} - (pqr + p - q + r)\,\mathbf{T}} \\ 1 \\ \dfrac{(pqr + p - q + r)\,\mathbf{Z} + (pq - pr + qr + 1)\,\mathbf{T}}{(pq - pr + qr + 1)\,\mathbf{Z} - (pqr + p - q + r)\,\mathbf{T}} \end{pmatrix}$$

Moreover the six points are on the circle $\left[-\left(\mathbf{T}^2 + \mathbf{Z}\mathcal{Z}\right), 2\,\mathbf{T}\left(\mathbf{Z} - \mathcal{Z}\right), \mathbf{T}^2 + \mathbf{Z}\mathcal{Z}, \mathbf{T}\left(\mathbf{Z} - \mathcal{Z}\right)\right]$. □

Proposition 14.8.9 (Monge)**.** *Centers $V_1V_2V_3$ are aligned, and also centers $V_jU_iU_k$ (even number of internal centers).*

Proof. Alignment of the V_j comes from: $r_1 (r_2 - r_3) V_1 + r_2 (r_3 - r_1) V_2 + r_3 (r_1 - r_2) V_3 = \overrightarrow{0}$. When changing r_j into $-r_j$, two circles are impacted, inducing the parity requirement. □

Proposition 14.8.10. *Starting by $M_0 \in \mathcal{C}_2$, the inversions $\gamma_1', \gamma_2', \gamma_3', \gamma_1', \gamma_2', \gamma_3'$ are leading to a set of six concyclic points where M_j belongs to $\mathcal{C}_{j+2}$ (indexes taken modulo 3) and $M_6 = M_0$. All circles (M_k) are centered on the perpendicular to line $V_1V_2V_3$ issued from z_4 (radical axis of circles $\gamma_1', \gamma_2', \gamma_3'$).*



$\mathcal{C}_1(z_1), \mathcal{C}_2(z_2), \mathcal{C}_3(z_3)$: black ; $\mathcal{C}_4$ (cenrad): blue ; $\gamma_j'(v_j)$: green ; $\gamma_j(u_j)$: dot-red

Figure 14.2: Three circles, six inversions

Proof. Circles $\mathcal{C}_j$ are orthogonal to $\mathcal{C}_4$ and circles γ_j inherit of this property. By Monge proposition, they are orthogonal to line $V_1V_2V_3$. Therefore circles γ_j belong to a same pencil, the lemma applies and the conclusion follows. When $M_0 \in \mathcal{C}_2 \cap \mathcal{C}_4$, all the M_j are concyclic on $\mathcal{C}_4$. See Figure 14.2 where the M_j are obtained using $V_1V_2V_3$. The inverse of this circle in $\mathcal{C}_4$ is also given (both in magenta). □

Fact 14.8.11. *N_0 can be chosen on $\mathcal{C}_2$ so that $N_3 = N_0$. And then circle $N_0N_1N_2$ is tangent to the $\mathcal{C}_j$ circles. See Figure 14.2 where the N_j are obtained using $V_1U_2U_3$. The inverse of this circle in $\mathcal{C}_4$ is also given (both in violet). And then line $V_1U_2U_3$ is the radical axis of these two circles. See Section 14.11*

14.8.4 Steiner porism

Definition 14.8.12. A n-Steiner chain of circles is a series of n circles, finite in number, each tangent to two fixed circles and to two other circles of the series (borrowed from Johnson, 1929, p. 113)).

Proposition 14.8.13. *If two circles α, β admit a n-Steiner chain, they admit an infinite number of such chains, and any circle tangent in the same way to α, β is a member of one chain.*

Proof. Use an inversion and transform α, β in two concentric circles. The whole chain transforms into another chain. □

14.9 Euler pencil and incircle

Consider $\mathcal{C}_1 = (X_1,\, r)$, $\mathcal{C}_3 = (X_3,\, R)$, $\mathcal{C}_5 = (X_5,\, R/2)$ and $\mathcal{C}_z = (X_z = X_{381},\, |GH|/2)$ i.e., respectively, the in-, circum- nine points and orthocentroidal circles (Figure 14.3a). Let U_j, $\underset{b}{\mathcal{V}}_j$, x_j, c_j be the respective representatives of centers and circles, together with their respective shadows (Figure 14.3b). Then :

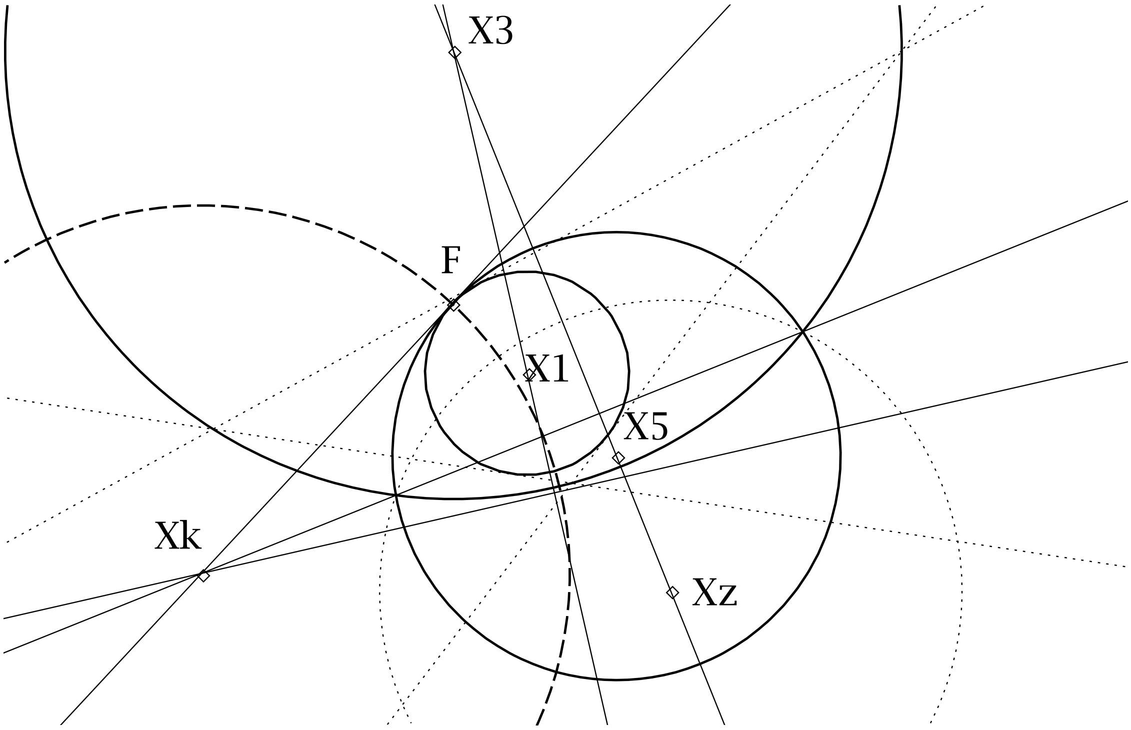
1. Circles (X_1), $\mathcal{C}_1$, $\mathcal{C}_\infty$ are concentric so that U_1, $\underset{b}{\mathcal{V}}_1$, *Sirius* are aligned and therefore x_1, c_1, G are aligned too. The same happens for $j = 5$ and $j = z$.
2. Cycles $\mathcal{C}_3$, $\mathcal{C}_5$, $\mathcal{C}_z$ belong to the same (Euler) pencil, together with their radical axis $AR_{3,5}$, so that representatives $\underset{b}{\mathcal{V}}_3$, $\underset{b}{\mathcal{V}}_5$, $\underset{b}{\mathcal{V}}_z$, $\underset{b}{\mathcal{V}}_{3,5}$ are aligned and therefore c_3, c_5, c_z, $ar_{3,5}$ are aligned too. Since c_3 is "far below the paper sheet", we have $c_5 = c_z = ar_{3,5}$. For the same reason, $c_1 = ar_{3,1}$.
3. Circles $\mathcal{C}_1$ and $\mathcal{C}_5$ are tangent at $F \doteq X_{11}$, the Feuerbach point. Thus cycles (F), $\mathcal{C}_1$, $\mathcal{C}_5$ belong to the same pencil, together with their common tangent $AR_{1,5}$. Representatives U_f, $\underset{b}{\mathcal{V}}_1$, $\underset{b}{\mathcal{V}}_5$, $\underset{b}{\mathcal{V}}_{1,5}$ are aligned and so are x_f, c_1, c_5, $ar_{1,5}$.
4. Cycles $AR_{1,3}$, $AR_{1,5}$, $AR_{3,5}$ are on the same pencil (they concur in the radical center X_k) and their shadows $ar_{1,3}$, $ar_{1,5}$, $ar_{3,5}$ are aligned.
5. In fact line c_1c_5 is not representative of a specific pencil, but rather of the bundle generated by $\mathcal{C}_1$, $\mathcal{C}_3$, $\mathcal{C}_5$. We have :

$$\underset{b}{\mathcal{V}}_1 \simeq {}^t\begin{bmatrix} (b+c-a)^2 \\ (c+a-b)^2 \\ (a+b-c)^2 \\ 4 \end{bmatrix}, \underset{b}{\mathcal{V}}_5 \simeq {}^t\begin{bmatrix} b^2+c^2-a^2 \\ c^2+a^2-b^2 \\ a^2+b^2-c^2 \\ 4 \end{bmatrix}, \underset{b}{\mathcal{V}}_3 \simeq {}^t\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

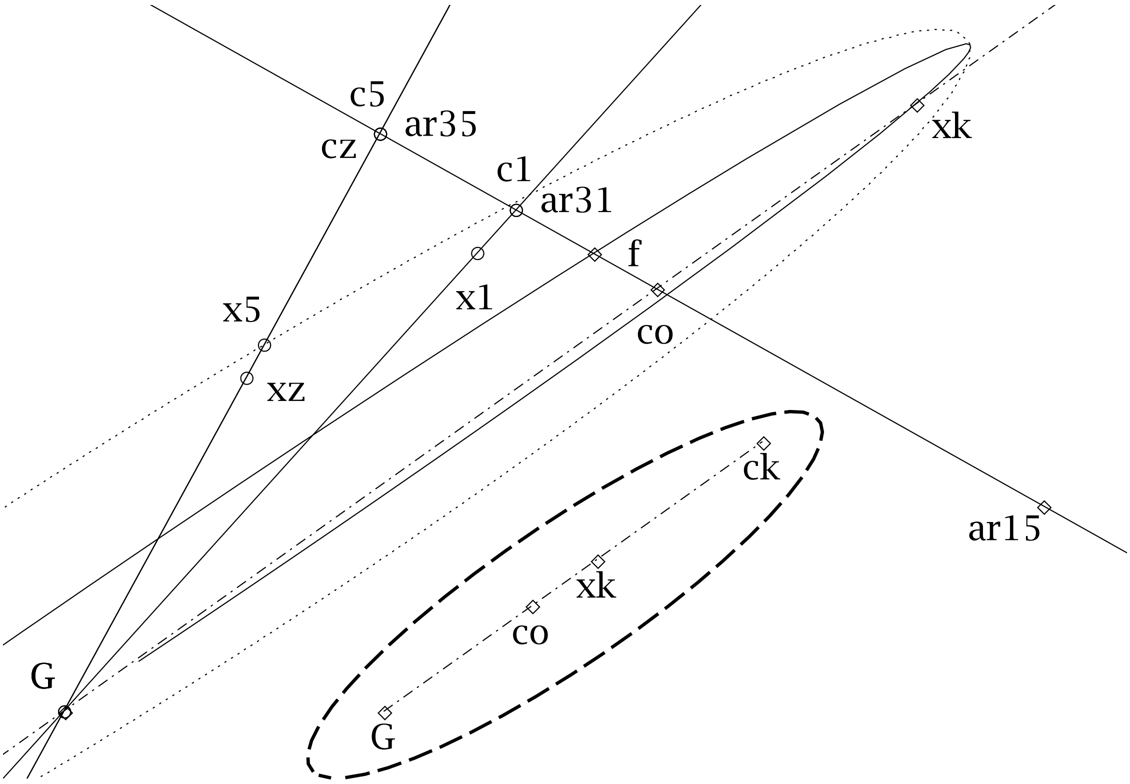
and therefore :

$$\underset{b}{\mathcal{V}}_k \simeq {}^t\begin{bmatrix} (c-b)(b+c-a)\left(b^2+c^2-a^2\right)\left(b^2+c^2-ab-ac\right) \\ (a-c)(c+a-b)\left(c^2+a^2-b^2\right)\left(c^2+a^2-bc-ba\right) \\ (b-a)(a+b-c)\left(a^2+b^2-c^2\right)\left(a^2+b^2-ca-cb\right) \\ 4\,(a-b)(b-c)(c-a)(a+b+c) \end{bmatrix}$$

From $\underset{b}{\mathcal{V}}_k$, the well-known result $X_k = X_{676}$ and the obvious $r_k = |X_kX_f|$ can be re-obtained.

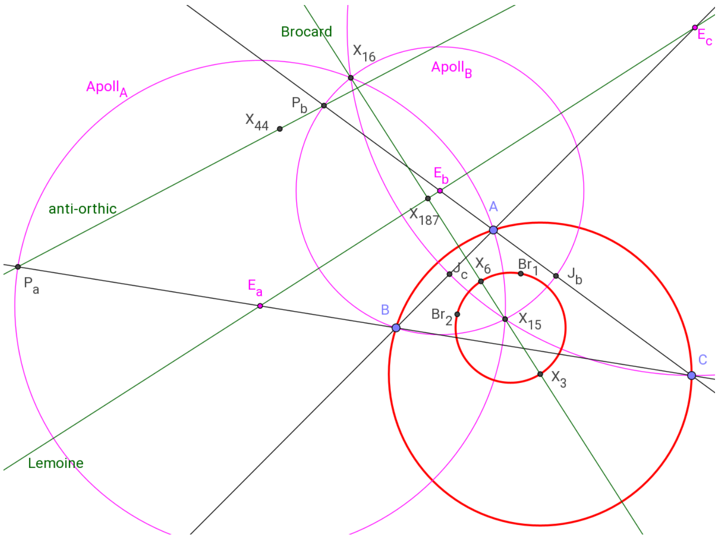


(a) lines and circles



(b) shadows

Figure 14.3: Euler pencil and incircle



each Apoll circle cuts the others at 60° (see (14.17))

Figure 14.4: Lemoine and Brocard pencils

6. As it should be, x_k, c_k, G are aligned (small insert, at the bottom of Figure 14.3b).

7. Consider W_k at intersection between *line* $(V_k Sirius)$ and *plane* (V_1, V_3, V_5). This points represents a circle that is both concentric and orthogonal to $\mathcal{C}_k$. This circle is therefore $(X_k,\ i\, r_k)$ and its shadow co belongs to both Gx_k and $c_5 c_1$. Moreover, division G, x_k, co, c_k is harmonic.

14.10 The Brocard-Lemoine pencils

Notation 14.10.1. The cevians of a point M are usually noted M_a, etc. When dealing with the incenter I_0, this would collide with I_a, etc, the usual notation of the excenters. Thus we will use J_a, etc to note the cevians of I_0 (i.e. the feet of the internal bisectors).

Proposition 14.10.2. *Let J_a and P_a be the points on sideline BC met by the interior and exterior bisectors of angle A. In other words, the J_j and P_j are respectively the cevians and the cocevians of I_0. The circle (E_a) having diameter $[J_a, P_a]$ goes through A and is called the A-**Apollonian circle**. The B- and C- Apollonian circles are similarly constructed, while the centers E_j are the cocevians of K =X(6).*

The Apollonian circles belong to a same (Lemoine) pencil whose base points are the isodynamic points X(15), X(16), while the radical axis is the Brocard line. The orthogonal (Brocard) pencil contains the circumcircle and the Brocard 3-6 circle, while the radical axis is the Lemoine line.

Proof. These assertions depend of the following Lemmas. □

Lemma 14.10.3. ***Lemoine pencil*** *(of Apollonian circles). Point J_a is the A cevian of $X(1)$. Thus $J_a \simeq 0 : b : c$ and $P_a \simeq 0 : b : -c$. Now, we take the wedge of the Veronese of A, J_a, P_a and obtain the column describing the A-Apollonian circle:*

$$\underset{b}{\mathcal{V}}_a \simeq \bigwedge_3 \left([1,0,0,0],\ [0, b\,(b+c)\,, (b+c)\,c, -a^2bc]\ [0, b\,(c-b)\,, c\,(b-c)\,, -a^2bc]\right) \simeq \begin{pmatrix} 0 \\ -a^2c^2 \\ a^2b^2 \\ b^2-c^2 \end{pmatrix}$$

Let us remember that $\underset{b}{\mathcal{V}}_a$ is a column that describe a point in $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^4)$, while each Veronese is a plane incident to this point. Here, it is obvious that the three $\underset{b}{\mathcal{V}}_j$ are not independent from each *other. Using the usual formalism to describe the corresponding pencil, one obtains :*

$$\boxed{Lemoine_{point}} \doteq \left(\underset{b}{\mathcal{V}_a} \underset{6}{\wedge} \underset{b}{\mathcal{V}_b}\right) \simeq \begin{pmatrix} 0 & a^2b^2\left(b^2-a^2\right) & a^2c^2\left(c^2-a^2\right) & a^4b^2c^2 \\ a^2b^2\left(a^2-b^2\right) & 0 & b^2c^2\left(c^2-b^2\right) & a^2b^4c^2 \\ a^2c^2\left(a^2-c^2\right) & b^2c^2\left(b^2-c^2\right) & 0 & a^2b^2c^4 \\ -a^4b^2c^2 & -a^2b^4c^2 & -a^2b^2c^4 & 0 \end{pmatrix}$$

where the index 'point' is used to remember that this object has to be multiplied by a point in $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^4\right)$, i.e. a column, to determine if the point belongs to the pencil.

Lemma 14.10.4. *In order to determine the point-circles that belong to the pencil, we solve $Ver(x:y:z)\cdot\boxed{\underset{b}{Q}^{-1}}\cdot\boxed{Lemoine_{point}}=0$. We obtain both umbilics, together with*

$$E_\pm \simeq \begin{pmatrix} a^2 \mp \boldsymbol{i}a^2\sqrt{3} \\ -2\,b^2 \\ c^2 \pm \boldsymbol{i}c^2\sqrt{3} \end{pmatrix} \simeq \begin{pmatrix} a^2\left(2\,a^2-b^2-c^2\right) \pm \boldsymbol{i}\ \sqrt{3}a^2\left(b^2-c^2\right) \\ b^2\left(2\,b^2-a^2-c^2\right) \pm \boldsymbol{i}\ \sqrt{3}b^2\left(c^2-a^2\right) \\ c^2\left(2\,c^2-b^2-a^2\right) \pm \boldsymbol{i}\ \sqrt{3}c^2\left(a^2-b^2\right) \end{pmatrix}$$

The fact that $E_\pm \simeq X\,(187) \pm \boldsymbol{i}\sqrt{3}\ X\,(512)$ is not real indicates that the pencil is an isoptic one. From these values, one sees that the line of centers is the Lemoine axis (eponymous property).

Lemma 14.10.5. *The polar planes Π_j of the $\underset{b}{\mathcal{V}_j}$ are obtained by $\Pi_j \doteq {}^t\underset{b}{\mathcal{V}_j}\cdot\boxed{\underset{b}{Q}}$. Their pencil is described by*

$$\boxed{Brocard_{plane}} \doteq \left(\Pi_a \underset{6}{\wedge} \Pi_b\right) \simeq \begin{pmatrix} 0 & 0 & 0 & b^2c^2 \\ 0 & 0 & 0 & a^2c^2 \\ 0 & 0 & 0 & a^2b^2 \\ -b^2c^2 & -a^2c^2 & -a^2b^2 & 0 \end{pmatrix}$$

where the index "plane" is to remember that this object has to be multiplied by a plane (i.e. a row) to determine if the plane belongs to the pencil of planes.

Lemma 14.10.6. ***Brocard pencil*** *(of Γ and 3-6 Brocard) . The dual of the $\boxed{Brocard_{plane}}$ pencil is called the Brocard pencil. Its matrix (acting over points, i.e. over representatives of cycles), is the dual of the former matrix:*

$$\boxed{Brocard_{point}} \simeq \begin{pmatrix} 0 & a^2b^2 & -a^2c^2 & 0 \\ -a^2b^2 & 0 & b^2c^2 & 0 \\ a^2c^2 & -b^2c^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This pencil contains both the circum-circle and the 3-6 Brocard circle (obvious).

Moreover, the point circles of the pencil are both umbilics, together with the isodynamic points points X(15),X(16) .

Proof. Using the same algorithm as before gives an expression which isn't symmetric... and rather complicated. The best thing to do is using the Kimberling's search keys. Thereafter, one can check that:

$$F_\pm = \begin{pmatrix} a^2\left(2\,a^2-b^2-c^2\right) + \frac{\sqrt{3}}{4S}\,a^2\left(\left(b^2+c^2\right)a^2-b^4-c^4\right) \\ b^2\left(2\,b^2-c^2-a^2\right) + \frac{\sqrt{3}}{4S}\,b^2\left(\left(c^2+a^2\right)b^2-c^4-a^4\right) \\ c^2\left(2\,c^2-a^2-b^2\right) + \frac{\sqrt{3}}{4S}\,c^2\left(\left(a^2+b^2\right)c^2-a^4-b^4\right) \end{pmatrix}$$

i.e. $F_\pm \simeq X\,(187) \pm \sqrt{3}\,\boxed{\text{OrtO}}\cdot X\,(512)$ (see Proposition 14.5.6). From these values, one sees that the line of centers is the Brocard axis (eponymous property). □

Remark 14.10.7. These pencils are reexamined at Section 19.3.2 (using Morley coordinates).

14.11 The Apollonius configuration

In the general case, it exists eight cycles Ω tangent to three given cycles Ω_1, Ω_2, Ω_3 (not from the same pencil). Two surveys of this question are Gisch and Ribando (2004) and Kunkel (2007). The usual decomposition into ten cases is Wikipedia: WillowW et al. (2006); Bogomolny (2009). The best space where this Apollonius problem can be discussed is $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^5\right)$ (cf Section 20). Nevertheless, most of the results can be formulated in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$... and it will appear that only one situation is really special (cycles through the same point), all the other belonging to the same general case.

14.11.1 Tangent cycles in the representative space

Proposition 14.11.1. *Two cycles are tangent when their pencil line is tangent to the fundamental quadric. Therefore, the locus of the representatives of all cycles tangent to a given (real) cycle Ω represented by V (not inside $\mathcal{Q}$) is the cone whose vertex is V and that goes through $\mathcal{Q} \cap polar\,(V)$.*

Proof. Two tangent cycles are defining a tangent pencil ! □

Remark 14.11.2. When Ω is a point circle, its representative U belongs to the fundamental quadric, and the cone of the tangent cycles degenerates into a doubly coated plane.

Definition 14.11.3. The Gram matrix $\mathrm{G}_{p,q,\cdots,r}$ of X_p, X_q, $\cdots$, $X_r \in \mathbb{R}^4$ is the matrix of all the products ${}^tX_p \boxed{\mathcal{Q}} X_q$. In this context, notation $W_{pq} = {}^tX_p \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot X_q$ and $w_p^2 = {}^tX_p \cdot \boxed{\underset{b}{\mathcal{Q}}} \cdot X_p$ will be used, leading to

$$\mathrm{G}_{pq} = \left(\begin{array}{cc} w_p^2 & W_{pq} \\ W_{pq} & w_q^2 \end{array}\right) \tag{14.24}$$

Proposition 14.11.4. *Two cycles Ω_1, Ω_2 are secant, tangent or external when* $\mathrm{signum}\det \mathrm{G}_{12}$ *is (respectively) $+1$, 0 or -1.*

Theorem 14.11.5. *Special cases of the Apollonius problem are (1) cycles from the same pencil and (2) cycles through the same point (tangent bundle). Otherwise, representatives V_j of the three given cycles and their common orthogonal cycle Ω_4 form a basis that splits the problem into four pairs of solutions. One of the solutions is given by $V_0 = \sum k_j\, V_j$ where :*

$$\begin{array}{rcl} k_1 & = & (w_2\, w_3 - W_{23})\,(-w_1\, w_2\, w_3 - w_1\, W_{23} + w_2\, W_{13} + w_3\, W_{12}) \\ k_2 & = & (w_1\, w_3 - W_{13})\,(-w_1\, w_2\, w_3 + w_1\, W_{23} - w_2\, W_{13} + w_3\, W_{12}) \\ k_3 & = & (w_1\, w_2 - W_{12})\,(-w_1\, w_2\, w_3 + w_1\, W_{23} + w_2\, W_{13} - w_3\, W_{12}) \\ k_4 & = & \sqrt{-2\,(w_2\, w_3 - W_{23})\,(w_1\, w_3 - W_{13})\,(w_1\, w_2 - W_{12})\;\mathrm{G}_{123}}\,/w_4 \end{array} \tag{14.25}$$

and the others are obtained by changing k_4 into $-k_4$ (inversion through Ω_4) or changing the signs of w_1, w_2, w_3. A solution is real/imaginary or "unimaginable" (object that would have a non real center) according to the sign of k_4^2. Globally, the number of "imaginable" solutions changes when the tangency condition $\prod \mathrm{G}_{jk}$ vanishes.

Proof. When Ω_j, $j = 1, 2, 3$ is a basis of a non tangent bundle, then Ω_j, $j = 1, 2, 3, 4$ is a basis of the whole representative space. The fundamental quadratic form is described, in this basis, by matrix G_{1234} where $W_{j4} = 0$ for $j = 1, 2, 3$. Computing, in this basis, the tangency condition of Ω_0 and any of the Ω_j leads to 0. Since the w_j are defined as $\sqrt{W_{jj}}$ we have 4 choices of signs leading, due to the possibility of a global proportionality factor, to eight different values. □

14.11.2 An example: the Soddy circles

Proposition 14.11.6. ***Soddy circles** are three mutually, externally, tangent circles. Let A, B, C be their centers. Then the common orthogonal circle of the Soddy's is the incircle of ABC (see Oldknow, 1996).*

Proof. Let x be the radius of circle (A) , etc. We have $a = y + z$, $b = z + x$, $c = x + y$. Therefore $x = b + c - a$, etc. The contact point of the $(B)\,,(C)$ circles is $G_a \simeq 0 : y : z$, etc. As a result, the G_j are the cevians of the Gergonne point G_e =X(7), and the conclusion follows. □

Proposition 14.11.7. *The Apollonius circles of the Soddy circles are twice each of them, a small circle (inside the intouch triangle) and an outer circle. The center of the smaller circle is called X(176), the other is called X(175). Let $\mathcal{H}_a$ be the branch of hyperbola that goes through A, G_a and has B, C as focuses. Then the three branches through a vertex of the hyperbolas concur at X(176), while the other three branches concur at X(175).*

Proof. This is clear from $a = y + z$, etc. □

Proposition 14.11.8. *Centers and radiuses of the Soddy circles are given by :*

$$nX(175) = \frac{2s}{2s-(4R+r_0)}\, nX(1) - \frac{4R+r_0}{2s-(4R+r_0)}\, nX(7) \simeq \left[\begin{array}{c} a - \dfrac{2S}{b+c-a} \\ \vdots \\ \vdots \end{array}\right]$$

$$nX(176) = \frac{2s}{2s+(4R+r_0)}\, nX(1) + \frac{4R+r_0}{2s+(4R+r_0)}\, nX(7) \simeq \left[\begin{array}{c} a + \dfrac{2S}{b+c-a} \\ \vdots \\ \vdots \end{array}\right] \qquad (14.26)$$

$$\frac{1}{\rho_6} - \frac{1}{\rho_5} = \frac{4}{r_0}\ ;\ \frac{1}{\rho_6} + \frac{1}{\rho_5} = 2\,\frac{4R+r_0}{s\,r_0}\ ;\ \rho_5 = \frac{r_0\, s}{2\,s-(4\,R+r_0)}$$

Proof. Using (14.10), the representatives of circles $\Omega_1 \cdots \Omega_4$ are :

$$\left(\begin{array}{ccc|c} (b+c-a)^2 & (b+c-a)(b-3c-a) & (b+c-a)(c-a-3b) & (b+c-a)^2 \\ (c+a-b)(a-b-3c) & (c+a-b)^2 & (c+a-b)(c-3a-b) & (c+a-b)^2 \\ (a+b-c)(a-3b-c) & (a+b-c)(b-c-3a) & (a+b-c)^2 & (a+b-c)^2 \\ -4 & -4 & -4 & 4 \end{array}\right)$$

Their Gram matrix is :

$$\left(\begin{array}{cccc} (b+c-a)^2 & -(b+c-a)(c+a-b) & -(a+b-c)(b+c-a) & 0 \\ -(b+c-a)(c+a-b) & (c+a-b)^2 & -(c+a-b)(a+b-c) & 0 \\ -(a+b-c)(b+c-a) & -(c+a-b)(a+b-c) & (a+b-c)^2 & 0 \\ 0 & 0 & 0 & \dfrac{16\,S^2}{(a+b+c)^2} \end{array}\right)$$

Then (14.25) gives the decomposition of the Soddy's circles on the Ω basis. We have :

$${}^tK \simeq \left[\frac{1}{b+c-a}; \frac{1}{c+a-b}; \frac{1}{a+b-c}; \frac{a+b+c}{2\,S}\right]$$

and the conclusion follows (since $a+b+c = 2s$). □

Remark 14.11.9. We have cross_ratio $(X_1, X_7, X_{175}, X_{176}) = -1$, while condition $4R + r_0 = 2s$ is not excluded. In this case, the X(1) and X(7) are on Ω_6, the inner Soddy circle, while X(175) is at infinity and Ω_5 is a straight line. When ABC $(a = b = c = 1)$, then $s = 3$, $R = \sqrt{1/3}$, $r_0 = \sqrt{1/12}$ and (14.26) provides a positive ρ_6.

Proposition 14.11.10. *The Soddy radiuses satisfy the following "curvature formula" :*

$$\left(\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} + \frac{1}{\rho_5}\right)^2 = 2\left(\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} + \frac{1}{\rho_3^2} + \frac{1}{\rho_5^2}\right)$$

Proof. For an elementary proof: substitute and simplify. For a stratospheric (and interesting !) proof, see Pedoe (1967). See Soddy (1936) for a poetic statement of this property. □

Proposition 14.11.11. *Let* ε_a, etc *be the cocevians and* G_a, etc *be the cevians of* G_e =*X(7). Draw circles E_j centered at ε_j and going through G_j. Then E_j cuts $\mathcal{C}_j$ at the contact points with the Soddy circles. Moreover the three E_j circles concur at X(3638) and X(3639).*

Proof. It is easy to check that each of the $E_j \cap \mathcal{C}_j$ points belongs to one Soddy circle. Moreover, one has

$$\begin{aligned} X(1323) &\simeq \frac{2a^2 - a(b+c) - (b-c)^2}{b+c-a} :: \\ X(516) &\simeq 2a^3 - b^3 - c^3 + \left(bc - a^2\right)(b+c) \\ X(3638) &= 3X(516) + 4\sqrt{3}SX(1323) \end{aligned}$$

And it is easy to check that X(3638) belongs to the three E_j circles. Caveat: the Soddy points are not the common points of the E_j circles, but they are the Poncelets of the Soddy circles. □

14.11.3 An other example: the not so Soddy circles

Proposition 14.11.12. *The* ***not so Soddy circles*** *of a triangle are the circles centered at A with radius $a = BC$*, etc. *Their common orthogonal circle is the Longchamps circle λ (i.e. the polar circle of the antimedial triangle).*

Proof. The circle γ_a, centered at A with radius a is described by :

$$yza^2 + b^2zx + c^2yx + (x+y+z)\left(a^2x + y\left(a^2-c^2\right) + z\left(a^2-b^2\right)\right) = 0$$

Its intersections with the circumcircle are :

$$Q_b \simeq a^2 - c^2 : -b^2 : c^2 - a^2 \quad and \quad Q_c \simeq a^2 - b^2 : b^2 - a^2 : -c^2$$

while its intersections with γ_b (resp. γ_c) are Q_c and $U_c \simeq 1 : 1 : -1$ (resp. Q_b and $U_b \simeq 1 : -1 : 1$). Points U_a, U_b, U_c are on circle $(H, 2R)$ and form the antimedial triangle. Lines Q_aU_a are the altitudes of $U_aU_bU_c$ and concur at $L \doteq H'$ =X(20). But they are also the radical axes of our circles. The radius r_L of the orthogonal circle is obtained by :

$$r_L^2 = |AL|^2 - a^2, \text{ etc} = \frac{-S_aS_bS_c}{S^2} = 4(2R+\rho+s)(2R+\rho-s)$$

□

Proposition 14.11.13. *The Apollonius circles of the not so Soddy circles are obtained by extraversions (i.e $a \mapsto -a$) from their central versions. These central circles are centered at the Soddy points X(175) and X(176).*

Proof. The representatives and the Gram matrix of γ_a, γ_b, γ_c, λ are :

$$\left(\begin{array}{cccc} -a^2 & c^2-b^2 & b^2-c^2 & a^2 \\ c^2-a^2 & -b^2 & a^2-c^2 & b^2 \\ b^2-a^2 & a^2-b^2 & -c^2 & c^2 \\ 1 & 1 & 1 & 1 \end{array}\right), \left(\begin{array}{cccc} a^2 & S_c & S_b & 0 \\ S_c & b^2 & S_a & 0 \\ S_b & S_a & c^2 & 0 \\ 0 & 0 & 0 & -S_a\,S_b\,S_c \div S^2 \end{array}\right)$$

Changing $w_1 \mapsto -w_1$ is $a \mapsto -a$, proving the extraversions. Formula(14.25) gives the coefficients :

$$K \simeq \left(\begin{array}{c} (bc - S_a)(-bca - aS_a + bS_b + cS_c) \\ (ca - S_b)(-bca - bS_b + cS_c + aS_a) \\ (ab - S_c)(-bca - cS_c + aS_a + bS_b) \\ 16\,S^3 \div (a+b+c) \end{array}\right)$$

Organizing the obtained equations, we have :

$$\begin{aligned} \gamma_6 &= \lambda + 2(x+y+z)(ax+by+cz)\,\rho_6 \qquad \text{where } \rho_6 = \frac{+2s(2R+\rho+s)}{4R+\rho+2s} \\ \gamma_5 &= \lambda + 2(x+y+z)(ax+by+cz)\,\rho_5 \qquad \text{where } \rho_5 = \frac{-2s(2R+\rho-s)}{4R+\rho-2s} \end{aligned}$$

This leads to $\gamma_6 = (X_{176}, \rho_6)$, etc. Additionally, this proves that $ax+by+cz = 0$ is the radical axis of the three circles. Moreover, the Soddy conic (through A, B, C, with focuses X(175), X(176) and perspector X(7)) is tangent to the Longchamps circle at the common points of λ, γ_6, γ_5 since we have :

$$conic = \lambda + (ax+by+cz)^2$$

□

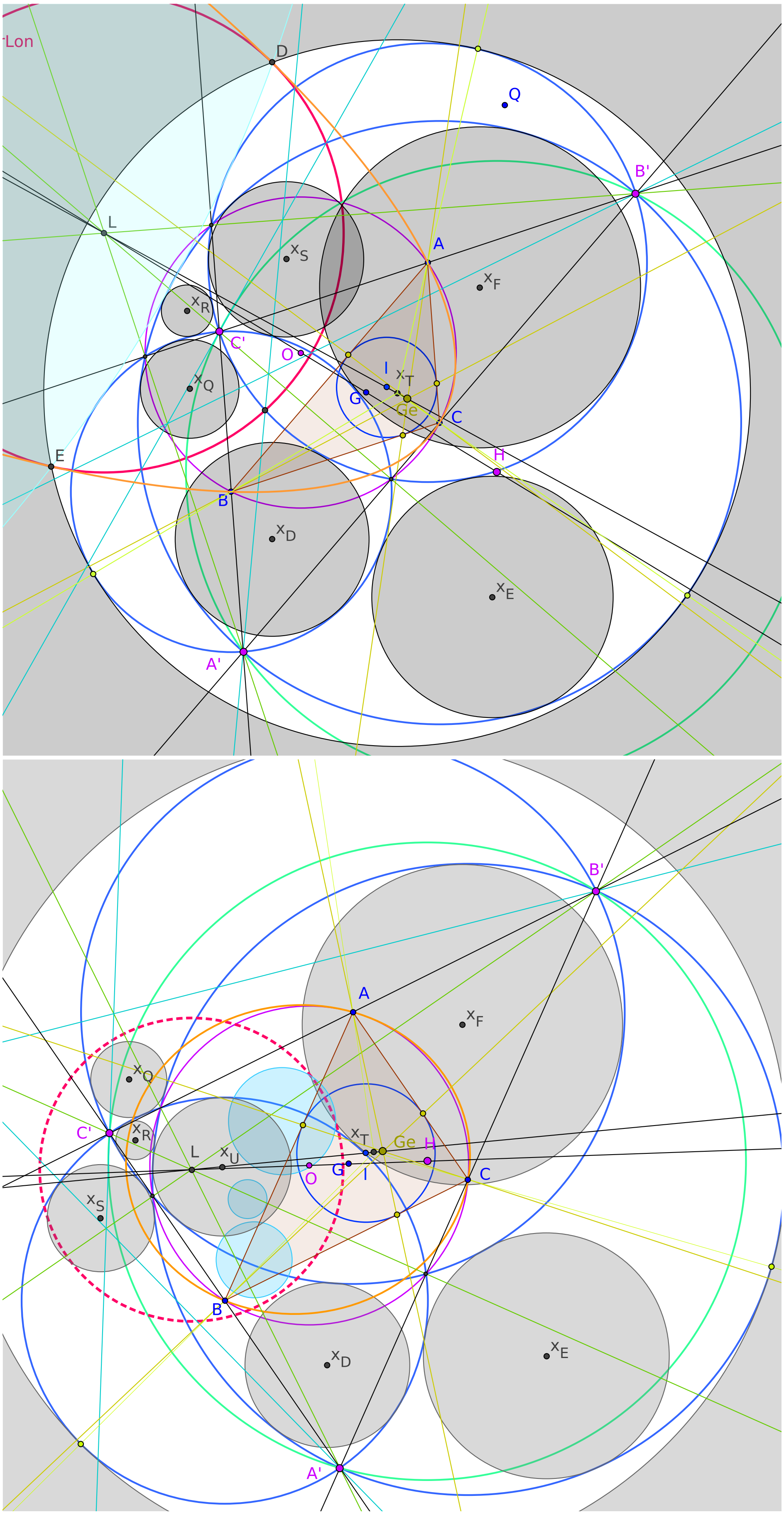


Figure 14.5: Apollonius circles of the not so Soddy configuration.

14.11.4 The three excircles

Taking the three excircles as $\Omega_1, \Omega_2, \Omega_3$ leads to a well-known situation (Stevanovic, 2003).

1. The representative of the point-circle $(X_1; 0)$ is :

$$U_0 = [bc\,(b+c-a)\,;ca\,(c+a-b)\,;ab\,(a+b-c)\,;a+b+c]$$

while radius of the incircle is :

$$r = \sqrt{\frac{(b+c-a)\,(c+a-b)\,(a+b-c)}{4\,(a+b+c)}}$$

2. The representative of the incircle, given by (14.10), is :

$$V_0 = \left[(b+c-a)^2\,;(c+a-b)^2\,;(a+b-c)^2\,;4\right]$$

3. Centers, radiuses and representatives V_a, V_b, V_c of the excircles are obtained by changing one of the sidelengths into its opposite in the respective formulas for the incircle.

4. The alt_Spieker circle is defined as the common orthogonal circle to the three excircles. From (14.18), this circle can be computed as :

$$V_4 = [(c+a-b)\,(a+b-c)\,;(a+b-c)\,(b+c-a)\,;(b+c-a)\,(c+a-b)\,;-4]$$

5. The radius of this circle, as computed from (14.15), is :

$$\begin{aligned} \omega_4 &= \sqrt{\frac{b^2c+ab^2+bc^2+a^2b+ac^2+a^2c+acb}{4\,(a+b+c)}} \\ &= \frac{1}{2}\sqrt{r_0^2+s^2} \end{aligned}$$

while the representative of the center is :

$$U_4 = {}^t\!\begin{pmatrix} 2\,a\left(b^2+c^2\right)-acb+b^3+c^3-a^3 \\ 2\,b\left(c^2+a^2\right)-acb+c^3+a^3-b^3 \\ 2\,c\left(a^2+b^2\right)-acb+a^3+b^3-c^3 \\ 4\;(a+b+c) \end{pmatrix}$$

and the center itself is :

$$b+c : a+c : a+b = X_{10}$$

6. The pairs of solutions of the Apollonius problem, as given by (14.25), are :

$$\begin{pmatrix} S_1 \\ S_5 \end{pmatrix} = \begin{pmatrix} b^2+c^2-a^2 & c^2+a^2-b^2 & a^2+b^2-c^2 & 4 \\ a+b+c+\dfrac{2\,bc}{a} & a+b+c+\dfrac{2\,ca}{b} & a+b+c+\dfrac{2\,ab}{c} & \dfrac{-4}{a+b+c} \end{pmatrix}$$

$$S_2 = {}^t\!\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\; S_6 = {}^t\!\begin{pmatrix} (a+b+c)\left(b^2+ab+ac+c^2\right) \\ (b+c)\,(a-b-c)\,(a+b-c) \\ (b+c)\,(a-b-c)\,(a-b+c) \\ 4\;(b+c) \end{pmatrix}$$

where point S_1 is the representative of the nine-points circle, centered at X_5 while S_5 is related to the Apollonius circle, centered ad X_{970}. Points S_2, S_3, S_4 are the representatives of lines BC, CA, AB while S_6 and S_7, S_8 (obtained cyclically) are the representatives of the last three solutions.

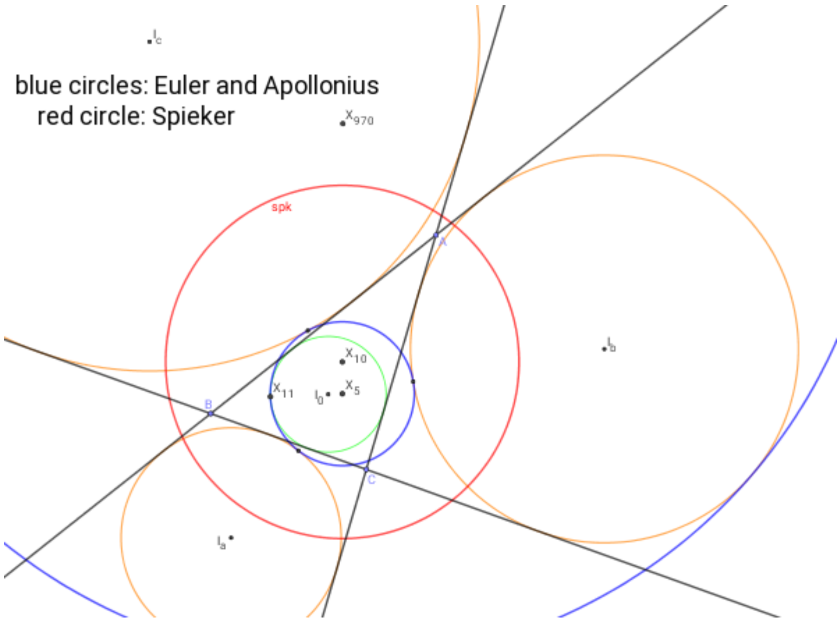


Figure 14.6: Alt-Spieker configuration

14.11.5 The special case

Proposition 14.11.14. *Let Ω_1, Ω_2, Ω_3 be three cycles generating a bundle whose common orthogonal cycle is a point-cycle (ω_5), and ω_4 be any other point. The representative of one of the cycles tangent to Ω_1, Ω_2, Ω_3 is given by $V_0 = \sum_1^3 k_j\, V_j + 4U_4$ where :*

$$k_1 = \left(\frac{w_2 w_3 - W_{2,3}}{(w_1 w_3 - W_{1,3})\,(w_1 w_2 - W_{1,2})}\ \mathrm{G}_{1,2,3,4} - 2\ \mathrm{G}_{2,3,4} \right) \div \Delta_{2,3,4}^{1,2,3} \tag{14.27}$$

$\Delta_{2,3,4}^{1,2,3}$ is the minor obtained by deleting row 1 and column 4 in $\mathrm{G}_{1,2,3,4}$, while k_2, k_3 are obtained cyclically. Three other cycles are obtained by changing one of the w_1, w_2, w_3 into its opposite. The other solutions are four times the point cycle ω_5.

Proof. In this special case, $\mathrm{G}_{1,2,3} = 0$ and Ω_4 is chosen so that $w_4 = 0$. When assuming that Ω_1, $\Omega_{2,}\Omega_3$ aren't pairwise tangent, a direct substitution shows that Ω_0 is tangent to any of the given cycles. □

Example 14.11.15. Using $\Omega_1 = 1 : 0 : 0 : 0$ (representative of line BC) etc, leads to $\omega_5 = Sirius$. An efficient choice for ω_4 is any vertex. Using, for example, $U_4 = 0 : c^2 : b^2 : 1$, one re-obtains easily the in/excircles.

Example 14.11.16. The Apollonius circles relative to the three circles (ABH), (BCH), (CAH) are $(H, 0)$ four times, $(H, 2R)$ once and three other circles, Ta, Tb, Tc.

Circles Ta, Tb, Tc are ever external to each other, and their common orthogonal circle To is real. Condition of (external) tangency is :

$$\left(a^2 - b^2\right)^2 - \left(a^2 + b^2\right) c^2 = 0$$

(etc) or ABC rectangular. The Apollonius circles of Ta, Tb, Tc are (ABH) etc, their inverses in To and two others. Center $X = x : y : z$ and radius ω of the first one are :

$$x = \left(b^2 + c^2 - a^2\right) \times \\ \left(a^8 - 2\left(b^2 + c^2\right) a^6 + 2\left(b^4 - b^2c^2 + c^4\right) a^4 - 2\left(b^2 + c^2\right)\left(b^2 - c^2\right)^2 a^2 + \left(b^2 - c^2\right)^4\right)$$

$$\omega = 2\,\frac{a^2 b^2 c^2 R}{a^6 + b^6 + c^6 - a^2b^4 - a^4b^2 - c^2b^4 - a^4c^2 - b^2c^4 - c^4a^2 + 4\,a^2b^2c^2}$$

while the second is less simple.

Chapter 15

Morley and complex numbers

Our aim in this chapter is to describe how to translate into complex numbers all of the ***methods*** we have described in the previous chapters. This is equivalent to give the complex version of all the operators which were described as of now.

In fact, some of these operators have a very simple form when using complex numbers and could have been introduced directly. In any case, we will prove the equivalence between "the Morley version" and "the barycentric version" of these operators.

15.1 Inclusive coordinates

When working with points $M = (\xi, \eta) \in \mathbb{R}^2$ and describing curves $\mathcal{C}$ by polynomials P so that $M \in \mathcal{C}$ when $P(\xi, \eta) = 0$, one has to face the following theorem:

Theorem 15.1.1 (Bezout)**.** *Two algebraic curves $\mathcal{C}(P_n)$ and $\mathcal{C}(P_m)$ of respective degrees m, n have exactly $m \times n$ common points when polynomials P_n and P_m have no non-constant common factor. To obtain this result, all the points have to be taken into account, including points with non real coordinates as well as points at infinity, and also considering the multiplicities of the solutions (Bezout, 1764).*

Obviously, a better formulation of this result can be obtained using the so called complex affixes

Definition 15.1.2. Complex affixes of a point. Let ξ_P, η_P be the Cartesian coordinates of a point P in the euclidian plane. The $\mathbb{C}$-affix of this point is defined as

$$z_P \doteq \xi_P + \boldsymbol{i}\,\eta_P$$

In this definition, quantity $\boldsymbol{i}$ is a quarter turn. Since two quarter turns performed one after another is nothing but one half turn, we have $\boldsymbol{i}^2 = -1$. This equation has another solution, namely $-\boldsymbol{i}$, the quarter turn in the opposite orientation. Obviously, the very choice of a frame to obtain Cartesian coordinates like ξ_P, η_P ensures a choice of orientation of the plane: when Bob looks at the plane from above, he measures angles by placing his protractor onto the plane, seeing $z_P = \xi_P + \boldsymbol{i}\,\eta_P$. But Alice, the girl who lives on the other side of the looking glass, will be watching from her seat. She will place her protractor on her side of the plane and, therefore, Alice will see

$$\zeta_P \doteq \xi_P - \boldsymbol{i}\,\eta_P$$

Remark 15.1.3. The Conjugate Coordinate System describes points as $(z_P, \zeta_P) \in \mathbb{C}^2$ and curves as $z = f(t)$ where $z \in \mathbb{C}$ and parameter t ranges in $\mathbb{R} \cup \{\infty\}$. See Carver (1956) for a survey.

Definition 15.1.4. Inclusive Coordinate System is the projective version of the Conjugate Coordinate System. When the projective cartesian coordinates of a point are $P \underset{c}{\simeq} X : Y : \mathbf{T}$, the **inclusive** coordinates of this point are defined by

$$P \underset{z}{\simeq} \begin{pmatrix} \mathbf{Z} \\ \mathbf{T} \\ \overline{\mathcal{Z}} \end{pmatrix} \doteq \begin{pmatrix} X + \boldsymbol{i}Y \\ \mathbf{T} \\ X - \boldsymbol{i}Y \end{pmatrix}$$

where $\mathbf{Z}, \mathbf{T}, \overline{\mathcal{Z}}$ are to be read as: "big z", "big tea" and "big zeta". In other words:

$$\begin{pmatrix} \mathbf{Z} \\ \mathbf{T} \\ \overline{\mathcal{Z}} \end{pmatrix} = \boxed{\mathrm{c\Phi m}} \cdot \begin{pmatrix} X \\ Y \\ \mathbf{T} \end{pmatrix} \qquad \text{where } \boxed{\mathrm{c\Phi m}} = \begin{pmatrix} 1 & +\boldsymbol{i} & 0 \\ 0 & 0 & 1 \\ 1 & -\boldsymbol{i} & 0 \end{pmatrix}$$

Remark 15.1.5. An algebraic variable is a placeholder used to write polynomials. It was therefore necessary to find a notation in order to satisfy the following constraints:

1. Use capital letters, since polynomials are usually written $P\left(X_1, X_2, \cdots, X_k\right)$

2. Avoid indices, and deal with the fact that the usual capitalization of the letter "ζ" has the same shape as the letter "big z" (nevertheless, letter $\overline{\mathcal{Z}}$ has to be read as "big zeta").

3. Have a robust cursive version, to facilitate hand computations: a Z is clearly a $\mathbf{Z}$, while a $\overline{Z}$ is clearly a $\overline{\mathcal{Z}}$!

4. Don't suggest the stupid feeling that the variable $\overline{\mathcal{Z}}$ could be the $\mathbb{C}$-conjugate of the variable $\mathbf{Z}$. What could be the $\mathbb{C}$-conjugate of a placeholder ?

The name "inclusive coordinates" has been coined in order to emphazise the fact that we are avoiding a choice of orientation by including both of the Bob's and Alice's viewpoints. [1]

Proposition 15.1.6. *When an algebraic curve is defined by a cartesian polynomial $P_n\left(X : Y : \mathbf{T}\right)$, it is also defined by the complex polynomial Q_n*

$$Q_n\left(\mathbf{Z},\, \mathbf{T},\, \overline{\mathcal{Z}}\right) = P_n\left(\frac{\mathbf{Z} + \boldsymbol{i}\,\overline{\mathcal{Z}}}{2},\, \frac{\mathbf{Z} - \boldsymbol{i}\,\overline{\mathcal{Z}}}{2\,\boldsymbol{i}},\, \mathbf{T}\right)$$

The complex polynomial associated to polynomial $\overline{P_n}$ (obtained by complex conjugation of the coefficients) is noted $\operatorname{conj}\left(Q_n\right)$ *and is therefore*

$$conj\left(\sum_{p+q+r=n} c_{p,\,q,\,r}\, \mathbf{Z}^p\, \overline{\mathcal{Z}}^q\, \mathbf{T}^r\right) = \sum_{p+q+r=n} \overline{c_{p,\,q,\,r}}\, \mathbf{Z}^q\, \overline{\mathcal{Z}}^p\, \mathbf{T}^r$$

in other words, conjugate the coefficients and exchange $\mathbf{Z}$ with $\overline{\mathcal{Z}}$.

Definition 15.1.7. Visible points. When a point $P \simeq \mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}} \in \mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ can be written as $z_P : 1 : \overline{z_P}$ for some $z_P \in \mathbb{C}$, then P will be referred as a visible finite point (aka an ordinary point). When P can be written as $\tau : 0 : 1/\tau$ for some $\tau = \exp\left(\boldsymbol{i}\vartheta\right), \vartheta \in \mathbb{R}$, then P will be referred as visible point at infinity (aka an ordinary direction). Taken together, ordinary points and ordinary directions are referred as the **visible point**s. All the other points of $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ are described as being **not-visible**.

Remark 15.1.8. The not-visible points of $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ correspond to the points that cannot be written as $(x : y : t)$ with $x, y, t \in \mathbb{R}$ in the cartesian projective representation. In the $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^3\right)$ context, they are referred as "non real" objects. In the context of $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$, this designation is no more suitable, and another name must be coined.

Proposition 15.1.9. *An algebraic curve is said to be "reduced to a point" when it contains only one visible point, and "visible" when it contains an infinite number of visible points. The (homogeneous) polynomial of an irreducible visible curve must be proportional to its conjugate.*

Remark 15.1.10. From an abstract point of view, using a constant factor to enforce $conj(P) = P$ is always possible. But in real life, this can only be done by carrying annoying factors and has to be avoided. Before any simplification, a polynomial obtained from a determinant, like the equation of a line or a circle, verifies $conj(P) = -P$.

[1] On the other hand, it cannot be totally excluded that such a name could has been coined in order to provide the additional result of infuriating some old salafs

15.2 Morley method to deal with complex conjugacy

As everybody knows, complex conjugacy is not a smoth transform. Using it poisons the well and kills all polynomial properties. As a result, complex conjugacy has to be avoided at all costs. Therefore all parameters should either

1. belong to $\mathbb{R} \cup \{\infty\}$, and then $\overline{k} = k$

2. belong to the unit circle, and then $\overline{\tau} = 1/\tau$

Definition 15.2.1. Following Morley and Morley (1933), we define a **turn** as a complex number τ used to describe a point on the unit circle, leading to $M \doteq \tau : 1 : 1/\tau$. On the other hand, we define a **clinant** as a complex number κ used to describe a point on the line at infinity, leading to $N \doteq \kappa : 0 : 1$. When M, N are supposed to be visible, the numbers τ, κ are supposed to be unimodular. But "strange" values like 0 or ∞ are allowed (spoiler: they are used to describe the umbilics).

Definition 15.2.2. In the Morley space, the ***line at infinity*** is defined as

$$\mathcal{L}_z \doteq [0, 1, 0] \tag{15.1}$$

Taken up to a proportionality factor, this would only be a a repetition of "$\mathbf{T} = 0$ implies $P \in \mathcal{L}_z$". Here, an hard equality is used.

Theorem 15.2.3. *In the Morley space, the* ***area of triangle****ABC is given by :*

$$area\,(ABC) \underset{z}{=} \left(-\frac{1}{4}\,\boldsymbol{i}\right) \begin{vmatrix} z_A & z_B & z_C \\ 1 & 1 & 1 \\ \overline{z_A} & \overline{z_B} & \overline{z_C} \end{vmatrix} \tag{15.2}$$

Proof. Despite it's great consequences, this result has a very short proof: formula (7.7) using Cartesian coordinates has to be modified by factor $-\boldsymbol{i}/2$ since this is the value of $1/\det \boxed{\text{c}\Phi\text{m}}$. □

Definition 15.2.4. $\boxed{\text{b}\Phi\text{m}}$**, the bartomor Matrix**. Let ABC be a non degenerate triangle and (O, R) its circumcircle. The complex Lubin's coordinate system associated to ABC is centered at O and we have

$$\boxed{\text{b}\Phi\text{m}} \doteq R^2 \begin{pmatrix} \alpha & \beta & \gamma \\ 1 & 1 & 1 \\ 1/\alpha & 1/\beta & 1/\gamma \end{pmatrix} \underset{z}{=} (ABC) \tag{15.3}$$

where α, β, γ are some turns.

Definition 15.2.5. Since our interest is directed toward central objects, we will largely use the so-called **elementary symmetric functions** :

$$\begin{gathered} \sigma_1 = z_A + z_B + z_C,\ \sigma_2 = z_A z_B + z_B z_C + z_A z_C,\ \sigma_3 = z_A z_B z_C \\ \sigma_4 = \boldsymbol{i}\,(z_A - z_B)\,(z_B - z_C)\,(z_C - z_A) \end{gathered} \tag{15.4}$$

Remark 15.2.6. Quantity s_4, i.e. $\boldsymbol{i}$ times the Vandermonde of the three numbers, is skew-symmetric and verifies :

$$s_4^2 = -s_1^2 s_2^2 + 4\,s_3\,s_1^3 + 4\,s_2^3 - 18\,s_3\,s_1\,s_2 + 27\,s_3^2$$

so that s_4 will not appear by a power greater than one. These "big" symmetric functions are not to be confused with the "small" ones, that will be defined later.

Exercise 15.2.7. Use the s_j to write the polynomial $\prod (X - I_k)$ where I_k ranges over the four in/excenters $\pm\beta\gamma \pm \gamma\alpha \pm \alpha\beta$. Then rewrite it using the σ_j and obtain

$$X^4 - 2\sigma_2\,X^2 + 8\sigma_3 X + \left(\sigma_2^2 - 4\sigma_1\sigma_3\right).$$

Proposition 15.2.8. *In the Morley space, the matrix* $\boxed{\mathrm{Pyth}_z}$ *of the quadratic form that gives the squared length of a vector in the* $\overrightarrow{\mathcal{V}}$ *space can be written as :*

$$\boxed{\mathrm{Pyth}_z} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{15.5}$$

Proof. Here, equalities are required since a length is not defined up to a proportionality factor. Remember that an element of $\overrightarrow{\mathcal{V}}$ is obtained as the difference of the normalized columns of two finite points, and therefore looks like $z_1/t_1 - z_2/t_2$, 0, $\overline{z_1}/t_1 - \overline{z_2}/t_2$. The resulting formula is nothing but the usual $|\zeta|^2 = \zeta\,\overline{\zeta}$. □

Proposition 15.2.9. *We have the following* **forward substitutions formulae:**

$$S = \frac{\boldsymbol{i}\,R^2\,(\alpha-\beta)\,(\beta-\gamma)\,(\gamma-\alpha)}{4\,\alpha\,\beta\,\gamma},\; a = \frac{R\,(\beta-\gamma)}{\sqrt{-\beta\,\gamma}},\text{ etc} \tag{15.6}$$

Proof. Direct examination. These partial results are required when trying to use

$$\boxed{\mathrm{Pyth}_z} = {}^t\boxed{\mathrm{b\Phi m}}^{-1}\cdot\boxed{\mathrm{Pyth}_b}\cdot\boxed{\mathrm{b\Phi m}}^{-1}$$

as a method to prove the $\boxed{\mathrm{Pyth}_z}$ formula from the $\boxed{\mathrm{Pyth}_b}$ formula. □

15.3 Morley version of the usual operators

Remark 15.3.1. The usual building process is cartesian coordinates $\mapsto$ complex coordinates $\mapsto$ barycentrics. In this book, barycentric coordinates have been treated first, and it only remains now to deduce the properties of complex coordinates from those of barycentric coordinates and from the $\mathcal{L}_z$, $\boxed{\mathrm{Pyth}_z}$ and $\boxed{\mathrm{b\Phi m}}$ formulae that were obtained in the previous section.

Proposition 15.3.2. *Let V be the point at infinity of a line Δ given by their z-coordinates. Then we have*

$$V \doteq \Delta \wedge \mathcal{L}_z \simeq \boxed{W_z}\cdot{}^t\Delta \qquad \text{where } \boxed{W_z} = 2\boldsymbol{i}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ +1 & 0 & 0 \end{pmatrix} \tag{15.7}$$

Proof. This assertion is nothing but the very definition of the $\wedge$ operator, while the $2\boldsymbol{i}$ coefficient is not involved in this property. This can also be seen as $\boxed{W_z} = \boxed{\mathrm{b\Phi m}}\cdot\boxed{W_b}\cdot{}^t\boxed{\mathrm{b\Phi m}}$ □

Proposition 15.3.3. ***Orthopoint***. *In the Morley space, the operator that transforms a direction $V \in \mathcal{L}_z$ into its orthogonal direction while transforming the circumcenter into $0:0:0$, is described by :*

$$\boxed{\mathrm{OrtO}_z} = \boldsymbol{i}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{15.8}$$

Proof. One can use $\boxed{\mathrm{OrtO}_z} = \boxed{\mathrm{b\Phi m}}\cdot\boxed{\mathrm{OrtO}_b}\cdot\boxed{\mathrm{b\Phi m}}^{-1}$ and then the usual substitutions. On the other hand, eigenvectors of $\boxed{\mathrm{OrtO}_z}$ are obviously both umbilics $\Omega_y \simeq 1:0:0$, $\Omega_x \simeq 0:0:1$ and $0:1:0$ (the circumcenter): umbilic Ω_y is rotated by a quarter of turn, and umbilic Ω_x is rotated by the opposite amount. □

Proposition 15.3.4. *The* **orthodir** *operator gives the orthopoint $V^\perp$ of the point at infinity of a line Δ. In the Morley space, this operator can be written using matrix* $\boxed{\mathcal{M}_z}$ *according to :*

$$V^\perp = \boxed{\mathcal{M}_z}\cdot{}^t\Delta \qquad \text{where } \boxed{\mathcal{M}_z} \doteq 2\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \boxed{\mathrm{OrtO}_z}\cdot\boxed{W_z} \tag{15.9}$$

Proof. Obvious from definitions. One can check that $|V| = |V^{\perp}|$. This can also be seen as $\boxed{\mathcal{M}_z} = \boxed{\mathrm{b\Phi m}} \cdot \boxed{\mathcal{M}_b} \cdot {}^t\boxed{\mathrm{b\Phi m}}$. □

Theorem 15.3.5. Tangent of two lines. *In the Morley space, the oriented angle from a visible line Δ_1 to a visible line Δ_2 is characterized by :*

$$\tan\left(\widehat{\Delta_1,\,\Delta_2}\right) = \frac{\Delta_1 \cdot \boxed{W_z} \cdot {}^t\Delta_2}{\Delta_1 \cdot \boxed{\mathcal{M}_z} \cdot {}^t\Delta_2} \tag{15.10}$$

where $\boxed{W_z}$, $\boxed{\mathcal{M}_z}$ *are exactly as given in* (15.7) *and* (15.9) *(not up to a proportionality factor).*

Proof. Formulas (7.22) and (15.10) have exactly the same shape and each of them generates the other through $\boxed{\mathrm{b\Phi m}}$. This can also be obtained from the fact that $\tan(D_1, D_2)$ is obviously

$$(p_2 - p_1) \div (1 + p_1 p_2)$$

when considering the two Cartesian lines $y = p_1 x + m_1$ and $y = p_2 w + m_2$. Indeed, the slope of a line is the tangent of the angle from the x axis to the line: it only remains to use the addition formula. Numerator tell us when lines are parallel, and denominator when they are orthogonal. □

Proposition 15.3.6 (Laguerre Formula)**.** *Consider two (distinct) lines Δ_1, Δ_2 and their common point M. Then*

$$\text{cross\_ratio}\,(M\Omega_y, M\Omega_x, \Delta_1, \Delta_2) = \exp\left(2\boldsymbol{i}\left(\widehat{\Delta_1,\,\Delta_2}\right)\right)$$

Proof. A tedious proof: compute and check using (15.10). □

Proof. A better proof: the four lines are cutting the line at infinity, and we have:

$$\text{cross\_ratio}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}\tau\\0\\1/\tau\end{bmatrix}, \begin{bmatrix}\sigma\\0\\1/\sigma\end{bmatrix}\right) = \frac{\sigma^2}{\tau^2}$$ □

Proposition 15.3.7. *In the Morley space,* ***distance from point*** P ***to line*** Δ *is given by :*

$$\mathrm{dist}\,(P,\,\Delta) = \frac{\Delta \cdot P}{(\mathcal{L}_z \cdot P)\,\sqrt{\Delta \cdot \boxed{\mathcal{M}_z} \cdot {}^t\Delta}} = \frac{fp + gq + hr}{2q\sqrt{hf}} \tag{15.11}$$

where $\mathcal{L}_z = [0, 1, 0]$ *and* $\boxed{\mathcal{M}_z}$ *is as given in* (15.9) *(not up to a proportionality factor).*

Proof. Immediate consequence of the barycentric formula (7.24). One can also transpose the proof given there. □

Remark 15.3.8. Formula (15.11) is invariant when coordinates of P or Δ are modified by a proportionality factor. Denominators are enforcing the fact that P is supposed to be at finite distance, and Δ is supposed not to be isotropic. The square root is the operator norm of the application

$$\phi :\ \mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}} \mapsto \left(z\,\mathbf{Z} + t\,\mathbf{T} + \zeta\,\overline{\mathcal{Z}}\right) / \left(\mathbf{T}\right)$$

Replace $\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}$ by $\mathbf{Z} + \mathbf{T}\,r\,\tau : \mathbf{T} : \overline{\mathcal{Z}} + \mathbf{T}\,r/\tau$. Derive wrt τ and obtain $r\,z - r\,\zeta/\tau^2$. Solve in τ and obtain $\tau = \sqrt{\zeta/z}$. So that $\Delta\phi = 2\sqrt{z\zeta}\,r$.

Proposition 15.3.9. *The Clawson-Schmidt homography is defined by:*

$$\Psi : A \mapsto A',\ B \mapsto B',\ C \mapsto C'$$

where A, B, C form a true triangle, while $A' \in BC$, etc. *Then A', B', C' are aligned if and only if Ψ is involutive.*

Proof. Let $A' = pB + (1-p)\,C$, etc. Compute Ψ, use the $\mathrm{a} + \mathrm{d} = 0$ rule and obtain the usual

$$pqr + (1-r)(1-q)(1-p) = 0$$ □

15.4 Lubin representation of first degree

Proposition 15.4.1. *The* **Morley-affix** *of a point P whose barycentrics are $p:q:r\in\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^{3}\right)$ with respect to triangle ABC is given by :*

$$\zeta_{P}\simeq\left(\begin{array}{ccc}z_{A} & z_{B} & z_{C}\\ 1 & 1 & 1\\ \overline{z_{A}} & \overline{z_{B}} & \overline{z_{C}}\end{array}\right)\cdot\left(\begin{array}{c}p\\ q\\ r\end{array}\right)\qquad(15.12)$$

Proof. When $p+q+r\neq0$, this is nothing but the usual definition of a barycenter, and ζ_{P} is a Morley finite point. When $p+q+r=0$, P is at infinity and ζ_{P} is a Morley direction. The condition $p,q,r\in\mathbb{R}$ ensures that no invisible points in the Morley-space can be generated from a real point in the Kimberling space. □

Definition 15.4.2. The Lubin parametrizations are obtained by assuming that the circumcircle of triangle ABC is nothing but the unit circle of the complex plane, together with the relations :

$$z_{A}=\alpha^{n},\;z_{B}=\beta^{n},\;z_{C}=\gamma^{n}$$

Remark 15.4.3. The $z_{A}=\alpha^{1}$, etc parametrization has already been introduced at (15.3) in order to avoid conjugacies. Here, an additionnal aim is to avoid the fractional powers that would otherwise appear from $a=\sqrt{a^{2}}$ and other angle division.

Definition 15.4.4. Since our interest is directed toward central objects, we will largely use the so-called **elementary symmetric functions**. The "big" ones are the already defined $\sigma_{1}=z_{A}+z_{B}+z_{C}$, etc, while the "small" ones are:

$$s_{1}\doteq\alpha+\beta+\gamma=,\;s_{2}\doteq\alpha\,\beta+\beta\,\gamma+\alpha\,\gamma,\;s_{3}\doteq\alpha\,\beta\,\gamma,\;s_{4}=\boldsymbol{i}\left(\alpha-\beta\right)\left(\beta-\gamma\right)\left(\gamma-\alpha\right)\qquad(15.13)$$

Theorem 15.4.5 (Newton)**.** *For any commutative ring A, every symmetric polynomial in n variables has a unique representation as a polynomial into the elementary symmetric functions of the said variables.*

**** listing the algorithm would be great ****

Theorem 15.4.6. Forward substitutions. *Suppose that barycentrics $p:q:r$ of point P depends rationally on a^{2}, b^{2}, c^{2}, S. Then Morley-affix of P is obtained by substituting the identities (15.6), i.e. :*

$$S=\frac{\boldsymbol{i}\,R^{2}\left(\alpha-\beta\right)\left(\beta-\gamma\right)\left(\gamma-\alpha\right)}{4\,\alpha\,\beta\,\gamma},\;a=\frac{R\,\left(\beta-\gamma\right)}{\sqrt{-\beta\,\gamma}},\;\text{etc}$$

into $p:q:r$ and premultiplying by $\boxed{\mathrm{b\Phi m}}$. *The result obtained is a rational fraction in α, β, γ whose degree is $+1$. When P is a triangle center, ζ_{P} depends only on σ_{1}, σ_{2}, σ_{3}. When P is invariant by circular permutation, but not by transposition, a σ_{4} term appears.*

Proof. Cancellation of radicals is assured by condition $p,q,r\in\mathbb{Q}\left(a^{2},\,b^{2},\,c^{2},\,S\right)$. Elimination of R comes from homogeneity. Symmetry properties are evident. □

Proposition 15.4.7. Backward substitutions. *Let $z_{P}\in\mathbb{C}\left(\alpha,\,\beta,\,\gamma\right)$ be an homogeneous rational fraction, supposed to be the complex affix of a finite point. Then $\deg\left(z_{P}\right)=1$ is required. Alternatively, let $\omega^{2}\in\mathbb{C}\left(\alpha,\,\beta,\,\gamma\right)$ be an homogeneous rational fraction, supposed to describe the Morley affix of a direction. Then $\deg\left(\omega^{2}\right)=2$ is required. When these conditions are fulfilled, the ABC-barycentrics $p:q:r$ of these objects can be obtained as follows. Compute the corresponding vector :*

$$\left(\begin{array}{c}p\\ q\\ r\end{array}\right)=\boxed{\mathrm{b\Phi m}}^{-1}\cdot\left(\begin{array}{c}z_{p}\\ 1\\ \overline{z_{p}}\end{array}\right)\quad\text{or}\quad\left(\begin{array}{c}p\\ q\\ r\end{array}\right)=\boxed{\mathrm{b\Phi m}}^{-1}\cdot\left(\begin{array}{c}\omega^{2}\\ 0\\ 1\end{array}\right)$$

then apply substitutions

$$\begin{aligned}\beta &= \left(\frac{a^{4}+b^{4}+c^{4}-2\left(a^{2}+b^{2}\right)c^{2}}{2\,a^{2}b^{2}}+2\boldsymbol{i}S\,\frac{a^{2}+b^{2}-c^{2}}{a^{2}b^{2}}\right)\alpha\\ \gamma &= \left(\frac{a^{4}+b^{4}+c^{4}-2\left(a^{2}+c^{2}\right)b^{2}}{2\,a^{2}c^{2}}-2\boldsymbol{i}S\,\frac{c^{2}+a^{2}-b^{2}}{a^{2}c^{2}}\right)\alpha\end{aligned}$$

to this vector and simplify the obtained expression using the Heron formula :

$$S^2 = -\frac{1}{16}\,(a+b+c)\,(b+c-a)\,(c+a-b)\,(a+b-c)$$

Proof. Transform α, β, γ into $\alpha\delta$, $\beta\delta$, $\gamma\delta$. Since this transform is a similarity, barycentrics must remain unchanged and the z_P affix is turned by δ. On the other hand, polynomial z_P is homogeneous and z_P is multiplied by δ^k where $k = \deg(z_P)$. Concerning the directions, $\arg(\omega^2)$ is twice the angle with the real axis, and degree 2 is required.

Alternatively, the degrees of rows of ζ_P are $+k$, 0, $-k$ while the degrees of columns of $\boxed{\mathrm{b\Phi m}}^{-1}$ are -1, 0, $+1$. Quantities p, q, r will therefore be a sum of terms whose degrees are respectively $k-1$, 0, $1-k$. But homogeneity is required in order that a transformation $\beta = B\,\alpha$, $\gamma = C\,\alpha$ can eliminate α, leading to $k = 1$.

To obtain the substitution formulas, compute β from $c^2\,\alpha\,\beta = -R^2\,(\alpha-\beta)^2$. A choice of branch (a sign for $\boldsymbol{i}$) has to be done. Exchange b and c (and therefore change S into $-S$) and obtain the corresponding γ. □

Remark 15.4.8. The substitution formulas can be written as:

$$\beta = \left(2\left(\frac{S_c}{a\,b}\right)^2 - 1 + 2\boldsymbol{i}\,\left(\frac{2\,S}{a\,b}\right)\left(\frac{S_c}{a\,b}\right)\right)\alpha\ ;\ \gamma = \left(2\left(\frac{S_b}{a\,c}\right)^2 - 1 - 2\boldsymbol{i}\,\left(\frac{2\,S}{a\,c}\right)\left(\frac{S_b}{a\,c}\right)\right)\alpha$$

i.e. $\beta = \alpha\exp(2\boldsymbol{i}\,C)$, $\gamma = \alpha\exp(-2\boldsymbol{i}\,B)$. This result is indeed symmetric, since $\widehat{B} = (BC,\ BA)$ while $\widehat{C} = (CA, CB)$.

Remark 15.4.9. When starting with a symmetric Morley-affix, the obtained $p : q : r$ remains symmetric in α, β, γ. The given substitutions are breaching the symmetry of individual coefficients p, q, r, that can only be reestablished by cancellation of asymmetric common factors between the p, q, r. Most of the time, its more efficient to proceed by numerical substitution and use the obtained search key to identify the point (and proceed back to obtain a proof of the result).

Proposition 15.4.10. *The Kimberling search key associated to a visible finite point defined by its Morley affix (short= Morley's search key) is obtained by substituting :*

$$z_A = 1\ ;\ z_B = -\frac{391}{729} - \boldsymbol{i}\,\frac{104}{729}\sqrt{35}\ ;\ z_C = \frac{401}{1521} - \boldsymbol{i}\,\frac{248}{1521}\sqrt{35}$$

into $\mathbf{Z}/\mathbf{T}$ *and then computing :*

$$searchkey\left(\frac{\mathbf{Z}}{\mathbf{T}}\right) \doteq \Re\left(\left(\frac{157}{840}\sqrt{35} - \boldsymbol{i}\,\frac{22}{3}\right)\frac{\mathbf{Z}}{\mathbf{T}}\right) + \frac{321}{280}\sqrt{35}$$

Proof. Kimberling's search keys are associated with triangle $a = 6$, $b = 9$, $c = 13$. The radius of the circumcircle is $R = (351/280)\sqrt{35}$. One can see that sidelengths of triangle $\alpha\beta\gamma$ are $6/R$, $9/R$, $13/R$. We apply these substitutions to obtain the numerical value of the "wayback" matrix, and then use (6.1) □

Remark 15.4.11. When using Lubin-n with $n > 1$, adequate substitutions have to be used to calculate $\mathbf{Z}/\mathbf{T}$.

Proposition 15.4.12. *The Morley's searchkey of a visible point at infinity (*$\mathbf{T} = 0$*) is obtained from* $\Omega = \mathbf{Z}/\overline{\mathcal{Z}}$ *by :*

$$\frac{1108809\,(241+16\,\boldsymbol{i}\sqrt{35})\,\Omega^2 - 907686\,(157+176\,\boldsymbol{i}\sqrt{35})\,\Omega + (-224394311+30270800\,\boldsymbol{i}\sqrt{35})}{389191959\,\Omega^2 - (106136082+118980576\,\boldsymbol{i}\sqrt{35})\,\Omega + (19397664\,\boldsymbol{i}\sqrt{35} - 371888361)}$$

Another method is identifying the isogonal conjugate of the given point, which is simply : $\sigma_3/\Omega : -1 : \Omega/\sigma_3$.

Proof. The searchkey of an point at infinity is : $\frac{x}{a} \times \left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right)$, leading to this tremendous expression. But after all, this formula is not designed for hand computation but rather to a floating evaluation by a computer... . □

15.5 Some examples of first degree

Example 15.5.1. The circumcenter O. By definition, $\zeta_O = 0:1:0$. The preceding transformations are giving :

$$\left(\begin{array}{c} p\\ q\\ r \end{array}\right)\simeq\left(\begin{array}{c} \alpha\,(\beta-\gamma)\,(\beta+\gamma)\\ \beta\,(\gamma-\alpha)\,(\alpha+\gamma)\\ \gamma\,(\alpha-\beta)\,(\alpha+\beta) \end{array}\right)\simeq\left(\begin{array}{c} a^2\left(b^2+c^2-a^2\right)\\ b^2\left(c^2+a^2-b^2\right)\\ c^2\left(a^2+b^2-c^2\right) \end{array}\right)$$

Example 15.5.2. Symmedian point X(6), aka Lemoine point.

1. Consider the middle A' of segment $[B,C]$ and define the A symmedian as the line Δ_A that goes through A and verifies $\angle(AB,\Delta_A)=\angle(AA',AC)$. We will use P instead ζ_P since this is more readable ... and hand-writable. A remark : symmetry wrt bisectors would be irrelevant, since bisectors are unreachable in the Lubin-1 representation !

2. We have $A'=B+C$, $AA'=A\wedge A'$ etc. and our equations are

$$\begin{array}{rcl} \Delta_A\cdot A &=& 0\\ \tan(AB,\,\Delta_A)+\tan(AC,\,AA') &=& 0 \end{array}$$

   solving this system, then permuting, gives :

$$\begin{array}{rcl} \Delta_A &\simeq& \left(\begin{array}{ccccc} 2\,\alpha-\beta-\gamma & ; & (\alpha\,\gamma+\alpha\,\beta-2\,\beta\,\gamma)\,\alpha & ; & -2\,\alpha^2+2\,\beta\,\gamma \end{array}\right)\\ \Delta_B &\simeq& \left(\begin{array}{ccccc} 2\,\beta-\gamma-\alpha & ; & (\alpha\,\beta+\beta\,\gamma-2\,\alpha\,\gamma)\,\beta & ; & -2\,\beta^2+2\,\alpha\,\gamma \end{array}\right) \end{array}$$

3. Intersecting two symmedians gives a symmetric result. Therefore, the three symmedians are concurrent at some point. This point is well-known as the Lemoine point, and we have :

$$K=\zeta\,(6)=\Delta_A\wedge\Delta_B=\left(\begin{array}{c} 2\,\sigma_2^2-6\,\sigma_3\,\sigma_1\\ \sigma_2\,\sigma_1-9\,\sigma_3\\ 2\,\sigma_1^2-6\,\sigma_2 \end{array}\right)$$

4. Going back to barycentrics, we obtain the well-known result :

$$X\,(6)\simeq\left(\begin{array}{c} \alpha\,(\gamma-\beta)^2\\ \beta\,(\alpha-\gamma)^2\\ \gamma\,(\alpha-\beta)^2 \end{array}\right)\simeq\left(\begin{array}{c} a^2\\ b^2\\ c^2 \end{array}\right)$$

Example 15.5.3. The Kiepert parabola.

1. Morley equation of the circumcircle is $\mathbf{Z}\,\overline{\mathcal{Z}}-\mathbf{T}^2=0$. The Morley affix Δ_P of the polar line of point $K=z:1:\zeta$ wrt the circumcircle is therefore given by :

$$\Delta_P\doteq\left[\begin{array}{ccc} z & 1 & \zeta \end{array}\right]\cdot\left(\begin{array}{ccc} 0 & 0 & 1\\ 0 & -2 & 0\\ 1 & 0 & 0 \end{array}\right)$$

2. The coefficients of the tangential conic determined by five given lines $[u_j,v_j,w_j]$ are obtained as :

$$\bigwedge_{j=1..5}\left[u_j^2,\,v_j^2,\,w_j^2,\,u_jv_j,\,v_jw_j,\,w_ju_j\right]$$

   by universal factorization of the corresponding 6×6 determinant. Let us consider the inconic tangent to the infinity line (parabola) and to the circumpolar of point K. Using $(BC)\simeq B\wedge C$, etc together with the previous equation, we obtain the symmetric matrix :

$$\boxed{\mathcal{C}^*}\simeq\left(\begin{array}{ccc} 2\,\sigma_3\,z^2-2\,\sigma_2\,\sigma_3\,z\,\zeta+4\,\sigma_3^2\,\zeta & qsp & -\sigma_1\,z^2+\sigma_2\,\sigma_3\,\zeta^2+2\,\sigma_2\,z-2\,\sigma_3\,\sigma_1\,\zeta\\ -\sigma_3\,\sigma_1\,z\zeta+\sigma_3^2\zeta^2+2\,\sigma_3\,z & 0 & -z^2+\sigma_2\,z\,\zeta-2\,\sigma_3\,\zeta\\ -\sigma_1\,z^2+\sigma_2\,\sigma_3\,\zeta^2+2\,\sigma_2\,z-2\,\sigma_3\,\sigma_1\,\zeta & qsp & 2\,\sigma_1\,z\,\zeta-2\,\sigma_3\,\zeta^2-4\,z \end{array}\right)$$

Degrees of all these expressions are :

$$\mathrm{dg}\left(\boxed{\mathcal{C}^*}\right)=\left(\begin{array}{ccc}5&4&3\\4&.&2\\3&2&1\end{array}\right)$$

3. A focus is a point such that both isotropic lines through that point are tangent to the conic. Writing that $Q=\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)\wedge(1:0:0)$ satisfies $Q\cdot\boxed{\mathcal{C}^*}\cdot{}^tQ=0$ and the similar with the other umbilic gives two equations whose solution is

$$F(z)\simeq\left(\begin{array}{c}\dfrac{z^2-\sigma_2\,z\zeta+2\,\sigma_3\,\zeta}{2\,z-\sigma_1\,z\zeta+\sigma_3\,\zeta^2}\\1\\\dfrac{2\,z-\sigma_1\,z\zeta+\sigma_3\,\zeta^2}{z^2-\sigma_2\,z\zeta+2\,\sigma_3\,\zeta}\end{array}\right)$$

and this point is on the circumcircle.

4. Take K at the Lemoine point X(6). Its circumpolar line is called the Lemoine axis. One obtains the Kiepert parabola, where :

$$\boxed{\mathcal{C}^*}\simeq\left(\begin{array}{ccc}2\,\sigma_2\,\sigma_3&\sigma_3\,\sigma_1&0\\\sigma_3\,\sigma_1&0&-\sigma_2\\0&-\sigma_2&-2\,\sigma_1\end{array}\right),\ \boxed{\mathcal{C}}\simeq\left(\begin{array}{ccc}-\sigma_2^2&2\,\sigma_3\,\sigma_1^2&-\sigma_2\,\sigma_3\,\sigma_1\\2\,\sigma_3\,\sigma_1^2&-4\,\sigma_2\,\sigma_3\,\sigma_1&2\,\sigma_2^2\,\sigma_3\\-\sigma_2\,\sigma_3\,\sigma_1&2\,\sigma_2^2\sigma_3&-\sigma_1^2\,\sigma_3^2\end{array}\right)$$

5. As stated in Proposition 12.3.15, the triangle of the circle-polars of the sidelines of triangle $\boxed{\mathcal{T}}$ is described by matrix $\boxed{\mathcal{C}^*}\cdot{}^t\boxed{\mathcal{T}^*}$. Both triangle are in perspective (lines AA', etc are concurrent). The perspector is obtained as $AA'\wedge BB'$ and concurrence is verified by the symmetry of the result. It is well-known that P =X(99), the Steiner point.

$$F=\frac{\sigma_2}{\sigma_1},\ P=\frac{\sigma_3\sigma_1^2-3\,\sigma_2\sigma_3}{\sigma_2^2-3\,\sigma_3\sigma_1}$$

6. Applying the preceding transformations, the $p:q:r$ associated with $z=\sigma_2/\sigma_1$ is obtained as :

$$\left(\begin{array}{c}p\\q\\r\end{array}\right)\simeq\left(\begin{array}{c}\alpha\,(\beta-\gamma)\left(\alpha\,\beta-\gamma^2\right)\left(\gamma\,\alpha-\beta^2\right)\\\beta\,(\gamma-\alpha)\left(\beta\,\gamma-\alpha^2\right)\left(\alpha\,\beta-\gamma^2\right)\\\gamma\,(\alpha-\beta)\left(\gamma\,\alpha-\beta^2\right)\left(\beta\,\gamma-\alpha^2\right)\end{array}\right)\simeq\left(\begin{array}{c}\dfrac{a^2}{(b+c)\,(b-c)}\\\dfrac{b^2}{(c+a)\,(c-a)}\\\dfrac{c^2}{(a+b)\,(a-b)}\end{array}\right)$$

and we can identify X(110), the focus of the Kiepert parabola.

Example 15.5.4. Isogonic, isodynamic and Napoleon.

1. Define $j=\exp\left(2\,\boldsymbol{i}\pi/3\right)$. Start from triangle ABC. Construct P_A such that triangle P_ABC is equilateral. More precisely, the z affix of P_A is such that $z+j\,\beta+j^2\,\gamma=0$, deciding of the orientation. The three lines AP_A, BP_B, CP_C are concurrent, leading to the first isogonic center X(13). Changing j into j^2 leads to the second isogonic point X(14). A simple computation leads to :

$$\frac{9\,\sigma_2\sigma_3-12\,\sigma_3\sigma_1^2+3\,\sigma_1\sigma_2^2}{6\,\sigma_2^2-18\,\sigma_3\sigma_1}\pm\sqrt{3}\frac{\sigma_4\,\sigma_2}{6\,\sigma_2^2-18\,\sigma_3\sigma_1}$$

2. When trying to transform the former expression into barycentrics, the formal computer is poisoned by the following fact. Quantity σ_4 describes the orientation of the triangle, while

the choice of $\pm\sqrt{3}$ depends on the orientation of the whole plane. We better generalize the problem using $\tan(AB,\, AA') = K$. This leads to :

$$\zeta_K \simeq \left(\begin{array}{c} 4\,K\left(\sigma_2^2 - 3\,\sigma_3\,\sigma_1\right) + \sigma_4\left(K^2+1\right)\sigma_1 \\ 2\,K\left(\sigma_2\,\sigma_1 - 9\,\sigma_3\right) + \sigma_4\left(3+K^2\right) \\ 4\,K\left(\sigma_1^2 - 3\,\sigma_2\right) + \sigma_4\left(K^2+1\right)\dfrac{\sigma_2}{\sigma_3} \end{array}\right)$$

$$\left(\begin{array}{c} p \\ q \\ r \end{array}\right) \simeq \left(\begin{array}{c} \dfrac{\gamma-\beta}{(\beta+\gamma)\,K + \boldsymbol{i}\,(\gamma-\beta)} \\ \dfrac{\alpha-\gamma}{(\alpha+\gamma)\,K + \boldsymbol{i}\,(\alpha-\gamma)} \\ \dfrac{\beta-\alpha}{(\alpha+\beta)\,K + \boldsymbol{i}\,(\beta-\alpha)} \end{array}\right) \simeq \left(\begin{array}{c} \dfrac{1}{2\,S - S_a K} \\ \dfrac{1}{2\,S - S_b K} \\ \dfrac{1}{2\,S - S_c K} \end{array}\right)$$

3. And we obtain a lot of results when changing K, and even more by isogonal conjugacy. In the following table, line K lists usual values for the tangent of an angle, while the other two lines give the Kimberling number of the corresponding points. The P_K points are on the Kiepert RH (more details in Proposition 13.22.2).

K	$-\sqrt{3}$	-1	$\frac{-1}{2}$	$\frac{-1}{\sqrt{3}}$	0	$\frac{+1}{\sqrt{3}}$	$\frac{+1}{2}$	1	$\sqrt{3}$	∞
P_K	13	485	3316	17	2	18	3317	486	14	4
$isog\,(P_K)$	15	371	3311	61	6	62	3312	372	16	3

15.6 Lubin representation of second degree

When dealing with half angles, we have to introduce the mid-arcs on the circumcircle of ABC, i.e. the circumcevians of the in-excenters.

Proposition 15.6.1. ***Lubin-2 parametrization***. *When using parametrization* $z_A = \alpha^2$, etc*, the mid-arcs* M_j *are* $\pm\beta\gamma, \pm\gamma\alpha, \pm\alpha\beta$. *But there are only four choices of sign since*

$$\text{product of midarcs} = (-1) \times \text{product of vertices}$$

must be enforced. When using the symmetric choice, i.e. $-\alpha\beta, -\beta\gamma, -\gamma\alpha$ *then the three lines* AM_a, BM_b, CM_c *concur at* $-\alpha\beta - \beta\gamma - \gamma\alpha = -s_2$.

Remark 15.6.2. **Lemoine transform**. As already stated at Theorem 2.1.9, the Lemoine transforms are obtained by $\alpha \mapsto -\alpha$ or $\beta \mapsto -\beta$ or $\gamma \mapsto -\gamma$ when using the Lubin-2 parametrization. And then, the Lubin-1 points (aka the strong points) remain unchanged under these actions.

Proof. Here again, the fact that $L_a \circ L_b = L_c$ comes from the homogeneity required for the formulas of interest. Remember: a theorem is a proposition with the biggest consequences, not something difficult to prove. □

Proposition 15.6.3. Forward-*2* and backward-*2* matrices. *Using the Lubin-2 parametrization, we have :*

$$\boxed{Lu_2} = \left(\begin{array}{ccc} \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \\ 1/\alpha^2 & 1/\beta^2 & 1/\gamma^2 \end{array}\right) \quad ; \quad \det\boxed{Lu_2} = \frac{\boldsymbol{i}\sigma_4}{\sigma_3} = \frac{\boldsymbol{i}\;s_4}{s_3}\;\frac{s_1\,s_2 - s_3}{s_3} = \frac{4\boldsymbol{i}}{R^2}\,S \tag{15.14}$$

$$\boxed{Lu_2^{-1}} = \frac{1}{\boldsymbol{i}\sigma_4}\left[\begin{array}{ccc} \alpha^2\left(\beta^2-\gamma^2\right) & \alpha^2\left(\gamma^4-\beta^4\right) & \sigma_3\left(\beta^2-\gamma^2\right) \\ \beta^2\left(\gamma^2-\alpha^2\right) & \beta^2\left(\alpha^4-\gamma^4\right) & \sigma_3\left(\gamma^2-\alpha^2\right) \\ \gamma^2\left(\alpha^2-\beta^2\right) & \gamma^2\left(\beta^4-\alpha^4\right) & \sigma_3\left(\alpha^2-\beta^2\right) \end{array}\right]$$

Proof. These formulas come from (15.2). Remember that $s_1 = \alpha + \beta + \gamma$ while $\sigma_1 = z_A + z_B + z_C$ etc. □

Theorem 15.6.4. Forward-2 substitutions. *Suppose that barycentrics $p:q:r$ of point P depends rationally on a, b, c, S. Then Morley-affix of P is obtained by substituting the identities :*

$$S=\frac{\boldsymbol{i}\;R^{2}\left(\alpha^{2}-\beta^{2}\right)\left(\gamma^{2}-\alpha^{2}\right)\left(\beta^{2}-\gamma^{2}\right)}{4\;\alpha^{2}\;\beta^{2}\;\gamma^{2}}\;\;;\;a=\boldsymbol{i}R\left(\frac{\gamma}{\beta}-\frac{\beta}{\gamma}\right),\text{ etc}\tag{15.15}$$

into $p:q:r$ and applying (15.12). The result obtained is a rational fraction in α, β, γ whose degree is $+2$. When P is a triangle center, ζ_P depends only on s_1, s_2, s_3. When P is invariant by circular permutation, but not by transposition, a s_4 term appears.

Proof. Elimination of R comes from homogeneity. Symmetry properties are evident. Sign chosen for a is irrelevant, but signs of b, c must be chosen accordingly. □

Proposition 15.6.5. Backward substitutions. *Let $z_P \in \mathbb{C}(\alpha,\ \beta,\ \gamma)$ be an homogeneous rational fraction, supposed to be the Lubin-2 affix of a finite point. Then $\deg(z_P)=2$ is required. Alternatively, let $\omega^2 \in \mathbb{C}(\alpha,\ \beta,\ \gamma)$ is an homogeneous rational fraction, supposed to describe the Lubin-2 affix of a direction. Then $\deg\left(\omega^2\right)=4$ is required. When these conditions are fulfilled, the ABC-barycentrics $p:q:r$ of these objects can be obtained as follows. Compute the corresponding vector :*

$$\left(\begin{array}{c}p\\q\\r\end{array}\right)=\boxed{Lu_2^{-1}}\cdot\left(\begin{array}{c}z_p\\1\\\overline{z_p}\end{array}\right)\quad\text{or}\quad\left(\begin{array}{c}p\\q\\r\end{array}\right)=\boxed{Lu_2^{-1}}\cdot\left(\begin{array}{c}\omega^2\\0\\1\end{array}\right)$$

then apply substitutions

$$\alpha=-1\;;\;\beta=\frac{S_c+2\,\boldsymbol{i}S}{ab}\;\;;\;\gamma=\frac{S_b-2\,\boldsymbol{i}S}{ac}\tag{15.16}$$

to this vector and simplify the obtained expression using the Heron formula :

$$S^2=-\frac{1}{16}\left(a+b+c\right)\left(b+c-a\right)\left(c+a-b\right)\left(a+b-c\right)$$

Proof. Result about degrees follows Proposition 15.4.7. Substitutions are $\alpha=-1,\ \beta=\exp\left(+\boldsymbol{i}C\right),\ \gamma=\exp\left(-\boldsymbol{i}B\right)$. This result is indeed symmetric, since $\widehat{B}=(BC,\ BA)$ while $\widehat{C}=(CA, CB)$. □

15.7 Poncelet representation

Notation 15.7.1. In this section z_M denotes the Lubin(2) affix of a point M, while ζ_M denotes the Poncelet affix of the same point M. The respective "coordinates in the view from below" will be noted as $\overline{z_M}$ and $\overline{\zeta_M}$, leading to

$$M\underset{Lubin}{\simeq}\left(\begin{array}{c}z_M\\1\\\overline{z_M}\end{array}\right)\underset{Poncelet}{\simeq}\left(\begin{array}{c}\zeta_M\\1\\\overline{\zeta_M}\end{array}\right)$$

Definition 15.7.2. The parameters of this representations are the contact points of the incircle, described as ρ, σ, τ in a frame using this circle as unit circle. Thus $\overline{\rho}=1/\rho$, etc.

Proposition 15.7.3. *This representation describes the triangle ABC by:*

$$\boxed{Pon}=\left[\begin{array}{ccc}\dfrac{2\tau\,\sigma}{\sigma+\tau}&\dfrac{2\,\rho\,\tau}{\tau+\rho}&\dfrac{2\,\rho\,\sigma}{\rho+\sigma}\\1&1&1\\2\,\left(\sigma+\tau\right)^{-1}&2\,\left(\tau+\rho\right)^{-1}&2\,\left(\rho+\sigma\right)^{-1}\end{array}\right]$$

The algebraic direct substitutions are:

$$a=\frac{2\,\boldsymbol{i}\rho\,\left(\sigma-\tau\right)}{\left(\tau+\rho\right)\left(\rho+\sigma\right)}\,r_0\;;\;b=\frac{2\,\boldsymbol{i}\sigma\,\left(\tau-\rho\right)}{\left(\rho+\sigma\right)\left(\sigma+\tau\right)}\,r_0\;;\;c=\frac{2\,\boldsymbol{i}\tau\,\left(\rho-\sigma\right)}{\left(\sigma+\tau\right)\left(\tau+\rho\right)}\,r_0\tag{15.17}$$

$$S=\boldsymbol{i}\,r_i^2\,\frac{\left(\rho-\sigma\right)\left(\sigma-\tau\right)\left(\tau-\rho\right)}{\left(\tau+\rho\right)\left(\rho+\sigma\right)\left(\sigma+\tau\right)}\;;\;R=-r_i\,\frac{2\,\tau\,\rho\,\sigma}{\left(\tau+\rho\right)\left(\rho+\sigma\right)\left(\sigma+\tau\right)}$$

while the backward substitutions are:

$$\sigma=\frac{-2\,\boldsymbol{i}\rho\,S}{ab}-\frac{\rho\,\left(a^{2}+b^{2}-c^{2}\right)}{2ab}\ ;\ \tau=\frac{2\,\boldsymbol{i}\rho\,S}{ac}-\frac{\rho\,\left(a^{2}-b^{2}+c^{2}\right)}{2ac} \tag{15.18}$$

Proof. Tangent T_ρ at ρ to the unit circle is $[\rho^{-1},-2,\rho]$, and $A=T_\sigma\cap T_\rho$, etc. Signum for a can be chosen at will, but the other two have to be synchronized. The σ,τ formulas are using $\sigma=\rho\,\exp\boldsymbol{i}\,(\pi+C)\,,\tau=\rho\,\exp\boldsymbol{i}\,(\pi-B)$ i.e. the $\pi-A$ property, and correct orientations. □

Proposition 15.7.4. *Going back from the Poncelet affix ζ_M of a point M to the Lubin-2 affix z_M of this point only requires the similarity:*

$$z_M=-s_2+\frac{1}{2}\ \left(s_2s_1-s_3\right)\overline{\zeta_M}=-s_2-s_3\frac{r_0}{R}\overline{\zeta_M}$$

Proof. Due to homogeneity, we can multiply all the ρ,σ,τ by a same non vanishing factor, and enforce $\rho=\alpha$. Then substituting (15.15) into (15.18) leads to:

$$\left\{\rho=\alpha\ ;\ \sigma=\beta\ ;\ \tau=\gamma\ ;\ r_0=-R\,\frac{(\beta+\gamma)\,(\alpha+\gamma)\,(\alpha+\beta)}{2\,\alpha\,\beta\,\gamma}=-R\,\frac{s_1\,s_2-s_3}{2\,s_3}\right\}$$

And then, it only remains a change of the projective basis, that is given by:

$$\operatorname{subs}\left(\rho=\alpha,\boxed{Lu}\cdot\boxed{Pon^{-1}}\right)=\left[\begin{array}{ccc}0 & -s_2 & \frac{1}{2}\left(s_2\,s_1-s_3\right)\\ 0 & 1 & 0\\ \frac{s_2\,s_1-s_3}{2\,s_3^{2}} & -\frac{s_1}{s_3} & 0\end{array}\right]$$

□

Fact 15.7.5. *The Poncelet affix of an ordinary point (i.e. ζ_M) is an homogeneous fraction whose total degree is +1. The Lubin-2 affix (i.e. z_M) of the same point has total degree 2... and this is verified in $z_M=-s_2+\frac{1}{2}\ (s_2s_1-s_3)\,\overline{\zeta_M}$ since the degree of $\overline{\zeta}$ is -1.*

Remark 15.7.6. In fact, formula $z_M=-s_2-s_3\left(\frac{r_0}{R}\right)\overline{\zeta_M}$ is rather obvious.

- The translation $-s_2$ is required to move I_0 from point $\zeta=0$ to its new place in the Lubin-2 frame, i.e. $z=-\beta\gamma-\gamma\alpha-\alpha\beta$.
- Homothety r_i/R is required to acknowledge that the radius of the "unit" circle has changed.
- Exchanging the umbilics is required by the orientations, i.e. the values of :

$$\det\boxed{Lu}/S=4\boldsymbol{i}/R^2\ ;\ \det\boxed{Pon}/S=-4\boldsymbol{i}/r_0^2$$

- And lastly, the $-s_3$ is required by the property used in the Acknowledgment of this book: the intouch triangle is perspective with the circumcevian triangle of the incenter, while the perspector X(56) is the similicenter of the incircle and the circumcircle: the mid-arcs are $-\beta\gamma$, etc. By complex conjugacy, they become $-1/\beta\gamma$ and multiplying by $-\alpha\beta\gamma$ is the rotation that gives the required α, etc.

Remark 15.7.7. The only notable difference between the Poncelet and the Lubin-2 parametrizations is their behavior wrt the Lemoine transform. Since Lubin-2 coordinates are relative to the strong points $\Omega^{\pm}$ and Ω_0=X(3), all of the strong points are invariant by the Lemoine transforms. When using Poncelet representation, the origin changes, and therefore the coordinates of the strong points don't remain unchanged, an annoying property.

15.8 Poulbot's points (using the Lubin-4 parametrization)

Proposition 15.8.1. *When using half angles, i.e.* $A/2$, etc, *we can introduce* α, etc *such that* $z_A = \alpha^4$, etc. *The intermediate points on the circumcircle can be described as:*

$$\beta^4 \;;\; \boldsymbol{i}\beta^3\gamma, -\beta^2\gamma^2, -\boldsymbol{i}\beta\,\gamma^3 \;;\; \gamma^4 \;;\; \gamma^3\,\alpha, \gamma^2\alpha^2, \gamma\alpha^3 \;;\; \alpha^4 \;;\; \alpha^3\beta, \alpha^2\beta^2, \alpha\,\beta^3 \;;\; \beta^4$$

And then we have:

$$\cot\frac{A}{2} = \frac{bc+S_a}{2\,S} \;;\; \cos\frac{A}{2} = \frac{\sqrt{b+c+a}\sqrt{b+c-a}}{2\,\sqrt{b}\sqrt{c}} \;;\; \sin\frac{A}{2} = \frac{\sqrt{a+b-c}\sqrt{a-b+c}}{2\,\sqrt{b}\sqrt{c}}$$

$$\cos\frac{A}{2} = \boldsymbol{i}\,\frac{\beta^2-\gamma^2}{2\,\beta\,\gamma} \;;\; \sin\frac{A}{2} = -\frac{\beta^2+\gamma^2}{2\,\beta\,\gamma}$$

Proof. Everything goes like in Lubin-2: product of points 2,5,8 must be $\boldsymbol{i}$ times the product of points 1,4,7, etc. □

Remark 15.8.2. The following quantity belongs to Lubin-2 :

$$\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) = \frac{r_i}{4\,R} = \frac{S_3 - S_1\,S_2}{8\,S_3}$$

Remember that $s_1 = \alpha+\beta+\gamma$, $S_1 = \alpha^2+\beta^2+\gamma^2$, $\sigma_1 = \alpha^4+\beta^4+\gamma^4$.

Example 15.8.3. Consider the circles going through the incenter I and tangent to AB and AC .

1. Clearly the centers of these circles, say A_j, must be on the AI line. Thus:

$$A_j = \mu A + (1-\mu)\,I = \begin{bmatrix} \mu\,\alpha^4 + (1-\mu)\left(-\alpha^2\beta^2 - \alpha^2\gamma^2 - \beta^2\gamma^2\right) \\ 1 \\ \dfrac{\mu}{\alpha^4} - (1-\mu)\;\dfrac{\left(\alpha^2+\beta^2+\gamma^2\right)}{\alpha^2\beta^2\gamma^2} \end{bmatrix}$$

2. Equating the distance to I and the distance to AB (15.11)we obtain :

$$\pm\mu\,\frac{\left(\alpha^2+\gamma^2\right)\left(\alpha^2+\beta^2\right)}{\alpha^2\beta\,\gamma} = (1-\mu)\;\frac{\left(\beta^2+\gamma^2\right)\left(\alpha^2+\gamma^2\right)\left(\alpha^2+\beta^2\right)}{2\;\alpha^2\beta^2\gamma^2}$$

leading to $\mu = \left(\beta^2+\gamma^2\right) \div \left(\beta^2 \pm 2\,\beta\,\gamma + \gamma^2\right)$, and to:

$$A_1 = \begin{bmatrix} \dfrac{S_2\,\alpha^3 - \alpha\,s_3^2 + 2\,S_2\,s_3}{-\alpha^3 + S_1\,\alpha - 2\,s_3} \\ 1 \\ \dfrac{S_1\,s_3 - \alpha^2 s_3 + 2\,S_1\,\alpha^3}{\alpha^3 s_3\,\left(-\alpha^3 + S_1\,\alpha - 2\,s_3\right)} \end{bmatrix} \;;\; A_0 = \begin{bmatrix} \dfrac{S_2\,\alpha^3 - \alpha\,s_3^2 - 2\,S_2\,s_3}{-\alpha^3 + S_1\,\alpha + 2\,s_3} \\ 1 \\ \dfrac{S_1\,s_3 - \alpha^2 s_3 - 2\,S_1\,\alpha^3}{\alpha^3 s_3\,\left(-\alpha^3 + S_1\,\alpha + 2\,s_3\right)} \end{bmatrix}$$

Here $s_3 = \alpha\beta\gamma$ while $S_1 = \alpha^2+\beta^2+\gamma^2$, etc. Point A_1 is unchanged by $\alpha \mapsto -\alpha$ (since this leads also to $s_3 \mapsto -s_3$). But $\beta \mapsto -\beta$ changes only s_3 and exchanges A_1 and A_0.

3. Obviously, circles (A_1) and (A_0) are tangent at I.

4. Let us note X_{jk} the second intersection of circle Y_j and circle Z_k. For example, B_{31} is $(C_3)\cap(A_1)$. We obtain:

$$z\left(A_{11}\right) = \frac{\begin{pmatrix} 2\,\alpha^3\beta\gamma\left(\beta^2+\gamma^2\right) + 2\,\alpha\beta^3\gamma^3 + \beta^2\gamma^2\left(\beta+\gamma\right)\left(\beta^2+\gamma^2\right) \\ +\alpha^2\left(\beta^2+\beta\gamma+\gamma^2\right)\left(\beta^2-\beta\gamma+\gamma^2\right)\left(\beta+\gamma\right) \end{pmatrix}}{\alpha^2\left(\beta+\gamma\right) - 2\,\alpha\beta\gamma}$$

And then, we use $\beta+\gamma = s_1-\alpha$ and $\beta\gamma = s_2 - \alpha s_1 + \alpha^2$. This leads to "huge" polynomials in α, that can be reduced using the relation $\alpha^3 - s_1\alpha^2 + s_2\alpha - s_3 = 0$. As a result:

$$z\left(A_{11}\right) = \frac{\begin{pmatrix} 2\,s_1^3 s_3 - s_1^2 s_2^2 + s_2^3 \\ -2\,s_1\,s_2\,s_3 + s_3^2 \end{pmatrix}\alpha + \begin{pmatrix} -2\,s_1^4 s_3 + s_1^3 s_2^2 + 4\,s_1^2 s_2\,s_3 \\ -2\,s_1\,s_2^3 - 7\,s_1\,s_3^2 + 3\,s_2^2 s_3 \end{pmatrix}}{\alpha\,s_2 - 3\,s_3}$$

α	β	γ	I_0	I_a	I_b	I_c	A_j	B_j	C_j	A_{jk}	B_{kn}	C_{nj}	A_{jk}	B_{kn}	C_{nj}
$i\alpha$	β	γ	I_a	I_0	I_c	I_b									
α	β	γ	I_0	I_a	I_b	I_c	A_1	B_1	C_1	A_{11}	B_{11}	C_{11}	A_{00}	B_{00}	C_{00}
$-\alpha$	$+\beta$	γ	I_0	I_a	I_b	I_c	A_1	B_0	C_0	A_{00}	B_{01}	C_{10}	A_{11}	B_{10}	C_{01}
$+\alpha$	$-\beta$	γ	I_0	I_a	I_b	I_c	A_0	B_1	C_0	A_{10}	B_{00}	C_{01}	A_{01}	B_{11}	C_{10}
α	β	$-\gamma$	I_0	I_a	I_b	I_c	A_0	B_0	C_1	A_{01}	B_{10}	C_{00}	A_{10}	B_{01}	C_{11}

Table 15.1: Action of the Klein group

5. And now consider circles (A, A_{jk}, I), (B, B_{kn}, I), (C, C_{nj}, I). On the Figure, we can see that these circles concur in a second point. Under the action of the Klein group of the $\alpha \mapsto -\alpha$ transforms, the parity of $j+k+n$ is kept, as described in Table 15.1: two situations are encountered.

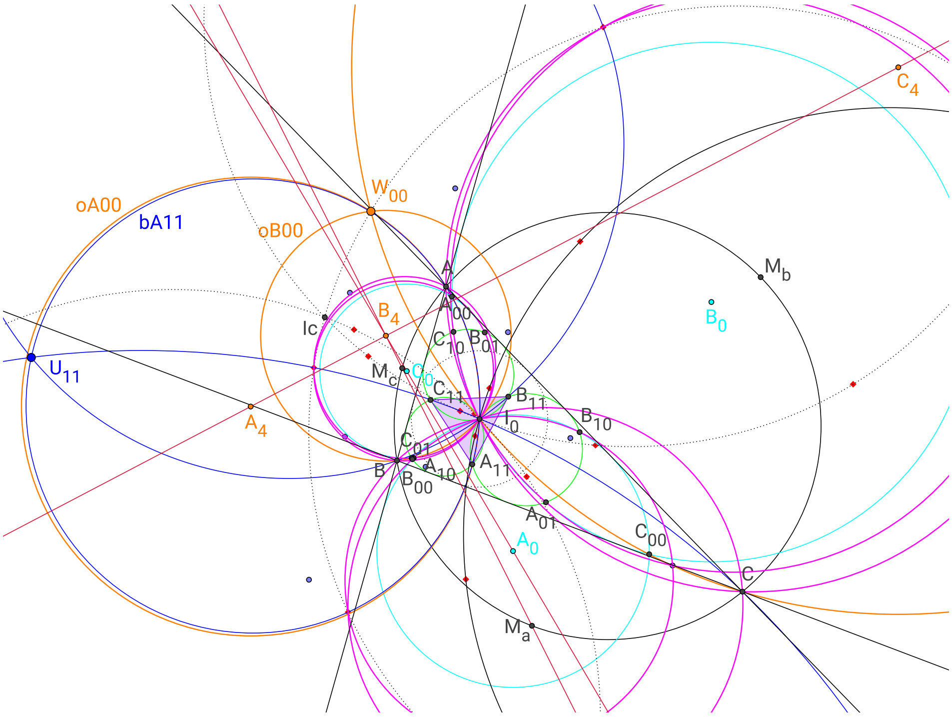
6. Construction. The center B_4 of circle (B, B_{00}, I) can be obtained as $A_0C_0 \cap M_aM_c$ where A_0, C_0 are the centers described above, and M_a, M_c are the mid-arcs relative to the incenter I_0.

7. Spoiler (Veronese map). Use $z(A) = \alpha^4$, $z(I) = 2s_1s_3 - s_2^2$ and $z(A_{11})$ as above. Take the Veronese of these points, and then the wedge of these three rows. Reduce this column, and take the remainder of each element wrt $\alpha^3 - s_1\alpha^2 + s_2\alpha - s_3$. Now, the representative of circle (I, A, A_{11}) is written as
$$V_a = V_1 + \alpha V_2 + \alpha^2 V_3$$
were V_1, V_2, V_3 are symmetric in α, β, γ. One can see that the family V_1, V_2, V_3 is not independent, so that the V_a, V_b, V_c belong to a same pencil.

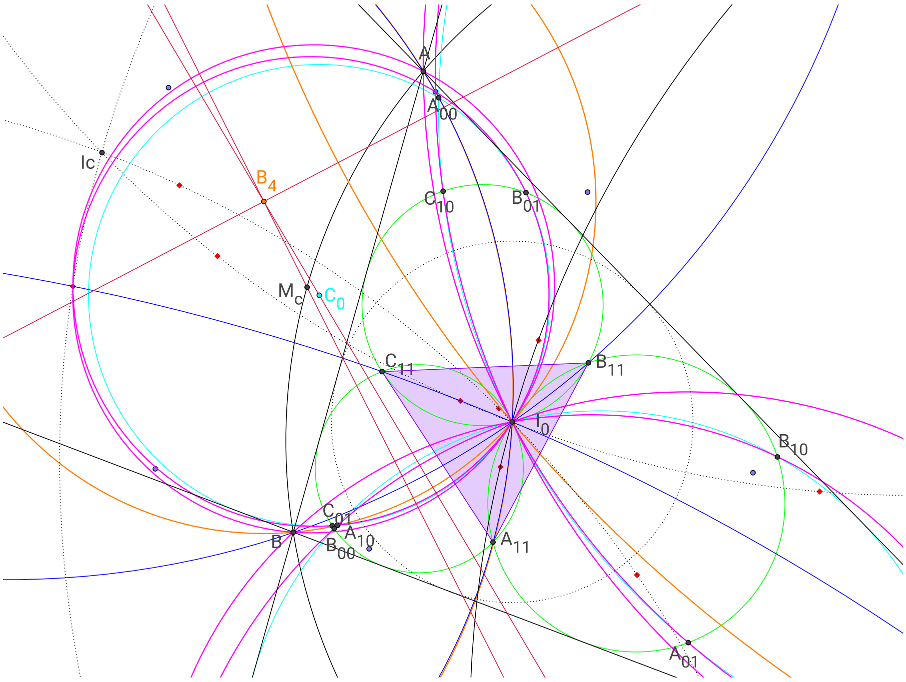
8. Spoiler (pencil of circles). Determine the point-circles in this pencil, i.e. determine K so that ${}^t(V_1 + K\,V_2) \cdot \boxed{\underset{z}{\mathcal{Q}}} \cdot (V_1 + K\,V_2) = 0$. This equation factors gently, giving K and therefore the point-circles. Going back to the equations in $\mathbf{Z}, \mathbf{T}, \overline{\mathcal{Z}}$, we obtain factored equations:
$$\begin{pmatrix} \left(s_1^2 - 2\,s_2\right)\mathbf{T} \\ +\overline{\mathcal{Z}}\,s_3^2 \end{pmatrix} \times \left(\begin{pmatrix} 2\,s_1^4 s_2\,s_3 - s_1^3 s_2^3 - 2\,s_1^3 s_3^2 - 3\,s_1^2 s_2^2 s_3 \\ +2\,s_1\,s_2^4 + 10\,s_1\,s_2\,s_3^2 - 4\,s_2^3 s_3 - 4\,s_3^3 \end{pmatrix} \mathbf{T} + \begin{pmatrix} s_1^3 s_2 - 2\,s_1\,s_2^2 \\ -3\,s_1^2 s_3 + 4\,s_2\,s_3 \end{pmatrix} \mathbf{Z} \right) = 0$$
and conjugate. This characterizes the base points of an *isoptic* pencil. Thus the $jkn = 111$ case leads to Poulbot's points of first kind (Ayme et al., 2014) :
$$z_U = \frac{-2\,s_1^4 s_2 s_3 + s_1^3 s_2^3 + 2\,s_1^3 s_3^2 + 3\,s_1^2 s_2^2 s_3 - 2\,s_1 s_2^4 - 10\,s_1 s_2 s_3^2 + 4\,s_2^3 s_3 + 4\,s_3^3}{s_1^3 s_2 - 2\,s_1 s_2^2 - 3\,s_1^2 s_3 + 4\,s_2 s_3}$$

9. In the same vein, the $jkn = 000$ case leads to Poulbot points of second kind:
$$z_W = \frac{s_1^2 s_2^3 - 2s_1^3 s_2 s_3 - 2s_2^4 + 3s_1 s_2^2 s_3 + 2s_1^2 s_3^2 - 2s_2 s_3^2}{s_1^2 s_2 - 2s_2^2 + s_1 s_3}$$
and it can be seen that this point is on the Bevan circle $J_aJ_bJ_c$. So are the other three.

15.9 More about the foci of a conic



(a) Whole figure



(b) Core figure

orange: 11 circles, blue 00 circles, magenta: 01 or 10 circles

Figure 15.1: Poulbot's points

Chapter 16

Collineations

16.1 Definition

Definition 16.1.1. A collineation is a reversible linear transformation of the barycentrics, i.e. $U = \phi(P)$ determined by :

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \simeq \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

where $\simeq$ is a reminder of the fact that barycentrics are determined up a proportionality factor.

Proposition 16.1.2. *A collineation is determined by two ordered lists four points:* $P_i,\ i = 1, 2, 3, 4,$ U_i, $i = 1, 2, 3, 4$ *such that no triples of* P *points are on the same line, and the same for the* U *points.*

Proof. If ϕ is reversible, then $\det M \neq 0$ is required and the $\phi(P_i)$ haven't alignments when the P_i haven't. Conversely, we have the following algorithm. □

Algorithm 16.1.3. Collineation algorithm. Let be given the two lists of points $P_i,\ i = 1, 2, 3, 4,$ U_i, $i = 1, 2, 3, 4$. With obvious notations, the question is to find the m_{ij} (not all being 0) and the k_i (none being 0) in order to ensure :

$$\begin{pmatrix} u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix} \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \\ r_1 & r_2 & r_3 & r_4 \end{pmatrix}$$

This system has 13 unknowns and 12 equations, since a global proportionality factor remains undetermined. The k_i are determined (up to a global proportionality factor) by

$$\frac{1}{k_i} \det \underset{\neq i}{U} = \det \underset{\neq i}{P} \det M$$

where a 3×4 matrix subscribed by an $\neq i$ refers to the square matrix obtained by deleting the i-th column. Thereafter, M is easily obtained. To summarize :

$$\boxed{k_i = \det \underset{\neq i}{U} \div \det \underset{\neq i}{P} \qquad M = \underset{\neq 4}{U} \cdot \underset{\neq 4}{K} \cdot \left(\underset{\neq 4}{P}\right)^{-1}}$$

With the given hypotheses, transformation ϕ is clearly reversible.

Remark 16.1.4. An efficient choice of the P_i, U_i is eight centers, or a central triangle and a center for the P and the corresponding U. In such a case, any center is transformed into a center, and homogeneous curves into homogeneous curves of the same degree.

16.2 Involutory collineations

Proposition 16.2.1. . *Let M_1, M_2, N_1, N_2 be four (different points). The collineation ψ that swaps the $(M_1,\ M_2)$ pair and also the $(N_1,\ N_2)$ pair is involutory. The line through the crossed intersections $M_1N_1 \cap M_2N_2$ and $M_1N_2 \cap M_2N_1$ is a line of fixed points (the axis of ψ). The paired intersection, i.e. point $P = M_1M_2 \cap N_1N_2$ is an isolated fixed point (the pole of ψ). Reciprocally, given an axis Δ and a pole P (outside of the axis), we obtain an involutory transform $U \mapsto X = \psi(U)$ by requiring P, U, X aligned together with $(P,\ U,\ PU \cap \Delta,\ X) = -1$ (harmonic conjugacy).*

Proof. Consider ψ defined by $A \longleftrightarrow P$ and $B \longleftrightarrow C$. Its matrix $\boxed{\psi}$ can be obtained by the general Alg. 16.1.3. Then the cevian triangle of P provides a diagonalization basis and we have :

$$\boxed{\psi} \simeq \left(\begin{array}{ccc} 1 & 0 & 0 \\ \dfrac{q}{p} & 0 & -\dfrac{q}{r} \\ \dfrac{r}{p} & -\dfrac{r}{q} & 0 \end{array}\right) \quad ; \quad \boxed{\mathcal{T}_P} = \left(\begin{array}{ccc} 0 & p & p \\ q & 0 & q \\ r & r & 0 \end{array}\right)$$

$$\boxed{\mathcal{T}_P}^{-1} \cdot \boxed{\psi} \cdot \boxed{\mathcal{T}_P} = \left(\begin{array}{ccc} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

In the general case, we can chose matrix $\boxed{\psi}$ to enforce $\det \boxed{\psi} = -1$. Then minimal polynomial is $\mu^2 - 1$ while characteristic polynomial is $\chi(\mu) = (\mu - 1)^2 (\mu + 1)$. □

16.3 Usual affine transforms as collineations

Remark 16.3.1. Umbilics have been defined in Subsection 14.1.2. A possible choice can be described as $\Omega^{\pm} \simeq abc\, X_{512} \pm \boldsymbol{i} R\, X_{511}$. The exact value is given in (14.2).

Proposition 16.3.2. ***Translation.*** *The matrix of the translation $U \mapsto U + \overrightarrow{V}$ where $\overrightarrow{V} = (p, q, r)$ is given by :*

$$\left(\begin{array}{ccc} 1+p & p & p \\ q & 1+q & q \\ r & r & 1+r \end{array}\right) = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) + \overrightarrow{V} \cdot \mathcal{L}_b \tag{16.1}$$

Proof. Use $P_i = A,\ B,\ \Omega^+,\ \Omega^-$ and $Q_i = C,\ D,\ \Omega^+,\ \Omega^-$. Characteristic polynomial is

$$\chi(\mu) = (\mu - 1)^2 (\mu - 1 - p - q - r)$$

For a translation, $p + q + r = 0$, and the matrix is not diagonalizable. □

Remark 16.3.3. The translation operator is linear, meaning that $M\left(\overrightarrow{V_1}\right) + M\left(\overrightarrow{V_2}\right) = M\left(\overrightarrow{V_1} + \overrightarrow{V_2}\right)$.

Proposition 16.3.4. ***Homothety.*** *When $p + q + r$ is different from 0 and -1 then (16.1) characterizes the homothety centered at point $P = p : q : r$ with ratio $\mu = 1/(1 + p + q + r)$.*

Proof. The factor is the reciprocal of the eigenvalue λ_P since a fixed point should be described by $\lambda = 1$: all the eigenvalues have to be divided by λ_P. This result can also be obtained by direct examination of $\overrightarrow{f(P)\, f(U)}$. □

Remark 16.3.5. When computing P, the column ${}^t(p, q, r)$ can be viewed as "defined up to a proportionality". This does not apply to the computation of μ. In any case, we are re-obtaining (7.29).

Remark 16.3.6. The matrix π_Δ of the orthogonal projector onto line $\Delta \simeq [p, q, r]$ is :

$$\pi_\Delta = \Delta \cdot \boxed{\mathcal{M}_b} \cdot {}^t\Delta - \boxed{\mathcal{M}_b} \cdot {}^t\Delta \cdot \Delta$$

while the matrix σ_Δ of the orthogonal reflection wrt line $\Delta \simeq [p, q, r]$ is :

$$\sigma_\Delta = \Delta \cdot \boxed{\mathcal{M}_b} \cdot {}^t\Delta - 2 \boxed{\mathcal{M}_b} \cdot {}^t\Delta \cdot \Delta$$

These formulas are recalled from Section 7.12, where more details are given.

Proposition 16.3.7. *The matrix of the rotation centered at finite point $P = p : q : r$ with angle ϕ is :*

$$(p+q+r)\boxed{\Phi} = \begin{pmatrix} p & p & p \\ q & q & q \\ r & r & r \end{pmatrix} + \begin{pmatrix} r+q & -p & -p \\ -q & r+p & -q \\ -r & -r & q+p \end{pmatrix} \cos\phi + \boxed{\text{OrtO}} (\mathcal{L}_b \cdot P - P.\mathcal{L}_b) \sin\phi$$

Proof. It suffices to check what happens to P, Ω^+, Ω^- : they are fixed points, with respective eigenvalues : $1, \exp(+i\phi), \exp(-i\phi)$, while the global factor $p+q+r$ is a remainder of the constraint $P \notin \mathcal{L}_b$. □

Stratospherical proof. A rotation with angle ϕ is multiplication by $\Phi = \cos\phi + \boldsymbol{i}\sin\phi$ in the complex plane. Therefore, rotation with center P and angle ϕ can be written as :

$$\Phi(X) = P + \left(\boxed{1}\cos\phi + \boxed{\boldsymbol{i}}\sin\phi\right)\overrightarrow{PX}$$

Since matrix $\boxed{\text{OrtO}}$ describes a "project and turn" action, we have $\boxed{\text{OrtO}}^3 = -\boxed{\text{OrtO}}$, so that $\boxed{\boldsymbol{i}} = \boxed{\text{OrtO}}$. Multiplying, we get : $\boxed{\text{OrtO}}^4 = -\boxed{\text{OrtO}}^2$ and $-\boxed{\text{OrtO}}^2$ is a projector onto space $\overrightarrow{\mathcal{V}}$. This gives : $\boxed{1} = -\boxed{\text{OrtO}}^2$. Canceling the denominators, we obtain :

$$\Phi\begin{pmatrix} x \\ y \\ z \end{pmatrix} \simeq (x+y+z)\begin{pmatrix} p \\ q \\ r \end{pmatrix} + \left(\sin\phi\boxed{\text{OrtO}} - \cos\phi\boxed{\text{OrtO}}^2\right)\left((p+q+r)\begin{pmatrix} x \\ y \\ z \end{pmatrix} - (x+y+z)\begin{pmatrix} p \\ q \\ r \end{pmatrix}\right)$$

leading to the required matrix :

$$\boxed{\Phi} = proj + \left(\sin\phi\boxed{\text{OrtO}} - \cos\phi\boxed{\text{OrtO}}^2\right).(\mathbf{1} - proj) \quad \text{where} \quad proj \doteq \frac{1}{\mathcal{L}_b \cdot P}(P \cdot \mathcal{L}_b)$$ □

Proposition 16.3.8. ***Similarity***. *When A, B, C, D are points at finite distance, with $A \neq B$, $C \neq D$ it exists two similarities ϕ, ψ, respectively called direct and skew, that sends $A \mapsto C$ and $B \mapsto D$. As collineations, they are characterized by :*

$$\begin{aligned} \phi &= \text{collineate}\left(A, B, \Omega^+, \Omega^- ; C, D, \Omega^+, \Omega^-\right) \\ \psi &= \text{collineate}\left(A, B, \Omega^+, \Omega^- ; C, D, \Omega^-, \Omega^+\right) \end{aligned}$$

Proof. The group of all the similarities is the stabilizer subgroup of the pair $\{\Omega^+, \Omega^-\}$ under the action of the group of all the collineations. This comes from the fact that any similarity transforms circles into circles, and therefore must preserve the umbilical pair. □

Proposition 16.3.9. ***Similarity (Morley plane)***. *Spoiler: in the Morley plane, the matrix of the similarity σ defined by center $M \simeq z : t : \zeta$, ratio k and turn $\tau$is:*

$$\boxed{\sigma} \simeq \begin{bmatrix} k\tau & \dfrac{z}{t}(1-k\tau) & 0 \\ 0 & 1 & 0 \\ 0 & \dfrac{\zeta}{t}\left(1-\dfrac{k}{\tau}\right) & \dfrac{k}{\tau} \end{bmatrix}$$

Stratospherical proof. Take Ω_y, M, Ω_x as basis and say that eigenvalues are $k\tau, 1, k/\tau$. □

Computational proof. Say that $\sigma = collineate\,(\Omega_y, \Omega_x, M, N, \Omega_y, \Omega_x, M, N')$ while $|MN'|^2 = k^2\,|MN|^2$ and $\tan(MN, MN') = \boldsymbol{i}\left(1-\tau^2\right)/\left(1+\tau^2\right)$. Solve for N' and obtain the former result. In fact there is another solution, obtained by $k\tau \mapsto -k\tau$. But we must obtain the unit matrix when $k\tau = 1$. □

16.4 Barycentric multiplication as a collineation

Proposition 16.4.1. *Barycentric multiplication by $P = p : q : r$ is what happens to the plane when using collineation $\phi : (A, B, C, X_2) \mapsto (A, B, C, P)$. In other words :*

$$X \underset{b}{*} P = \boxed{P} \cdot X \qquad \text{where } \boxed{P} \doteq \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{pmatrix}$$

Remark 16.4.2. Obviously, trilinear multiplication can be described using collineations involving X_1.

Proposition 16.4.3. *The collineation whose matrix is diagonal, with elements $U \underset{b}{\div} P$ transforms (A, B, C, P) into (A, B, C, U) and circumconic $CC(P)$ into $CC(U)$.*

Proof. Direct examination. One obtains :

$$\frac{pqr}{uvw} {}^t\begin{pmatrix} \frac{u}{p} & 0 & 0 \\ 0 & \frac{v}{q} & 0 \\ 0 & 0 & \frac{w}{r} \end{pmatrix} \cdot \begin{pmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{u}{p} & 0 & 0 \\ 0 & \frac{v}{q} & 0 \\ 0 & 0 & \frac{w}{r} \end{pmatrix} = \begin{pmatrix} 0 & r & q \\ r & 0 & p \\ q & p & 0 \end{pmatrix}$$

□

Construction 16.4.4. *The following recipe constructs $F \doteq D \underset{b}{*} E$.*

1	*Points*	A, B, C, D, E	*given*
2	*Line*	ab	*through* $A,\ B$
3	*Line*	bc	*through* $B,\ C$
4	*Line*	ca	*through* $C,\ A$
5	*Point*	X_1	*Intersection of* bc, *Line*$[A,\ D]$
6	*Point*	X_2	*Intersection of* bc, *Line*$[A,\ E]$
7	*Point*	K_x	*Intersection of* ab, *Line*$[X_2,\ ca]$
8	*Point*	H_x	*Intersection of* ca, *Line*$[X_1,\ ab]$
9	*Point*	Y_1	*Intersection of* ca, *Line*$[B,\ D]$
10	*Point*	Y_2	*Intersection of* ca, *Line*$[B,\ E]$
11	*Point*	H_y	*Intersection of* ab, *Line*$[Y_1,\ bc]$
12	*Point*	K_y	*Intersection of* bc, *Line*$[Y_2,\ ab]$
13	*Point*	Q_x	*Intersection of Line*$[B,\ H_x]$, *Line*$[C,\ K_x]$
14	*Point*	Q_y	*Intersection of Line*$[C,\ H_y]$, *Line*$[A,\ K_y]$
15	*Point*	F	*Intersection of Line*$[A,\ Q_x]$, *Line*$[B,\ Q_y]$

Proof. The idea is to construct parallelograms $AK_1X_1H_1$, $AK_2X_2H_2$ and use them as pantographs. Using $D \simeq p : q : r$ and $E \simeq u : v : w$, we have

X_1	X_2	K_x	H_x	Y_1	Y_2	H_y	K_y	Q_x	Q_y	F
0	0	w	q	p	u	p	0	wq	up	up
q	v	v	0	0	0	r	u	qv	ur	qv
r	w	0	r	r	w	0	w	wr	wr	wr

□

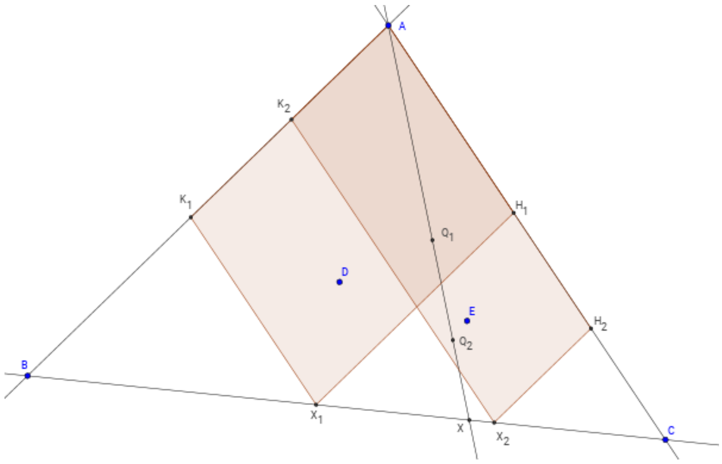


Figure 16.1: Construction of barymul

16.5 Complement and anticomplement as collineations

Proposition 16.5.1. *Complement is what happens to the plane when using collineation* $(A, B, C, X_2) \mapsto (A_2B_2C_2, X_2)$ *where* $A_2B_2C_2$ *is the medial triangle. In other words :*

$$complem\,(X) = \boxed{\mathcal{C}} \cdot X \qquad \text{where} \quad \boxed{\mathcal{C}} \doteq \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$anticomplem\,(X) = \boxed{\mathcal{C}^{-1}} \cdot X \qquad \text{where} \quad \boxed{\mathcal{C}^{-1}} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Proof. Direct computation. □

Proposition 16.5.2. *The* ***cevian collineation*** *wrt point* P *is defined as collineation* $(A, B, C, P) \mapsto (A_P B_P C_P, P)$. *Its matrix is :*

$$\phi_P = \boxed{P} \cdot \boxed{\mathcal{C}} \cdot \boxed{P^{-1}} = \begin{pmatrix} 0 & p/q & p/r \\ q/p & 0 & q/r \\ r/p & r/q & 0 \end{pmatrix}$$

where $\boxed{P^{-1}}$ *is to be understood as the reciprocal of matrix* $\boxed{P}$.

Proof. Composition of the two former collineations. □

Proposition 16.5.3. *We have the following relation between conics :*

$$complem\,(X) \in conicev\,(isotom\,(U)\,,\, X_2) \Longleftrightarrow X \in conicir\,(complem\,(U))$$

16.6 Collineations and cevamul, cevadiv, crossmul, crossdiv

In this Section 16.6, the start point will ever be Table 3.2 (II) i.e. $\mathcal{T}_1 = \mathcal{C}_P$ (the cevian of P), $\mathcal{T}_2 = ABC$, $\mathcal{T}_3 = \mathcal{A}_U$ (the anticevian of U).

Proposition 16.6.1. *Start as described, and use* $\phi_U = \boxed{U} \cdot \boxed{\mathcal{C}} \cdot \boxed{U^{-1}}$. *This collineation is tailored so that :* $\phi\,(\mathcal{T}_3) = ABC$, $\phi\,(\mathcal{T}_2) = \mathcal{C}_U$ *and* $\phi\,(\mathcal{T}_1)$ *is the cevian triangle of* $\phi\,(P)$ *wrt* $\mathcal{C}_U$. *Then :*

$$\begin{array}{rcccl} \phi \cdot cevadiv\,(P,\, U) & = & crossdiv\,(\phi \cdot P,\, \phi \cdot U) & = & \frac{u^2}{p} : \frac{v^2}{q} : \frac{w^2}{r} = P_U^{\#} \\ cevamul\left(\phi^{-1}.X,\, \phi^{-1}.U\right) & = & \phi^{-1} \cdot crossmul\,(X,\, U) & = & \frac{u^2}{x} : \frac{v^2}{y} : \frac{w^2}{z} = X_U^{\#} \end{array}$$

Proof. Direct computation. The symmetry between U, X is broken by using ϕ_U. □

Exercise 16.6.2. Explain why one obtains $sqrtdiv\left(U,\,P\right)$ and $sqrtdiv\left(U,X\right)$, reverting the symmetry.

Proposition 16.6.3. *Start as described and use* $\psi_P = \boxed{P} \cdot \boxed{\mathcal{C}^{-1}} \cdot \boxed{P^{-1}}$. *This collineation is tailored so that* $\psi\left(\mathcal{T}_1\right) = ABC$, $\psi\left(\mathcal{T}_2\right) = \mathcal{A}_P$ *and* $\psi\left(\mathcal{T}_3\right)$ *is the anticevian of* $\psi\left(U\right)$ *wrt* $\mathcal{A}_P$. *Triangles* $\psi\left(\mathcal{T}_3\right)$ *and* $\psi\left(\mathcal{T}_2\right)$ *are perspective wrt* $\psi\left(U\right)$ *while* $\psi\left(\mathcal{T}_3\right)$ *and* $\psi\left(\mathcal{T}_1\right)$ *are perspective wrt* $\psi\left(X\right)$ *and :*

$$\psi \cdot cevadiv\left(\psi^{-1} \cdot P, \psi^{-1} \cdot U\right) = sqrtdiv\left(P,\,U\right) = \frac{p^2}{u} : \frac{q^2}{v} : \frac{r^2}{w} = U_P^{\#}$$

Proposition 16.6.4. *Start as before, but use instead collineation* $\left(ABC, X_2\right) \mapsto \left(ceva\left(P\right), P\right)$ *i.e.* $\psi_P = \boxed{\mathcal{C}^{-1}} \cdot \boxed{P^{-1}}$. *Then* $\psi\left(\mathcal{T}_1\right)$ *is* ABC *while* $\psi\left(\mathcal{T}_2\right)$ *is the anticomplementary triangle. Perspector between* $\psi\left(\mathcal{T}_1\right)$ *and* $\psi\left(\mathcal{T}_2\right)$ *is* $\psi\left(U\right) = X_2$, *perspector between* $\psi\left(\mathcal{T}_2\right)$ *and* $\psi\left(\mathcal{T}_3\right)$ *is* $\psi\left(P\right) = anticomplem\left(U/P\right)$ *while perspector between* $\psi\left(\mathcal{T}_1\right)$ *and* $\psi\left(\mathcal{T}_3\right)$ *is isotomic conjugate of the former. In other words :*

$$X = cevadiv\left(P,\,U\right) = \left(\psi^{-1} \circ isotom \circ \psi\right)\left(U\right)$$

16.7 Cevian conjugacies

Definition 16.7.1. The psi-Kimberling collineation of pole P is the collineation ψ_P such that $ABC \mapsto cevian\left(P\right)$ and $X_1 \to P$. Therefore :

$$\begin{aligned} \psi_P\left(U\right) &= P \underset{b}{*} complem\left(U \div_b X_1\right) \qquad (16.2)\\ &= p\left(\frac{v}{b} + \frac{w}{c}\right) : q\left(\frac{u}{a} + \frac{w}{c}\right) : r\left(\frac{u}{a} + \frac{v}{b}\right) \\ \psi_P^{-1}\left(U\right) &= X_1 \underset{b}{*} anticomplem\left(U \div_b P\right) \\ &= a\left(-\frac{u}{p} + \frac{v}{q} + \frac{w}{r}\right) : b\left(\frac{u}{p} - \frac{v}{q} + \frac{w}{r}\right) : c\left(\frac{u}{p} + \frac{v}{q} - \frac{w}{r}\right) \end{aligned}$$

Remark 16.7.2. It is clear that ψ_P, ψ_P^{-1} are type-keeping when $X_1\left(a:b:c\right)$, $P(p:q:r)$ and $U(u:v:w)$ are transformed. Moreover, $\psi_P\left(A\right) = A_p = 0:q:r$, $\psi_P\left(X_1\right) = P$ (from the very definition) while $\psi_P\left(-a:b:c\right) = A$ is obvious.

This ψ_P collineation has been used by Kimberling (2002a) to construct some new functions, following the patterns :

$$\phi \mapsto \psi_P \circ \phi \circ \psi_P^{-1} \quad or \quad \phi \mapsto \psi_P^{-1} \circ \phi \circ \psi_P$$

1. **cevadivision** of P by U can be re-obtained as $\psi_P \circ isogon \circ \psi_P^{-1}$. The result X is the perspector of $cevian\left(P\right)$ and $anticevian\left(U\right)$. More about this operation in Section 3.11. One has the formulas :

$$X = \left(-uqr + vrp + wqp\right)u \,:\, \left(uqr - vrp + wqp\right)v \,:\, \left(uqr + vrp - wqp\right)w$$
$$P = \left(vz + wy\right)^{-1} \,:\, \left(uz + wx\right)^{-1} \,:\, \left(yu + xv\right)^{-1}$$

   (a) cevadivision and ceva-multiplication are both type-keeping with respect to P and U. Using isogonal conjugacy result in the disappearing of a, b, c from the equations.

   (b) fixed points are $P = \psi_P\left(X_1\right)$ and the three vertices $A = \psi_P\left(-a:b:c\right)$, since $\pm a : \pm b : \pm c$ are the fixed points of *isogon*. A brute force resolution leads also to the cevians of P. A Taylor expansion around $1:0:0$ shows that vertices are really fixed points of cevadivision, while a Taylor expansion around $0:p:q$ shows that undetermined $\psi_P\left(0:q:r\right) = 0:0:0$ must be determined as $\psi_P\left(0:q:r\right) = 0:-q:r$

2. **alephdivision** of P by U : $\psi_P^{-1} \circ isogon \circ \psi_P$ (Hyacinthos #4111, Oct. 11, 2001). Formulas (cyclically) :

$$x \simeq a\left(p^2r^2v^2 + q^2p^2w^2 - q^2r^2u^2\right) + \frac{p^2r^2b^2 + q^2p^2c^2 - q^2r^2a^2}{bc}\,(vaw + ubw + cuv)$$

$$p^2 : q^2 : r^2 = \frac{1}{(bw+cv)\,(bz+cy)} : \frac{1}{(cu+aw)\,(cx+az)} : \frac{1}{(av+bu)\,(ay+bx)}$$

Therefore, the alephmultiplication gives four result, one inside the triangle ABC and three outside

3. **bethdivision** of P by U : $\psi_P \circ sym3 \circ \psi_P^{-1}$ (Hyacinthos #4146, Oct. 26, 2001) where involution $sym3$ is the reflection in the circumcenter X_3. This involution $sym3$ is related to the Darboux cubic. Barycentrics are :

$$\begin{aligned} x &\simeq a\,u - \frac{p\,(c+b)\,(a+c-b)}{q\,(-a+b+c)}\,v - \frac{p\,(c+b)\,(b+a-c)}{r\,(-a+b+c)}\,w \\ &= -a\,u + \frac{\widehat{p}}{\widehat{q}}\,(c+b)\,v + \frac{\widehat{p}}{\widehat{r}}\,(c+b) \qquad where\ \widehat{p} : \widehat{q} : \widehat{r} = P * X_7 \end{aligned}$$

   (a) This operation is type-keeping with respect to P, U.
   (b) The fixed points of $U \mapsto \beta\,(P,U)$ are obtained by ψ_P from the fixed points of $sym3$. They are $\psi_P\,(X_3)$ together with $\psi_P\,(\mathcal{L}_b)$, namely the line : $u/\widehat{p} + v/\widehat{q} + w/\widehat{r} = 0$.
   (c) Bethdivision of $X_{21} = a\,(b+c-a)\,/\,(b+c)$ by the circumcircle gives the circumcircle.
   (d) Bethdivision of P by U is P if and only if $U = P \underset{b}{*} X_{57}$.
   (e) Bethmultiplication is not simple (equation of third degree).

Exercise 16.7.3. What are the situations where the discriminant vanishes ?

4. **gimeldivision** of P by U : $\psi_P^{-1} \circ sym3 \circ \psi_P$. Using barycentrics, one obtains :

$$\begin{aligned} F\,(U) &= 16\sigma^2\,U - \alpha\,\beta\,X_1 + 2\,\alpha\left(X_{48} \underset{b}{*} isot\,(P)\right) \quad where \\ 16\sigma^2 &= (b+c-a)\,(a+c-b)\,(b+a-c)\,(b+a+c) \\ \alpha &= \frac{(q+r)\,u}{a} + \frac{(r+p)\,v}{b} + \frac{(p+q)\,w}{c} \\ \beta &= \frac{\left(b^2+c^2-a^2\right)a^2}{p} + \frac{\left(c^2+a^2-b^2\right)b^2}{q} + \frac{\left(a^2+b^2-c^2\right)c^2}{r} \end{aligned}$$

   (a) The fixed points of $U \mapsto \gamma\,(P,U)$ are obtained by ψ_P^{-1} from the fixed points of $sym3$. They are $\psi_P^{-1}\,(X_3)$ together with $\psi_P\,(\mathcal{L}_b)$, namely the line : $u\,bc\,(q+r) + v\,ca\,(r+p) + w\,ab\,(p+q) = 0$.
   (b) Gimel multiplication leads to three points on the triangle sides, and three other points.

5. **mimosa** aka "much ado about nothing" X(1707)-X(1788). As with other names in ETC, the name Mimosa is that of a star. Define $mimosa\,(P)$ as $\psi_P^{-1}\,(X_3)$. Using barycentrics, one obtains :

$$mimosa\,(p:q:r) = u : v : w \qquad where$$

$$u = a\left(-\frac{\left(-a^2+b^2+c^2\right)a^2}{p} + \frac{b^2\left(a^2-b^2+c^2\right)}{q} + \frac{c^2\left(a^2+b^2-c^2\right)}{r}\right)$$

and also :

$$\begin{aligned} mimosa\,(P) &= cevadiv\left(X_{92} \underset{b}{*} P,\ X_1\right) \\ mimosa^{-1}\,(U) &= cevamul\,(U,\ X_1) \underset{b}{*} X_{63} \end{aligned}$$

Then, marvelously, the Mimosa transform M(X) arises in connection with the equation $gimeldiv(P, X) = X$. And there are too many such cases of gimel conjugates for all to be itemized in ETC... Here is a list of pairs (I,J) for which X(J) = M(X(I)).

1	46	20	1712	48	43	71	846	85	1729
2	19	21	4	54	47	72	191	86	1730
3	1	27	1713	55	1721	73	1046	88	1731
4	920	28	1714	56	1722	74	1725	89	1732
6	1707	29	1715	57	1723	75	1726	90	90
7	1708	31	1716	58	1724	77	57	95	92
8	1158	35	1717	59	109	78	40	96	91
9	1709	36	1718	60	580	80	1727	97	48
10	1710	37	1719	63	9	81	579	98	1733
19	1711	40	1720	69	63	84	1728	99	1577

6. **zosma**, yet another star. X(1824)-X(1907). The Zosma transform of a point X is the isogonal conjugate of the inverse mimosa transform of X.

7. **dalethdivision** of P by U : $\psi_P \circ hirst_1 \circ \psi_P^{-1}$ where $hirst_1\,(X) = hirstpoint\,(X_1,\, X)$ and thus $U \neq P$. Using barycentrics, one obtains :

$$x \simeq \left(\frac{w}{r} - \frac{v}{q}\right)^2 p - \left(\frac{u}{p} + \frac{v}{q} + \frac{w}{r}\right) u - 3\,\frac{u^2}{p}$$

   (a) This operation is type-keeping with respect to P, U.

   (b) The locus of fixed points of $hirst_1$ is the circumconic $cc\,(X_1)$. Therefore, the locus of fixed points of $daleth_P$ is the conic $cvc\,(P, P)$ tangent to the sidelines of ABC at the cevian points of P.

8. **hedivision** of P by U : $\psi_P^{-1} \circ \phi \circ \psi_P$ where $\phi\,(X) = hirstpoint\,(X_1,\, X)$ and thus $U \neq \psi_P^{-1}\,(P)$. Using barycentrics, one obtains :

$$x \simeq -p\left(\frac{v}{b} + \frac{w}{c}\right)^2 + \frac{qa}{b}\left(\frac{u}{a} + \frac{w}{c}\right)^2 + \frac{ra}{c}\left(\frac{u}{a} + \frac{v}{b}\right)^2 + \frac{rqa^2}{cbp}\left(\frac{u}{a}+\frac{v}{b}\right)\left(\frac{u}{a}+\frac{w}{c}\right) - \frac{qcp}{br}\left(\frac{u}{a}+\frac{w}{c}\right)\left(\frac{v}{b}+\frac{w}{c}\right) - \frac{brp}{qc}\left(\frac{u}{a}+\frac{v}{b}\right)\left(\frac{v}{b}+\frac{w}{c}\right)$$

   (a) The locus of fixed points is a conic, but not a conic of cevians.

16.8 Miscellany

16.8.1 Poles-of-lines and polar-of-points triangles

In what follows, indexes are to be taken modulo 3.

Definition 16.8.1. Polars-of-points triangle. Consider the general triangle $\mathcal{T}$ with vertices T_i, for $i = 1, 2, 3$. Taking the tripolars, we obtain a trigone. Taking the dual, we obtain the (may be degenerate) polars-of-points triangle $\mathcal{T}_U = pntpoltri\,(\mathcal{T})$. Its vertices are :

$$U_i = tripolar\,(T_{i+1}) \wedge tripolar\,(T_{i+2}) \tag{16.3}$$

Definition 16.8.2. Poles-of-lines triangle. Consider the general triangle $\mathcal{T}$ with vertices T_i, for $i = 1, 2, 3$. Taking the dual trigone and then the tripoles, we obtain the (may be degenerate) poles-of-lines triangle $\mathcal{T}_P = linpoltri\,(\mathcal{T})$. Its vertices are :

$$P_i = tripole\,(T_{i+1} \wedge T_{i+2}) \tag{16.4}$$

Remark 16.8.3. An example of flat line-polar triangle is given by triangles sharing the circumcircle of ABC

Lemma 16.8.4. *The determinants of these triangles are:*

$$\det \mathcal{T}_U = (\det \mathcal{T}_{isot})^2 \;\; ; \; \det \mathcal{T}_P = \frac{(\det \mathcal{T})^2 \, \det \mathcal{T}_{isot}}{\prod_9 \, Adjoint\,(\mathcal{T})}$$

where $\det \mathcal{T}_{isot}$ *is either the determinant of the triangle of the isotomics or the determinant of the trigone of the tripolars, and* Π_9 *is the condition expressing that two vertices of* $\mathcal{T}$ *are aligned with a vertex of* ABC.

Proposition 16.8.5. *When isotomic conjugates are collinear,* $\mathcal{T}_U$ *is totally degenerate. Otherwise* $pntpoltri\,(\mathcal{T})$ *is a triangle. Point-polarity is type-keeping (and both tribes share the same formula).*

Proof. For example, the polar of P_1 is the line $x/p_1 + y/q_1 + z/r_1 = 0$, and the polars of P_2 and P_3 are defined cyclically. Then U_1 is obtained as the common point of the last two lines. □

Proposition 16.8.6. *When* $\mathcal{T}$ *is flat (aligned points), then* $\mathcal{T}_P$ *is totally degenerate. When isotomic conjugates are collinear,* $\mathcal{T}_P$ *is flat, i.e. simply degenerate. Otherwise,* $linpoltri\,(\mathcal{T})$ *is a triangle. The line-polarity transform is type-keeping (and both tribes share the same formula).*

Proof. For example $U_2 \wedge U_3$ gives the barycentrics of line U_2U_3, while tripole is transpose and invert : being the product of two type-crossing transforms, *linpoltri* is type-keeping. □

Proposition 16.8.7. *Line-polar and point-polar transforms are converse of each other... in the generic case. More precisely,* $pntpoltri\,(linpoltri\,(\mathcal{T}))$ *gives* $\mathcal{T}$ *times* $\det \mathcal{T}$*, going back to any non degenerate triangle. On the contrary,* $linpoltri\,(pntpoltri\,(\mathcal{T}))$ *gives* $\mathcal{T}$ *times* $1/\det \mathcal{T}_{isot}$*: the converse relation holds certainly when isotomic conjugates of points* P_i *aren't collinear* and *points* P_i *aren't collinear either* and *points* P_i *aren't on the sidelines.*

16.8.2 Unary cofactor triangle, eigencenter

Definition 16.8.8. The **unary cofactor triangle** of triangle U_i $(i = 1, 2, 3)$ is the triangle whose vertices are the isoconjugates of the vertices of the line-polar triangle of the points U_i. This operator is type-crossing over the U_i, but is nevertheless type-keeping over all involved points when using any fixed point F instead of $P = F^2$. Using barycentrics :

$$X_i = {}^t(U_{i+1} \wedge U_{i+2}) \underset{b}{*} P$$

Proposition 16.8.9. *When triangle* $U_1U_2U_3$ *is degenerate (collinear vertices), then triangle* $X_1X_2X_3$ *is totally degenerate (reduced to a point). Apart this situation, the unary cofactor transform is involutory.*

Definition 16.8.10. Eigencenter. Any triangle $U_1U_2U_3$ and its unary cofactor $X_1X_2X_3$ are perspective. Their perspector is called the eigencenter of these triangles (formula don't simplify, and has Maple-length 945).

When the original triangle is the cevian or the anticevian of a point U, formula shorten into :

$$\begin{aligned} eigencenter\,(\mathcal{C}_U) &= anticomplem\left(\left(U \underset{b}{*} U\right)^*_P\right) \underset{b}{*} U^*_p = cevadiv\,(U, U^*) \\ eigencenter\,(\mathcal{A}_U) &= anticomplem\left(U \underset{b}{*} U \underset{b}{\div} P\right) \underset{b}{*} U \end{aligned}$$

These points are called, respectively, the eigentransform and the antieigentransform of point U (see Section 22.4.8).

Chapter 17

Perspective Drawing

17.1 Working out an example

In the Euclidean plane, let us consider an equally divided square, where $A = (-1, -1)$ and $C = (+2, +2)$. And apply the following rules:

1. Perspective preserves alignment.
2. Real world parallels intersect in the drawing at a "vanishing point". All such points are on a same line (the horizon).
3. On a parallel to the horizon, the ratios of algebraic measurements are preserved.

In the drawing plane, let us take three points at random: $A\,(12, 0)$, $B\,(19, 6)$, $D\,(9, 5)$. Here, "at random" means: change these coordinates if you want to, but dont expect simpler numerics...).

1. /home/douillet/docs/Forums/Phorum/0_Geometrie_projective/Perspective_in_art
2. /home/douillet/docs/Forums/Phorum/0_Geometrie_projective/Stfj_cobars
   (a) figures gauche - droite
   (b) en particulier macros ABCx, ABCLx (à la fin)
3. Enseigner_projective_2024 : Swingmustard msg-2479049
4. Enseigner_projective_2024: Swingmustard msg-2479253 – 1 May 2024
5. même quand @pldx1 disait (p.3 de Cobars): — extrait de Swingmustard msg-2479577 – 3 May 2024 7:48PM

Eléments de biblio

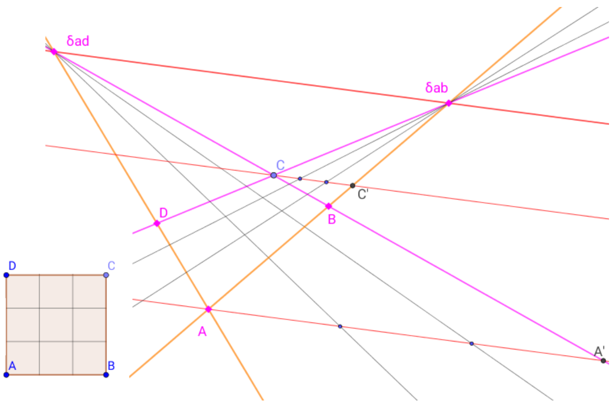


Figure 17.1: Drawing a divided square

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   (d) 095 (103) De la peinture
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    (b) https://drive.google.com/file/d/0Bz4Gx3D4ZlgGZzlsaG1hb0hxX1U/view?resourcekey=0-m59GO8_MKDzwPf_hut3Ggg
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Chapter 18

Cremona group and isoconjugacies

Notation 18.0.1. From now on, the four upright bold letters $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}$ are four complex numbers involved in a Cremona homographic transforms, not to be confused with the slanted letters a, b, c which are used to note the sidelengths of a given triangle. In this context, $\mathrm{a}', \mathrm{b}', \mathrm{c}', \mathrm{d}'$ are four other (independent) complex variables, the relations $\mathrm{a}' = \overline{\mathrm{a}}$, etc being assumed only for visible objects.

The context should be sufficient to avoid any confusion between a and a, etc. Moreover, a careful reader will not confuse upright characters with italic ones.

18.1 Homographic Cremona transforms of the projective plane

Definition 18.1.1. The upper spherical map of the Morley space $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ is the projection $\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}} \mapsto \mathbf{Z} : \mathbf{T}$, while the lower spherical map is the projection $\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}} \mapsto \overline{\mathcal{Z}} : \mathbf{T}$. Each of them sends $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ onto (yet another copy) of the Riemann sphere $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^2\right)$.

Definition 18.1.2. (reminder) An homography ψ is an element of $\mathbb{PGL}_{\mathbb{C}}\left(\mathbb{C}^2\right)$. Such object can be seen as

$$z \mapsto \psi\left(z\right) \doteq \frac{\mathrm{a}\, z + \mathrm{b}}{\mathrm{c}\, z + \mathrm{d}}$$

acting on the Riemann sphere $\overline{\mathbb{C}}$.

Construction 18.1.3. ***Construct the middle of a subtangent*** $[J, K]$***,*** *even if E is inside the conic (so that only pol$E$ and BC are available, see Figure 18.1). Obtain T, the contact point and join to the center O. Then OT cuts $FG = \mathrm{pol}E$ at some point L. And finally, EL cuts BC at M which is the required middle of $[J, K]$. See pappus (2017) for some context, and Construction 27.10.12 for an application.*

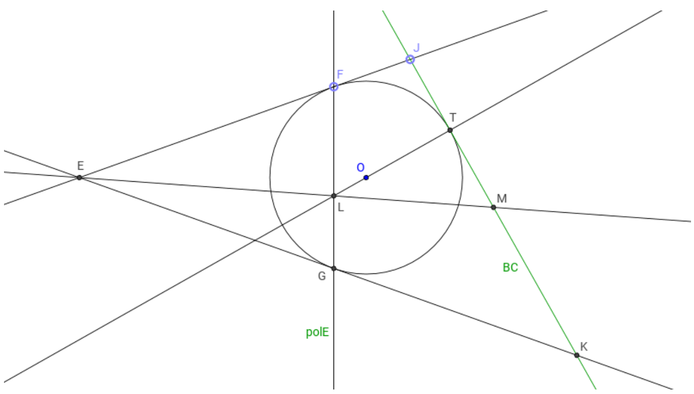

Figure 18.1: Construct the middle of a subtangent

Proof. A first step is to prove the result for the unit circle when E lies outside the circle and nothing goes at infinity, so that ordinary complex numbers can be used. Let F be a point on the unit circle, note $z_F = \alpha$ and let $z_G = 1/\alpha$. The tangents at F and G cut at a point$E$ on the real axis, and $x_E = 2\alpha/\left(1+\alpha^2\right)$. We have $FG \simeq \left[\alpha : -1-\alpha^2 : \alpha\right]$.

Then define $J = E + k\overrightarrow{EF}$, $K = E + k'\overrightarrow{EG}$. If we require that JK remains tangent to $\mathfrak{C}$, this induces an homography between parameters:

$$k' = \frac{-4k+4}{\left(\alpha^{+2}-2+\alpha^{-2}\right)k+4} = \frac{x_E^2\,k - x_E^2}{\left(x_E^2-1\right)k - x_E^2}$$

Straightforward computations are leading to:

$$z_T = \frac{k\,\alpha\left(1-\alpha^2\right)-2\alpha}{k\left(\alpha^2-1\right)-2\,\alpha^2}\ ;\ z_L = \alpha\,\frac{\left(\alpha^2-1\right)^2k^2+4\left(\alpha^2-1\right)k+4}{\left(\alpha^2-1\right)^2k^2+4\,\alpha^2}$$

$$z_M = \frac{\alpha\,\left(\left(\alpha^2-1\right)k+2\right)\left(\left(\alpha^2-1\right)^2k+6\,\alpha^2+2\right)}{2\left(\alpha^2+1\right)\left(\left(\alpha^2-1\right)^2k+4\,\alpha^2\right)}$$

And we can check that $z_J + z_K = 2z_M$.

The key point is that line FG is the polar of E and therefore remains visible even if E goes inside the circle, while α ceases to be a "true" turn and J, K become conjugate invisible points on the visible line $BC \simeq \left[\dfrac{k\left(\alpha^2-1\right)-2\,\alpha^2}{\alpha\,\left(k\left(\alpha^2-1\right)+2\right)}, 2, \dfrac{\alpha\,\left(k\left(\alpha^2-1\right)+2\right)}{k\left(\alpha^2-1\right)-2\,\alpha^2}\right]$. □

Definition 18.1.4. A **Cremona homography** combines two ordinary homographies, each of them acting on its own Riemann sphere. This can be seen as:

$$(\psi,\psi')\begin{pmatrix}\mathbf{Z}\\ \mathbf{T}\\ \overline{\mathcal{Z}}\end{pmatrix} \doteq \begin{pmatrix}\dfrac{\mathrm{a}\,\mathbf{Z}+\mathrm{b}\,\mathbf{T}}{\mathrm{c}\,\mathbf{Z}+\mathrm{d}\,\mathbf{T}}\\ 1\\ \dfrac{\mathrm{a}'\,\overline{\mathcal{Z}}+\mathrm{b}'\,\mathbf{T}}{\mathrm{c}'\,\overline{\mathcal{Z}}+\mathrm{d}'\,\mathbf{T}}\end{pmatrix} \simeq \begin{pmatrix}\left(\mathrm{a}\,\mathbf{Z}+\mathrm{b}\,\mathbf{T}\right)\left(\mathrm{c}'\,\overline{\mathcal{Z}}+\mathrm{d}'\,\mathbf{T}\right)\\ \left(\mathrm{c}\,\mathbf{Z}+\mathrm{d}\,\mathbf{T}\right)\left(\mathrm{c}'\,\overline{\mathcal{Z}}+\mathrm{d}'\,\mathbf{T}\right)\\ \left(\mathrm{a}'\,\overline{\mathcal{Z}}+\mathrm{b}'\,\mathbf{T}\right)\left(\mathrm{c}\,\mathbf{Z}+\mathrm{d}\,\mathbf{T}\right)\end{pmatrix}$$

A visible homography is such that visible points are transformed into visible points. This implies the obvious relations of complex conjugacy between the coefficients of ψ and ψ'.

Theorem 18.1.5. *Consider a proper conic $\mathcal{C}$ in the projective plane and two fixed tangents Δ_1, Δ_2 to that conic. A moving tangent Δ cuts Δ_1 at M and Δ_2 at N. If we adopt two linear parametrization, k on Δ_1 and K on Δ_2, the relation $M \mapsto N$ induces an homography between parameters k and K. Moreover correspondence $\Delta_1 \hookrightarrow \Delta_2 : M \mapsto N$ can be extended into a transform Ψ that acts into $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ and looks like a pair of homographies ψ, $\overline{\psi}$, each of them acting onto one of the spherical maps.*

Proof. Consider the inscribed conic whose auxiliary line is $Q = [u, v, w]$. Consider lines AB, AC and parametrize by $M = k\,A + (1-k)\,B$, $N = K\,B + (1-K)\,C$. Assume that MN is tangent to $\mathcal{C}$ and obtain :

$$K = \frac{kw}{(u+v+w)\,k - v}$$

This proves the first part. Using the Lubin transmutation, we have :

$$M_z \doteq \begin{pmatrix}\mathbf{Z}\\ \mathbf{T}\\ \overline{\mathcal{Z}}\end{pmatrix} \simeq \boxed{aller}\cdot\begin{pmatrix}k\\ 1-k\\ 0\end{pmatrix} \simeq \begin{pmatrix}\alpha\,k+\beta\,(1-k)\\ 1\\ \dfrac{k}{\alpha}+\dfrac{1-k}{\beta}\end{pmatrix}$$

$$N_z \doteq \begin{pmatrix}\mathbf{Z}'\\ \mathbf{T}'\\ \overline{\mathcal{Z}}'\end{pmatrix} \simeq \boxed{aller}\cdot\begin{pmatrix}K\\ 0\\ 1-K\end{pmatrix} \simeq \begin{pmatrix}\alpha\,kw+\gamma\,(ku+vk-v)\\ k\,(u+v+w)-v\\ \dfrac{kw}{\alpha}+\dfrac{ku+vk-v}{\gamma}\end{pmatrix}$$

Identifying with respect to parameter k, we are conducted to define

$$\boxed{\psi} = \begin{pmatrix} (\gamma\,(u+v)+\alpha\,w) & -\left(\beta\gamma\,u+\gamma\alpha\,v+\alpha\beta\,w\right) \\ u+v+w & -\left(\alpha\,v+\beta\,(u+w)\right) \end{pmatrix}$$

in order to have :

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{Z}' \\ \mathbf{T}' \\ \overline{\mathcal{Z}}' \end{pmatrix} = \begin{pmatrix} \psi_{11} & \psi_{12} & 0 \\ \psi_{21} & \psi_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{Z} \\ \mathbf{T} \\ \overline{\mathcal{Z}} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{Z}' \\ \mathbf{T}' \\ \overline{\mathcal{Z}}' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \overline{\psi_{22}} & \overline{\psi_{21}} \\ 0 & \overline{\psi_{12}} & \overline{\psi_{11}} \end{pmatrix}\begin{pmatrix} \mathbf{Z} \\ \mathbf{T} \\ \overline{\mathcal{Z}} \end{pmatrix}$$

□

Theorem 18.1.6. *(Continued). Finally, the four focuses of $\mathcal{C}$ are the fixed points of the transform Ψ that acts into $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ while the projections of the focuses onto the upper spherical map are the two ordinary fixed points of $\psi \in \mathbb{PGL}_{\mathbb{C}}(\mathbb{C}^2)$.*

Proof. Fixed points of ψ are the roots of :

$$(u+v+w)\,\mathbf{Z}^2 - \left(\alpha\,(v+w)+\beta\,(w+u)+\gamma\,(u+v)\right)\mathbf{ZT} + \left(\beta\gamma\,u+\gamma\alpha\,v+\alpha\beta\,w\right)\mathbf{T}^2 = 0$$

This equation can be rewritten into :

$$\frac{u}{\mathbf{Z}-\alpha\mathbf{T}} + \frac{v}{\mathbf{Z}-\beta\mathbf{T}} + \frac{w}{\mathbf{Z}-\gamma\mathbf{T}} = 0$$

and this equation characterizes the focuses of the conic in the upper map. □

Notation 18.1.7. We will use concurently $f_1, f_2 \in \mathbb{C}$, but also $z_1 : t_1 : \zeta_1$ and $z_2 : t_2 : \zeta_2 \in \mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ to describe the fixed points of ψ.

Proposition 18.1.8. *Let $z \mapsto (\mathrm{a}z+\mathrm{b})/(\mathrm{c}z+\mathrm{d})$ be an homography ψ of the Riemann sphere. If we assume that the fixed points are not equal, then ψ is characterized by a number $k \in \mathbb{C}$ that is neither 0 nor ∞. We call it the **multiplier** of ψ and we have :*

$$k \doteq \text{cross\_ratio}\ (f_1, f_2, z, \psi(z)) = \frac{\mathrm{c}\,f_1+\mathrm{d}}{\mathrm{c}\,f_2+\mathrm{d}} = \psi'(f_2) = \frac{1}{\psi'(f_1)}$$

Conversely, when f_1, f_2, k are given, then ψ is given by:

$$\mathrm{a} = k\,\frac{z_1}{t_1} - \frac{z_2}{t_2}, \mathrm{b} = \frac{z_1 z_2}{t_2 t_1} - k\,\frac{z_1 z_2 k}{t_2 t_1}, \mathrm{c} = k-1, \mathrm{d} = \frac{z_1}{t_1} - k\,\frac{k z_2}{t_2}$$

So that $\psi \in \mathbb{PGL}_{\mathbb{C}}(\mathbb{C}^2)$ induces a transformation $\Psi : \mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right) \hookrightarrow \mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ defined by

$$\Psi\begin{pmatrix} \mathbf{Z} \\ \mathbf{T} \\ \overline{\mathcal{Z}} \end{pmatrix} \simeq \begin{bmatrix} \dfrac{(k t_2 z_1 - z_2 t_1)\,\mathbf{Z} - z_1 z_2\,(k-1)\,\mathbf{T}}{(k-1)\,t_2 t_1\,\mathbf{Z} - (z_2 t_1 k - z_1 t_2)\,\mathbf{T}} \\ 1 \\ \dfrac{(\kappa t_2 \zeta_1 - t_1 \zeta_2)\,\overline{\mathcal{Z}} - \zeta_1 \zeta_2\,(\kappa-1)\,\mathbf{T}}{(\kappa-1)\,t_2 t_1 \overline{\mathcal{Z}} - (\kappa \zeta_2 t_1 - \zeta_1 t_2)\,\mathbf{T}} \end{bmatrix} \tag{18.1}$$

and,in turn, Ψ has four fixed points:

$$F_1 \simeq \frac{z_2}{t_2} : 1 : \frac{\zeta_2}{t_2}\ ;\ F_2 \simeq \frac{z_2}{t_2} : 1 : \frac{\zeta_1}{t_1}\ ;\ F_3 \simeq \frac{z_1}{t_1} : 1 : \frac{\zeta_2}{t_2}\ ;\ F_4 \simeq \frac{z_1}{t_1} : 1 : \frac{\zeta_1}{t_1} \tag{18.2}$$

Proof. To see that cross_ratio $(f_1, f_2, z_1, \psi(z_1))$ = cross_ratio $(f_1, f_2, z_2, \psi(z_2))$, use the fact that cross_ratio (f_1, f_2, z_1, z_2) is invariant by ψ. The rest is obvious from definitions. □

Lemma 18.1.9. *A more symmetric quantity is :*

$$\sigma \doteq k + \frac{1}{k} - 2 = \frac{(k-1)^2}{k} = \frac{(\mathrm{a}-\mathrm{d})^2 + 4\,\mathrm{b}\,\mathrm{c}}{\mathrm{a}\,\mathrm{d} - \mathrm{b}\,\mathrm{c}}$$

Proof. Direct computation. □

Remark 18.1.10. An homography $z \mapsto (\mathrm{a}z+\mathrm{b}) / (\mathrm{c}z+\mathrm{d})$ of the Riemann sphere is involutory when $\mathrm{a}+\mathrm{d}=0$.

Proposition 18.1.11. *The focal transform described at* (18.1) *is transmuted into the following standardized transform :*

$$\Psi_0 \doteq \mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}} \mapsto \frac{(1+\mu)\,\mathbf{Z}+(1-\mu)\,\mathbf{T}}{(1-\mu)\,\mathbf{Z}+(1+\mu)\,\mathbf{T}} : 1 : \frac{(1+\nu)\,\overline{\mathcal{Z}}+(1-\nu)\,\mathbf{T}}{(1-\nu)\,\overline{\mathcal{Z}}+(1+\nu)\,\mathbf{T}}$$

by similarity :

$$\begin{pmatrix} f_1-f_2 & f_1+f_2 & 0 \\ 0 & 2 & 0 \\ 0 & \overline{f_1}+\overline{f_2} & \overline{f_1}-\overline{f_2} \end{pmatrix}$$

while Ψ *can be factored into :*

$$\Psi = trans \circ \sigma \circ multi \circ trans$$

where the Cremona transform σ*, the multiplication and the translation (the same translation is applied once and again, not once and the reverse afterward) are defined by :*

$$\sigma \begin{pmatrix} \mathbf{Z} \\ \mathbf{T} \\ \overline{\mathcal{Z}} \end{pmatrix} \simeq \begin{pmatrix} 1/\mathbf{Z} \\ 1/\mathbf{T} \\ 1/\overline{\mathcal{Z}} \end{pmatrix}, \boxed{multi} = \begin{pmatrix} \sigma_\mu & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & \sigma_\nu \end{pmatrix}, \boxed{trans} = \begin{pmatrix} 1 & \dfrac{\mu+1}{\mu-1} & 0 \\ 0 & 1 & 0 \\ 0 & \dfrac{\nu+1}{\nu-1} & 1 \end{pmatrix}$$

Proof. Direct computation. □

18.2 Defining the general Cremona transforms

Definition 18.2.1. The Cremona group is defined as the set of the bi-rational transforms of $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$. Therefore a transformation $\Psi \in$ Cremona can be written as :

$$\Psi\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right) = \psi_1\left(\mathbf{Z},\mathbf{T},\overline{\mathcal{Z}}\right) : \psi_2\left(\mathbf{Z},\mathbf{T},\overline{\mathcal{Z}}\right) : \psi_3\left(\mathbf{Z},\mathbf{T},\overline{\mathcal{Z}}\right)$$

where the ψ_j are three homogeneous polynomials of the same degree. And the existence of another transform $\Phi \in$ Cremona is assumed so that, at least formally, $(\Phi\circ\Psi)\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right) \simeq \left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)$.

Exercise 18.2.2. What can be said about the degrees when two transforms are inverse of each other ? See Diller (2011)

Definition 18.2.3. Given a Cremona transform, we define the *indeterminacy points* and the *exceptional curves* by :

$$\begin{aligned} \operatorname{Ind}(\Psi) &= \{ M \mid \psi_1(M) = \psi_2(M) = \psi_3(M) = 0 \} \\ \operatorname{Exc}(\Psi) &= \left\{ M \,\middle|\, \det \frac{\partial(\psi_1,\psi_2,\psi_3)}{\partial\left(\mathbf{Z},\mathbf{T},\overline{\mathcal{Z}}\right)} = 0 \right\} \end{aligned}$$

Exercise 18.2.4. What do you think about : *A quadratic transformation* $f \in$ Cremona *acts by blowing up three (indeterminacy) points* $\operatorname{Ind}(f) = \left\{p_1^+, p_2^+, p_3^+\right\}$ *and blowing down the (exceptional) lines joining them. Typically, the points and the lines are distinct, but they can occur with multiplicity. Then* f^{-1} *is also a quadratic transformation and* $\operatorname{Ind}\left(f^{-1}\right) = \left\{p_1^-, p_2^-, p_3^-\right\}$ *consists of the images of the three exceptional lines ?*

Remark 18.2.5. This simple relation between indeterminacy points and exceptional curves does not hold for higher degree transforms.

Theorem 18.2.6. *The Cremona group is generated by collineations and the "inverse everything" transform i.e.* $\sigma : \left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right) \mapsto \left(\mathbf{T}\overline{\mathcal{Z}}:\overline{\mathcal{Z}}\mathbf{Z}:\mathbf{Z}\mathbf{T}\right)$.

Proof. A detailed proof can be found in Alberich-Carramiñana (2002), and an historical sketch is given in Déserti (2009a). The idea is to separate infinitely neighbor points in $\operatorname{Ind}(\psi)$ if required and then proceed to a descending recursion over the cardinal of $\operatorname{Ind}(\psi)$. □

18.3 Working out some examples

*** references Alexander (1916) Coble (1922) Trkovska (2008) should be introduced here ***

Example 18.3.1. Transform ρ is $\rho\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)=\left(\mathbf{Z}\overline{\mathcal{Z}}:\mathbf{T}\overline{\mathcal{Z}}:\mathbf{T}^2\right)$. This is an involution. The set $\operatorname{Ind}(\rho)$ contains $1:0:0$ (twice) and $0:0:1$. The exceptional locus is the reunion of the contraction lines $\mathbf{T}=0$ and $\overline{\mathcal{Z}}=0$. The decomposition of this transform can be conducted as described in Table 18.1.

	result	$\operatorname{Ind}(\psi)$	transform
1	$\begin{pmatrix} xz \\ yz \\ y^2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$
2	$\begin{pmatrix} (x+y)\,z \\ yz \\ (y-z)\,y \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	σ
3	$\begin{pmatrix} (y-z)\,y \\ (x+y)\,(y-z) \\ (x+y)\,z \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\begin{pmatrix} +1 \\ -1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$
4	$\begin{pmatrix} (y-z)\,y \\ (x+y)\,(y-z) \\ (x+y)\,y \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\begin{pmatrix} +1 \\ -1 \\ -1 \end{pmatrix}$	σ
5	$\begin{pmatrix} x+y \\ y \\ y-z \end{pmatrix}$	none	$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$
6	$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$	none	

Table 18.1: Reduction of a Cremona transform

Example 18.3.2. Transform μ is $\mu\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)=\left(\mathbf{Z}\overline{\mathcal{Z}}:\mathbf{Z}^2-\mathbf{T}\overline{\mathcal{Z}}:\overline{\mathcal{Z}}^2\right)$. This is an involution. The points of indeterminacy form a sequence of infinitely close neighbor $\mu_3^+\ll\mu_2^+\ll\mu_1^+=0:1:0$. Similarly, the exceptional locus is line $\overline{\mathcal{Z}}=0$, counted three times. Due to this specificity, its Cremona factorization is longer. A description of this process is given in Déserti (2008–2009), leading to a nine steps process $(\phi_5\sigma\phi_4\sigma\phi_3\sigma\phi_2\sigma\phi_1)$.

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18.4 Isoconjugacy and sqrtdiv operator

Definition 18.4.1. Let us recall the **heuristic definition of the sqrtdiv operator** that was already given at Definition 1.4.10. We start from triangle ABC and 'fix' a point F not on the sidelines. If we call $f:g:h$ it's ABC-barycentrics, then $U_F^{\#}$, the sqrtdiv image of $U\simeq u:v:w$ is defined by :

$$sqrtdiv_F\left(U\right)\doteq U_F^{\#}\doteq\frac{f^2}{u}:\frac{g^2}{v}:\frac{h^2}{w} \tag{18.3}$$

Remark 18.4.2. Maps *sqrtdiv* and *sqrtmul* are related with cevian nests (Table 3.2, case III).

Proposition 18.4.3. *The operator sqrtdiv is globally type-keeping and therefore is a pointwise transform. Seen as a* $U \mapsto U_F^{\#}$ *transform, we have clearly a Cremona-wise involution. The fixed points are the four* $\pm f : \pm g : \pm h$, *i.e.* F *and its associates under the Lemoine transforms wrt triangle* ABC... *In other words, any of the* F_j *and the vertices of its anticevian triangle.*

Proof. Obvious from definition. □

Corollary 18.4.4. *When* U *is on line* F_jF_k, *so is* $U_F^{\#}$ *and we have :*

$$\text{cross\_ratio}\left(F_j,\, F_k,\, U,\, U_F^{\#}\right) = -1$$

Proof. We have : $\det\left(F_1,\, F_2, U_F^{\#}\right) = (-gh/vw) \times \det\left(F_1,\, F_2,\, U\right)$. □

Construction 18.4.5. . *When a pair* P, $P^{\#}$ *is known, the image* $M^{\#}$ *of a given point* M *can be constructed by ruler only. As a proof, the coordinates of the corresponding points are given, using* $P \simeq p : q : r$, $P^{\#} \simeq u : v : w$, $M \simeq x : y : z$)

1. $E_0 \doteq AM \cap CP^{\#} \simeq [uy,\, vy,\, vz]$
2. $E_1 \doteq PM \cap BC \simeq [0,\, qx - py,\, rx - pz]$
3. $E_2 \doteq AP \cap E_0E_1 \simeq [puy,\, qvx,\, rvx]$
4. $E_3 \doteq AB \cap E_0E_2 \simeq [uy\,(pz - rx)\,,\, vx\,(qz - ry)\,,\, 0]$
5. $M^{\#} \doteq CE_2 \cap E_3P^{\#} \simeq \left[\dfrac{pu}{x},\, \dfrac{qv}{y},\, \dfrac{rw}{z}\right]$

Theorem 18.4.6. Formal definition of the sqrtdiv operator. *Start from four independent points* F_1, F_2, F_3, F_4 *(three of them are never on the same line). Call* ABC *their diagonal triangle, i.e. define*

$$A = F_1F_4 \cap F_2F_3 \;;\; B = F_2F_4 \cap F_1F_3 \;;\; C = F_3F_4 \cap F_1F_2$$

For any point U *in the plane, draw both conics :*

$$\mathcal{C}_{12} \doteq \mathcal{C}\left(U,\, F_1,\, F_2,\, F_1F_4 \cap F_2F_3,\, F_2F_4 \cap F_1F_3\right) = \mathcal{C}\left(U,\, F_1,\, F_2,\, A,\, B\right)$$
$$\mathcal{C}_{34} \doteq \mathcal{C}\left(U,\, F_3,\, F_4,\, F_1F_4 \cap F_2F_3,\, F_2F_4 \cap F_1F_3\right) = \mathcal{C}\left(U,\, F_3,\, F_4,\, A,\, B\right)$$

Their fourth intersection is independent of the order chosen for the set $\{F_j\}$ *and is the* $sqrtdiv_F$ *image of* U *that was formerly defined wrt* ABC, *the diagonal triangle of the* F_j.

Proof. Choose an order over set $\{F_j\}$, use the above defined triangle ABC as the reference triangle and let $f : g : h$ be the barycentrics of F_4 in this context. Then F_1, F_2, F_3 is the anticevian triangle of F_4 wrt ABC, enforcing $F_1 = -f : g : h$, etc. Compute the conics using the usual 6×6 determinant and obtain :

$$\boxed{\mathcal{C}_{12}} \simeq \left[\begin{array}{ccc} 0 & h^2w\,(fv + gu) & -g\left(fgw^2 + h^2uv\right) \\ h^2w\,(fv + gu) & 0 & -f\left(fgw^2 + h^2uv\right) \\ -g\left(fgw^2 + h^2uv\right) & -f\left(fgw^2 + h^2uv\right) & 2\,fgw\,(fv + gu) \end{array}\right]$$

$$\boxed{\mathcal{C}_{34}} \simeq \left[\begin{array}{ccc} 0 & h^2w\,(gu - fv) & g\left(fgw^2 - h^2uv\right) \\ h^2w\,(gu - fv) & 0 & f\left(h^2uv - fgw^2\right) \\ g\left(fgw^2 - h^2uv\right) & f\left(h^2uv - fgw^2\right) & 2\,fgw\,(fv - gu) \end{array}\right]$$

Computing their re-intersection is straightforward, and the symmetry of the result implies the independence from the way the set $\{F_j\}$ was ordered. □

Construction 18.4.7. ***The fixed points***. *Given generic* A, B, C, U, V *the fixed points of the* ABC*-isoconjugacy that exchanges* U, V *are obtained as intersection of conics* γ_U *and* γ_V *where* γ_U *is the conic that goes through* U, *the anticevians of* U *and* cevadiv (V, U), *etc. Remark: line* UV *is tangent at* U *to* γ_U *and at* V *to* γ_V.

Proof. Equation of γ_U is $\left(g^2r^2-h^2q^2\right)x^2+\left(h^2p^2-f^2r^2\right)y^2+\left(f^2q^2-g^2p^2\right)z^2=0$. It is obvious that the four F_j belong to this conic. □

Construction 18.4.8. ***The fourth harmonic*** *(harmonic conjugate of point $U\in(F_1,F_2)$ wrt points $F_1,\ F_2$) can be constructed by choosing two arbitrary points F_3F_4 and then using the sqrtdiv operator having the four F_j as fixed points. If we chose F_3,F_4 in order to obtain the middle of F_1F_2 and the two umbilics as triangle ABC, we obtain the usual construction using one circle from the pencil admitting F_1F_2 as base points and one from the pencil admitting F_1F_2 as limit points.*

Definition 18.4.9. Isoconjugacy. Forget that $f^2:g^2:h^2$ are the squares of the barycentrics of four points, and consider them as the barycentrics of a new point, called the pole $P=p:q:r$ of the transform. Then the isoconjugate of $U=u:v:w$ wrt pole P is obtained by:

$$U_P^*=(u:v:w)_P^*\doteq p\,vw:q\,wu:r\,uv \tag{18.4}$$

Remark 18.4.10. This transform was introduced in order to unify isotomic conjugacy (Section 3.4) and isogonal conjugacy into a common frame... and also to deal with the reluctance towards imaginary focuses. When using barycentrics, isotomic conjugacy is obtained with $P=X_2$ and isogonal conjugacy with $P=X_6$. When using trilinears, you have to use (respectively) $P=X_{75}$ and $P=X_1$. When X_2 is special, its isogonal conjugate X_6 is special too. When X_1 is special, its isotomic conjugate X_{75} is special too.

Remark 18.4.11. Isoconjugacy $U\mapsto U^*$, considered as a function of U alone is type-crossing, so that it is not so clear to say that this mapping has four fixed points (real or not), namely the points $F_i\doteq\pm\sqrt{p}:\pm\sqrt{q}:\pm\sqrt{r}$.

Remark 18.4.12. Most of the time, barycentric multiplication appears as the result of two successive isoconjugacies, according to :

$$\left(X_U^*\right)_P^*=X\underset{b}{*}U_P^*$$

For example, $isot\,(isog\,(X))=\left(X_6^*\right)_2^*=X\underset{b}{*}isot\,(X_6)$ while $isog\,(isot\,(X))=\left(X_2^*\right)_6^*=X\underset{b}{*}isog\,(X_2)$

18.4.1 Some other constructions

Remark 18.4.13. A construction has already be given at Construction 3.12.1. It "suffices" to draw $\mathcal{T}_2$ as the anticevian of F and then $\mathcal{T}_3$ as the anticevian of U wrt triangle $\mathcal{T}_2$. Then $sqrtdiv_F\,(U)$ is the perspector of $\mathcal{T}_1=ABC$ with $\mathcal{T}_3$. But constructing anti-cevian triangles is not so easy.

Fact 18.4.14. *Let be given A,B,C,U,V in generic position (without alignments). Then the ABC-isoconjugacy ψ that exchanges U,V can be constructed as follows. Once for ever, point R is chosen on line UV and its cevian triangle $A_RB_RC_R$ is obtained, and triangle $\mathcal{T}_1$ is constructed as :*

$$A_1=VA\cap UA_R,\ B_1=VB\cap UB_R,\ C_1=VC\cap UC_R$$

Then consider a point X not on the sidelines, define triangle $\mathcal{T}_2$ by $A_2=A_1X\cap BC$ (etc) and triangle $\mathcal{T}_3$ as cross$\,(\mathcal{T}_1,\,\mathcal{T}_2)$ i.e. $A_3=B_1C_2\cap C_1B_2$ (etc). It happens that triangle ABC and $A_3B_3C_3$ are perspective, and this perspector is the required point $\psi\,(X)$.

Proof. Put $V=p:q:r$ and $U=u:v:w$. Express R as $R=V-\rho U$. Existence of A_1 requires $wq-vr\neq0$, i.e. A,U,V not collinear. The result is :

$$\mathcal{T}_1=\begin{pmatrix}\rho u & p & p\\ q & \rho v & q\\ r & r & \rho w\end{pmatrix},\qquad \mathcal{T}_2=\begin{pmatrix}0 & \rho vx-py & \rho wx-pz\\ \rho uy-qx & 0 & \rho wy-qz\\ \rho uz-rx & \rho vz-ry & 0\end{pmatrix}$$

$$\mathcal{T}_3=\begin{pmatrix}p\,(yw+zv)-\rho vwx & puz & puy\\ qvz & q\,(zu+xw)-\rho wuy & qxv\\ rwy & rwx & r\,(xv+yu)-\rho uvz\end{pmatrix}$$

and the perspector is : $pu/x:qv/y:rw/z$ as required. In (Dean and van Lamoen, 2001), ψ was called **reciprocal conjugacy**. □

Fact 18.4.15. *Another construction, with same hypotheses (A, B, C, U, V given, without alignments). Consider the conic γ that goes through A, B, C, U, V, and the traces of UV on the sidelines, i.e. $T_a \doteq UV \cap BC$, etc. For a moving point X, call shadows of X the re-intersections X_a, X_b, X_c of lines AX, BX, CX and the conic γ. Then traces and shadows are collinear, i.e. Y_a is the second intersection of $T_a X_a$ with γ, etc.*

Proof. Direct computation. Here again $wq - vr \neq 0$, etc is required. □

18.4.2 Morley point of view

Proposition 18.4.16. Isogonal conjugacy. *The Morley affix of the isogonal conjugate of point $P = \mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}$ is given by :*

$$isog \begin{pmatrix} \mathbf{Z} \\ \mathbf{T} \\ \overline{\mathcal{Z}} \end{pmatrix} \simeq \begin{pmatrix} \sigma_3 \overline{\mathcal{Z}}^2 - \mathbf{Z}\mathbf{T} - \sigma_2 \overline{\mathcal{Z}}\mathbf{T} + \sigma_1 \mathbf{T}^2 \\ \mathbf{T}^2 - \mathbf{Z}\overline{\mathcal{Z}} \\ \dfrac{1}{\sigma_3}\mathbf{Z}^2 - \dfrac{\sigma_1}{\sigma_3}\mathbf{Z}\mathbf{T} - \overline{\mathcal{Z}}\mathbf{T} + \dfrac{\sigma_2}{\sigma_3}\mathbf{T}^2 \end{pmatrix} \tag{18.5}$$

Proof. This formula can be stated using the representation of first degree. Start from point P and obtain Δ_A, the A-isogonal of line AP, by solving :

$$\begin{aligned} \Delta_A \cdot A &= 0 \\ \tan(AB,\, \Delta_A) + \tan(AC,\, AP) &= 0 \end{aligned}$$

Compute $\Delta_A \wedge \Delta_B$ and obtain a symmetric expression, proving that Δ_C goes also through this point. One can check that $isog\,(\Omega^+) = \Omega^-$ and vice versa. □

Exercise 18.4.17. Use unimodular α, β, γ to describe the reference triangle of the complex plane. Let $P \simeq \mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}$ be the generic point, P^* its isogonal conjugate wrt ABC and $U = (P + P^*)/2$. Let O be the circum-center, and N =X(5) the Euler center. Use formula (18.5), (15.10), and the usual law $s \star t = (s+t)/(1 - s\,t)$ to check that quantity :

$$\psi(P) \doteq \mathrm{tandrs}\,(Ox, OP) \star \mathrm{tandrs}\,(Ox, OP^*) \star \mathrm{tandrs}\,(Ox, NU)$$

is independent of P. One can also use barycentrics, and use BC instead of Ox as reference line. The question is: what does this prove ?

Proposition 18.4.18. *The four **in-excenters** of triangle ABC are enumerated by the polynomial :*

$$\zeta^4 - 2\,\sigma_2\,\zeta^2 + 8\,\sigma_3\,\zeta + \left(\sigma_2^2 - 4\,\sigma_1\,\sigma_3\right) \tag{18.6}$$

Proof. The fixed points of the isogonal transform are obtained by solving the equation $\mathbf{T}\,isog\,(M) - \left(\mathbf{T}^2 - \mathbf{Z}\overline{\mathcal{Z}}\right) M = 0$. This gives two polynomials of degree 3, that are not self-conjugate but conjugate of each other. The corresponding algebraic curves are not visible. Intersecting these curves, we obtain nine points. Among them, are both umbilics. The umbilical pair is fixed, but a given umbilic is not fixed. They are nevertheless appearing since both $\mathbf{T}$ and $\mathbf{T}^2 - \mathbf{Z}\overline{\mathcal{Z}}$ are vanishing here. When computing the $\overline{\mathcal{Z}}$ resultant of these polynomials, we obtain an eight degree polynomial that factors into :

$$\mathbf{T} \times \prod_3 (\mathbf{Z} - \alpha\mathbf{T}) \times \mathrm{poly}_4\,(\mathbf{Z},\, \mathbf{T})$$

The umbilical pair is represented by $\mathbf{T}$, vertices are appearing and it remains the required polynomial of degree 4: this gives the $2 + 3 + 7 = 9$ intersections of two degree 3 curves. To be sure of what happens, we can compute the $\mathbf{T}$ resultant of both polynomials and obtain :

$$\mathbf{Z}\overline{\mathcal{Z}} \times \prod_3 \left(\mathbf{Z} - \alpha^2 \overline{\mathcal{Z}}\right) \times \mathrm{poly}_4\,\left(\mathbf{Z},\, \overline{\mathcal{Z}}\right)$$

where each point is represented by a specific factor. □

Remark 18.4.19. When substituting $\alpha = \alpha^2$, etc (i.e. using the Lubin representation of second degree), polynomial (18.6) splits, with roots $\pm\beta\gamma \pm \gamma\alpha \pm \alpha\beta$ as required.

18.4.3 The isogonal Morley formula

Proposition 18.4.20. *Points* $\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}$ *and* $z : t : \zeta$ *are isogonal conjugate of each other if and only if :*

$$\left\{ \begin{array}{lcl} \sigma_3 \zeta \overline{\mathcal{Z}} + t\mathbf{Z} + z\mathbf{T} - \sigma_1 t\mathbf{T} & \underset{1}{=} & 0 \\ z\mathbf{Z} + \left(\zeta\mathbf{T} + t\overline{\mathcal{Z}}\right)\sigma_3 - \sigma_2\, t\mathbf{T} & \underset{1}{=} & 0 \end{array} \right. \tag{18.7}$$

Since these equations are complex conjugates of each other, only one relation is required for visible points.

Proof. This is Corollary 10.6.3. A method to find such an expression ϑ is as follows. We search coefficients such that :

$$z\mathbf{Z}c_{00} + t\mathbf{Z}c_{01} + \zeta\mathbf{Z}c_{02} + z\mathbf{T}c_{10} + t\mathbf{T}c_{11} + \zeta\mathbf{T}c_{12} + z\overline{\mathcal{Z}}c_{20} + t\overline{\mathcal{Z}}c_{21} + \zeta\overline{\mathcal{Z}}c_{22} = 0$$

when conjugacy occurs. This formula must be symmetric in M and M^*, so that $c_{10} = c_{01}$, $c_{20} = c_{02}$, $c_{12} = c_{21}$. Writing that in-excenters are fixed points gives us 4 equations, leaving two indeterminate coefficients. One obtains, for example, $c_{01}\vartheta + c_{12}\,\vartheta'$. And we can go back to Lubin-1. But ϑ' happens to be the complex conjugate of ϑ. Exactly as written, equation $\vartheta = 0$ defines a line. But this equation is not self-conjugate, and only a point of the line is visible. When cutting by conjϑ, we obtain a point (and a 2nd degree equation, as it should be). □

Exercise 18.4.21. Apply the same method to the isotomic conjugacy, obtain the only possible formula... and conclude.

18.4.4 Isoconjkim

In the old ancient times, Kimberling (1998) introduced another definition of the isoconjugacy, in an attempt to unify isotomic and isogonal conjugacies into a broader concept. Thereafter, this definition was changed into the one given above. To avoid confusion with (18.4), we will use the term **isoconjkim** to describe the older concept.

Definition 18.4.22. isoconjkim. For points outside the sidelines of ABC, the Kimberley P-**isoconjkim** of U is the point X such the product of the trilinears of P, U, X gives $1 : 1 : 1$. Restated into barycentrics, this gives :

$$P \underset{b}{*} U \underset{b}{*} X = X_1 \underset{b}{*} X_1 \underset{b}{*} X_1 \tag{18.8}$$

In this transformation, P and U play the same role so that *isoconjkim* acts like a multiplication. On the other hand U and X play also the same role, so that $isoconjkim_P$ is involutory. Description $X = isoconjkim_P\,(U)$ reflects the fact that second point of view is the more useful.

Example 18.4.23. Here is a list of various isoconjkim transformations. The *isogonal conjugation* is X_1-*isoconjkim* while the *isotomic conjugation* is X_{31}- *isoconjkim*.

P	barycentrics	trilinears	pole	bar(pole)	fixed
$X(1)$	$a^2\,\frac{1}{u}$	$1\,\frac{1}{u}$	$X(6)$	a^2	$X(1)$
$X(2)$	$a^3\,\frac{1}{u}$	$a\,\frac{1}{u}$	$X(31)$	a^3	$X(365)$
$X(3)$	$\left(a^2/\cos A\right)\frac{1}{u}$	$(1/\cos A)\,\frac{1}{u}$	$X(19)$	a/S_a	$X(???)$
$X(4)$	$a^2\cos A\,\frac{1}{u}$	$\cos A\,\frac{1}{u}$	$X(48)$	$a^3 S_a$	$X(???)$
$X(6)$	$a\,\frac{1}{u}$	$(1/a)\,\frac{1}{u}$	$X(1)$	a	$X(366)$
$X(19)$	$a\,\cos A\,\frac{1}{u}$	$(\cos(A)/a)\,\frac{1}{u}$	$X(3)$	$a^2 S_a$	$X(???)$
$X(31)$	$1\,\frac{1}{u}$	$\left(1/a^2\right)\frac{1}{u}$	$X(2)$	1	$X(2)$
$X(48)$	$(a/\cos A)\,\frac{1}{u}$	$1/(a\,\cos A)\,\frac{1}{u}$	$X(4)$	$1/S_a$	$X(???)$

18.5 Angular coordinates

18.5.1 The general case

Remark 18.5.1. In this section, α, etc are not the later defined Lubin affixes. This remark can be understood as an anti-spoiler !

Definition 18.5.2. Let M be a point neither at infinity nor on the circumcircle. The three angles $\alpha = (MB, MC)$, etc are called the **angular coordinates** of M. According to the following proposition, the point M is characterized by α, β, γ.

Proposition 18.5.3. *Let be given three finite numbers* $\cot\alpha$, $\cot\beta$, $\cot\gamma$. *The locus of the points such that* $\cot(MB, MC) = \cot\alpha$ *is a circle* μ_a. *The three circles* μ_j *are concurrent in a point* M *if and only if*

$$\begin{aligned}(\cot\alpha - \cot A)(\cot\beta - \cot B)(\cot\gamma - \cot C) &\neq 0\\ \alpha+\beta+\gamma &= 0\end{aligned}$$

and then we have : $M \simeq \dfrac{1}{\cot(\alpha)-\cot(A)} : \dfrac{1}{\cot(\beta)-\cot(B)} : \dfrac{1}{\cot(\gamma)-\cot(C)}$

Conversely, if $M = p : q : r$ *then*

$$\cot(\alpha) - \cot(A) = -\frac{a^2qr + b^2pr + c^2pq}{2S(p+q+r)}\,\frac{1}{p} \tag{18.9}$$

Proof. Obtain $\mu_a \simeq S_a - 2S\cot\alpha : 0 : 0 : 1$, etc from the very definition of μ_a. Then compute $\nu \doteq \bigwedge_3(\mu_a, \mu_b, \mu_c)$ and require that $\nu \cdot \boxed{\underset{b}{Q^{-1}}} \cdot {}^t\nu = 0$. This gives a product of four factors. Condition $\cot\alpha = \cot A$ leads to A or to the whole circumcircle, and is to be discarded. The last factor is the denominator of the well known addition formula. Thereafter, obtaining M is straightforward. ☐

Example 18.5.4. Here are some points having simple angular coordinates

	X(1)	X(80)	X(36)	X(1)	X(265)	X(3)	X(4)	X(186)
	$\frac{A+\pi}{2}$	$\frac{\pi-A}{2}$	$\frac{3A-\pi}{2}$		$-2A$	$2A$	$-A$	$3A$
a	aX(80)	aX(1)	33599		aX(3)	aX(265)	x	5962
i		10260	iX(1)	iX(36)	5961	x	iX(186)	iX(4)
g	gX(1)	36	80	gX(1)	gX(186)	gX(4)	gX(3)	gX(265)

	X(14)	X(13)	X(15)	X(16)				
	$+\frac{\pi}{3}$	$-\frac{\pi}{3}$	$A+\frac{\pi}{3}$	$A-\frac{\pi}{3}$	α	-α	$A+\alpha$	$A-\alpha$
a	aX(13)	aX(14)	11600	11601	*	*		
i	6105	6104	iX(16)	iX(15)			*	*
g	gX(16)	gX(15)	gX(13)	gX(14)4		*	*	

Proposition 18.5.5. *When* M, M' *are isogonal conjugates (g), then* $\alpha + \alpha' = A$, etc. *When* M, M' *are inverse in the circumcircle (i), then* $\alpha + \alpha' = 2A$, etc.

Proof. Obvious from the definitions. One can also substitute the isogonal formula into $1/\cot(\alpha + \alpha' - A)$, or the invincircum formula into $1/\cot(\alpha + \alpha' - 2A)$... and conclude using 18.9. ☐

18.5.2 Steiner triangle

Definition 18.5.6. Let P_a, P_b, P_c be the orthogonal reflections of a point P in the sidelines BC, CA, AB of the reference triangle. Due to Proposition 10.2.1, this triangle is called the **Steiner triangle** P.

Proposition 18.5.7. *The complex coordinates of triangle* $[P_j]$ *are:*

$$\mathrm{Steiner}\left(\begin{array}{c} z \\ t \\ \zeta \end{array}\right) \underset{z}{\simeq} \left[\begin{array}{ccc} \beta+\gamma-\beta\gamma\dfrac{\zeta}{t} & \alpha+\gamma-\gamma\alpha\dfrac{\zeta}{t} & \alpha+\beta-\alpha\beta\dfrac{\zeta}{t} \\ 1 & 1 & 1 \\ \dfrac{1}{\gamma}+\dfrac{1}{\beta}-\dfrac{z}{\gamma\beta t} & \dfrac{1}{\gamma}+\dfrac{1}{\alpha}-\dfrac{z}{\gamma\alpha t} & \dfrac{1}{\beta}+\dfrac{1}{\alpha}-\dfrac{z}{\beta\alpha t} \end{array}\right] \tag{18.10}$$

while the direct and the skew homographies ϕ and ψ defined by $A\mapsto P_a$, $B\mapsto P_b$, $C\mapsto P_c$ are

$$\begin{array}{rcl} \phi\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right) & \simeq & \dfrac{t^2\,\mathbf{Z}-\left(s_1\,t^2-s_2\zeta t+s_3\,\zeta^2\right)\mathbf{T}}{t\zeta\,\mathbf{Z}-t^2\,\mathbf{T}}:1:\dfrac{s_3\,t^2\,\overline{\mathcal{Z}}-\left(s_2\,t^2-s_1zt+z^2\right)\mathbf{T}}{\left(s_3zt\,\overline{\mathcal{Z}}-s_3t^2\,\mathbf{T}\right)} \\ \psi\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right) & \simeq & \dfrac{\left(s_1\,t^2-s_2\zeta t+s_3\,\zeta^2\right)\overline{\mathcal{Z}}-t^2\,\mathbf{T}}{t\left(t\overline{\mathcal{Z}}-\zeta\,\mathbf{T}\right)}:1:\dfrac{\left(s_2\,t^2-s_1zt+z^2\right)\mathbf{Z}-s_3\,t^2\mathbf{T}}{s_3t\left(t\,\mathbf{Z}-z\,\mathbf{T}\right)} \end{array}$$

Their poles are, respectively $\mathrm{i}P\simeq tz:z\zeta:t\zeta$ and P itself. Moreover ϕ is involutive.

Proof. Direct computation. □

Proposition 18.5.8. *The circles $AP_bP_c, P_aBP_c, P_aP_bC$ are going through a common point on the unit circle, the celebrated Miquel point.*

$$M_{\mathrm{q}}\simeq\frac{s_3\zeta-s_2t}{z-s_1t}:1:\frac{z-s_1t}{s_3\zeta-s_2t}$$

Therefore the circles $P_aP_bP_c, P_aBC, AP_bC, ABP_c$ are going through a same point, namely $\phi\left(M_{\mathrm{q}}\right)$. See Figure 18.2 (Schoute, 1882).

Fact 18.5.9. *One has the result:*

$$\mathrm{Steiner}\left(M\right).\mathrm{isogon}\left(M\right)\simeq M$$

18.5.3 Antigonal conjugacy

Theorem 18.5.10. *Points P and $Q\doteq\left(\phi\circ M_{\mathrm{q}}\right)(P)$ from 18.5.8 are* **antigonal conjugates**, *i.e. satisfy $(PB,PC)+(QB,QC)=0$, etc. Using the formerly given notations, this amounts to $\alpha+\alpha'=0$, , etc. Thus the* antigon *transform is involutive and satisfies*

$$\mathrm{antigon}=\mathrm{isogon}\,@\,\mathrm{invincircum}\,@\,\mathrm{isogon}$$

Proof. One has $\mathrm{clasim}\left(\left(\phi\circ M_{\mathrm{q}}\right)(P),B,C\right)\underset{b}{*}\mathrm{clasim}\left(P,B,C\right)\simeq 1:0:1$. □

Proposition 18.5.11. *When point P is given either by $P_b\simeq p:q:r$ (barycentrics) or by $P_z\simeq\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}$ (Lubin affixes), its antigonal Q is given by :*

$$Q\underset{b}{\simeq}\left(\begin{array}{c} \dfrac{p}{\left(a^2-b^2-c^2\right)p^2+\left(a^2-b^2\right)pq+\left(a^2-c^2\right)pr+a^2qr} \\ \dfrac{q}{\left(b^2-c^2-a^2\right)q^2+\left(b^2-c^2\right)qr+\left(b^2-a^2\right)qp+b^2rp} \\ \dfrac{r}{\left(c^2-a^2-b^2\right)r^2+\left(c^2-a^2\right)rp+\left(c^2-b^2\right)rq+c^2pq} \end{array}\right)$$

$$Q\underset{z}{\simeq}\left(\begin{array}{c} \dfrac{-\sigma_3\mathbf{Z}\overline{\mathcal{Z}}^2+\left(\sigma_2\mathbf{Z}\overline{\mathcal{Z}}+\sigma_3\sigma_1\overline{\mathcal{Z}}^2\right)\mathbf{T}-\sigma_1\mathbf{Z}\mathbf{T}^2+\left(\sigma_3-\sigma_1\sigma_2\right)\overline{\mathcal{Z}}\mathbf{T}^2+\left(\sigma_1^2-\sigma_2\right)\mathbf{T}^3}{\left(\sigma_3\overline{\mathcal{Z}}^2-\mathbf{Z}\mathbf{T}-\sigma_2\overline{\mathcal{Z}}\mathbf{T}+\sigma_1\mathbf{T}^2\right)\mathbf{T}} \\ 1 \\ \dfrac{-\sigma_3\mathbf{Z}^2\overline{\mathcal{Z}}+\left(\sigma_2\mathbf{Z}^2+\sigma_1\sigma_3\mathbf{Z}\overline{\mathcal{Z}}\right)\mathbf{T}-\left(\sigma_1\sigma_2-\sigma_3\right)\mathbf{T}^2\mathbf{Z}-\sigma_2\sigma_3\overline{\mathcal{Z}}\mathbf{T}^2+\left(\sigma_2^2-\sigma_1\sigma_3\right)\mathbf{T}^3}{\sigma_3\left(\mathbf{Z}^2-\sigma_1\mathbf{Z}\mathbf{T}-\sigma_3\overline{\mathcal{Z}}\mathbf{T}+\sigma_2\mathbf{T}^2\right)\mathbf{T}} \end{array}\right)$$

Proof. Direct computation. □

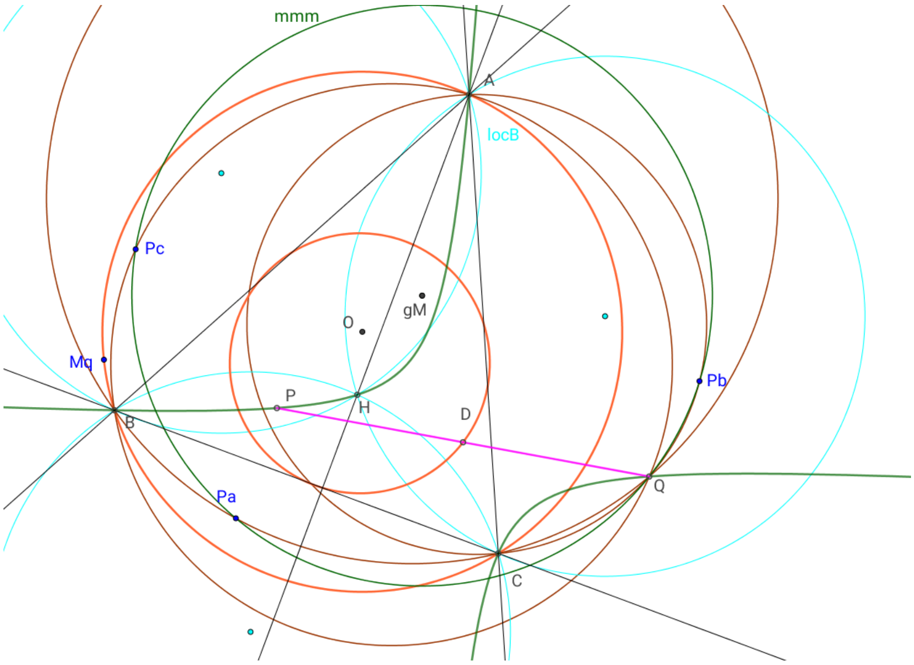


Figure 18.2: Antigonal conjugacy

Proposition 18.5.12. *The conic that goes through* $A, B, C, P, Q \doteq \operatorname{antigon}(A, B, C, P)$ *is a rectangular hyperbola (and therefore, its center is on the NPC). It goes not only through* Q *and through* $H \doteq X(A, B, C, 4)$ *but it goes also through* $\mathrm{aA} \doteq \operatorname{antigon}(P, B, C, A)$, etc *and* $X(P, B, C, 4)$, etc. *Moreover,* (P, Q) *are antipodes on this conic (and so are* (A, aA), etc. *Circle* $P_a P_b P_c$ *is centered at* $\operatorname{isogon}(P)$ *and goes also through* Q.

Proof. Direct computation. One obtains, inter alia,

$$\mathrm{aA} \simeq 1 \; : \; \frac{2\,(pS_a - rS_c)\,q}{q\,(p+r)\,a^2 - p\,(q+r)\,b^2} \; : \; \frac{2r\,(pS_a - qS_b)}{r\,(p+q)\,a^2 - p\,(q+r)\,c^2}$$

These results confirm the involutory nature of this transform. □

Proposition 18.5.13. *Antigonal conjugacy is a 5th degree Cremona transform. The indeterminacy set contains six points:* A, B, C, H *and both umbilics. The exceptional locus is the reunion of six conics. Each of them goes through five points of* $\operatorname{Ind}(\operatorname{antigon})$*: circumcircle is blown-down into* H*, circle* BCH *(centered at* $B + C - O$*) is blown-down into* A*, idem for the other two vertices. Finally, curve :*

$$\gamma_y = \mathbf{Z}^2 - \sigma_1 \mathbf{Z}\mathbf{T} - \sigma_3 \overline{\mathcal{Z}}\mathbf{T} + \sigma_2 \mathbf{T}^2$$

that goes through A, B, C, H, Ω_y *is blown-down into* Ω_y *(the same umbilic), idem for the other umbilic.*

Proof. Direct computation. □

Proposition 18.5.14. *Define the ig transform as* $\operatorname{ig} = \operatorname{invincircum} \circ \operatorname{isogon}$ *then point* $\operatorname{ig}(A, B, C, P)$ *is independent of the ordering of points* A, B, C, P *(Hyacinthos 20929).*

Proof. Transform triangle ABC into ABP, and thus C into $\phi^{-1}(P)$. Then use the usual change of triangle formula. Since ig only involves even powers of the sidelengths, everything goes fine... and the result follows. □

Proposition 18.5.15. *Seen as a Cremona map, the* ig *transform has the same indeterminacy locus as the antigonal transform. The exceptional locus contains three lines and three conics. A sideline like* BC *blows-down into the opposite vertex* A*. Curve* γ_x *(the same as before) now blows down to* Ω_y*, while* γ_y *blows-down to* Ω_x*. Finally, the circumcircle blows-down to its center X(3), while* $X(3) \underset{\mathrm{ig}}{\mapsto} X(186)$ *is regular and X(265) is the only regular point that maps onto H =X(4).*

Proof. This lack of symmetry (a curve that blows down to a regular point, a point of indeterminacy that does not blows-up under the reverse transform) is related to the fact that ig is not involutory. □

18.6 Isogonality and perspectivity

Lemma 18.6.1. *Results from Proposition 3.8.11. Let $\mathcal{T}_2$ be a triangle perspective wrt $\mathcal{T}_1 = ABC$. Then triangle $\mathcal{T}_2$, perspector P and perspectrix Δ can be written as :*

$$\mathcal{T}_2 = \left(\begin{array}{ccc} u & p & p \\ q & v & q \\ r & r & w \end{array} \right) \; ; \; P \simeq p:q:r \; ; \; \Delta_2 = \left[\frac{1}{p-u} \; ; \; \frac{1}{q-v} \; ; \; \frac{1}{r-w} \right]$$

Point $U \simeq u : v : w$ is the perspector of $\mathcal{T}_1$ and $\mathcal{T}_3 \doteq cross\,(\mathcal{T}_1, \mathcal{T}_2)$. But, in these formulas, the same proportionality factor must be applied to (p,q,r) and (u,v,w), so that factor $k \doteq (u+v+w)\,/\,(p+q+r)$ is a projective quantity.

Proposition 18.6.2. *Assume that vertices of $\mathcal{T}_2$ are not on the sidelines of $\mathcal{T}_1$, and perspectivity as in Lemma 18.6.1. Define $\mathcal{T}_2^*$ as the triangle whose vertices are the isogonal images of the $\mathcal{T}_2$ vertices. Then triangles $\mathcal{T}_1$ and $\mathcal{T}_2^*$ are perspective, and P, U are replaced by P^* and U^*. Moreover, $\mathcal{T}_2$ and $\mathcal{T}_2^*$ share the same perspectrix if and only if $U = k\,P^*$. In this case, the isogon $\mathcal{T}_2^*$ and the crosstri $\mathcal{T}_3$ are equal.*

Proof. Computations are easy from the lemma. If A_1 is on BC then $A_2 = A$ and everything degenerates, etc. □

Remark 18.6.3. This obviously occurs with the 27 Taylor-Marr triangles since, for example, A, B_1, C_2 are aligned on the same trisector (and cyclically).

Chapter 19

Pencils of cycles in the complex plane

In this chapter we will transpose to the Complex Plane what has already been done for the Triangle Plane at Chapter 14. There are three usual ways for this purpose, each of them using its own basis for the cycles space.

Veronese	$\underset{z}{Ver}\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)\simeq\left[\mathbf{ZT},\mathbf{T}^2,\overline{\mathcal{Z}}\mathbf{T},\mathbf{Z}\overline{\mathcal{Z}}\right]$
Pedoe	$\underset{p}{Ver}\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)\simeq\left[\mathbf{ZT},\mathbf{T}^2,\overline{\mathcal{Z}}\mathbf{T},\mathbf{Z}\overline{\mathcal{Z}}-\mathbf{T}^2\right]$
Spherical	$\underset{s}{Ver}\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)\simeq\left[2X\mathbf{T},2Y\mathbf{T},\overline{\mathcal{Z}}\mathbf{Z}-\mathbf{T}^2,\mathbf{Z}\overline{\mathcal{Z}}+\mathbf{T}^2\right]$

The first choice is the simpler, the better. Dividing by four the size of each element of a 4×4 matrix divides the whole size by 64. The second choice is for comparison with the illuminating Pedoe (1970).

The third choice is rather an introduction to the stereographic formalism. This allows the most nice and intuitive figures (but there is a price to pay: computations are slower!).

Notation 19.0.1. For the the further use of the reader, we summarize here all the notations that will be introduced throughout this chapter.

index	**b**arycent.	**z** Morley	**p** Pedoe	**s** Spherical
Veronese	Ver	Verz	Verp	Vers
Verx(∞)	Sirius	Sirius	Sirius	South pole
		Verb2Verz	Verp2Verb	Verb2Vers
$\mathcal{Q}$	mQQ	zQQ	pQQ	sQQ
$\mathcal{Q}^{-1}$	mQQI	zQQI	pQQI	sQQI
$[C_j]\mapsto$ G	mkgram	mkzgram	mkpgram	mksgram
$eq\mapsto\mathcal{V}$	eq2colu	eq2coluz	eq2colup	eq2colus
$\mathcal{V}\mapsto\left(M,\rho^2\right)$	colu2bar	coluz2mor	colup2mor	colus2mor
$\left(M,\rho^2\right)\mapsto\mathcal{V}$	bar2colu	mor2coluz	mor2colup	mor2colus
$\boxed{\mathcal{C}}\mapsto\mathcal{V}$		mmz2colu	mmp2colu	mms2colu
$h\mapsto$ action		mhatz	mhatp	mhats:

The induced objects will be named using the relevant index (z,p,s). For example, the (barycentric) matrix $\underset{b}{\boxed{\mathcal{Q}}}$ will be declined as $\underset{z}{\boxed{\mathcal{Q}}}$, $\underset{p}{\boxed{\mathcal{Q}}}$ and $\underset{s}{\boxed{\mathcal{Q}}}$.

Fact 19.0.2. *In the Morley space, the equation*

$$p\,\mathbf{Z}\overline{\mathcal{Z}}+\left(q_1\mathbf{Z}+q_2\overline{\mathcal{Z}}\right)\mathbf{T}+r\,\mathbf{T}^2=0$$

describes an ordinary, visible circle when $p=1$, $q \doteq q_2 = \overline{q_1}$ and $\rho^2 \doteq q\overline{q} - r > 0$ (center is $-q$, radius is ρ). When $p=0$, this describes the union of an ordinary line and the line at infinity (or the line at infinity where each point is counted twice, i.e. the horizon circle).

Definition 19.0.3. We will say that four points M_j are concyclic when

$$\det_{j=1}^{4}\left(\underset{z}{Ver}\,(M_j)\right) = 0$$

Due to linearity, the $\underset{z}{Ver}$ can be replaced (all the four at the same time) by the $\underset{p}{Ver}$ or the $\underset{s}{Ver}$.

Definition 19.0.4. Alternatively, we can say that a **cycle** is a conic that goes through the **umbilics**, i.e. the points :

$$\Omega_x \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \Omega_y = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We call it a "circle" when the conic is either $\mathbf{T}^2$ (the horizon circle) or isn't degenerate, and a "line" when it degenerates into the union of the line at infinity and another line.

Remark 19.0.5. When searching the intersection of two circles, we better subtract the normalized equations, and obtain $\mathbf{T}\Delta$ where Δ is the "flat" radical axis (first degree equation).

19.1 Pencil of cycles in the complex plane

19.1.1 Veronese map

lubo $\boxed{Lu}$ luboo $\boxed{Lu}$ luboq $\boxed{Lv}$ lurett luretq

Definition 19.1.1. The Veronese map used in the barycentric triangle plane was defined at (14.4). The Veronese map used in the Morley space is simply:

$$\underset{z}{Ver}\left(\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}\right) \simeq \left[\mathbf{ZT}, \mathbf{T}^2, \overline{\mathcal{Z}}\mathbf{T}, \mathbf{Z}\overline{\mathcal{Z}}\right] \tag{19.1}$$

Proposition 19.1.2. *Both umbilics are mapped to* $[0,0,0,0]$ *(points of indeterminacy) while all other points at infinity are mapped to* $[0,0,0,1]$*, called Sirius. When using* $\left(\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}\right) = \boxed{\mathrm{b\Phi m}}\,.\,(x : y : z)$*, we have*

$$\underset{z}{Ver}\left(\boxed{\mathrm{b\Phi m}}\,.\,(x:y:z)\right) \simeq \underset{b}{Ver}\,(x:y:z) \cdot {}^t\boxed{\mathrm{b\Psi m}} \qquad \text{where } \boxed{\mathrm{b\Psi m}} = \left(\begin{array}{c|c} \boxed{\mathrm{b\Phi m}} & 0 \\ \hline R^2, R^2, R^2 & 1 \end{array}\right)$$

Proof. The $\boxed{\mathrm{b\Phi m}}$ part is obvious, while the R^2 acknowledge the fact that (X_3, R) is the model of all the circles in the triangle plane. □

Maple 19.1.3. One can check this result by:

```
subs(les1lon, VerB(1/bar2mor.vz)).Tr(vbartomor):  FActor(%);
```

Verz(norz(vz1)).zQQI. Tr(Verz(norz(vz2))); factor(% / zpytha(vz1,vz2));

Remark 19.1.4. The Veronese map amounts to generate the projective space of all the cycles from four of them, namely the horizon circle $\mathbf{T}^2$ (i.e. the line at infinity described twice), the fundamental isotropic lines (each of them completed by the line at infinity to obtain $\mathbf{ZT}$ and $\overline{\mathcal{Z}}\mathbf{T}$) and the "factored" circle $\mathbf{Z}\overline{\mathcal{Z}}$.

Proposition 19.1.5. *The polar hyperplanes related to the* $\underset{z}{Ver}\,(M_\tau)$ *of all the points* M_τ *of a given cycle* $\mathcal{C}$ *are going through a same point of* $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^4\right)$*, called the circle representative, that will be noted* $\underset{c}{\mathcal{V}}\,(\mathcal{C})$.

Proof. When the cycle is a circle, then it's generic point can be written as

$$\tau \simeq \frac{1}{t}\left(z:t:\zeta\right)+\rho\left(\tau:0:\frac{1}{\tau}\right)$$

and we only have to check that $\bigwedge_3\left(\underset{z}{Ver}M_\tau,\,\underset{z}{Ver}M_{+1},\,\underset{z}{Ver}M_{-1}\right)$ doesn't depends on τ. As a result, we have:

$$\underset{c}{\mathcal{V}}\left[\left(z:t:\zeta\right),\rho^2\right]\doteq\left(-\zeta:\frac{z\,\zeta}{t}-t\,\rho^2:-z:+t\right)$$

When the cycle is the line $\Delta\simeq[a,b,c]$ then $\underset{c}{\mathcal{V}}(\Delta)=a:b:c:0$. □

Corollary 19.1.6. *The locus of the representatives* $\underset{c}{\mathcal{V}}$ *of the* $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ *lines is the plane*

$$\mathcal{P}_z\simeq[0:0:0:1]$$

Proposition 19.1.7. *The Veronese row-images of the ordinary* $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ *points belong to a 3D quadric:*

$$\underset{z}{Ver}(P)\cdot\boxed{\underset{c}{\mathcal{Q}^{-1}}}\cdot{}^t\underset{z}{Ver}(P)=0 \qquad \text{where}\ \boxed{\underset{c}{\mathcal{Q}^{-1}}}=\left(\begin{array}{cccc}0&0&-1&0\\0&0&0&1\\-1&0&0&0\\0&1&0&0\end{array}\right) \tag{19.2}$$

Proof. We even have a more precise result: for any points at finite distance M_1, M_2, one has:

$$\underset{z}{Ver}\left(\frac{M_1}{\mathcal{L}_z\cdot M_1}\right)\cdot\boxed{\underset{c}{\mathcal{Q}^{-1}}}\cdot{}^t\underset{z}{Ver}\left(\frac{M_2}{\mathcal{L}_z\cdot M_2}\right)=\left|M_1M_2\right|^2$$

when $\underset{z}{Ver}$ and $\boxed{\underset{c}{\mathcal{Q}^{-1}}}$ are taken as exactly egal to their definitions (i.e. not up to a proportionality factor). □

Maple 19.1.8. One can check this result by:

```
Verz(norz(vz1)).  zQQI. Tr(Verz(norz(vz2))); factor(% / zpytha(vz1,vz2));
```

Proposition 19.1.9. *The column-representatives of point-circles, i.e. the* $\underset{c}{\mathcal{V}}(P)\simeq\boxed{\underset{c}{\mathcal{Q}^{-1}}}\cdot{}^t\underset{z}{Ver}(P)$, *belong to the 3D paraboloid defined by* $\boxed{\underset{z}{\mathcal{Q}}}$. *When two circles are involved, we have the more precise result :*

$$\frac{{}^t\underset{c}{\mathcal{V}}(M_1,\rho_1)\cdot\boxed{\underset{z}{\mathcal{Q}}}\cdot\underset{c}{\mathcal{V}}(M_2,\rho_2)}{\left(\mathcal{P}_z\cdot\underset{c}{\mathcal{V}}_1\right)\times\left(\mathcal{P}_z\cdot\underset{c}{\mathcal{V}}_2\right)}=\left|M_1M_2\right|^2-\rho_1^2-\rho_2^2 \tag{19.3}$$

$$\text{where}\ \boxed{\underset{z}{\mathcal{Q}}}=\left(\begin{array}{cccc}0&0&-1&0\\0&0&0&1\\-1&0&0&0\\0&1&0&0\end{array}\right)$$

Proof. We have $\boxed{\underset{c}{\mathcal{Q}^{-1}}}=\boxed{\mathrm{b\Psi m}}^{-1}\cdot\boxed{\underset{b}{\mathcal{Q}^{-1}}}\cdot{}^t\boxed{\mathrm{b\Psi m}}^{-1}$ and $\boxed{\underset{z}{\mathcal{Q}}}=\boxed{\mathrm{b\Psi m}}\cdot\boxed{\underset{b}{\mathcal{Q}}}\cdot{}^t\boxed{\mathrm{b\Psi m}}$. □

Remark 19.1.10. A failed attempt has been done in the past to use a set of factors -2 and -1/2 in order to obtain the squared radius itself in formula 19.3. But our final choice is to keep the value $d^2-\rho_1^2-\rho_2^2$ as the result of **??**.

Maple 19.1.11. One can check these results by:

```
(vbar2mor).(FActor@subs)(les1lon, mQQ).Tr(vbar2mor):  FActor(%); zipd(zQQ,%);
Tr(1/vbar2mor).(FActor@subs)(les1lon, mQQI).(1/vbar2mor):  FActor(%):  zipd(zQQI,%);
(mor2coluz)(vz,K^2):  %/(plinfz.%):  factor(Tr(%).zQQ.%)/(-2);
```

Proposition 19.1.12. *Two point-circles are orthogonal wrt the quadric when their centers share one of their two coordinates.*

Proof. Quite obvious from (19.3) and $|M_1M_2|^2 = \left(\frac{z_2}{t_2} - \frac{z_1}{t_1}\right)\left(\frac{\zeta_2}{t_2} - \frac{\zeta_1}{t_1}\right)$ □

Maple 19.1.13. The Maple package 'faisceaux' contains:

constants: "zQQ"= $\underset{z}{\boxed{\mathcal{Q}}}$, "zQQI"= $\underset{c}{\boxed{\mathcal{Q}^{-1}}}$,"plinfz"= $\mathcal{P}_z$

functions: "Verz"= $\underset{z}{Ver}$, "mor2coluz"= $\underset{c}{\mathcal{V}}$, "coluz2mm", "coluz2mor", "eq2coluz", "mhatz", "mm2coluz", "mkzgram", "zaction"

Theorem 19.1.14. ***Common orthogonal cycle.*** *Let be given three cycles $\Omega_1, \Omega_2, \Omega_3$. If they don't belong to the same pencil, the bundle they generate is exactly the set of all the cycles orthogonal to a fixed cycle $\Omega_\perp$. We have the formulas:*

$$\begin{aligned} W &\doteq \bigwedge_3 \left(\underset{c}{\mathcal{V}_1}, \underset{c}{\mathcal{V}_2}, \underset{c}{\mathcal{V}_3}\right) \quad \textit{(a 4-sized row)} \\ \underset{c}{\mathcal{V}_\perp} &= \underset{c}{\boxed{\mathcal{Q}^{-1}}} \cdot {}^tW \\ \text{center} &= W_1 : W_2 : W_3 \\ \text{squared radius} &= \left(\frac{-1}{2}\right) \frac{W \cdot \underset{c}{\mathcal{V}_\perp}}{\left(\mathcal{P}_z \cdot \underset{c}{\mathcal{V}_\perp}\right)^2} = \left(\frac{-1}{2}\right) \frac{{}^t\underset{c}{\mathcal{V}_\perp} \cdot \underset{z}{\boxed{\mathcal{Q}}} \cdot \underset{c}{\mathcal{V}_\perp}}{\left(\mathcal{P}_z \cdot \underset{c}{\mathcal{V}_\perp}\right)^2} \end{aligned} \tag{19.4}$$

Remark 19.1.15. These formulas are –and should remain– exactly the same as the barycentric formulas (14.18). Don't even think of using any trick to "simplify" anything. Only remember that

$${}^t\underset{c}{\mathcal{V}_1} \cdot \underset{z}{\boxed{\mathcal{Q}}} \cdot \underset{c}{\mathcal{V}_2} = \left(d_{12}^2 - r_1^2 - r_2^2\right) \times \left(\underset{c}{\mathcal{V}_1}[4]\ \underset{c}{\mathcal{V}_2}[4]\right)$$

Proposition 19.1.16. *When the 4×4 matrix $\underset{z}{\boxed{\Delta}} = \left(\underset{c}{\mathcal{V}_1} \underset{6}{\wedge} \underset{c}{\mathcal{V}_2}\right)$ describes the pencil generated by cycles $\mathcal{C}_1, \mathcal{C}_2$, the orthogonal pencil is described by:*

$$\underset{z}{\boxed{\Delta^\perp}} \simeq \underset{z}{\boxed{\mathcal{Q}}} \cdot \underset{z}{\boxed{\Delta^*}} \cdot \underset{z}{\boxed{\mathcal{Q}}} \tag{19.5}$$

Proof. See the proof of (14.20). □

19.1.2 Homographic actions over the cycles' space

CAVEAT: in this subsection, letters $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{a}', \mathrm{b}', \mathrm{c}', \mathrm{d}', k, \kappa$ are general complex numbers, while $\mathrm{a}' = \bar{\mathrm{a}}$, etc is intended for visible objects.

Proposition 19.1.17. *We will say that H, a Cremona transform H acting over $\mathbb{P}_\mathbb{C}\left(\mathbb{C}^3\right)$, is an homography when H is seen as $h : z \mapsto (\mathrm{a}\, z + \mathrm{b}) / (\mathrm{c} z + \mathrm{d})$ on the upper sphere $\mathbf{Z} : \mathbf{T}$ and seen as $\overline{h} : \zeta \mapsto (\mathrm{a}'\zeta + \mathrm{b}') / (\mathrm{c}'\zeta + \mathrm{d}')$ on the lower sphere $\mathcal{Z} : \mathbf{T}$. As already said, $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{a}', \mathrm{b}', \mathrm{c}', \mathrm{d}' \in \mathbb{C}$, together with $\mathrm{a\,d} - \mathrm{b\,c} \neq 0$, $\mathrm{a}'\,\mathrm{d}' - \mathrm{b}'\,\mathrm{c}' \neq 0$ are assumed. Therefore, we have:*

$$H : \begin{pmatrix} \mathbf{Z} \\ \mathbf{T} \\ \mathcal{Z} \end{pmatrix} \simeq \begin{pmatrix} (\mathrm{a}\,\mathbf{Z} + \mathrm{b}\,\mathbf{T})\,(\mathrm{c}'\,\mathcal{Z} + \mathrm{d}'\,\mathbf{T}) \\ (\mathrm{c}\,\mathbf{Z} + \mathrm{d}\,\mathbf{T})\,(\mathrm{c}'\,\mathcal{Z} + \mathrm{d}'\,\mathbf{T}) \\ (\mathrm{c}\,\mathbf{Z} + \mathrm{d}\,\mathbf{T})\,(\mathrm{a}'\,\mathcal{Z} + \mathrm{b}'\,\mathbf{T}) \end{pmatrix} \tag{19.6}$$

Assuming that $\mathrm{c\,c}' \neq 0$, the points of indeterminacy are both umbilics, and the so-called pole: $P \doteq -\mathrm{d}/\mathrm{c} : 1 : -\mathrm{d}'/\mathrm{c}'$. The exceptional locus is the union of $\mathcal{L}_z$, $P\Omega_x$, $P\Omega_y$ while $H(\mathcal{L}_z)$ collapses to a single point, the elop $E \simeq \mathrm{a}/\mathrm{c} : 1 : \mathrm{a}'/\mathrm{c}'$ (i.e. the pole of H^{-1}).

Proof. This is nothing but the usual formulas: $h^{-1}(\infty) = -\mathrm{d}/\mathrm{c}$ and $h(\infty) = \mathrm{a}/\mathrm{c}$. □

Proposition 19.1.18. *Homography H, as an action over the points of $\mathbb{P}_\mathbb{C}(\mathbb{C}^3)$, induces an action which is linear over the columns of $\mathbb{P}_\mathbb{C}(\mathbb{C}^4)$. Its matrix is :*

$$\boxed{\widehat{H_z}} \simeq \frac{1}{|det|}\begin{bmatrix} +\mathrm{d\,a'} & -\mathrm{c\,a'} & +\mathrm{c\,b'} & -\mathrm{d\,b'} \\ -\mathrm{b\,a'} & +\mathrm{a\,a'} & -\mathrm{a\,b'} & +\mathrm{b\,b'} \\ +\mathrm{b\,c'} & -\mathrm{a\,c'} & +\mathrm{a\,d'} & -\mathrm{b\,d'} \\ -\mathrm{d\,c'} & +\mathrm{c\,c'} & -\mathrm{c\,d'} & +\mathrm{d\,d'} \end{bmatrix} \tag{19.7}$$

$$\text{and we have } {}^t\boxed{\widehat{H_z}} \cdot \underset{z}{\boxed{\mathcal{Q}}} \cdot \boxed{\widehat{H_z}} = \underset{z}{\boxed{\mathcal{Q}}} \quad \text{where } |det| = \sqrt{(\mathrm{a\,d}-\mathrm{b\,c})(\mathrm{a'\,d'}-\mathrm{b'\,c'})} \tag{19.8}$$

As a result, the image of a pencil of cycles is a pencil of cycles, while orthogonality is preserved.

Proof. H^{-1} is obtained by the substitution $\mathrm{a} \leftrightarrow -\mathrm{d}$; $\mathrm{a'} \leftrightarrow -\mathrm{d'}$ into (19.6). And then one identifies $\underset{z}{Ver}\left(H^{-1}\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)\right) = \underset{z}{Ver}\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)\cdot\boxed{\widehat{H_z}}$. □

Proposition 19.1.19. ***Inversion of cycles in a cycle****. Let Ω_0 be a fixed cycle, and define the transformation σ by*

$$\boxed{\sigma} \doteq \mathrm{Id} - 2\,\frac{\underset{c}{\mathcal{V}}(\Omega_0)\cdot{}^t\underset{c}{\mathcal{V}}(\Omega_0)\cdot\underset{z}{\boxed{\mathcal{Q}}}}{{}^t\underset{c}{\mathcal{V}}(\Omega_0)\cdot\underset{z}{\boxed{\mathcal{Q}}}\cdot\underset{c}{\mathcal{V}}(\Omega_0)} \tag{19.9}$$

Then a point circle is mapped onto a point circle, and the corresponding action is the reflection into the circle Ω_0.

Proof. Compute ${}^t\left(\boxed{\sigma}.\overrightarrow{X}\right).\underset{z}{\boxed{\mathcal{Q}}}\cdot\left(\boxed{\sigma}.\overrightarrow{X}\right)$ and re-obtain ${}^t\overrightarrow{X}.\underset{z}{\boxed{\mathcal{Q}}}\cdot\overrightarrow{X}$. Moreover, γ is invariant ($\lambda = -1$) and so is any cycle orthogonal to γ ($\lambda = +1$). One can also use (14.22) . □

Proposition 19.1.20. *When the $\mathbb{P}_\mathbb{C}(\mathbb{C}^2)$ homography has exactly **two fixed points** $f_1, f_2 \in \mathbb{C}$, we introduce the multipliers k, κ as described at (18.1). As already said,*

$$\Psi\begin{pmatrix}\mathbf{Z}\\ \mathbf{T}\\ \overline{\mathcal{Z}}\end{pmatrix} \simeq \begin{bmatrix} \dfrac{(kt_2z_1 - z_2t_1)\,\mathbf{Z} - z_1z_2\,(k-1)\,\mathbf{T}}{(k-1)\,t_2t_1\,\mathbf{Z} - (z_2t_1k - z_1t_2)\,\mathbf{T}} \\ 1 \\ \dfrac{(\kappa t_2\zeta_1 - t_1\zeta_2)\,\overline{\mathcal{Z}} - \zeta_1\zeta_2\,(\kappa-1)\,\mathbf{T}}{(\kappa-1)\,t_2t_1\overline{\mathcal{Z}} - (\kappa\zeta_2t_1 - \zeta_1t_2)\,\mathbf{T}} \end{bmatrix} \tag{19.10}$$

This Ψ has four fixed points F_j and $\boxed{\widehat{H_z}}$ admits a basis of eigencolumns, namely the $\mathfrak{F}_j \simeq \underset{c}{\mathcal{V}}(F_j)$. And we have:

$$\mathfrak{F}_z \simeq \begin{bmatrix} -t_2\,\zeta_2 & -\zeta_1\,t_2 & -t_1\,\zeta_2 & -t_1\,\zeta_1 \\ z_2\,\zeta_2 & z_2\,\zeta_1 & z_1\,\zeta_2 & z_1\,\zeta_1 \\ -t_2\,z_2 & -z_2\,t_1 & -t_2\,z_1 & -t_1\,z_1 \\ t_2^2 & t_2\,t_1 & t_2\,t_1 & t_1^2 \end{bmatrix}$$

$$\mathfrak{F}_z^{-1}\cdot\boxed{\widehat{H_z}}\cdot\mathfrak{F}_z \simeq \begin{bmatrix} k\kappa & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & \kappa & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \;;\; {}^t\mathfrak{F}_z\cdot\underset{z}{\boxed{\mathcal{Q}}}\cdot\mathfrak{F}_z \simeq \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Proof. Straightforward computations. □

Proposition 19.1.21. *When the ordinary homography h, acting over $\mathbb{P}_\mathbb{C}(\mathbb{C}^2)$, has exactly **one fixed point** z_1 then conjugating by $g : z \mapsto 1 \div (z - z_1)$ leads to $\left(g^{-1}\circ h\circ g\right)(z) = z + k$ (where*

$k\in\mathbb{C}\setminus\{0\}$). Acting over $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$, H has only one fixed point $F\simeq z_1:t_1:\zeta_1$, while $\boxed{\widehat{H_z}}$ can be reduced in Jordan form according to:

$$\mathfrak{F}_z\simeq\left[\begin{array}{c|ccc} -k & \dfrac{k\kappa\,\zeta_1}{t_1} & -k & 0\\ \dfrac{kz_1}{t_1}-\dfrac{\kappa\,\zeta_1}{t_1} & -\dfrac{k\kappa\,z_1\,\zeta_1}{t_1^2} & \dfrac{kz_1}{t_1}+\dfrac{\kappa\,\zeta_1}{t_1} & 1\\ \kappa & \dfrac{k\kappa\,z_1}{t_1} & -\kappa & 0\\ 0 & -k\kappa & 0 & 0\end{array}\right]$$

$$\mathfrak{F}^{-1}\cdot\boxed{\widehat{H_z}}^n\cdot\mathfrak{F}\simeq\left[\begin{array}{c|ccc}1&0&0&0\\\hline 0&1&-2&-1\\0&0&1&1\\0&0&0&1\end{array}\right]^n=\left[\begin{array}{c|ccc}1&0&0&0\\\hline 0&1&-2\,n&-n^2\\0&0&1&n\\0&0&0&1\end{array}\right]\;;\;{}^t\mathfrak{F}\cdot\boxed{\underset{z}{\mathcal{Q}}}\cdot\mathfrak{F}\simeq k\kappa\left[\begin{array}{cccc}-2&0&0&0\\0&0&0&1\\0&0&2&0\\0&1&0&0\end{array}\right]$$

Proof. The factorization is easy to check. Let us give some rationales for this choice of the matrix $\mathfrak{F}$. One has cross_ratio $(z-k,z+k,z,\infty)=-1$. By conjugacy, one has also cross_ratio $\left(h^2M,M,hM,F\right)=-1$. Therefore, each cycle (F,M,hM) is globally invariant.

Let P and E be the pole and the elop, i.e. $P=h^{-1}(\infty)$ and $E=h(\infty)$. From the previous result, the line (PFE) is invariant (our first column). The tangent pencil generated by the circle point (F) and line (PE) is therefore invariant. So is its orthogonal pencil, so that $med\,[E,F]$ is a good candidate for the third column. And we use $\mathcal{L}_z$ for the last one.

Finally, the expansion of $\mathfrak{F}^{-1}\cdot\boxed{\widehat{H_z}}^n\cdot\mathfrak{F}$ is obtained using $1/\left(1-X\times\mathfrak{F}^{-1}\cdot\boxed{\widehat{H_z}}\cdot\mathfrak{F}\right)$. □

Exercise 19.1.22. For $n\in\mathbb{N}$, let $\gamma_n=h^n(\gamma_0)$ be the iterated images of a cycle γ_0. Describe this circle by $x:y:z:t$, the coordinates of $\underset{c}{\mathcal{V}_0}$ wrt $\mathfrak{F}$ as a basis, and find the condition such that γ_0 and γ_1 are tangent. Show that, in this case, cycles γ_n and γ_{n+1} are tangent for all n . A first case is when all the circles are going through F.

Otherwise, let N_n be the contact point of γ_n and γ_{n+1}. Show that all the N_n belong to a same circle q_0 while all the circles γ_n are tangent to two fixed circles q_1,q_2 that are inverse wrt q_0. Check your results by inversion into $\mathcal{C}_4$.

Exercise 19.1.23. Explore the following situation. A similitude (M,k,τ) acts over the cycles space. And the four proper spaces are:

$$k:\mathcal{L}_b\;;\;\frac{1}{k}:\mathcal{C}(M,0)\;;\;\tau:\Omega_y\wedge M\;;\;\frac{1}{\tau}:\Omega_x\wedge M$$

19.2 Revisiting the Euler pencil

Remark 19.2.1. Cycles representatives $\mathcal{V}$ are columns. Thus $\left(\mathcal{V}_1\underset{6}{\wedge}\mathcal{V}_2\right)$ is a square matrix.

The Euler pencil has been treated in detail at Section 14.9. Let us examine again this pencil, in the light of the new formalism.

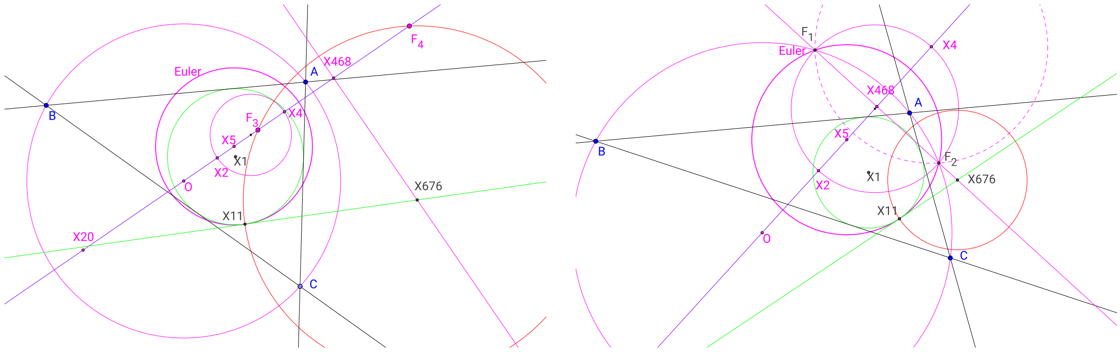
1. We use the Lubin-1 representation, i.e. $A^z\simeq\alpha:1:1/\alpha$. The representative of Γ is known to be $V_{cir}\simeq 0:-1:0:1$. The Euler circle goes through the midpoints. It's normalized representative is therefore:

$$\bigwedge_3\left(\underset{z}{Ver}\left(A^z+B^z\right),\,\text{etc}\right)\simeq\underset{c}{\mathcal{V}_{eul}}\simeq\frac{1}{4\,\sigma_3}\left(\begin{array}{c}-2\,\sigma_2\\ \sigma_1\sigma_2-\sigma_3\\ -2\,\sigma_1\sigma_3\\ 4\,\sigma_3\end{array}\right)$$

2. Applying (19.3), we have

$${}^t\underset{c}{\mathcal{V}_{eul}}\cdot\boxed{\underset{z}{\mathcal{Q}}}\cdot\left(\underset{c}{\mathcal{V}_{eul}}\right)=\left(\frac{\sigma_1}{2},\,1,\,\frac{\sigma_2}{2\sigma_3},\,\frac{\sigma_1\sigma_2-\sigma_3}{4\,\sigma_3}\right).\underset{c}{\mathcal{V}_{eul}}=\frac{-1}{2}$$

i.e. a confirmation of center X(5) $z=\sigma_1/2$ and radius $1/2$.



(a) Acute triangle (b) Obtuse triangle

Figure 19.1: The Euler pencil

3. The line representative of the Euler pencil (generated by circum- and Euler circles) is therefore:

$$\boxed{Euler^{z}_{colu}} \doteq \left(\underset{c}{\mathcal{V}_{cir}} \underset{6}{\wedge} \underset{c}{\mathcal{V}_{eul}}\right) \simeq \begin{pmatrix} 0 & 2\,\sigma_1\sigma_3 & \sigma_1\sigma_2+3\,\sigma_3 & 2\,\sigma_1\sigma_3 \\ -2\,\sigma_1\sigma_3 & 0 & 2\,\sigma_2 & 0 \\ -\sigma_1\sigma_2-3\,\sigma_3 & -2\,\sigma_2 & 0 & -2\,\sigma_2 \\ -2\,\sigma_1\sigma_3 & 0 & 2\,\sigma_2 & 0 \end{pmatrix}$$

4. Consider a point $F \simeq z : t : \zeta$. Its Veronese image is $\left[zt,\ t^2,\ \zeta t,\ z\zeta\right]$ and the associated point circle is represented by $V_F \simeq -t\zeta : z\zeta : -tz : t^2$. This circle belongs to the Euler pencil when ${}^{t}\underset{c}{\mathcal{V}_F} \cdot \boxed{Euler^{z}_{colu}} = \overrightarrow{0}$. One obtains the so-called **Walsmith points** X(5000) and X(5001).

$$F_{\pm} \simeq \begin{pmatrix} \dfrac{\sigma_1\sigma_2+3\,\sigma_3}{\sigma_2} \pm \dfrac{\sigma_3 W}{\sigma_2} \\ 4 \\ \dfrac{\sigma_1\sigma_2+3\,\sigma_3}{\sigma_1\sigma_3} \pm \dfrac{W}{\sigma_1} \end{pmatrix} \qquad \text{where } W^2 \doteq \left(9-\frac{\sigma_1\sigma_2}{\sigma_3}\right)\left(1-\frac{\sigma_1\sigma_2}{\sigma_3}\right)$$

5. The midpoint of F_+, F_- is X(468). This is recognizable to the fact that $z_{468} = \left(z_4 + 3z_{186}\right)/4$ where $z_4 = \sigma_1$ is the affix of the orthocenter X(4) and $z_{186} = \sigma_3/\sigma_2$ is the affix of the inverse of X(4) in the circumcircle, i.e. X(186).

6. The direction of line F_+F_- is $\sigma_1\sigma_3 : 0 : \sigma_2$ i.e. X(30)... the direction of the Euler line. This was obvious from the beginning.

7. Both factors of quantity W^2 are real. This can be seen from $\overline{\sigma_1} = \sigma_2/\sigma_3$. Since $|\sigma_1| < 3$, the first factor is ever positive. The second factor can be split into :

$$+1 - \frac{\sigma_1\sigma_2}{\sigma_3} = -\frac{(\beta+\gamma)\,(\alpha+\gamma)\,(\beta+\alpha)}{\alpha\,\beta\,\gamma}$$

Thus $W^2 = 1$ for equilateral triangles ($\sigma_1 = 0$) and vanishes only for right-angled triangles, so that $W^2 > 0$ characterizes acute triangles. One can also use:

$$9 - \frac{\sigma_1\,\sigma_2}{\sigma_3} = \frac{32\,S_\omega\,S^2}{a^2b^2c^2}\ ;\ 1 - \frac{\sigma_1\,\sigma_2}{\sigma_3} = \frac{8\,S_a\,S_c\,S_b}{a^2b^2c^2}$$

8. Points $F_\pm$ are the base points of the Euler pencil, seen as an *isotomic* pencil. Consider now the *isoptic* pencil of the cycles that are orthogonal to all of the cycles of the Euler pencil (i.e. the orthic pencil). When $\underset{c}{\mathcal{V}_1}, \underset{c}{\mathcal{V}_2}$ are the representatives of two orthogonal cycles, then

${}^{t}\underset{c}{\mathcal{V}}_1 \cdot \boxed{\underset{z}{\mathcal{Q}}} \cdot \underset{c}{\mathcal{V}}_2 = 0$. Thus ${}^{t}\underset{c}{\mathcal{V}}_{eul} \cdot \boxed{\underset{z}{\mathcal{Q}}}$ is an hyper-plane that describes the bundle of cycles orthogonal to Ω_{eul}. And therefore

$$\boxed{Orthic^{z}_{colu}} \doteq \text{dual}\left(\left(\left({}^{t}\underset{c}{\mathcal{V}}_a \cdot \boxed{\underset{z}{\mathcal{Q}}}\right) {}^{t}\underset{6}{\wedge} \left({}^{t}\underset{c}{\mathcal{V}}_b \cdot \boxed{\underset{z}{\mathcal{Q}}}\right)\right)\right) \simeq \begin{pmatrix} 0 & -2\,\sigma_1\,\sigma_3 & 0 & 2\,\sigma_1\,\sigma_3 \\ 2\,\sigma_1\,\sigma_3 & 0 & 2\,\sigma_2 & \sigma_1\,\sigma_2 + 3\,\sigma_3 \\ 0 & -2\,\sigma_2 & 0 & 2\,\sigma_2 \\ -2\,\sigma_1\,\sigma_3 & -\sigma_1\,\sigma_2 - 3\,\sigma_3 & -2\,\sigma_2 & 0 \end{pmatrix}$$

9. Using the same method as above, we search the centers of the point-circles that belong to the Orthic pencil. We obtain:

$$E_{\pm} \simeq \begin{pmatrix} \dfrac{\sigma_1\sigma_2 + 3\,\sigma_3}{\sigma_2} \pm \dfrac{\sigma_3 W}{\sigma_2} \\ 4 \\ \dfrac{\sigma_1\sigma_2 + 3\,\sigma_3}{\sigma_1\sigma_3} \mp \dfrac{W}{\sigma_1} \end{pmatrix} \qquad \text{where W is as above}$$

10. The midpoint of E_+, E_- is X(468) again, while the direction of E_+E_- is $\sigma_1\sigma_3 : 0 : -\sigma_2$, i.e. X(511).
11. When the triangle is acute, W^2 is positive, and W is real. Thus *conjugate* $(W) = W$ so that the $F_\pm$are visible, while the $E_\pm$are not. When the triangle is obtuse, W^2 is negative, and W is imaginary. Thus *conjugate* $(W) = -W$ so that the $E_\pm$are visible, while the $F_\pm$are not... as can be seen at Figure 19.1.

19.3 Isodynamic points

19.3.1 Equianharmonic points

Definition 19.3.1. A set of four points A, B, C, D are said to be equianharmonic set when one of their cross-ratio is either J or J^2, where $J^2 + J + 1 = 0$.

Proposition 19.3.2. *All the 24 cross-ratios of an equianharmonic set are either J or J^2. When a set $\{A, B, C, \infty\}$ is equianharmonic, then the triangle ABC is equilateral (with one or the other orientation)..*

Proof. Brute force. □

19.3.2 Revisiting the Brocard-Lemoine pencil

Notation 19.3.3. Here, J_a, etc are the cevians of the incenter I_0 while P_a, etc are its cocevians. See Section 14.10 for more details.

The Brocard-Lemoine pencils are build from the so-called **Apollonian circles**, whose diameters are the segments $[J_a, P_a]$, etc. They have been treated in detail at Section 14.10. Let us examine again these pencils, in the light of the new formalism.

1. We start from the Lubin-2 representation i.e. $A^z \simeq \alpha^2 : 1 : \alpha^{-2}$ and obtain :

$$J_a^z, P_a^z \underset{2}{\simeq} \begin{pmatrix} \beta\,\gamma\,\left(\alpha^2\beta^2 + \alpha^2\beta\,\gamma + \alpha^2\gamma^2 - \beta^2\gamma^2\right) \\ \left(\alpha^2 + \beta\,\gamma\right)\beta\,\gamma \\ \beta^2 - \alpha^2 + \beta\,\gamma + \gamma^2 \end{pmatrix}, \begin{pmatrix} \beta\,\gamma\,\left(\alpha^2\beta^2 - \alpha^2\beta\,\gamma + \alpha^2\gamma^2 - \beta^2\gamma^2\right) \\ \left(\alpha^2 - \beta\,\gamma\right)\beta\,\gamma \\ \alpha^2 - \beta^2 + \beta\,\gamma - \gamma^2 \end{pmatrix}$$

2. Now, we take the wedge of the Veronese of A, J_a, P_a. Since J_a, P_a are Lemoine-conjugates, the result can be expressed in the Lubin-1 frame, leading to:

$$\underset{c}{\mathcal{V}}_a \underset{2}{\simeq} \bigwedge_3 \left(\underset{z}{Ver} A^z, \underset{z}{Ver} P_a^z, \underset{z}{Ver} J_a^z\right) \underset{1}{\simeq} \begin{pmatrix} \beta + \gamma - 2\,\alpha \\ \alpha^2 - \beta\,\gamma \\ \alpha\left(2\,\beta\gamma - \alpha\beta - \alpha\gamma\right) \\ \alpha^2 - \beta\,\gamma \end{pmatrix} \simeq \boxed{Lv_1^{-1}} \cdot \underset{b}{\mathcal{V}}_a$$

3. When computing the line describing the pencil generated by $\underset{c}{\mathcal{V}_a}, \underset{c}{\mathcal{V}_b}$, one obtains a symmetric expression, proving that $\underset{c}{\mathcal{V}_c}$ belongs to the pencil.

$$\boxed{Lemoine^z_{colu}} \doteq \left(\underset{c}{\mathcal{V}_a} \underset{6}{\wedge} \underset{c}{\mathcal{V}_b}\right) \underset{1}{\simeq} {}^t\boxed{Lv_1} \cdot \boxed{Lemoine^b_{colu}} \cdot \boxed{Lv_1}$$
$$\underset{1}{\simeq} \begin{pmatrix} 0 & 3\,\sigma_1\,\sigma_3 - \sigma_2^2 & 0 & \sigma_2^2 - 3\,\sigma_1\,\sigma_3 \\ \sigma_2^2 - 3\,\sigma_1\,\sigma_3 & 0 & \sigma_1^2 - 3\,\sigma_2 & \sigma_1\,\sigma_2 - 9\,\sigma_3 \\ 0 & 3\,\sigma_2 - \sigma_1^2 & 0 & \sigma_1^2 - 3\,\sigma_2 \\ 3\,\sigma_1\,\sigma_3 - \sigma_2^2 & 9\,\sigma_3 - \sigma_1\,\sigma_2 & 3\,\sigma_2 - \sigma_1^2 & 0 \end{pmatrix}$$

4. All these circles are going through X(15), X(16), the isodynamic points (as described §6 below).

5. The conjugate pencil is obtained by:

$$\boxed{Brocard^z_{colu}} \doteq \boxed{\underset{z}{\mathcal{Q}}} \cdot \operatorname{dual}\left(\boxed{Lemoine^z_{colu}}\right) \cdot \boxed{\underset{z}{\mathcal{Q}}} \simeq {}^t\boxed{Lv} \cdot \boxed{Brocard_{colu}} \cdot \boxed{Lv}$$
$$\underset{1}{\simeq} \begin{pmatrix} 0 & \sigma_2^2 - 3\,\sigma_1\sigma_3 & \sigma_1\sigma_2 - 9\,\sigma_3 & \sigma_2^2 - 3\,\sigma_1\sigma_3 \\ 3\,\sigma_1\sigma_3 - \sigma_2^2 & 0 & \sigma_1^2 - 3\,\sigma_2 & 0 \\ 9\,\sigma_3 - \sigma_1\sigma_2 & 3\,\sigma_2 - \sigma_1^2 & 0 & 3\,\sigma_2 - \sigma_1^2 \\ 3\,\sigma_1\sigma_3 - \sigma_2^2 & 0 & \sigma_1^2 - 3\,\sigma_2 & 0 \end{pmatrix}$$

6. Solving $Ver^z\left(\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}\right) \cdot \boxed{\underset{z}{\mathcal{Q}}}^{-1} \cdot \boxed{Brocard^z_{colu}} = \overrightarrow{0}$ give the point-circles that generate the Brocard pencil. One obtains :

$$F^z_{\pm} \underset{1}{\simeq} \begin{pmatrix} \dfrac{\sigma_1\,\sigma_2 - 9\,\sigma_3}{\sigma_1^2 - 3\,\sigma_2} \pm \dfrac{\sqrt{3}\sigma_4}{\sigma_1^2 - 3\,\sigma_2} \\ 2 \\ \dfrac{\sigma_1\,\sigma_2 - 9\,\sigma_3}{\sigma_2^2 - 3\,\sigma_1\,\sigma_3} \pm \dfrac{\sqrt{3}\sigma_4}{\sigma_2^2 - 3\,\sigma_1\,\sigma_3} \end{pmatrix} \simeq \begin{pmatrix} \dfrac{\beta\,\gamma + \alpha\,\beta\,J + \alpha\,\gamma\,J^2}{-\left(\alpha + \gamma\,J + \beta\,J^2\right)} \\ 1 \\ \dfrac{-\left(\alpha + \beta\,J + \gamma\,J^2\right)}{\beta\gamma + \alpha\,\gamma\,J + \alpha\,\beta\,J^2} \end{pmatrix} \tag{19.11}$$

where $J \doteq J_+ \doteq \left(-1 + \boldsymbol{i}\sqrt{3}\right)/2$ (and thus $J_- \doteq J^2$). One can check that F_+ is X(15) while F_- is X(16), i.e. the isodynamic points already obtained using barycentrics.

7. Solving $Ver^z\left(\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}\right) \cdot \boxed{\underset{z}{\mathcal{Q}}}^{-1} \cdot \boxed{Lemoine^z_{point}} = \overrightarrow{0}$ give the point-circles that generate the Lemoine pencil. One obtains :

$$E^z_{\pm} \simeq \begin{pmatrix} \dfrac{\sigma_1\,\sigma_2 - 9\,\sigma_3}{\sigma_1^2 - 3\,\sigma_2} \pm \dfrac{\sqrt{3}\sigma_4}{\sigma_1^2 - 3\,\sigma_2} \\ 2 \\ \dfrac{9\,\sigma_3 - \sigma_1\,\sigma_2}{3\,\sigma_1\,\sigma_3 - \sigma_2^2} \pm \dfrac{\sqrt{3}\sigma_4}{3\,\sigma_1\,\sigma_3 - \sigma_2^2} \end{pmatrix}$$

8. Obviously, pairs $F_\pm$ and $E_\pm$ share the same midpoint, while their directions are orthogonal. But there is a huge difference between both pairs: points of the F pair are visible, the others are not. Finally, one can check the usual relation: if one pair is noted $z_1 : 1 : \overline{z_1}$, $z_2 : 1 : \overline{z_2}$, the other one is $z_1 : 1 : \overline{z_2}$, $z_2 : 1 : \overline{z_1}$.

19.3.3 Homographic stabilizer

Remark 19.3.4. Isodynamic points X(15),X(16) were characterized at Subsection 7.11.4 as the centers whose pedal triangle is equilateral. Just above, they appeared as the base points of the Brocard pencil, and therefore as the Poncelet points of the Lemoine pencil. Another point of view is as follows.

Definition 19.3.5. In the complex plane $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$, the set of all the Cremona homographies which are fixing a given set of three distinct points is called the homographic stabilizer of this set.

Proposition 19.3.6. *The homographic stabilizer of a given triple is a copy of the dihedral group* $D_6 = \mathfrak{S}_3 \rtimes \mathfrak{S}_2$.

Proof. Let ρ be the inversion wrt the circle $\Gamma(z_1, z_2, z_3)$. Then for each $\widehat{\mu} \in \mathfrak{S}_3$, the relations $z_j \mapsto \widehat{\mu}(z_j)$ are defining a direct Cremona-homography μ of the whole plane, while $\mu \circ \rho = \rho \circ \mu$ is a skew Cremona-homography. □

Proposition 19.3.7. *The skew homography* τ_a *defined by* $\alpha \mapsto \alpha$, $\beta \longleftrightarrow \gamma$ *is the inversion into the circle* γ_A *through* A, A' *and orthogonal to* $\Gamma(A, B, C)$, *where* A' *is defined by*

$$\text{cross\_ratio}_z(A, A', B, C) = \text{cross\_ratio}_\zeta(A, A', B, C) = -1$$

Proof. Orthogonality is required since Γ has to be invariant. □

Theorem 19.3.8. *Circles* γ_A, etc *are the already defined Apollonian circles. As a result, their common points (the isodynamic centers) are conveyed by any Cremona-homography of the stabilizer. In other words,*

$$\begin{cases} \mu(X(A, B, C, 15)) = X(\mu A, \mu B, \mu C, 15) & \text{when } \mu \text{ is direct} \\ \mu(X(A, B, C, 15)) = X(\mu A, \mu B, \mu C, 16) & \text{when } \mu \text{ is skew} \end{cases}$$

Proof. All the circles involved are conveyed by the group of the Cremona homographies. □

Proposition 19.3.9. *When* A, B, C *are defined by their inclusive coordinates* z_j, t_1, z_j, *then*

$$X(15), X(16) = \begin{bmatrix} \dfrac{-t_1 z_3 z_2 - t_3 z_1 z_2 J - t_2 z_1 z_3 J^2}{t_2 t_3 z_1 + t_1 t_2 z_3 J + t_3 t_1 z_2 J^2} \\ 1 \\ \dfrac{-t_1 \zeta_2 \zeta_3 - t_2 \zeta_3 \zeta_1 J - t_3 \zeta_1 \zeta_2 J^2}{t_2 t_3 \zeta_1 + t_3 t_1 \zeta_2 J + t_1 t_2 \zeta_3 J^2} \end{bmatrix}$$

where $J = \left(-1 \pm \sqrt{3}\right)/2$.

Proof. Consider the direct homography $M_1 \mapsto M_2 \mapsto M_3 \mapsto M_1$. Its fixed points are the required isodynamic points. One can check that any direct homography h transports $X(15)$ onto $h(X(15))$. □

Proposition 19.3.10. *When applying these properties to the standard triangle, one obtains the following table, where* σ *is the cycle* $\alpha \mapsto \beta \mapsto \gamma \mapsto \alpha$ *and* $\rho\tau_{bc}$ *is the inversion into the* ***A-Apollonian circle*** *through A,X(15),X(16). Moreover,* $h(\infty)$ *is the image of* $\infty \simeq 0 : 1 \in (\mathbf{Z} : \mathbf{T})$.

		elop				elop		
	h	name	$h_z(\infty)$		ρh	name	$h_b(\infty)$	$h_z(\infty)$
1	1_E		∞	7	ρ	X_3	circ	0
2	σ		$\dfrac{9\,s_3 - s_1\,s_2 - \boldsymbol{i}s_4}{6\,s_2 - 2s_1^2}$	8	$\rho\sigma$	Br_1	a^2b^2 ::	$\dfrac{6\,s_1\,s_3 - 2s_2^2}{9\,s_3 - s_1\,s_2 + \boldsymbol{i}s_4}$
3	σ^2		$\dfrac{9\,s_3 - s_1\,s_2 + \boldsymbol{i}s_4}{6\,s_2 - 2s_1^2}$	9	$\rho\sigma^2$	Br_2	a^2c^2 ::	$\dfrac{6\,s_1\,s_3 - 2s_2^2}{9\,s_3 - s_1\,s_2 - \boldsymbol{i}s_4}$
4	τ_{bc}	E'_a	$\dfrac{\kappa s_1 - s_2}{3\,\kappa - s_1}$	10	$\rho\tau_{bc}$	E_a		$\dfrac{\kappa s_2 - 3\,s_3}{\kappa s_1 - s_2}$
5	τ_{ca}	E'_b		11	$\rho\tau_{ca}$	E_b		
6	τ_{ab}	E'_c		12	$\rho\tau_{ab}$	E_c		

Proof. Inversion wrt the A-Apollonian circle maps the values α, γ, β onto α, β, γ. □

Fact 19.3.11. *The following properties are left as exercises.*

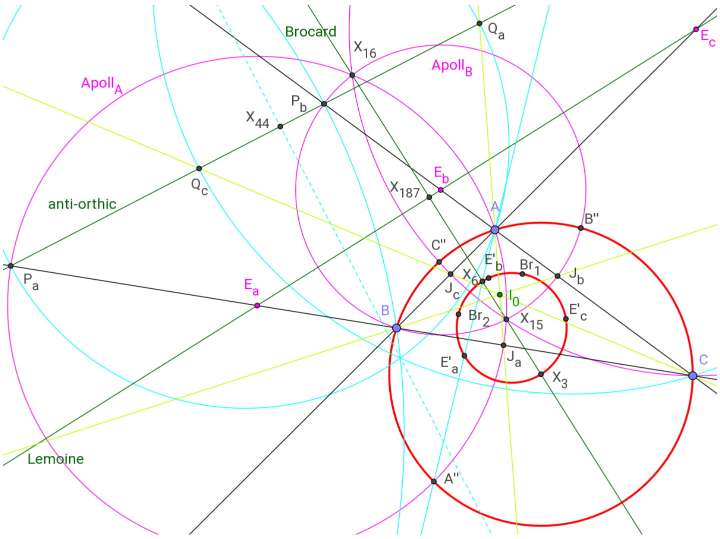


Figure 19.2: Lemoine and Brocard revisited

1. Points $4, 5, 6, 7, 8, 9$ are on the Brocard 3-6 circle (diameter $[O, K]$). Points 8,9 are the Brocard points themselves.
2. Points $1, 2, 3, 10, 11, 12$ are on the Lemoine axis (tripolar of X_6). In fact, 10,11,12 are the cocevians E_j of X_6.
3. Points $n, n+6$ are inverse in the circum- circle (and thus aligned with O).
4. Points $A, X_6, (4) = E'_a$ and A'' are aligned, etc.
5. The fixed points of homographies σ, σ^2 are the isodynamics points $X_{15},\ X_{16}$, while the fixed points of homography τ_{bc} are A and A''.
6. The Apollonian circles are at 60° from each other (consider the multiplier of an homography such that $\sigma^3 = id$).

Remark 19.3.12. The previous results aren't invalidated when points ABC are aligned while remaining distincts.

19.4 The Pedoe formalism

19.4.1 The Pedoe map

Definition 19.4.1. The Pedoe map, as defined in Pedoe (1970), uses the unit circle instead of the "factored" one. One has:

$$\underset{p}{Ver}\left(\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}\right) \simeq \left[\mathbf{ZT}, \mathbf{T}^2, \overline{\mathcal{Z}}\mathbf{T}, \mathbf{Z}\overline{\mathcal{Z}} - \mathbf{T}^2\right] \tag{19.12}$$

Proposition 19.4.2. *Both umbilics are mapped to* $[0, 0, 0, 0]$ *(points of indeterminacy) while all other points at infinity are mapped to* $[0, 0, 0, 1]$*, called Sirius. When using* $\left(\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}\right) = \boxed{Lu}.(x : y : z)$*, we have*

$$\underset{p}{Ver}\left(\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}\right) \simeq Ver\,(x : y : z) \cdot {}^t\boxed{Lv_p} \qquad \text{where } \boxed{Lv_p} = \begin{pmatrix} \boxed{Lu} & 0 \\ 0 & 1/R^2 \end{pmatrix}$$

Proof. The $\boxed{Lu}$ part is obvious. The $1/R^2$ acknowledges the fact that (X_3, R) is the model of all circles in the triangle plane. □

Proposition 19.4.3. *The polar hyperplanes related to the* $\underset{p}{Ver}\,(M_\tau)$ *of all the points* M_τ *of a same cycle are going through a same point of* $\mathbb{P}_\mathbb{C}\left(\mathbb{C}^4\right)$*, called the circle Pedoe-representative, that will be noted* $\underset{p}{\mathcal{V}}$*.*

Proof. Similar to the $\underset{c}{\mathcal{V}}$ proof. And we have:

$$\underset{p}{\mathcal{V}}\left[(z:t:\zeta)\,,\,\rho^2\right] \doteq \left(-\zeta : \frac{z\,\zeta}{t} + t - t\,\rho^2 : -z : +t\right)$$

When the cycle is the line $\Delta \simeq [a,b,c]$ then $\underset{c}{\mathcal{V}}(\Delta) = a : b : c : 0$. ☐

Corollary 19.4.4. *The locus of the representatives $\underset{p}{\mathcal{V}}$ of the $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ lines is the plane*

$$\mathcal{P}_p \simeq [0:0:0:1]$$

Proposition 19.4.5. *The Pedoe row-images of the $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ points belong to a 3D quadric:*

$$\underset{p}{Ver}(P)\cdot \boxed{\underset{p}{\mathcal{Q}^{-1}}}\cdot {}^t\underset{p}{Ver}(P) = 0 \qquad \text{where } \boxed{\underset{p}{\mathcal{Q}^{-1}}} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & +2 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix} \tag{19.13}$$

And the columns representative of point-circles, i.e. the $\underset{p}{\mathcal{V}}(P) \simeq \boxed{\underset{p}{\mathcal{Q}^{-1}}}\cdot {}^t\underset{p}{Ver}(P)$, belong to the 3D paraboloid defined by $\boxed{\underset{p}{\mathcal{Q}}}$. For a circle (γ), we have the more precise result :

$$\rho^2 = \frac{-1}{2} \times \frac{{}^t\underset{p}{\mathcal{V}}(\gamma)\cdot \boxed{\underset{p}{\mathcal{Q}}}\cdot \underset{p}{\mathcal{V}}(\gamma)}{\left(\mathcal{P}_p \cdot \underset{p}{\mathcal{V}}(\gamma)\right)^2} \tag{19.14}$$

$$\text{where } \boxed{\underset{p}{\mathcal{Q}}} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & -2 \end{pmatrix}$$

Proof. See the proof of (14.9). ☐

Proposition 19.4.6. *Two point-circles are orthogonal wrt the quadric when their centers share one of their two coordinates.*

Proof. Assuming $M_j \simeq z_j : t_j : \zeta_j$, we have the more precise result:

$$\frac{{}^t\left(\underset{p}{\mathcal{V}}\left[M_1,\, r_1^2\right]\right)\cdot \boxed{\underset{p}{\mathcal{Q}}}\cdot \left(\underset{p}{\mathcal{V}}\left[M_2,\, r_2^2\right]\right)}{\left(\mathcal{P}_p \cdot \underset{p}{\mathcal{V}_1}\right) \times \left(\mathcal{P}_p \cdot \underset{p}{\mathcal{V}_2}\right)} = \left(\frac{z_2}{t_2} - \frac{z_1}{t_1}\right)\left(\frac{\zeta_2}{t_2} - \frac{\zeta_1}{t_1}\right) - r_1^2 - r_2^2 \tag{19.15}$$

☐

Remark 19.4.7. And then, business as usual, using the adapted matrices.

Maple 19.4.8. The Maple package 'faisceaux' contains:

constants: "pQQ"= $\boxed{\underset{p}{\mathcal{Q}}}$, "pQQI"= $\boxed{\underset{p}{\mathcal{Q}^{-1}}}$, "plinfp"= $\mathcal{P}_p$

functions: "Verp"= $\underset{p}{Ver}$, "mor2colup"= $\underset{p}{\mathcal{V}}$, "colup2mm", "colup2mor", "eq2colup", "mhatp", "mm2colup", "mkpgram", "paction"

19.4.2 Pedoe version of the homographic actions

CAVEAT: in this subsection, letters $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{a}', \mathrm{b}', \mathrm{c}', \mathrm{d}', k, \kappa$ are general complex numbers, while $\mathrm{a}' = \bar{\mathrm{a}}$, etc is intended for visible objects.

Proposition 19.4.9. *The homography H, defined at (19.6), is an action over the points of $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$. It induces an action $\widehat{H_p}$ which is linear over the Pedoe-columns. Its matrix is :*

$$\boxed{\widehat{H_p}} \simeq \frac{1}{|det|} \left[\begin{array}{cccc} \mathrm{d\,a}' & -\mathrm{c\,a}' & \mathrm{c\,b}' & \mathrm{c\,a}' - \mathrm{d\,b}' \\ -\mathrm{b\,a}' - \mathrm{d\,c}' & \mathrm{a\,a}' + \mathrm{c\,c}' & -\mathrm{a\,b}' - \mathrm{c\,d}' & -\mathrm{a\,a}' + \mathrm{b\,b}' - \mathrm{c\,c}' + \mathrm{d\,d}' \\ \mathrm{b\,c}' & -\mathrm{a\,c}' & \mathrm{a\,d}' & \mathrm{a\,c}' - \mathrm{b\,d}' \\ -\mathrm{d\,c}' & \mathrm{c\,c}' & -\mathrm{c\,d}' & -\mathrm{c\,c}' + \mathrm{d\,d}' \end{array} \right] \tag{19.16}$$

and we have $\,{}^t\boxed{\widehat{H_p}} \cdot \boxed{\underset{p}{\mathcal{Q}}} \cdot \boxed{\widehat{H_p}} = \boxed{\underset{p}{\mathcal{Q}}}$ where $|det| = \sqrt{(\mathrm{a\,d} - \mathrm{b\,c})\,(\mathrm{a}'\,\mathrm{d}' - \mathrm{b}'\,\mathrm{c}')}$

Proof. Same proof as for (19.7). □

Proposition 19.4.10. ***Inversion of cycles in a cycle***. *Consider is the reflection s into cycle γ which acts over $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$. Then s induces a linear action over the affixes, described by the matrix:*

$$\boxed{\sigma} \doteq \mathrm{Id} - 2 \frac{\underset{p}{\mathcal{V}_\gamma} \cdot {}^t\underset{p}{\mathcal{V}_\gamma} \cdot \boxed{\underset{p}{\mathcal{Q}}}}{{}^t\underset{p}{\mathcal{V}_\gamma} \cdot \boxed{\underset{p}{\mathcal{Q}}} \cdot \underset{p}{\mathcal{V}_\gamma}} \tag{19.17}$$

Proof. Same proof as for (14.22). □

Proposition 19.4.11. *When the ordinary homography h, acting over $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^2\right)$, has exactly two fixed points f_1, f_2 then, using notations of (18.1), a basis of eigencolumns for $\boxed{\widehat{H_p}}$ is made of the $\underset{p}{\mathcal{V}}(F_j)$, i.e.*

$$\mathfrak{F}_p \simeq \left[\begin{array}{cccc} -t_2\,\zeta_2 & -\zeta_1\,t_2 & -t_1\,\zeta_2 & -t_1\,\zeta_1 \\ t_2^2 + z_2\,\zeta_2 & t_2\,t_1 + z_2\,\zeta_1 & t_2\,t_1 + z_1\,\zeta_2 & t_1^2 + z_1\,\zeta_1 \\ -t_2\,z_2 & -z_2\,t_1 & -t_2\,z_1 & -t_1\,z_1 \\ t_2^2 & t_2\,t_1 & t_2\,t_1 & t_1^2 \end{array} \right] \quad \text{leading to}$$

$$\mathfrak{F}_p^{-1} \cdot \boxed{\widehat{H_p}} \cdot \mathfrak{F}_p \simeq \left[\begin{array}{cccc} k\kappa & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & \kappa & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \;;\; {}^t\mathfrak{F}_p \cdot \boxed{\underset{p}{\mathcal{Q}}} \cdot \mathfrak{F}_p \simeq \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right]$$

Proof. Same as the proof for $\mathfrak{F}_z$ □

Proposition 19.4.12. *.When homography h has exactly one fixed point z_1 then a basis of triangulation is *****

$$\mathfrak{F}_p \simeq \left[\begin{array}{cccc} -k & \dfrac{k\kappa\,\zeta_1}{t_1} & -k & 0 \\ \dfrac{kz_1}{t_1} - \dfrac{\kappa\,\zeta_1}{t_1} & -k\kappa - \dfrac{k\kappa\,z_1\,\zeta_1}{t_1^2} & \dfrac{kz_1}{t_1} + \dfrac{\kappa\,\zeta_1}{t_1} & 1 \\ \kappa & \dfrac{k\kappa\,z_1}{t_1} & -\kappa & 0 \\ 0 & -k\kappa & 0 & 0 \end{array} \right] \simeq \left[\underset{p}{\mathcal{V}}_{EP},\ \underset{p}{\mathcal{V}}_{F},\ \underset{p}{\mathcal{V}}_{med[E,P]},\ \underset{p}{\mathcal{V}}_{\mathcal{L}_z} \right]$$

$$\mathfrak{F}^{-1} \cdot \boxed{\eta}^n \cdot \mathfrak{F} \simeq \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right]^n = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & -2\,n & -n^2 \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{array} \right] \;;\; {}^t\mathfrak{F} \cdot \boxed{\underset{z}{\mathcal{Q}}} \cdot \mathfrak{F} \simeq \frac{k\kappa}{2} \left[\begin{array}{cccc} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right]$$

Proof. Having a basis made of the fixed point and three independent lines is clearly interesting. To understand this specific choice, one can start from cross_ratio $(z-k, z+k, z, \infty) = -1$, and obtain cross_ratio $\left(h^2 M, M, hM, F\right) = -1$. Therefore, each cycle (F, M, hM) is globally invariant, the line (PFE) among them. Since two of them cannot have any other common point apart F, these cycles form a tangent pencil $\mathcal{F}$, generated by the circle-point (F) and the line (PFE). Since $\mathcal{F}$ is invariant as a pencil, then $\mathcal{F}^{\perp}$ is globally invariant. For the expansion of $\mathfrak{F}^{-1} \cdot \boxed{\eta}^n \cdot \mathfrak{F}$, consider the generating series $1/\left(1 - X \times \mathfrak{F}^{-1} \cdot \boxed{\eta} \cdot \mathfrak{F}\right)$. □

Exercise 19.4.13. Consider the homographic transposition $\beta \leftrightarrow \gamma$, $\alpha \leftrightarrow \infty$. Determine the associated Cremona transform, and its action over the cycles space. Determine a basis of $\ker(+1)$ and a basis of $\ker(-1)$. A line and a circle centered somewhere on BC would be a great choice. Check for the (14.20)(19.4). The images of the four in-ex-circles are the so-called four **A-mixtilinear circles** tangent to the circumcircle and to the AB, AC sidelines.

19.5 The Spherical formalism

On 2021-10-05, it has been decided to adopt the South pole point of view. As a result, the hyperplane containing the line representatives is the North plane !

19.5.1 The Spherical map

Definition 19.5.1. The projective sphere of $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^4\right)$ is defined as the locus of the points V such that $v_4^2 - v_1^2 - v_2^2 - v_3^2 = 0$. In other words

$$V \in \mathbb{S}_4 \rightleftharpoons \left({}^t V \cdot \boxed{Mink} \cdot V = 0\right) \qquad \text{where } \boxed{Mink} := \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & +1 \end{pmatrix}$$

By analogy with the Pedoe formalism, we will also use $\boxed{\underset{s}{\mathcal{Q}}} \doteq \boxed{\underset{s}{\mathcal{Q}^{-1}}} \doteq \boxed{Mink}$

Remark 19.5.2. When restricted to $\mathcal{E}_3$, i.e. to the real points where $v_4 \neq 0$, this is nothing but the ordinary unit sphere of the elementary geometry.

Definition 19.5.3. The Spherical version of the Veronese map uses both the unit visible circle and the unit imaginary circle and is defined by:

$$\underset{s}{Ver}\left(\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}\right) \simeq \left[2\mathbf{T}\, X,\, 2\mathbf{T}\, Y,\, \mathbf{T}^2 - \mathbf{Z}\overline{\mathcal{Z}},\, \mathbf{T}^2 + \mathbf{Z}\overline{\mathcal{Z}}\right] \tag{19.18}$$
$$\text{where } 2X \doteq \mathbf{Z} + \overline{\mathcal{Z}}\ ;\ 2Y \doteq -i\left(\mathbf{Z} - \overline{\mathcal{Z}}\right) \qquad \text{so that } X + iY = \mathbf{Z}\ ;\ X - iY = \overline{\mathcal{Z}}$$

Remark 19.5.4. These four cycles are orthogonal to each other, since $d^2 - r_1^2 - r_2^2$ applied to the last two gives obviously zero. And thus, the associated matrices $\boxed{\mathcal{Q}}$ will be diagonal.

Remark 19.5.5. Due to the relation:

$$\underset{s}{Ver}(M) = \underset{z}{Ver}(M) \cdot {}^t\boxed{\text{Z2S}} \qquad \text{where } \boxed{\text{Z2S}} = \begin{bmatrix} +1 & 0 & +1 & 0 \\ -\boldsymbol{i} & 0 & +\boldsymbol{i} & 1 \\ 0 & +1 & 0 & -1 \\ 0 & +1 & 0 & +1 \end{bmatrix},$$

we shall not expect some breaking results compared to Section 19.1. Nevertheless, the Veronese quadric $v_4\, v_2 - v_3\, v_1$, with signature $(+2, -2)$, has been replaced by the spherical quadric $v_4^2 - v_1^2 - v_2^2 - v_3^2$, with signature $(+1, -3)$.

Proposition 19.5.6. *Both umbilics are mapped to* $[0, 0, 0, 0]$ *(points of indeterminacy) while all other points at infinity are mapped to* $[0, 0, -1, 1]$*, the so-called South pole. When using*

$\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)=\boxed{Lu}.\left(x:y:z\right)$, *we have*

$$\underset{s}{Ver}\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)\simeq Ver\left(x:y:z\right)\cdot{}^{t}\boxed{Lv_s}\qquad\text{where }\boxed{Lv_s}=\left(\begin{array}{cccc}\dfrac{1}{\alpha}+\alpha & \dfrac{1}{\beta}+\beta & \dfrac{1}{\gamma}+\gamma & 0\\ \dfrac{\boldsymbol{i}}{\alpha}-\boldsymbol{i}\,\alpha & \dfrac{\boldsymbol{i}}{\beta}-\boldsymbol{i}\,\beta & \dfrac{\boldsymbol{i}}{\gamma}-\boldsymbol{i}\,\gamma & 0\\ 0 & 0 & 0 & -1/R^2\\ 2 & 2 & 2 & +1/R^2\end{array}\right)$$

Proof. One has $\boxed{Lv_s}=\boxed{\mathrm{Z2S}}\cdot\boxed{Lv_z}$. □

Proposition 19.5.7. *The polar hyperplanes related to the* $\underset{s}{Ver}\left(M_\tau\right)$ *of all the points* M_τ *of a same cycle are going through a same point of* $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^4\right)$, *called the cycle spherical-representative, that will be noted* $\underset{s}{\mathcal{V}}$.

Proof. Similar to the $\underset{c}{\mathcal{V}}$ proof. And we have:

$$\underset{s}{\mathcal{V}}\left[\left(\begin{array}{c}z\\ t\\ \zeta\end{array}\right),\rho^2\right]\doteq\left(\begin{array}{c}\zeta+z\\ \boldsymbol{i}\zeta-\boldsymbol{i}z\\ \dfrac{+t^2-z\,\zeta}{t}+t\,\rho^2\\ \dfrac{-t^2-z\,\zeta}{t}+t\,\rho^2\end{array}\right)\simeq\left(\begin{array}{c}\dfrac{z}{t}+\dfrac{\zeta}{t}\\ \boldsymbol{i}\dfrac{\zeta}{t}-\boldsymbol{i}\dfrac{z}{t}\\ +1-\dfrac{z\,\zeta}{t^2}\\ -1-\dfrac{z\,\zeta}{t^2}\end{array}\right)+\rho^2\left(\begin{array}{c}0\\ 0\\ +1\\ +1\end{array}\right)$$

When the cycle is the line $[f,g,h]$, it's generic point is $M_t\simeq\dfrac{1}{f}:\dfrac{t}{g}:-\dfrac{1+t}{h}$. Like above, the wedge of three of the $\underset{s}{Ver}\left(M_j\right)$ doesn't depend on the t_j and we obtain:

$$\underset{s}{\mathcal{V}}\left[[f,g,h]\right]\doteq\left(\begin{array}{c}f+h\\ \boldsymbol{i}\,f-\boldsymbol{i}\,h\\ g\\ g\end{array}\right)\tag{19.19}$$

□

Corollary 19.5.8. *The locus of the representatives of the non-circles among the cycles (i.e. the completed lines) is the plane*

$$\mathcal{P}_s\doteq\left(1/\sqrt{2}\right)\times[0,0,-1,+1]$$

whose equation is $V_4-V_3=0$. *This plane goes through the North pole... and will therefore be called the North plane. The* $1/\sqrt{2}$ *factor comes from* $\sqrt{2}=\sqrt{(-1)^2+(+1)^2}$ *and will be required when dealing with measurements..*

Proposition 19.5.9. *The spherical row-images of the* $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ *points belong to the 3D sphere*

$$\underset{s}{Ver}\left(P\right)\cdot\boxed{\underset{s}{\mathcal{Q}}^{-1}}\cdot{}^{t}\underset{s}{Ver}\left(P\right)=0\tag{19.20}$$

For a circle (γ), *we have the more precise radius formula:*

$$\rho^2=\frac{-1}{2}\times\frac{{}^{t}\left(\underset{s}{\mathcal{V}}\left(\gamma\right)\right)\cdot\boxed{\underset{s}{\mathcal{Q}}}\cdot\left(\underset{s}{\mathcal{V}}\left(\gamma\right)\right)}{\left(\mathcal{P}_s\cdot\underset{s}{\mathcal{V}}\left(\gamma\right)\right)^2}\tag{19.21}$$

On the other hand, the columns representative of point-circles, i.e. the $\underset{s}{\mathcal{V}}\left(P\right)\simeq\boxed{\underset{s}{\mathcal{Q}}^{-1}}\cdot{}^{t}\underset{s}{Ver}\left(P\right)$, *belong to the 3D sphere defined by* $\boxed{\underset{s}{\mathcal{Q}}}$.

Proof. Obviously, the two spheres are two copies of $\mathbb{S}_4$. But one of them is a punctual object, while the other is a tangential one, whose elements are hyper-planes. In a later step (stereographic formalism), we will only consider one sphere. □

Proposition 19.5.10. *Two point-circles are orthogonal wrt the quadric when their centers share one of their two coordinates. And then each center lies on one of the isotropic lines through the other center.*

Proof. We have the more precise formula:

$$\frac{{}^t\left(\underset{s}{\mathcal{V}}\left[M_1,\, r_1^2\right]\right)\cdot\boxed{\underset{s}{\mathcal{Q}}}\cdot\left(\underset{s}{\mathcal{V}}\left[M_2,\, r_2^2\right]\right)}{\left(\mathcal{P}_s\cdot\underset{s}{\mathcal{V}_1}\right)\times\left(\mathcal{P}_s\cdot\underset{s}{\mathcal{V}_2}\right)}=\left(\frac{z_2}{t_2}-\frac{z_1}{t_1}\right)\left(\frac{\zeta_2}{t_2}-\frac{\zeta_1}{t_1}\right)-r_1^2-r_2^2 \tag{19.22}$$

□

Maple 19.5.11. The Maple package 'faisceaux' contains:

constants: "sQQ"= $\boxed{\underset{s}{\mathcal{Q}}}$, "sQQI"= $\boxed{\underset{s}{\mathcal{Q}^{-1}}}$, "plinfs"= $\mathcal{P}_s$

functions: "Vers"= $\underset{s}{Ver}$, "mor2colus"= $\underset{s}{\mathcal{V}}$, "colus2mm", "colus2mor", "eq2colus", "mhats", "colus2mm", "mm2colus", "mksgram", "saction"

19.5.2 Spherical version of the homographic actions

CAVEAT: in this subsection, letters $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{a}', \mathrm{b}', \mathrm{c}', \mathrm{d}', k, \kappa$ are general complex numbers, while $\mathrm{a}' = \bar{\mathrm{a}}$, etc is intended for visible objects.

Proposition 19.5.12. *The Cremona-homography H, defined at (19.6), is an action over the points of $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$. It induces an action $\widehat{H_s}$ wich is linear over the Spherical-columns. Its matrix is :*

$$\boxed{\widehat{H_s}}\simeq\frac{1}{2\,|det|}\times \tag{19.23}$$

$$\left[\begin{array}{cccc} -\mathrm{ad}'-\mathrm{da}'-\mathrm{bc}'-\mathrm{cb}' & \boldsymbol{i}\,(\mathrm{da}'-\mathrm{ad}'+\mathrm{bc}'-\mathrm{cb}') & \mathrm{ac}'+\mathrm{ca}'-\mathrm{bd}'-\mathrm{db}' & \mathrm{ac}'+\mathrm{ca}'+\mathrm{bd}'+\mathrm{db}' \\ \boldsymbol{i}\,(\mathrm{ad}'-\mathrm{da}'+\mathrm{bc}'-\mathrm{cb}') & -\mathrm{ad}'-\mathrm{da}'+\mathrm{bc}'+\mathrm{cb}' & \boldsymbol{i}\,(\mathrm{ca}'-\mathrm{ac}'+\mathrm{bd}'-\mathrm{db}') & \boldsymbol{i}\,(\mathrm{ca}'-\mathrm{ac}'-\mathrm{bd}'+\mathrm{db}') \\ \mathrm{ab}'+\mathrm{ba}'-\mathrm{cd}'-\mathrm{dc}' & \boldsymbol{i}\,(\mathrm{ab}'-\mathrm{ba}'-\mathrm{cd}'+\mathrm{dc}') & -\mathrm{aa}'+\mathrm{bb}'+\mathrm{cc}'-\mathrm{dd}' & -\mathrm{aa}'-\mathrm{bb}'+\mathrm{cc}'+\mathrm{dd}' \\ \mathrm{ab}'+\mathrm{ba}'+\mathrm{cd}'+\mathrm{dc}' & \boldsymbol{i}\,(\mathrm{ab}'-\mathrm{ba}'+\mathrm{cd}'-\mathrm{dc}') & -\mathrm{aa}'+\mathrm{bb}'-\mathrm{cc}'+\mathrm{dd}' & -\mathrm{aa}'-\mathrm{bb}'-\mathrm{cc}'-\mathrm{dd}' \end{array}\right]$$

$$\text{moreover } {}^t\boxed{\widehat{H_s}}\cdot\boxed{\underset{s}{\mathcal{Q}}}\cdot\boxed{\widehat{H_s}}=\boxed{\underset{s}{\mathcal{Q}}} \qquad \text{where } |det|=\sqrt{(\mathrm{a\,d}-\mathrm{b\,c})\,(\mathrm{a}'\,\mathrm{d}'-\mathrm{b}'\,\mathrm{c}')}$$

Proof. Same as the $\boxed{\widehat{H_z}}$ proof. Mind the following fact: as it should be, a point-circle is mapped onto a point-circle. □

Proposition 19.5.13. *Assume that s, acting over $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$, is the reflection into cycle γ. This induces a linear action over the spherical affixes, described by the matrix: *****

$$\boxed{\sigma}\doteq\mathrm{Id}-2\,\frac{\underset{s}{\mathcal{V}}(\gamma)\cdot{}^t\underset{s}{\mathcal{V}}(\gamma)\cdot\boxed{\underset{s}{\mathcal{Q}}}}{{}^t\underset{s}{\mathcal{V}}(\gamma)\cdot\boxed{\underset{s}{\mathcal{Q}}}\cdot\underset{s}{\mathcal{V}}(\gamma)} \tag{19.24}$$

Proof. Same proof as for (14.22). Moreover, one can check that, in $\mathbb{C}^4$, $\underset{s}{\mathcal{V}}(\gamma)$ is changed into its opposite, while the representative of any cycle orthogonal to γ is unchanged. □

Proposition 19.5.14. *When the ordinary homography h, acting over $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^2)$, has exactly two fixed points f_1, f_2 then, using notations of (18.1), a basis of eigencolumns for $\boxed{\widehat{H_s}}$ is made of the $\underset{s}{\mathcal{V}}(F_j)$, i.e.* ****

$$\mathfrak{F}_s \simeq \frac{1}{|F_1F_2|}\begin{bmatrix} -\dfrac{z_2}{t_2}-\dfrac{\zeta_2}{t_2} & -\dfrac{z_2}{t_2}-\dfrac{\zeta_1}{t_1} & -\dfrac{\zeta_2}{t_2}-\dfrac{z_1}{t_1} & -\dfrac{z_1}{t_1}-\dfrac{\zeta_1}{t_1} \\ \boldsymbol{i}\left(\dfrac{z_2}{t_2}-\dfrac{\zeta_2}{t_2}\right) & \boldsymbol{i}\left(\dfrac{z_2}{t_2}-\dfrac{\zeta_1}{t_1}\right) & \boldsymbol{i}\left(\dfrac{z_1}{t_1}-\dfrac{\zeta_2}{t_2}\right) & \boldsymbol{i}\left(\dfrac{z_1}{t_1}-\dfrac{\zeta_1}{t_1}\right) \\ \dfrac{z_2\,\zeta_2}{t_2^2}-1 & \dfrac{z_2\,\zeta_1}{t_1\,t_2}-1 & \dfrac{z_1\,\zeta_2}{t_1\,t_2}-1 & \dfrac{z_1\,\zeta_1}{t_1^2}-1 \\ \dfrac{z_2\,\zeta_2}{t_2^2}+1 & \dfrac{z_2\,\zeta_1}{t_1\,t_2}+1 & \dfrac{z_1\,\zeta_2}{t_1\,t_2}+1 & \dfrac{z_1\,\zeta_1}{t_1^2}+1 \end{bmatrix} \quad \text{leading to}$$

$$\det\mathfrak{F}_s = +4i\ ;\ \mathfrak{F}_s^{-1}\cdot\boxed{\widehat{H_s}}\cdot\mathfrak{F}_s = \begin{bmatrix} k & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 \\ 0 & 0 & 1/\kappa & 0 \\ 0 & 0 & 0 & 1/k \end{bmatrix}\ ;\ {}^t\mathfrak{F}_s\cdot\boxed{\underset{s}{\mathcal{Q}}}\cdot\mathfrak{F}_s \simeq 4\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Proof. Direct computation using

$$\mathrm{a} = k\,\frac{z_2}{t_2}-\frac{z_1}{t_1}\ ;\ \mathrm{b} = \frac{z_1\,z_2}{t_1\,t_2}\,(1-k)\ ;\ \mathrm{c} = k-1\ ;\ \mathrm{d} = -k\,\frac{z_1}{t_1}+\frac{z_2}{t_2},\ \text{etc}$$
$$|det| = \sqrt{(\mathrm{a}'\,\mathrm{d}'-\mathrm{b}'\,\mathrm{c}')\,(\mathrm{a}\,\mathrm{d}-\mathrm{b}\,\mathrm{c})} = |F_1F_2|^2\,\sqrt{k\,\kappa}$$

□

Example 19.5.15. The following table describes how to generate the four actions $v_j \mapsto -v_j$:

eqn	*param*	$\underset{s}{\mathcal{V}}$	*action*
X	$z:1:-z$	$1:0:0:0$	$v_1 \mapsto -v_1$
Y	$z:1:+z$	$0:1:0:0$	$v_2 \mapsto -v_2$
$-\mathbf{T}^2+\mathbf{Z}\overline{\mathcal{Z}}$	$+\tau:1:1/\tau$	$0:0:1:0$	$v_3 \mapsto -v_3$
$+\mathbf{T}^2+\mathbf{Z}\overline{\mathcal{Z}}$	$-\tau:1:1/\tau$	$0:0:0:1$	$v_4 \mapsto -v_4$

19.6 Stereographic projection

On 2021-10-05, it has been decided to adopt the South pole point of view. Vae victis !

Definition 19.6.1. When $\mathbb{C}$ is identified with the $z=0$ plane in $\mathcal{E}_3$, the 'tangential' Riemann sphere is defined by its diameter $[S,O]$ where $S=[0,0,-1]$ is the South pole and $O=[0,0,0]$ is the origin. And the 'tangential' stereography is defined by M on the plane, P_t on the tangential sphere, and S,M,P_t aligned.

Remark 19.6.2. This stereography is the projection used by cartographers to draw maps of the circumpolar places. Obviously, both local metrics in the vicinity of O (on the plane and on the sphere) are the same. Nevertheless, having the center of the sphere at the origin is more handy when studying the isometries of the sphere itself.

Definition 19.6.3. The sphere in $\mathcal{E}_3$ whose equator is the trigonometric circle (in the $z=0$ plane) should be called the equatorial Riemann sphere. But we will rather call it as ***the*** Riemann sphere. And the correspondence where A is on the plane, while P is on the sphere and S,A,P are aligned will be called ***the*** stereographic projection.

Proposition 19.6.4. *The stereographic correspondence can be computed as*

$$\widehat{\pi}:\begin{pmatrix} z \\ t \\ \zeta \end{pmatrix} \in \mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right) \mapsto \frac{1}{t^2+z\zeta}\begin{pmatrix} t\,(z+\zeta) \\ -i\,t\,(z-\zeta) \\ +t^2-z\zeta \end{pmatrix} \tag{19.25}$$

$$\widehat{\pi}^{-1}:\begin{pmatrix} x \\ y \\ r \end{pmatrix} \in \mathbb{R}^3 \mapsto \begin{pmatrix} x+i\,y \\ 1+r \\ x-i\,y \end{pmatrix} \in \mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$$

where ζ is the conjugate of z and $x^2+y^2+r^2=1$ is assumed.

Proof. Write $P=\mu A+(1-\mu)\,S$ and obtain μ using $x^2+y^2+r^2=1$. □

Theorem 19.6.5. *The* ***spherical map*** ${}^t\underset{s}{V}er$ *introduced at 19.18 is the projective version of the* ***stereographic map*** *(from the South pole) introduced at 19.25. While the map $M \mapsto \underset{s}{\mathcal{V}}\,(2O-M,\,0)$ is the projective version of the stereographic map relative to the North pole.*

Proof. Simple comparison between both formula. Remember: a theorem is characterized among the propositions by its efficiency, not by the difficulty of the proof. □

Example 19.6.6. Let us consider two visible points: $z_1=1+2i$, $z_2=3+i$. We have

$$F_1 \doteq \underset{s}{\mathcal{V}}\,(z_1) \simeq \begin{pmatrix} 2 \\ 4 \\ -4 \\ -6 \end{pmatrix} \; ; \; F_4 \doteq \underset{s}{\mathcal{V}}\,(z_2) \simeq \begin{pmatrix} 6 \\ 2 \\ -9 \\ -11 \end{pmatrix}$$

One computes the family :

$$\begin{aligned}(1-\mu)\,F_1+(1+\mu)\,F_4 \mapsto Ponc\,(\mu) &= \underset{s}{\mathcal{V}}\left[(1-\mu)\,z_1+(1+\mu)\,z_2,\; R^2=\frac{5}{4}\left(\mu^2-1\right)\right] \\ &= \underset{s}{\mathcal{V}}\left[\begin{pmatrix} 4+3\,i \\ 2 \\ 4-3\,i \end{pmatrix}+\mu\begin{pmatrix} 2-i \\ 0 \\ 2+i \end{pmatrix},\; R^2=\frac{5}{4}\left(\mu^2-1\right)\right]\end{aligned}$$

When μ is real and $|\mu|>1$ we obtain a set of visible circles.

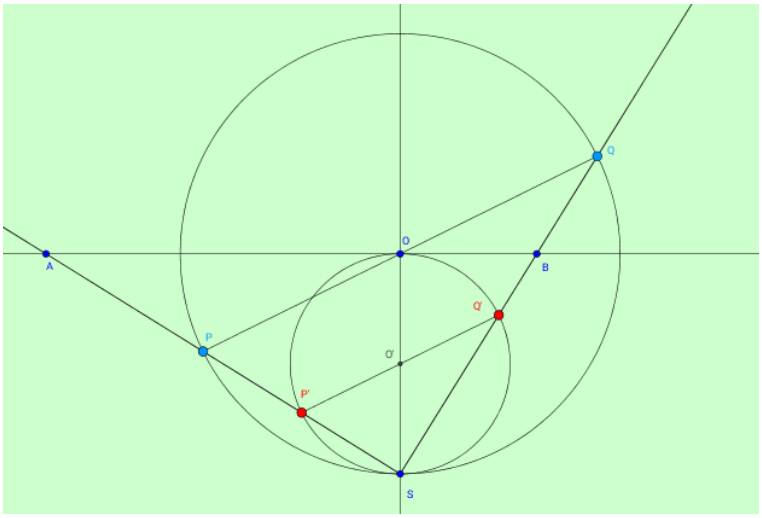


Figure 19.3: Both projections of A, B in the plane.

Let us consider the orthogonal point circles and their representatives:

$$F_2 \doteq \underset{s}{\mathcal{V}}\left[\begin{pmatrix} z_1/t_1 \\ 1 \\ \zeta_2/t_2 \end{pmatrix}, 0\right] \simeq \begin{pmatrix} 4+i \\ 3+2\,i \\ -4-5\,i \\ -6-5\,i \end{pmatrix} \;;\; \mathrm{F}_3 \doteq \underset{s}{\mathcal{V}}\left[\begin{pmatrix} z_2/t_2 \\ 1 \\ \zeta_1/t_1 \end{pmatrix}, 0\right] \simeq \begin{pmatrix} 4-i \\ 3-2\,i \\ -4+5\,i \\ -6+5\,i \end{pmatrix}$$

One computes the family:

$$\begin{aligned}(1+i\,\mu)\,F_2+(1-i\,\mu)\,F_3 \mapsto Arcs\,(\mu) &\simeq \begin{pmatrix} 4 \\ 3 \\ -4 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -2 \\ +5 \\ +5 \end{pmatrix} \\ &= \underset{s}{\mathcal{V}}\left[\begin{pmatrix} 4+3\,i \\ 2 \\ 4-3\,i \end{pmatrix} + i\,\mu \begin{pmatrix} -(2-i) \\ 0 \\ +(2+i) \end{pmatrix},\; R^2=\frac{5}{4}\left(\mu^2+1\right)\right]\end{aligned}$$

When μ is real, we obtain a set of visible circles (visible center, real radius). All of them are orthogonal to F_1 and F_4: these circles are going through points z_1 and z_2. And therefore, the first pencil is the *isotomic* (Poncelet) pencil of points z_1 and z_2, while the second one is their *isoptic* pencil.

19.7 Quaternary

Proposition 19.7.1. *Consider the visible Cremona transform $\widehat{\sigma}$ defined by*

$$\sigma\begin{pmatrix} \mathbf{Z} \\ \mathbf{T} \end{pmatrix} \simeq \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \mathbf{Z} \\ \mathbf{T} \end{pmatrix} \qquad \text{where } \det\sigma = 1$$

and suppose that $\widehat{\sigma}$ induces a rotation $\widehat{H}$ in $\mathcal{E}_3$, then it exists reals A, f, g, h such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \simeq \cos A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin A \begin{bmatrix} -h & -f+ig \\ -f-ig & +h \end{bmatrix} \tag{19.26}$$

Proof. In a rotation of the sphere, the antipodal relation is preserved. **** This amounts to the invariance by $v_4 \mapsto -v_4$, and matrix $\boxed{\widehat{H_s}}$ as to commute with $\boxed{\underset{s}{\mathcal{Q}}}$, leading to $\alpha = \dfrac{\delta\,d}{a}$, $\beta = -\dfrac{\delta\,c}{a}$, $\gamma = -\dfrac{\delta\,b}{a}$ where $\delta = a$ can be chosen. This leads to

$$a = c' - ih' \;;\; b = g' - if' \;;\; c = -g' - if' \;;\; d = c' + ih'$$

It only remains to put $c' = \cos A$ and then normalize $f' : g' : h'$. □

$$\begin{array}{ccc} \overline{\mathbb{C}} & \xrightarrow{h} & \overline{\mathbb{C}} \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{S}^2 & \xrightarrow{\widehat{h}} & \mathbb{S}^2 \end{array} \qquad \begin{array}{ccc} \mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right) & \xrightarrow{H} & \mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right) \\ \downarrow \widehat{\pi} & & \downarrow \widehat{\pi} \\ \mathbb{S}^2 & \xrightarrow{\widehat{h}} & \mathbb{S}^2 \end{array} \qquad \begin{array}{ccc} \mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right) & \xrightarrow{H} & \mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right) \\ \downarrow Arn & & \downarrow Arn \\ \mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^4\right) & \xrightarrow{\widehat{H}} & \mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^4\right) \end{array}$$

Proposition 19.7.2. *Conversely, the equation defined at (19.26) induces a rotation of the unit sphere in $\mathcal{E}_3$. Seen from the unit vector ${}^t[f, g, h]$, its angle is $+2A$... while seen from ${}^t[-f, -g, -h]$, its angle is $-2A$.*

Proof. Substitute (19.26) into (19.23), use:

$$\sin A^2 = U - \cos A^2 \ ; \ \sin A = \frac{1}{2}\frac{\sin B}{\cos A} \ ; \ \cos A^2 = \frac{1}{2}\cos B + \frac{U}{2}$$

and collect in $U, \sin, \cos$. Restore $U = 1$ and obtain:

$$\boxed{\widehat{H_s}} = m_U + \sin 2A \, m_S - \cos 2A \, {m_S}^2$$
$$\text{where } m_U = \begin{bmatrix} f^2 & fg & fh & 0 \\ fg & g^2 & gh & 0 \\ fh & gh & h^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ ; \ m_S = \begin{bmatrix} 0 & h & -g & 0 \\ -h & 0 & f & 0 \\ g & -f & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

□

Remark 19.7.3. One has: $\chi\left(\boxed{\widehat{H_s}},\ X\right) = (X-1)^2 (X-k)(X-\kappa)$ where $\overline{\kappa} = k = \exp iB\pi$, while $\chi\left(\boxed{h},\ X\right) = (X-\mu_1)(X-\mu_2)$ where $\overline{\mu_2} = \mu_1 = \exp iA\pi$. But the multipliers of h are the ratios of the eigenvalues, giving again k and κ.

Theorem 19.7.4. *When rotations are composed in $\mathcal{E}_3$, quaternions are multiplied as 2×2 matrices. This amounts to use the Hamilton's rule:* $+\overrightarrow{j} \cdot \overrightarrow{k} = \overrightarrow{i} = -\overrightarrow{k} \cdot \overrightarrow{j}$, etc.

19.8 Stereographic formalism

Here, we will use the stereographic projection from the South pole, and apply the Theorem 19.6.5 to obtain a visual formalism using:

Definition 19.8.1. We define the **apex** of a point, or of a cycle, as the columns:

$$\begin{aligned} \mathcal{A}(M) &\doteq {}^t\underset{s}{Ver}(M) \\ \mathcal{A}(\mathcal{C}) &\doteq \boxed{Mink} \cdot \underset{s}{\mathcal{V}}(\mathcal{C}) \end{aligned}$$

Proposition 19.8.2. *The apex of a point* $\mathbf{Z} = X + iY$ *in the* xOy *plane is the stereographic projection of this point. All apexes are columns and live in the same copy of* $\mathbb{P}_\mathbb{C}(\mathbb{C}^4)$*. Moreover, the apex of a point is the same as the apex of the point circle centered at this point.*

Proof. From former results, ${}^t\mathcal{A}(M) \cdot \boxed{Mink} \cdot \mathcal{A}(M) = 0$, so that $\mathcal{A}(M) \in \mathbb{S}_4$. One can check easily that $M \doteq X : Y : 0 : \mathbf{T}$ and the apex $\mathcal{A}(M)$ are aligned with *South*, proving the stereographic property. Moreover

$$\mathcal{A}(M) \doteq {}^t\underset{s}{Ver}(M) \simeq \begin{pmatrix} (z+\zeta)\,t \\ -i\,(z-\zeta)\,t \\ t^2 - z\,\zeta \\ t^2 + z\,\zeta \end{pmatrix} \simeq \boxed{Mink} \cdot \underset{s}{\mathcal{V}}(M,0) \doteq \mathcal{A}[M,0]$$

This is the rationale for using the same name for both objects. □

Example 19.8.3. Consider the points

$$M_j = 0.22 + 0.84\,i \ ; \ 1.34 - 0.28\,i \ ; \ 0.22 - 0.44\,i$$

They define a circle with center $B = 0.7 + 0.2i$ and radius $\rho = 0.8$. Computing the apexes, we obtain:

$$A_B, A_0, A_1, A_2 \simeq \begin{pmatrix} 1.40 \\ 0.40 \\ 0.47 \\ 1.53 \end{pmatrix}, \begin{pmatrix} 0.440 \\ 1.680 \\ 0.246 \\ 1.754 \end{pmatrix}, \begin{pmatrix} +2.680 \\ -0.560 \\ -0.874 \\ +2.874 \end{pmatrix}, \begin{pmatrix} +0.440 \\ -0.880 \\ +0.758 \\ +1.242 \end{pmatrix}$$

and therefore,

$$U \doteq \mathcal{A}\left(\mathcal{C}\left(M_j\right)\right) = \boxed{Mink} \cdot {}^t\bigwedge_3 \left(A_j\right) = \begin{pmatrix} 1.40 \\ 0.40 \\ 0.47 \\ 1.53 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ +0.64 \\ -0.64 \end{pmatrix}$$

Proposition 19.8.4. *The apex of a line lies in the South plane, and we have:*

$$\mathcal{A}\left([f, g, h]\right) \doteq \begin{pmatrix} (f+h) \\ i\,(f-h) \\ +g \\ -g \end{pmatrix} \tag{19.27}$$

Proposition 19.8.5. *When using apexes, the orthogonality formula between two circles is now:*

$$\frac{{}^t\left(\mathcal{A}\left[M_1,\, r_1^2\right]\right) \cdot \boxed{Mink} \cdot \left(\mathcal{A}\left[M_2,\, r_2^2\right]\right)}{{}^t\left(\mathcal{A}\left[M_1,\, r_1^2\right]\right) \cdot \boxed{\underset{k}{\mathcal{N}}} \cdot \left(\mathcal{A}\left[M_2,\, r_2^2\right]\right)} = \left(\frac{z_2}{t_2} - \frac{z_1}{t_1}\right)\left(\frac{\zeta_2}{t_2} - \frac{\zeta_1}{t_1}\right) - r_1^2 - r_2^2 \tag{19.28}$$

$$\text{where } \boxed{\underset{k}{\mathcal{N}}} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

so that $M \in \mathcal{C}$ can be restated as $\mathcal{A}(M) \perp \mathcal{A}(\mathcal{C})$.

Proof. In (19.21) the normalization has to be changed due to the product by $\boxed{Mink}$. □

Remark 19.8.6. Sending the plane $\mathcal{A}_4 = 0$ at infinity "discards" the circles orthogonal to $\mathbf{Z}\overline{\mathcal{Z}} + \mathbf{T}^2 = 0$ and the quadric appears as a sphere in the remaining space $\mathcal{E}_3$. On the contrary, sending the South plane $\mathcal{A}_4 + \mathcal{A}_3 = 0$ at infinity "discards" the (completed) straight lines and allows for a normalization of the circle-representatives.

Proposition 19.8.7. *Define the* ***shadow*** *of a cycle $\mathcal{C}$ as the locus of the apexes of the points of $\mathcal{C}$. The shadow of a cycle is a circle on the sphere $\mathbb{S}_4$. Its center belongs to line $[O, U]$ and, in fact, is the inverse of U wrt the sphere. When $\mathcal{C}$ is a circle, the line $[S, B]$ describes the pencil of all circles centered at B, and therefore goes through $\mathcal{A}(B)$ and $U = \mathcal{A}(\mathcal{C})$ (see Figure 19.4).*

Proposition 19.8.8. *Let $\mathcal{A}_j$, $j = 1..4$ be the apexes of four cycles $\mathcal{C}_j$. Describe the pencil generated by $\mathcal{C}_1, \mathcal{C}_2$ using the matrix $\boxed{\Delta_{12}} = \left(\mathcal{A}_1 \underset{6}{\wedge} \mathcal{A}_2\right)$, and the pencil generated by $\mathcal{C}_3, \mathcal{C}_4$ using the matrix $\boxed{\Delta_{34}} = \left(\mathcal{A}_3 \underset{6}{\wedge} \mathcal{A}_4\right)$. When each pencil is orthogonal to the other, then*

$$\boxed{\Delta_{34}} = \boxed{Mink} \cdot \boxed{\Delta_{12}^*} \cdot \boxed{Mink}$$

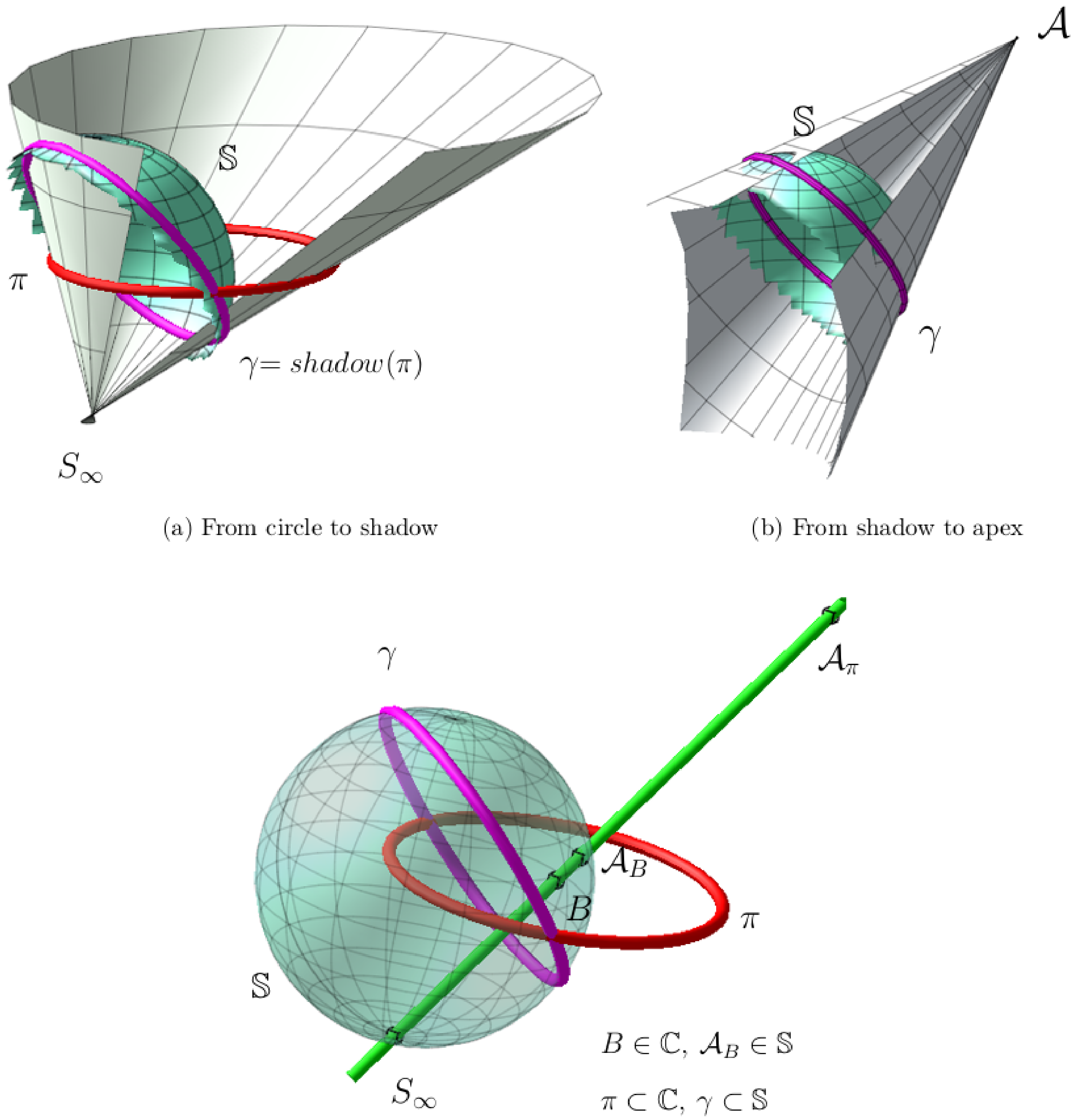
Using electrical notation (see Definition 8.1.5), if $\boxed{\Delta_{12}} = \left(\overrightarrow{E},\, \overleftarrow{B}\right)$ then $\boxed{\Delta_{34}} = \left(\overleftarrow{B},\, -\overrightarrow{E}\right)$.

Proof. Cut Δ_{12} by the four base hyperplanes. Among the four expressions obtained, at most two are $0 : 0 : 0 : 0$. Do the same with the proposed matrix. And check that all the 16 orthogonality relations are fulfilled (this computation is rather easy since most results are obviously 0, while the others involve $E_x B_x + E_y B_y + E_z B_z$. □

19.9 Comparison with Cartesian and Artinian metrics

To be written.

For all metrics, $\boxed{\mathrm{OrtO}}^* = \left(\boxed{\mathrm{OrtO}}^2\right)^*$ while $\operatorname{trace}\left(\boxed{\mathrm{OrtO}}^*\right) = 1$.



(a) From circle to shadow

(b) From shadow to apex

(c) Back from apex to center

Figure 19.4: Stereographic projection and apexes

Chapter 20

The Lie Sphere

20.1 Elementary properties

Remark 20.1.1. In this chapter, the radius of a circle is noted either as r_j or as ρ_j. The r_j radiuses are unsigned quantities (only r_j^2 is meaningful). On the contrary, the ρ_j are signed quantities. The first point of view is more adapted to orthogonality, the second one is more adapted to contact properties.

Remark 20.1.2. Let us recall the following matrices

$$\boxed{\underset{b}{\mathcal{Q}}} \doteq -\frac{1}{8S^2}\begin{bmatrix} a^2 & -S_c & -S_b & -a^2 S_a \\ -S_c & b^2 & -S_a & -b^2 S_b \\ -S_b & -S_a & c^2 & -c^2 S_c \\ -a^2 S_a & -b^2 S_b & -c^2 S_c & b^2 a^2 c^2 \end{bmatrix} \; ; \; \boxed{\underset{z}{\mathcal{Q}}} \doteq \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and how they are applied (see Remark 19.1.15)

$$\left(d_{12}^2 - \rho_1^2 - \rho_2^2\right) \quad = \quad \frac{{}^t\mathcal{V}_1 \cdot \boxed{\mathcal{Q}} \cdot \mathcal{V}_2}{(\mathcal{P}_\infty \cdot \mathcal{V}_1) \times (\mathcal{P}_\infty \cdot \mathcal{V}_2)}$$

As a result, the $\mathbb{P}_\mathbb{C}\left(\mathbb{C}^4\right)$ formalism only deals with squared radiuses. And therefore the $r^2 \in \mathbb{R}$ property allows the existence of imaginary radiuses (i.e. $r^2 < 0$).

Definition 20.1.3. The Lie Sphere formalism assigns a sign to the radius of a circle, and stores this signed radius into the fifth coordinate of the **Lie sphere representative** of this cycle, according to

$$\widehat{\mathcal{V}}\,(P, \rho) \doteq (\mathcal{V}\,(P, |\rho|) : \rho) \in \mathbb{P}_\mathbb{C}\left(\mathbb{C}^5\right)$$

where the $\mathcal{V}$ is normalized using $\mathcal{P}_\infty \cdot \mathcal{V} = 1$. How to deal with completed lines will be stated later.

Proposition 20.1.4. *All the $\widehat{\mathcal{V}}$ belong to a quadric, the so-called* ***Lie sphere quadric****.*

$${}^t\widehat{\mathcal{V}} \cdot \boxed{\widehat{\mathcal{Q}}} \cdot \widehat{\mathcal{V}} = 0 \qquad \text{where } \boxed{\widehat{\mathcal{Q}}} \doteq \begin{bmatrix} \boxed{\mathcal{Q}} & 0 \\ 0 & +2 \end{bmatrix}$$

Here, the $\boxed{\mathcal{Q}}$ matrix is one of the "official" matrices, i.e. exactly $\boxed{\underset{b}{\mathcal{Q}}}$ or $\boxed{\underset{z}{\mathcal{Q}}}$ etc.

Proof. Obvious from ${}^t\mathcal{V} \cdot \boxed{\mathcal{Q}} \cdot \mathcal{V} = -2r^2$. □

Definition 20.1.5. Oriented contacts: we will say that two circles are gearing when ${}^t\widehat{\mathcal{V}_1} \cdot \boxed{\widehat{\mathcal{Q}}} \cdot \widehat{\mathcal{V}_2} = 0$ and that they are anti-gearing when

$${}^t\widehat{\mathcal{V}_1} \cdot \boxed{\widehat{\mathcal{Q}^x}} \cdot \widehat{\mathcal{V}_2} = 0 \qquad \text{where } \boxed{\widehat{\mathcal{Q}^x}} \doteq \begin{bmatrix} \boxed{\mathcal{Q}} & 0 \\ 0 & -2 \end{bmatrix}$$

Theorem 20.1.6. *Assume that a circle is clockwise oriented when $\rho > 0$, and counterclockwise oriented when $\rho < 0$. When two circles are gearing, we have $d_{12}^2 - (\rho_1 - \rho_2)^2 = 0$. If , additionally, we assume that both circles have the same orientation, this implies $(d_{12} - r_1 + r_2)(d_{12} + r_1 - r_2)$ and one circle is internal to the other. If, on the contrary, we assume that both circles are oriented differently, this implies $(d_{12} - r_1 - r_2)(d_{12} + r_1 + r_2) = 0$ and the two circles are external to eachother.*

When the two circles are anti-gearing, we have $d_{12}^2 - (\rho_1 + \rho_2)^2 = 0$ and the conclusions are reversed.

Proof. Obvious from definitions. This is nevertheless the key property here. □

Remark 20.1.7. Let us recall that the distance from a point to a line is given by (7.24), (15.11) i.e. by

$$\operatorname{dist}(P, \Delta) = \frac{\Delta \cdot P}{(\mathcal{L}_\infty \cdot P)\sqrt{\Delta \cdot \boxed{\mathcal{M}} \cdot {}^t\Delta}}$$

Moreover, due to the choice of the b and z bases, we have

$$\boxed{\mathcal{M}_b} \doteq -2\,\text{submatrix}\left(\underset{b}{\boxed{\mathcal{Q}}}, 1..3, 1..3\right) \;;\; \boxed{\mathcal{M}_z} \doteq -2\,\text{submatrix}\left(\underset{z}{\boxed{\mathcal{Q}}}, 1..3, 1..3\right)$$

Proposition 20.1.8. *In order to use ${}^t\widehat{\mathcal{V}_1} \cdot \boxed{\widehat{\mathcal{Q}}} \cdot \widehat{\mathcal{V}_3} = 0$ as a condition for an "oriented contact" between circle $\mathcal{C}_1$ and line Δ_3, the representative $\widehat{\mathcal{V}_3}$ as to be defined as:*

$$[f, g, h] \underset{z}{\mapsto} \left(f : g : h : 0 : \pm\sqrt{fh}\right)$$

and more generally by using : $\left(2\,\widehat{\mathcal{V}_3}\,[5]\right)^2 \underset{b}{=} \Delta \cdot \boxed{\mathcal{M}} \cdot {}^t\Delta$ i.e. by replacing $\boxed{\mathcal{M}_z}$ with its equivalent in the other formalisms.

Proof. Direct computation. For two lines, "contact" means parallelism ! □

Remark 20.1.9. When using the Lie representation, objects that don't belong to $\mathcal{Q}_5$ are meaningless, while $\mathcal{Q}_5$ itself is obtained by a double coating of the outside of $\mathcal{Q}$ in $\mathbb{P}_\mathbb{C}(\mathbb{C}^4)$. Therefore, imaginary circles are lost : when the radius decreases to 0 in an *isotomic* pencil, the differentiable continuation is going back to real radiuses (with the other orientation) and not escaping to imaginary values.

Definition 20.1.10. When $\widehat{\mathcal{V}_0}$ is not a point-circle, the **inversion wrt** $\widehat{\mathcal{V}_0}$ is the linear application defined by the matrix

$$\boxed{\Sigma_0} \doteq \begin{bmatrix} \boxed{\sigma_0} & 0 \\ 0 & -1 \end{bmatrix}$$

where the 4×4 matrix $\boxed{\sigma}$ is defined as in Theorem 14.8.4, i.e. by

$$\boxed{\sigma_0} = \text{Id} - 2\left(\mathcal{V}_0 \cdot {}^t\mathcal{V}_0 \cdot \boxed{\mathcal{Q}}\right) \div \left({}^t\mathcal{V}_0 \cdot \boxed{\mathcal{Q}} \cdot \mathcal{V}_0\right)$$

Proposition 20.1.11. *Assuming that $\mathcal{V}_0$ is not a point-circle, we have the following properties:*

1. *Σ_0 is involutive.*
2. $\ker(\Sigma_0 + 1)$ *is generated by* $\widehat{\mathcal{V}}$ *and* $0 : 0 : 0 : 0 : 1$,
   while $\ker(\Sigma_0 - 1)$ *is the plane* $[-uz, ut, -ux, uy, 0], [0, 0, 0, 0, 1]$.
3. *As a result, $\widehat{\mathcal{V}_0}$ and it's opposite are invariant (with their orientation), i.e. $\Sigma_0(\mathcal{V}_0 : +\rho_0) \simeq (\mathcal{V}_0 : +\rho_0)$ and $\Sigma_0(\mathcal{V}_0 : -\rho_0) \simeq (\mathcal{V}_0 : -\rho_0)$ while orthogonal circles are reverted.*

Proof. Direct examination. □

Theorem 20.1.12. *Following Searby (2009), we define*

$$\mathrm{Searby}\left(\widehat{\mathcal{V}_1},\widehat{\mathcal{V}_2}\right)=\frac{{}^t\widehat{\mathcal{V}_1}\cdot\boxed{\widehat{\mathcal{Q}}}\cdot\widehat{\mathcal{V}_2}}{2\,\widehat{\mathcal{V}_1}[5]\times\widehat{\mathcal{V}_2}[5]}-1=\frac{d_{12}^2-\rho_1^2-\rho_2^2}{2\,\rho_1\rho_2}$$

Then this quantity is projective, and invariant by inversion. Moreover, Searby $\left(\widehat{\mathcal{V}_1},\widehat{\mathcal{V}_2}\right)$ *equals respectively* $+1,0,-1$ *when the cycles are respectively gearing, orthogonal and anti-gearing.*

Proof. Direct examination. One can remark that $\rho_j\mapsto-\rho_j$ so that a squared difference remains a squared difference. □

20.2 Example 1: the incircle

1. Using barycentrics, we have $BC\simeq[1,0,0]$ so that $\mathrm{sid}_A\underset{b}{\simeq}1:0:0:0:-\dfrac{a}{4S}$. Solving

$$\left\{{}^tX\cdot\boxed{\underset{b}{\widehat{\mathcal{Q}}}}\cdot J\mid J=\mathrm{sid}_A,\mathrm{sid}_B,\mathrm{sid}_C,X\right\}$$

   we obtain two solutions

$$(1:1:1:0:0)\ ;\ \left(\frac{(b+c-a)^2}{4}:\frac{(c+a-b)^2}{4}:\frac{(a+b-c)^2}{4}:1:\frac{2S}{a+b+c}\right)$$

   i.e Sirius and the usual incenter.

2. Using Lubin2 coordinates, we have

$$A\underset{2}{\simeq}\alpha^2:1:\frac{1}{\alpha^2}\ ;\ BC\underset{2}{\simeq}[1,-\beta^2-\gamma^2,\gamma^2\beta^2]\ ;\ \mathrm{sid}_A\underset{2}{\simeq}1:-\beta^2-\gamma^2:\gamma^2\beta^2:0:-\beta\gamma$$

   Solving the z-system, we obtain again Sirius (aka $0:1:0:0:0$) together with:

$$\widehat{\Omega_0}\underset{z}{\simeq}\frac{s_1}{s_3}:-\frac{s_2^2s_1^2}{4s_3^2}+\frac{3s_2s_1}{2s_3}-\frac{1}{4}:s_2:1:-\frac{(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)}{2\alpha\beta\gamma}$$

3. Solving the s-system, we obtain $0:0:1:1:0$ together with:

$$\widehat{\Omega_0}\underset{s}{\simeq}\begin{bmatrix}-4s_3(s_1+s_2s_3)\\-4\,\boldsymbol{i}s_3(s_1-s_2s_3)\\s_2^2s_1^2-6s_2s_3s_1+5s_3^2\\s_2^2s_1^2-6s_2s_3s_1-3s_3^2\\2\sqrt{2}\,s_3(s_1s_2-s_3)\end{bmatrix}$$

4. And, obviously, the three formalisms z,p,s lead to the same

$$I_0\underset{z}{\simeq}-s_2:1:-\frac{s_1}{s_3}\ ;\ r_0\underset{z}{\simeq}-\frac{(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)}{2\alpha\beta\gamma}$$

20.3 Exemple 2: the three excenters

1. Let's use Lubin-2 and the z-formalism. We have $\widehat{A}\underset{z}{\simeq}\left(-\dfrac{1}{\alpha^2}:1:-\alpha^2:1:0\right)$ and $BC\underset{5}{\simeq}\left(1:-\beta^2-\gamma^2:\gamma^2\beta^2:0:\beta\gamma\right)$.

2. The incircle is

$$\widehat{\mathcal{V}}\underset{z}{\simeq}\left[\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}:\frac{6s_2s_3s_1-s_2^2s_1^2-s_3^2}{4s_3^2}:\alpha\beta+\alpha\gamma+\beta\gamma:1:\frac{(\beta+\gamma)(\alpha+\gamma)(\alpha+\beta)}{2\alpha\beta\gamma}\right]$$

3. The ex-circles $\widehat{\mathcal{V}_j}$ are obtained by using $\alpha \mapsto -\alpha$, etc.
4. Solving the Apollonius equations

$$\left\{ {}^{t}X \cdot \boxed{\underset{z}{\widehat{\mathcal{Q}}}} \cdot J \mid J = \widehat{\mathcal{V}_A}, \widehat{\mathcal{V}_B}, \widehat{\mathcal{V}_C}, X \right\}$$

one re-obtains $S_1 \simeq -\dfrac{\sigma_2}{2\sigma_3} : \dfrac{\sigma_1\sigma_2 - \sigma_3}{4\sigma_3} : -\dfrac{\sigma_1}{2} : 1 : \dfrac{1}{2}$ (i.e. the Euler circle[1]) together with S_5, the Apollonius circle itself (see Subsection 14.11.4).

5. The common orthogonal circle is the alt-Spiecker circle $\widehat{\mathcal{V}_4} \underset{z}{\simeq}$

$$\left[\begin{array}{c} 2s_1s_3 - 2s_2^2 \\ -7s_3^2 + \left(-2s_1^3 + 6s_1s_2\right)s_3 + s_2^2\left(s_1^2 - 2s_2\right) \\ -2s_3^2\left(s_1^2 - s_2\right) \\ 4s_3^2 \\ 2Ws_3 \end{array}\right] \quad \text{where } W = \sqrt{s_1^3s_3 - 5s_2s_3s_1 + s_2^3 + 7s_3^2}$$

and one can check that the S_2 solution is the sideline BC, etc, while the four S_{4+j} are the inverses of the S_j wrt $\widehat{\mathcal{V}_4}$.

6. And we have the following Searby products:

$$\text{Searby}\left(\left[\widehat{\mathcal{V}_a}, \widehat{\mathcal{V}_b}, \widehat{\mathcal{V}_c}\right] \otimes \left[S_1, S_5, S_a, S_6, S_b, S_7, S_c, S_8, \widehat{\mathcal{V}_4}\right]\right)$$
$$= \left[\begin{array}{ccccccccc} -1 & +1 & -1 & +1 & +1 & -1 & +1 & -1 & 0 \\ -1 & +1 & +1 & -1 & -1 & +1 & +1 & -1 & 0 \\ -1 & +1 & +1 & -1 & +1 & -1 & -1 & +1 & 0 \end{array}\right]$$

7. The same computations cans also be conducted within the b-formalism. One only has to deal with the usual radicals.

20.4 Mixtilinear circles

1. Use barycentrics and consider cycles Γ, AB, AC. Their representants are:

$$0:0:0:1:-\frac{abc}{4S} \ ; \ 0:0:1:0:-\frac{c}{4S} \ ; \ 0:1:0:0:-\frac{b}{4S}$$

2. Solutions of the Apollonius equations are $0 : c^2 : b^2 : 1 : 0$ (i.e. the point-circle A) and

$$\frac{4b^2c^2}{(b+c-a)^2} : \frac{c^2(a+b-c)^2}{(b+c-a)^2} : \frac{b^2(a-b+c)^2}{(b+c-a)^2} : 1 : \frac{bc(a+b-c)(a-b+c)}{2S(b+c-a)}$$

3. Using the z-formalism, the three cycles are

$$0:-1:0:+1:1 \ ; \ -1:\alpha^2+\beta^2:-\beta^2\alpha^2:0:\alpha\beta \ ; \ -1:\alpha^2+\gamma^2:-\gamma^2\alpha^2:0:\alpha\gamma$$

and one obtains $(A, 0)$ together with

$$\left(\left[\begin{array}{c} \alpha^2\left(\alpha\beta+\alpha\gamma+2\beta\gamma\right)^2 \\ \alpha^2\left(\beta-\gamma\right)^2 \\ \left(2\alpha+\beta+\gamma\right)^2 \end{array}\right], \frac{2(\beta+\gamma)(\alpha+\gamma)(\alpha+\beta)}{(\beta-\gamma)^2\alpha}\right)$$

4. When using the s-formalism, we have

$$\Gamma \simeq 0:0:-2:0:\sqrt{2} \ ; \ AB \simeq -\beta^2\alpha^2 - 1 : \boldsymbol{i}\beta^2\alpha^2 - \boldsymbol{i} : \alpha^2+\beta^2 : \alpha^2+\beta^2 : \sqrt{2}\,\alpha\beta, \text{ etc}$$

and obviously, one obtains the same results.

5. The difficulty here is the choice of the orientations of the circles...

[1] caveat: $\sigma_1 = \sum \alpha^2 \neq s_1$!

20.5 Arbelos

Proposition 20.5.1. *Consider points* $I_0 \simeq +\boldsymbol{i}h : 0 : -\boldsymbol{i}h$, $A \simeq a : 1 : a$, etc *and circles* $\mathcal{C}_0 \doteq \mathcal{C}(I_0, 1)$ *and* $\mathcal{C}_a \simeq \mathcal{C}([B, C])$, etc, *where* $h, a, b, c \in \mathbb{R}$. *If we assume that the four circles are tangent by pairs, we have*

$$h = \pm 2 \; ; \; \{a, b, c\} \in \left\{x, \psi(x), \psi^2(x)\right\} \qquad \text{where } \psi(x) = \frac{x-3}{x+1}$$

Chapter 21

Hyperbolic geometry

In a first step, hyperbolic geometry was created to prove that a geometry can be build which satisfies all of the usual axioms except from the Euclidean one about the parallel lines. As a result, it becomes proven that the Euclidean axiom is independent from the other axioms. See Cannon et al. (1997) and Arlan Ramsay (1995).

Notation 21.0.1. For the further use of the reader, we summarize here all the notations that will be introduced throughout this chapter.

- Prefixes "$\mathbb{C}$-", "P-" and "K-" are used to distinguish the usual cartesian/complex objects from their Poincare or Klein counterparts.
- Letter O is ever the common origin to all spaces. The $\mathbb{C}$-unit circle, i.e. $\gamma_C(O,1)$, is noted Γ. The same circle, when used as the P- or the K- horizon circle, is noted $\partial\mathbb{H}$, while $\mathbb{H}$ denotes the open disk limited by $\partial\mathbb{H}$.
- A line through points A, B is noted AB, a circle centered at A and going through M is noted $(A; M)$. When enforcing the brand of a specific object seems useful, notations $\Delta_X(A,B)$ or $\gamma_X(A;M)$ are used.
- Coordinates $A_P \simeq z : t : \zeta$ of a P-point are the usual $\mathbb{C}$-coordinates, while a widehat denote the $\mathbb{C}$-reflection of a point into Γ, i.e. $\widehat{A} \underset{C}{\simeq} t/\zeta : 1 : t/z$.
- Coordinates $A_K \simeq k : 1 : \kappa$ are used for a K-point. From

  $$W = \sqrt{1 - k\kappa} = (1 - z\zeta) \, / \, (1 + z\zeta) \in \mathbb{R}$$

  the K-points are not supposed to cross the boundary Γ.

21.1 The Poincaré plane

Remark 21.1.1. Our strategy is as follows. We start from the $\mathbb{C}$-objects which occur in the vicinity of O, and then we convey everything to the vicinity of the generic point A_P. Therefore, the P-lines through O are the $\mathbb{C}$-diameters of Γ, while the P-circles $(O;M)$ are identified with the $\mathbb{C}$-circles $(O;M)$. Moreover, the space is assumed to be locally Euclidian at O, i.e.

$$(\mathrm{d}s_O)^2 = \mathrm{Cte}^2 \; (\mathrm{d}z \, \mathrm{d}\zeta)$$

Definition 21.1.2. The Poincaré model is what is obtained by using homographies as conveyors.

Theorem 21.1.3. *The* ***Poincaré conveyor*** *is the Cremona homography defined by:*

$$\tau_A : O \mapsto A_P \simeq \left(\begin{array}{c} z \\ t \\ \zeta \end{array} \right) \; ; \; \left(\begin{array}{c} \mathbf{Z} \\ \mathbf{T} \\ \overline{\mathcal{Z}} \end{array} \right) \mapsto \left[\begin{array}{c} \dfrac{t\,\mathbf{Z} + z\,\mathbf{T}}{\zeta\,\mathbf{Z} + t\,\mathbf{T}} \\ 1 \\ \dfrac{t\,\overline{\mathcal{Z}} + \zeta\,\mathbf{T}}{z\,\overline{\mathcal{Z}} + t\,\mathbf{T}} \end{array} \right]$$

Proof. Conveyor τ_A has to keep globally invariant both the horizon circle $\partial\mathbb{H}$ and the line OA. Therefore points $OA\cap\Gamma$, i.e. $\pm\sqrt{z/\zeta}:1:\pm\sqrt{\zeta/z}$, are the fixed points. Knowing 3 points and their images, we conclude using the cross-ratio. □

Remark 21.1.4. Seen as a quadratic Cremona transform, τ_A has two more fixed points (on $\partial\mathbb{H}$, but not visible) $\pm\sqrt{z/\zeta}:1:\mp\sqrt{\zeta/z}$, while the three points of indeterminacy are the umbilics $\Omega_x\simeq 0:0:1$ $\Omega_y\simeq 1:0:0$ together with $1/\zeta:-1/t:1/z$ (the reflection of A into the unit imaginary circle). Moreover, τ_A^{-1} is obtained by $t\mapsto -t$.

Theorem 21.1.5. *The* ***P-metric*** *near the* P*-point* A_P *is induced by the* P*-metric near* O *and we have :*

$$\left(\mathrm{d}s_P\right)^2(A)=\mathrm{Cte}^2\,\frac{\mathrm{d}z\mathrm{d}\zeta}{\left(1-z\zeta\right)^2} \tag{21.1}$$

Proof. Use the τ_A^{-1} conveyor to carry back $A+\mathrm{d}A\simeq z+\mathrm{d}z:1:\zeta+\mathrm{d}\zeta$ and then use the P-metric at O. □

Proposition 21.1.6. *The* P*-****line*** *through points* A,B *is the visible part of the* $\mathbb{C}$*-circle through* $A,B,\widehat{A},\widehat{B}$ Goodman-Strauss *(2001, p. 42). When* A *is fixed, the supports of the* P*-lines through* A *are the members of the* $\mathbb{C}$*-isoptic*$\left(A,\widehat{A}\right)$ *pencil.*

Proof. Since τ_A is conformal, we have $\Delta_P(A,B)\perp\Gamma$ and therefore $\widehat{A}$ belongs to the $\mathbb{C}$-circle which supports the P-line. □

Proposition 21.1.7. *The* P*-circles* $\gamma_P(A,\rho)$ *are the members of the* $\mathbb{C}$*-isotomic*$\left(A_P,\widehat{A_P}\right)$ *pencil, orthogonal to the former* $\mathbb{C}$*-pencil of the* P*-lines through* A_P*.*

When $M_1\simeq z_1:t_1:\zeta_1$*, then* $\gamma_P(A,M_1)=\gamma_C(U,\rho_C)$ *where*

$$\begin{aligned}
U &\underset{C}{\simeq} \left[\begin{array}{c} t\,z\,\left(t_1^2-z_1\,\zeta_1\right)\\ t_1\,\left(t^2t_1-t\,z\,\zeta_1-t\,\zeta\,z_1+z\,\zeta\,t_1\right)\\ t\,\zeta\,\left(t_1^2-z_1\,\zeta_1\right)\end{array}\right]\\
\rho_C^2 &= \frac{\left(\zeta_1\,t-t_1\,\zeta\right)\left(z_1\,t-z\,t_1\right)\left(t\,t_1-z_1\,\zeta\right)\left(t\,t_1-\zeta_1\,z\right)}{t_1^2\left(t^2t_1-t\,z\,\zeta_1-t\,\zeta\,z_1+z\,\zeta\,t_1\right)^2}\\
\rho_C &= \frac{\mathrm{distance}(M,A)\mathrm{distance}\left(M,\widehat{A}\right)}{2\,\mathrm{distance}\left(M,\mathrm{med}\left(A,\widehat{A}\right)\right)}
\end{aligned}$$

Proof. Adjust the value of p in $\underset{c}{\mathcal{V}}(\gamma)\underset{C}{\simeq}p\underset{c}{\mathcal{V}}(A,0)+(1-p)\underset{c}{\mathcal{V}}\left(\widehat{A},0\right)$ so that γ goes through M_1 and obtain

$$\gamma_P \underset{C}{\simeq} \left(\begin{array}{c} t\,\zeta\left(\zeta_1\,z_1-t_1^2\right)\\ -\left(z\,\zeta+t^2\right)z_1\,\zeta_1+t\,t_1\left(z\,\zeta_1+\zeta\,z_1\right)\\ z\,t\,\left(\zeta_1\,z_1-t_1^2\right)\\ \left(z\,\zeta+t^2\right)t_1^2-t\,t_1\left(z\,\zeta_1+\zeta\,z_1\right)\end{array}\right) \qquad \square$$

Proposition 21.1.8. *As points on the* $\mathbb{C}$*-line* $A\widehat{A}$*, the* $\mathbb{C}$*-center* U *and the diametrical points* $u_1,u_2=\gamma_C\cap A\widehat{A}$ *are given by the following Geogebra commands*

```
U =barycenter({A,A'},{distance(M,A')^2, -distance(M,A)^2})
u1=barycenter({A,A'},{distance(M,A'),   +distance(M,A)})
u2=barycenter({A,A'},{distance(M,A'),   -distance(M,A)})
```

Proof. Simple substitution. □

Remark 21.1.9. Spoiler: the line $\mathrm{med}\left(A_P,\widehat{A_P}\right)$ is the Γ polar of the later defined A_K.

Proposition 21.1.10. *The* P*-symmetry* $\sigma_A : M \mapsto M'$ *where* M' *belongs together to the* P*-line* AM *and to the* P*-circle* $(A; M)$ *is the following homography:*

$$\mathbf{Z} \mapsto \frac{(z\zeta+1)\,\mathbf{Z} - 2\,z}{2\,\mathbf{Z}\zeta - (z\zeta+1)} \;;\; \begin{pmatrix} \mathbf{Z} \\ \mathbf{T} \\ \overline{\mathcal{Z}} \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{(z\zeta+t^2)\,\mathbf{Z} - 2tz\,\mathbf{T}}{2t\zeta\,\mathbf{Z} - (z\zeta+t^2)\,\mathbf{T}} \\ 1 \\ \dfrac{(z\,\zeta+t^2)\,\overline{\mathcal{Z}} - 2t\zeta\,\mathbf{T}}{2tz\,\overline{\mathcal{Z}} - (z\,\zeta+t^2)\,\mathbf{T}} \end{pmatrix}$$

Proof. One has $\sigma_A = \tau_A \circ \sigma_O \circ \tau_A^{-1}$ where $\sigma_O = \mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}} \mapsto -\mathbf{Z} : \mathbf{T} : -\overline{\mathcal{Z}}$. As a result, $\widehat{A}$ is also invariant and we have $\text{cross\_ratio}_{\,C}\left(A, \widehat{A}, M, M'\right) = -1$. □

Construction 21.1.11. *Construct the* P*-circle when a* P*-diameter* $[M, N]$ *is known*

1. *Draw the* $\mathbb{C}$*-circle* $(B) \doteq \left(M, N, \widehat{M}, \widehat{N}\right)$. *This is the* P*-line* MN.
2. *Draw the* $\mathbb{C}$*-tangents* MC *and* NC *to the* $\mathbb{C}$*-circle* (B) *and obtain* C
3. *The required* P*-circle* γ *is the* $\mathbb{C}$*-circle* $(C; M)$.
4. *Add point* Q *on* γ. *The* $\mathbb{C}$*-tangent to* γ *at* Q *cut the* $\mathbb{C}$*-mediatrix of* $\left[Q, \widehat{Q}\right]$ *at some point* B_Q. *Then the* $\mathbb{C}$*-circle* $(B_Q; Q)$ *is another* P*-diameter of* γ
5. *The intersection of both diameters gives* A*(visible, in* $\mathbb{H}$*) and* $\widehat{A}$ *(visible, but transfinite).*

Construction 21.1.12. *Construct the* P*-midpoint* J*of the* P*-segment* $[O, A]$.

1. *Draw the* $\mathbb{C}$*-circle* δ *having* $[A, O]$ *for diameter.*
2. *Add a point* R *on this circle. Draw the* $\mathbb{C}$*-tangent at* R. *Cut by the* $\mathbb{C}$*-mediatrix of* $\left[R, \widehat{R}\right]$ *and obtain* B_R.
3. *Then the* $\mathbb{C}$*-circle* $(B_R; R)$ *is another diameter of* δ*, obtaining* J.

21.2 The Klein plane

Definition 21.2.1. The Klein model is what is obtained by using collineations as conveyors.

Theorem 21.2.2. *The* ***Klein conveyor*** *is the collineation defined by:*

$$\vartheta_A : O \mapsto A \simeq \begin{pmatrix} k \\ 1 \\ \kappa \end{pmatrix} \;;\; \begin{pmatrix} \mathbf{Z} \\ \mathbf{T} \\ \overline{\mathcal{Z}} \end{pmatrix} \mapsto \begin{bmatrix} \dfrac{1+W}{2} & k & \dfrac{k}{\kappa}\left(\dfrac{1-W}{2}\right) \\ \kappa/2 & 1 & k/2 \\ \dfrac{\kappa}{k}\left(\dfrac{1-W}{2}\right) & \kappa & \dfrac{1+W}{2} \end{bmatrix} \cdot \begin{pmatrix} \mathbf{Z} \\ \mathbf{T} \\ \overline{\mathcal{Z}} \end{pmatrix}$$
$$\text{where } W = \sqrt{1 - k\,\kappa}$$

Proof. Matrix $\boxed{\vartheta_A}$ fulfills its requirements. Moreover, this matrix diagonalizes as

$$\begin{pmatrix} 1+W_a \\ 1-W_a \\ \sqrt{1-W_a^2} \end{pmatrix} \;;\; \begin{bmatrix} k & k & -k \\ W_a & -W_a & 0 \\ \kappa & \kappa & \kappa \end{bmatrix} \qquad \text{where } W_a = \sqrt{k\kappa}$$

The proper columns are related to the unavoidable fixed points, namely the intersections $OA \cap \Gamma$ and the Γ-pole of the OA line, while one has also $\lambda_1\lambda_2 = \lambda_3^2$. As a result, we have not only the existence, but also the unicity of ϑ_A. Moreover, here again, $\boxed{\vartheta_A}^{-1}$ is nothing but $\boxed{\vartheta_{-A}}$. □

Theorem 21.2.3. *The **K-metric** near the* K*-point* $A_K \simeq k:1:\kappa$ *is:*

$$\left(\mathrm{d}s_K\right)^2 \left(\mathfrak{A}_K\right) = \frac{\mathrm{Cte}^2}{4}\left(\frac{\mathrm{d}k\mathrm{d}\kappa}{1-k\kappa}+\frac{\left(\kappa\,\mathrm{d}k+k\,\mathrm{d}\kappa\right)^2}{4\left(1-k\,\kappa\right)^2}\right)$$

See Theorem 21.2.3 for the fact that $\mathrm{Cte}=2$.

Proof. Use the ϑ_A^{-1} conveyor to carry back $A_K+\mathrm{d}A \simeq k+\mathrm{d}k:1:\kappa+\mathrm{d}\kappa$ near the origin and then use the K-metric at O. □

Remark 21.2.4. The K-line through A, B is nothing but the $\mathbb{C}$-line through the same points. Circles centered at A_K will be studied later.

Proposition 21.2.5. *The* K*-symmetry* $\mathfrak{s}_A : M \mapsto M'$ *centered at* A_K *is the involutive collineation defined by:*

$$\boxed{\mathfrak{s}_A} \simeq \frac{1}{1-k\,\kappa}\left[\begin{array}{ccc} -1 & 2\,k & -k^2 \\ -\kappa & 1+k\kappa & -k \\ -\kappa^2 & 2\,\kappa & -1 \end{array}\right] \tag{21.2}$$

Proof. This comes from $\mathfrak{s}_A = \vartheta_A \circ \mathfrak{s}_O \circ \vartheta_A^{-1}$. Alternate proof: diagonalize and obtain:

$$\left(\begin{array}{c} +1 \\ -1 \\ -1 \end{array}\right) ; \left[\begin{array}{ccc} k & \kappa^{-1} & -k \\ 1 & 1 & 0 \\ \kappa & k^{-1} & \kappa \end{array}\right]$$

The proper columns are related to the unavoidable fixed points, namely A_K itself together with $\widehat{A_K}$ and the Γ-pole of OA. As a result $\mathfrak{s}_A$ is an homology. □

21.3 From a model to the other

Proposition 21.3.1. *Consider* Δ_P *and* Δ_K*, the* P*- and* K*-lines sharing the same* $\mathbb{C}$*-turns* α, β *as points at horizon* $\partial\mathbb{H}$*. When written in* $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^4\right)$*, the projective space of the* $\mathbb{C}$*-cycles, the maps* $\mathfrak{P}\mathrm{to}\mathfrak{K} : \Delta_P \mapsto \Delta_K$ *and* $\mathfrak{K}\mathrm{to}\mathfrak{P} : \Delta_K \mapsto \Delta_P$ *(Klein to Poincaré and conversely) are:*

$$\begin{aligned} \underset{c}{\mathcal{V}}\left(\mathfrak{K}\mathrm{to}\mathfrak{P}\left(\Delta_K\right)\right) &= \underset{c}{\mathcal{V}}\left(\Delta_K\right)+\underset{c}{\mathcal{V}}\left(horz\right) \\ \underset{c}{\mathcal{V}}\left(\mathfrak{P}\mathrm{to}\mathfrak{K}\left(\Delta_P\right)\right) &= \underset{c}{\mathcal{V}}\left(\Delta_P\right)-\underset{c}{\mathcal{V}}\left(horz\right) \end{aligned}$$

(when normalizing the $\mathbb{C}$*-circles by* $U_4=1$ *and the* $\mathbb{C}$*-lines by* $U_2=2$*).*

Proof. We have $\Gamma \simeq 0:-1:0:1$ while $\underset{c}{\mathcal{V}}\left(\Delta_P\right) \simeq *:1:*:1$ and $\underset{c}{\mathcal{V}}\left(\Delta_K\right) \simeq *:*:*:0$. □

Proposition 21.3.2. *The map* $\mathfrak{P}\mathrm{to}\mathfrak{K}$ *transforms the pencil of all the* P*-lines through a given* A_P *into the pencil of all the* K*-lines through a same point* A_K*. And one has the punctual maps:*

$$\mathfrak{P}\mathrm{to}\mathfrak{K}\left(A_P\right) = \mathfrak{P}\mathrm{to}\mathfrak{K}\left(\begin{array}{c} z \\ t \\ \zeta \end{array}\right) = \left(\begin{array}{c} 2\,z\,t \\ t^2+z\,\zeta \\ 2\,\zeta\,t \end{array}\right) = A_K \tag{21.3}$$

$$\mathfrak{K}\mathrm{to}\mathfrak{P}\left(A_K\right) = \mathfrak{K}\mathrm{to}\mathfrak{P}\left(\begin{array}{c} k \\ 1 \\ \kappa \end{array}\right) = \left(\begin{array}{c} \left(1-\sqrt{1-\kappa\,k}\right)\div\kappa \\ 1 \\ \left(1-\sqrt{1-\kappa\,k}\right)\div k \end{array}\right) = A_P$$

Proof. Straightforward computation. Moreover: $k = \dfrac{2z}{1+z\zeta} = \left(z+\widehat{\widehat{z}}\right)/2$. □

Proposition 21.3.3. *The Poincaré* $\leftrightarrow$ *Klein maps can be illustrated by involving the* $3D$ *sphere* $\mathbb{S}$ *having* Γ *as equatorial circle. Let* N *be the North pole of* $\mathbb{S}$*. Start from a point* A_P *in the* P*-plane. Draw the* $\mathbb{C}$*-line* NA*. It cuts the sphere at a point* Q *(the stereographic projection). And then draw the vertical of* Q*, intersecting the equatorial plane at* A_K*.*

Proof. Assume $t = 1$ and describe Q as a point in $\mathbb{C}^2 \times \mathbb{R}$. Then

$$\left(\begin{array}{c} z_Q \\ t_Q \end{array}\right) = \frac{1}{1+z\zeta}\left(\begin{array}{c} 2\,z \\ 1-z\zeta \end{array}\right) = \left(\begin{array}{c} k \\ \sqrt{1-k\kappa} \end{array}\right)$$

The first $=$ sign is the stereographic formula, the second one is (21.3). □

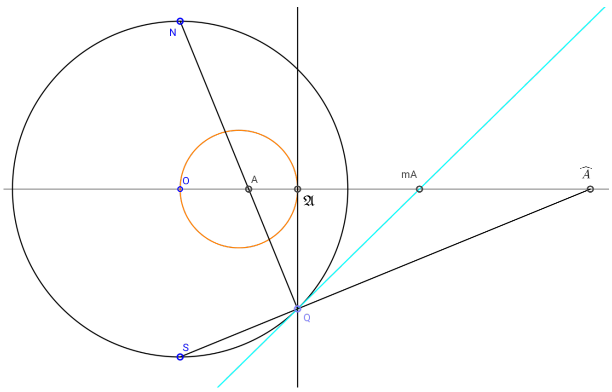


Figure 21.1: The Poincaré to Klein transform

Proposition 21.3.4. *The value of the* Cte *at Theorem 21.1.5 and Theorem 21.2.3 is* Cte $= 2$.

Proof. At Figure 21.1, one sees that $A_P \approx O$ implies $\mathrm{d}S = 2\mathrm{d}A_P$ together with $\mathrm{d}S = \mathrm{d}A_K$. □

Remark 21.3.5. The P-symmetry wrt the P-point $z : t : \zeta$ induces the following involutive map on $\partial\mathbb{H}$

$$\alpha \mapsto \beta = \frac{\left(t^2 + z\,\zeta\right)\alpha - 2\,z\,t}{2\,\zeta\,t\,\alpha - \left(t^2 + z\,\zeta\right)} = \frac{\alpha - k}{\alpha\,\kappa - 1}$$

while the P-symmetry wrt the $\mathbb{H}$-line (γ, δ) induces the map

$$\alpha \mapsto \beta = \frac{(\gamma+\delta)\,\alpha - 2\,\gamma\,\delta}{2\,\alpha - (\gamma+\delta)}$$

Remark 21.3.6. One has the following relation

$$\mathrm{cross\_ratio}\,(a, b, u, v) \times \mathrm{cross\_ratio}\,(b, c, u, v) = \frac{(u-a)\,(v-b)}{(u-b)\,(v-a)} \times \frac{(u-b)\,(v-c)}{(u-c)\,(v-b)} = \mathrm{cross\_ratio}\,(a, c, u, v)$$

and therefore the function $\ln \circ \mathrm{cross\_ratio}$ is additive.

Proposition 21.3.7. *Consider the* $\mathbb{C}$*-circles corresponding to the* P*-lines* (α, β) *and* (γ, δ)*. And then define the four points*

$$U_d^u \simeq \left(\begin{array}{c} \dfrac{\alpha\,\beta - \delta\,\gamma \pm W\,(\beta-\gamma)\,(\alpha-\delta)}{\alpha+\beta-\delta-\gamma} \\ 1 \\ \dfrac{\alpha\,\beta - \delta\,\gamma \pm W\,(\beta-\gamma)\,(\alpha-\delta)}{\beta\,\delta\,\alpha + \alpha\,\gamma\,\beta - \alpha\,\gamma\,\delta - \beta\,\delta\,\gamma} \end{array}\right) \qquad \text{where } W = \sqrt{\frac{(\beta-\delta)\,(\alpha-\gamma)}{(\beta-\gamma)\,(\alpha-\delta)}}$$

Then the points U_+^+, U_-^- *are* Γ*-inverse of each other and belong to the given* P*-lines (seen as full* $\mathbb{C}$*-circles), while the points* U_-^+, U_+^- *belong to* $\partial\mathbb{H}$ *and define a* P*-line orthogonal to the two given* P*-lines.*

When $W^2 < 0$*, points* (α, β) *and* (γ, δ) *are tangled, points* U_+^+, U_-^- *are* $\mathbb{C}$*-visible and one of them belongs to* $\mathbb{H}$*. When* $W^2 > 0$*, points* (α, β) *and* (γ, δ) *are untangled, points* U_-^+, U_+^- *are* $\mathbb{C}$*-visible and define a (visible)* $\mathbb{H}$ *line.*

Proof. Being the cross-ratio of the four given turns, quantity W^2 is real. Everything else comes from direct computation. Remark: when the given lines are orthogonal to each other, then $W^2 = -1$. □

21.4 More about hyperbolic distance

Notation 21.4.1. In what follows, δ_{AB} is distance(A,B), i.e. $\delta_{AB}^2 =$

Proposition 21.4.2. *In the* K*-plane, the distance between the* K*-points* M_1, M_2 *is*

$$\left|\frac{1}{2}\ln \text{cross\_ratio}\ (M_1, M_2, M_3, M_4)\right| \tag{21.4}$$

where $M_3, M_4 \in \Gamma$ *and the* M_j *are* K*-aligned (in any order).*

Proof. Use $M_1, M_2, M_3, M_4 = x_1, x_2, -1, +1$ on the x-axis. Substitute $k = \kappa = x$ and $\mathrm{d}k = \mathrm{d}\kappa = \mathrm{d}x$ into

$$(\mathrm{d}s_K)^2 \quad = \quad \frac{\mathrm{Cte}^2}{4}\left(\frac{\mathrm{d}k\mathrm{d}\kappa}{1-k\kappa} + \frac{(\kappa\,\mathrm{d}k + k\,\mathrm{d}\kappa)^2}{4\,(1-k\,\kappa)^2}\right)$$

and obtain $\mathrm{d}s = \dfrac{\mathrm{Cte}}{4}\left(\dfrac{1}{1-x} + \dfrac{1}{x+1}\right)\mathrm{d}x$. Integrating, we obtain

$$\delta = \frac{\mathrm{Cte}}{4}\ln\left(\frac{(x_2+1)\,(x_1-1)}{(x_2-1)\,(x_1+1)}\right) = \frac{\mathrm{Cte}}{4}\ln\left(\frac{(x_1-x_4)\,(x_2-x_3)}{(x_2-x_4)\,(x_1-x_3)}\right)$$

i.e. the required cross-ratio formula. This applies to all pair of points since the cross-ratio is invariant under any conveyor. When the M_j are not in this order, it becomes necessary to take the absolute value of the expression. □

Proposition 21.4.3. *In the* P*-plane, the distance between the* P*-points* M_1, M_2 *is*

$$|\ln \text{cross\_ratio}\ (M_1, M_2, M_3, M_4)| \tag{21.5}$$

where $M_3, M_4 \in \Gamma$ *and the* M_j *are* P*-aligned.*

Proof. Let $C, D \simeq \gamma : 1 : \gamma^{-1}, \delta : 1 : \delta^{-1} \in \Gamma$. They define the circle $\gamma \simeq \left[\dfrac{-2}{\gamma+\delta}, 1, \dfrac{-2\,\delta\,\gamma}{\delta+\gamma}, 1\right]$. Therefore the points A_P, B_P on $\Delta_P\,(C,D)$ can be written as

$$A_P, B_P \simeq \begin{pmatrix} \dfrac{2\,\gamma\,\delta}{\delta+\gamma} \\ 1 \\ \dfrac{2}{\delta+\gamma} \end{pmatrix} + \frac{i\,(\delta-\gamma)}{\delta+\gamma}\begin{pmatrix} \alpha \\ 0 \\ \dfrac{1}{\alpha} \end{pmatrix}, \text{ etc}$$

where α, β are some turns. And a straightforward computation leads to:

$$\text{cross\_ratio}\ (A_K, B_K, C, D) = \left(\frac{(i\alpha+\gamma)\,(i\beta-\delta)}{(i\beta+\gamma)\,(i\alpha-\delta)}\right)^2 = \text{cross\_ratio}\ (A_P, B_P, C, D)^2$$ □

Proposition 21.4.4. *With the same hypotheses, we have*

$$d_P\,(M_1, M_2) = \left|\ln \frac{d_{\mathbb{C}}\,(M_1, M_3) \times d_{\mathbb{C}}\,(M_2, M_4)}{d_{\mathbb{C}}\,(M_1, M_4) \times d_{\mathbb{C}}\,(M_2, M_3)}\right|$$

Proof. Continue using the same method as above. Rem: method used in geogebra. □

Theorem 21.4.5. *There are algebraic expressions for the* cosh *of the hyperbolic distances:*

$$\begin{aligned} \cosh\left(d_P\,(A_P, B_P)\right) &= 1 + \frac{2\,\delta_{AB}^2}{\mathfrak{p}_A\,\mathfrak{p}_B} \\ \cosh\left(d_K\,(A_K, B_K)\right) &= \frac{\delta_{AB}^2 - \mathfrak{p}_A - \mathfrak{p}_B}{2\sqrt{\mathfrak{p}_A}\sqrt{\mathfrak{p}_B}} = \frac{\left(t_2\,t_1 - \frac{1}{2}\,(z_1\,\zeta_2 + z_2\,\zeta_1)\right)}{\sqrt{t_2^2 - z_2\,\zeta_2}\sqrt{t_1^2 - \zeta_1\,z_1}} \end{aligned} \tag{21.6}$$

Proof. For the P formula: obtain M_3, M_4 from $\Delta_P \cap \Gamma$. This involves the radical

$$W = \sqrt{(t_1\,t_2 - \zeta_1\,z_2)\,(t_1\,t_2 - z_1\,\zeta_2)\,(t_1\,z_2 - t_2\,z_1)\,(t_1\,\zeta_2 - \zeta_1\,t_2)}$$

Then substitute into (21.5) and take the cosh. For the K formula, the same method can be used, but using $\mathfrak{P}$to$\mathfrak{K}$ is another possibility. □

Corollary 21.4.6. *In the Poincaré model, we have the additional formula:*

$$\cosh^2\left(d_P\left(A_P, B_P\right)/2\right) = \frac{(t_2\,t_1 - z_1\,\zeta_2)\,(t_2\,t_1 - \zeta_1\,z_2)}{(t_1^2 - z_1\,\zeta_1)\,(t_2^2 - z_2\,\zeta_2)} = 1 + \frac{\delta_{AB}^2}{\mathfrak{p}_A\,\mathfrak{p}_B}$$

$$\sinh^2\left(d_P\left(A_P, B_P\right)/2\right) = \frac{(\zeta_2\,t_1 - \zeta_1\,t_2)\,(z_2\,t_1 - z_1\,t_2)}{(t_1^2 - z_1\,\zeta_1)\,(t_2^2 - z_2\,\zeta_2)} = \frac{\delta_{AB}^2}{\mathfrak{p}_A\,\mathfrak{p}_B}$$

$$\tanh\left(d_P\left(A_P, B_P\right)/2\right) = \frac{|z_1 - z_2|}{|1 - z_1\,\zeta_2|} = \frac{\delta_{AB}}{\sqrt{\delta_{AB}^2 + \mathfrak{p}_A\,\mathfrak{p}_B}}$$

Remark 21.4.7. Another way to write (21.6) is

$$\cosh\left(d_K\left(M, N\right)\right) = \frac{1 - \langle M \mid N\rangle}{\sqrt{(1 - \langle M \mid M\rangle)\,(1 - \langle N \mid N\rangle)}}$$

Proposition 21.4.8. *Hyperbolic Pythagoras Theorem. When a, b are the right-angled sides and c is the third side, one has*

$$\cosh c = \cosh a\,\cosh b$$

Proof. Consider the triangle $0, x, iy$ and use either the Klein or the Poincaré formulas. □

Proposition 21.4.9. *The $\mathbb{C}$-homology whose center A_K and axis Δ are Γ-polar of each other describes the K-symmetry wrt $A_K \simeq k : 1 : \kappa$ when this point is inside the Klein disk $\mathbb{H}$ but describes the K-symmetry wrt Δ when A_K is in the outer world (and Δ_K is a real K-line).*

Proof. The matrix $\boxed{\mathfrak{s}_A}$ of this homology is given at (21.2). We have:

$$\cosh^2\left(d_K\left(A_K, M_j\right)\right) = \frac{\left(\kappa\,\mathbf{Z} + k\overline{\mathcal{Z}} - 2\,\mathbf{T}\right)^2}{4\left(\mathbf{T}^2 - \mathbf{Z}\overline{\mathcal{Z}}\right)(1 - \kappa\,k)}$$

$$\cosh^2\left(d_K\left(U_K, M_j\right)\right) = 1 + \frac{\left(\kappa\,\mathbf{Z} + k\overline{\mathcal{Z}} - 2\,\mathbf{T}\right)^2}{4\left(\mathbf{T}^2 - \mathbf{Z}\overline{\mathcal{Z}}\right)(\kappa\,k - 1)}$$

where $M_2 \doteq \boxed{\mathfrak{s}_A} \cdot M_1$ and $U_K = \Delta \cap M_1 M_2$. □

Proposition 21.4.10. *In the P-plane, the previous proposition becomes:*

$$M_2, M_2' \simeq \left(\begin{array}{c} \dfrac{\left(t^2 + \zeta\,z\right)\mathbf{Z} - (2\,t\,z)\,\mathbf{T}}{(2\,t\,\zeta)\,\mathbf{Z} - \left(t^2 + \zeta\,z\right)\mathbf{T}} \\ 1 \\ \dfrac{\left(t^2 + \zeta\,z\right)\overline{\mathcal{Z}} - (2\,t\,\zeta)\,\mathbf{T}}{(2\,t\,z)\,\overline{\mathcal{Z}} - \left(t^2 + \zeta\,z\right)\mathbf{T}} \end{array}\right), \left(\begin{array}{c} \dfrac{(2\,t\,z)\,\overline{\mathcal{Z}} - \left(t^2 + \zeta\,z\right)\mathbf{T}}{\left(t^2 + \zeta\,z\right)\overline{\mathcal{Z}} - (2\,t\,\zeta)\,\mathbf{T}} \\ 1 \\ \dfrac{(2\,t\,\zeta)\,\mathbf{Z} - \left(t^2 + \zeta\,z\right)\mathbf{T}}{\left(t^2 + \zeta\,z\right)\mathbf{Z} - (2\,t\,z)\,\mathbf{T}} \end{array}\right)$$

where M_2 is the P-symmetric of M_1 wrt $A_P \simeq z : t : \zeta$ when $A_P \in \mathbb{H}$, while M_3 is the P-symmetric of M_1 wrt the P-line which is $\mathbb{C}$-centered at A_P when $A_P \notin \mathbb{H}$.

Proof. Substitute $k : 1 : \kappa$ with $\mathfrak{P}$to$\mathfrak{K}\,(z : t : \zeta)$, etc, compute and then goes back using $\mathfrak{K}$to$\mathfrak{P}$. Remark: M_2, M_2' are $\mathbb{C}$-inverse wrt Γ. □

Exercise 21.4.11. When the radius ρ varies, the K-circles $\gamma_K\left(\mathfrak{A}, \rho\right)$ form a family of $\mathbb{C}$-ellipses. Find the locus of their $\mathbb{C}$-focuses

Exercise 21.4.12. Consider the P-points A, B. Determine the parameter R corresponding to the K-circles $(\mathfrak{A}, \mathfrak{B})$.

21.5 Hyperbolic triangle

(list of results)

21.5.1 Sideline

1. The K-line is equal to the $\mathbb{C}$-line, while the P-line is $\mathcal{C}\left(B, \widehat{B}, C, \widehat{C}\right)$
2. The point $O_{BC} \doteq \text{Polar}\left(klinA, horz\right) = \text{Center}\left(plinA\right)$ is outside of $\mathbb{H}$.

21.5.2 Line-bisectors

1. The P-formula are:

$$\underset{c}{\mathcal{V}}\left(\mu_P\right) \simeq \mathfrak{p}_B \underset{c}{\mathcal{V}}\left(C_P\right) - \mathfrak{p}_C \underset{c}{\mathcal{V}}\left(B_P\right)$$
$$\omega = \left(\mathfrak{p}_2 \frac{z_3}{t_3} - \mathfrak{p}_3 \frac{z_2}{t_2}\right) / \left(\mathfrak{p}_2 - \mathfrak{p}_3\right) \ ; \ \rho^2 = \omega\,\overline{\omega} - 1$$

2. The K-formula are (where lines are normalized using $\Delta_2 = 1$)

$$med_A \mathrel{\dot{\simeq}} \sqrt{\mathfrak{p}_C}\,\text{Polar}\left(B\right) - \sqrt{\mathfrak{p}_B}\text{Polar}\left(C\right)$$

3. The three $\mathbb{H}$-line-bisectors are ever collinear. As a result, the K-line-bisectors are ever going through some point $O_K \in \mathbb{C}$. When this point belongs to $\mathbb{H}$, there exists an $\mathbb{H}$-circle going through A, B, C (and centered at $O_{\mathbb{H}}$). Otherwise, $O_{\mathbb{H}}$ is the pole of an $\mathbb{H}$-line and this line is the common perpendicular to the three line-bisectors.

Exercise 21.5.1. Let B, C be given in $\mathbb{H}$. Find all the $A \in \mathbb{H}$ such that the $\mathbb{H}$-circle $\left(A, B, C\right)$ exist. Hint: use the K-model together with $B, C = \pm k$; $A = x + iy$ and obtain the horicycles:

$$\left(k^2y^2 + k^2 - x^2 - 2\,y^2\right) \pm 2\,y\left(k^2 - 1\right)$$

as boundaries. One obtains also

$$\left(2\,k^2y^2 - k^2 + x^2 - y^2\right) \pm 2\,ixy\left(k^2 - 1\right)$$

Any opinion on this extra-locus ?

21.5.3 Medians

Definition 21.5.2. Cut $\text{med}_P\left(B, C\right)$ with $\Delta_P\left(B, C\right)$ and obtain G_A. Then $\Delta_P\left(A, G_A\right)$ is called the Amedian.

Proposition 21.5.3. *The three medians of a triangle form a pencil.*

Proof. Computations are straightforward. □

21.5.4 Altitudes

1. $paltA = \mathcal{C}\left(A, \widehat{A}, \text{Reflect}\left(A, plinA\right)\right)$
2. $kaltA = \Delta\left(A, O_{BC}\right)$
3. the three $\mathbb{C}$-lines $kaltX$ are concurrent (inside $\mathbb{H}$ or outside !)
4. The three $\mathbb{H}$-altitudes are ever collinear. As a result, the K-altitudes are ever going through some point $H_K \in \mathbb{C}$. When this point belongs to $\mathbb{H}$, the triangle admits an $\mathbb{H}$-orthocenter. Otherwise, $H_{\mathbb{H}}$ is the pole of an $\mathbb{H}$-line and this line is the common perpendicular to the three altitudes.

Exercise 21.5.4. Let B, C be given in $\mathbb{H}$. Find all the $A \in \mathbb{H}$ such that $\left(A, B, C\right)$ admits an $\mathbb{H}$-orthocenter. Hint: use the K-model together with $B, C = \pm k$; $A = x + iy$ and obtain

$$\left(k^2y^2 + k^2 - x^2 - 2\,y^2\right) \pm 2\,y\left(k^2 - 1\right)$$

as boundary.

21.5.5 Trigonometry

Proposition 21.5.5. *When triangle A, B, C is rectangular in C then:*

$$\sin A = \frac{\sinh a}{\sinh c} \; ; \; \cos A = \frac{\tanh b}{\tanh c} \; ; \; \tan A = \frac{\tanh a}{\sinh b}$$

$$\cosh(\mathrm{b}) = \frac{\cos B}{\sin A}, \cosh(\mathrm{c}) = \cosh(\mathrm{a})\cosh(\mathrm{b}) = \frac{\cos A \cos B}{\sin A \sin B}$$

Proof. Consider the standard rectangular triangle $z_A = k, z_B = iK, z_C = 0$ and use the usual formula for the angle between two circles. □

21.6 The Upper-half plane

Taking a point P on Γ and reflecting everything from the P-plane wrt the circle $\gamma_C(P, 2)$ leads to another model of the hyperbolic plane, where U-circles are C-circles and U-lines are half-C-circles, ending at the horizon line.

Exercise 21.6.1 (Soland's porism)**.** (2018)Take n points a_j on circle Γ, with the intent to construct a sequence of n circles, sequentially tangent to each other and internally tangent to Γ. When n is odd, there is exactly one solution. When n is even, the n-th point is determined by the others. And then, one of the radiuses can be chosen at will.

Hint: at Figure 21.2, P is taken at a_0 and cirP, the inversion circle, is tangent to Γ. This leads to $a_j \mapsto A_j$. Find a relation between $d_{12} = d_C(A_1, A_2)$ and r_1, r_2, the C-radiuses of the C-circles O_1, O_2.

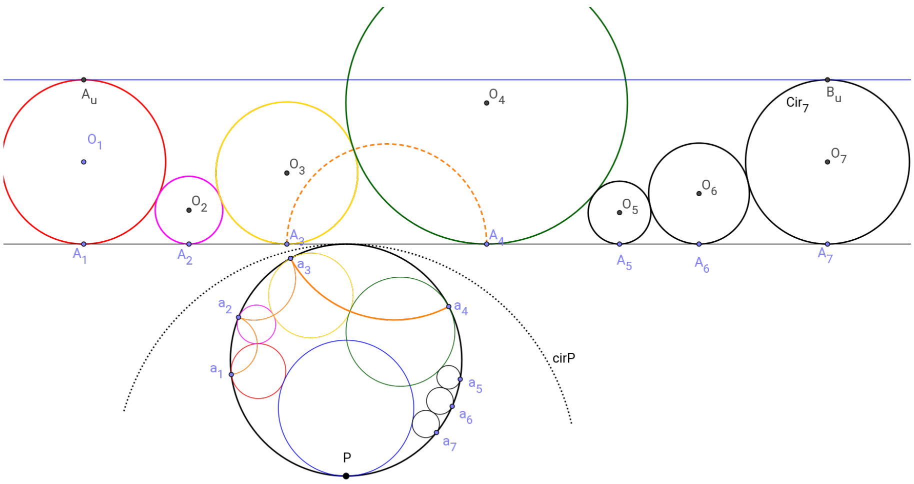


Figure 21.2: The Poincaré to Upper-half transform

21.7 Teaching tensors to a computer

Informally, tensors are friendly multi-dimensional arrays implementing the "two" Einstein's rules:
(0_1) in X_j^k, there aren't exponents, but only indices, the j being "down" while the k is "up"
(0_2) indices are implying ranges and variables
(1): repeated index (one up, one down): X_{jk}^k means $(j) \mapsto \sum_{k \in \mathrm{range}(k)} X_{jk}^k$
(2): comma: $X_{j,k}$ means $(j, k) \mapsto \partial X_j / \partial x_k$ (assuming that index k implies variable x)

Notation 21.7.1. In the end, we will use four set of variables, two external and two internal. Using an index will imply which variable is indexed. For example, $\mathfrak{N}_{,\nu}^m$ (where $\mathfrak{N}$ means Jacobian) is to be read as $\frac{\partial x^m}{\partial u^\nu}$ while $\mathfrak{N}_{,\sigma}^\nu$ is to be read as $\frac{\partial u^\nu}{\partial t^\sigma}$. Everything will be introduced in details but, for the reader's convenience, these associations are summarized here, in Table 21.1.

index	range	tag	variables	comment	here
m, n, p, q, r	1..3	x	x, y, z	external	cartesian
a, b, c, d, e	1..3	w	u, v, w	extended	
$\mu, \nu, \phi, \psi, \rho$	1..2	u	u, v	1° internal	stereographic
$\sigma, \tau, \epsilon, \omega, \kappa$	1..2	t	t, s	2° internal	spheric

Table 21.1: Indices and the associated variables

Definition 21.7.2. We **define** a tensor as a triple `[index_list,updo_list,Maple_rtable]`. The `index_list` is a Maple list of unassigned names, where repetitions are allowed. The `updo_list` is a list of booleans, coded 1 for "up" and 0 for "down". When a name is repeated in the `index_list`, only two occurrences are allowed, one being tagged as "up" and the other as "down".

Remark 21.7.3. Another method would have been to build a stratospheric theory using the marvelous operator $\otimes$ (read it as otimes). But we need something else, i.e. a practical computing tool. Using Mathematica, Sage or any other formal computing tool instead of Maple is probably possible... mind the details, they are where the devil lives.

Fact 21.7.4. *When X is a Maple_rtable, the command* `ArrDim`*, defined by*

```
macro(ArrDim=ArrayTools[Dimensions])
```

returns a list of Maple `ranges`, i.e. something like `1..3,1..2,1..4` and then `X[tut1,tut2,tut3]` returns some value, assuming that variables `tut1,tut2,tut3` contain integer numbers inside the right ranges. Here, the `tutj` are freshly build Maple variables, and not the names given in the `index_list`.

Maple 21.7.5. In order to implement Einstein's (1) or (2), the naming conventions must be explicitly stated. When assuming Table 21.1 the following procedure will inform the computer of our choices:

```
SETVARS := proc (var) ; global glodex
if member(var, [m, n, p, q, r])  then glodex := [m, n, p, q, r] ; return [x, y, z]
else if member(var, [a, b, c, d, e])  then glodex := [a, b, c, d, e] ; return [u, v, w]
else if member(var, [μ, ν, φ, ψ, ρ])  then glodex := [μ, ν, φ, ψ, ρ] ; return [u, v]
else  glodex := [σ, τ, ϵ, ω, κ] return [s, t]
end if
```

Listing 21.1: Procedure **setvars** tells our naming conventions to the computer

Maple 21.7.6. Applying some process to each element of the internal array is done by procedure Alg. 21.2. As examples, `action` can be `factor` or `U-> simplify(U,symbolic)`.

```
ATENS := proc (qui, action) ; local tt1, tt2, tt3
tt1, tt2, tt3 := op(qui)
return [tt1, tt2, map(action, tt3)]
```

Listing 21.2: The atens procedure (A means Action)

Maple 21.7.7. While the reader is supposed to decipher $X^j_{,\mu}$ as $\partial X^j / \partial u^\mu$, this has to be explained to a computer by procedure Alg. 21.3.

Maple 21.7.8. While the reader is supposed to decipher $a^m b^n$ as another tensor build using $(m, n) \mapsto a^m \times b^n$, this has to be explained to a computer by procedure Alg. 21.4.

```
Require: qui depends on the var indexed variables
DTENS := proc (qui, var, updo)
local laproc, tt1, tt2, tt3, nn, lesdex, ttq, vars
tt1, tt2, tt3 := op(qui) ; vars := setvars(var)
nn := nops(tt1) ; lesdex := seq(Catenate(tut, j), j = 1..nn)
(lesdexx, ttq) ↦ diff(tt3[lesdexx], vars[ttq])
laproc := subs(lesdexx = lesdex, %)
tt1 := [op(tt1), var] ; tt2 := [op(tt2), updo]
return [tt1, tt2, rtable(op(ArrDim(tt3)), 1..nops(vars), eval(laproc))]
```

LISTING 21.3: The dtens procedure (D means Derivation)

```
1: CTENS := proc (v1, v2)
2: local ss1, ss2, ss3, tt1, tt2, tt3, tmp, lesdex1, lesdex2, lesdex, laproc
3: ss1, ss2, ss3 := op(v1) ; tt1, tt2, tt3 := op(v2)
4: lesdex1 := seq(Catenate(tut, j), j = 1..nops(ss1))
5: lesdex2 := seq(Catenate(tvt, j), j = 1..nops(tt1))
6: lesdex := lesdex1, lesdex2
7: lesdexx ↦ ss3[lesdexx1] * tt3[lesdexx2]
8: laproc := subs(lesdexx = lesdex, lesdexx1 = lesdex1, lesdexx2 = lesdex2, %)
9: tmp := [op(ss1), op(tt1)], [op(ss2), op(tt2)]
10: return [tmp, rtable(op(ArrDim(ss3)), op(ArrDim(tt3)), eval(laproc))]
Ensure: Some tests are done on indices: repetition requires one index up, one down.
```

——

The "arrow" procedure defined at line 7 must go through the cryptic substitutions of line 8 in order to be accepted by the Maple constructor `rtable` at line 10.
Accept this state of affairs... or rewrite the whole `array` package !

LISTING 21.4: The ctens procedure (C means Catenate)

Maple 21.7.9. While the reader is supposed to decipher $a_{n\mu}^{mn}$ as a two indices tensor, build using $(m,\mu) \mapsto \sum_n a_{n\mu}^{mn}$, this new tensor has to be explicitly created by procedure ALG. 21.5.

```
RTENS := proc (qui, vas) ; global tt3, laproc
local tt1, tt2, ou1, ou2, lemu, lesdex, lesdey, lesdim, lestt2, lestt1
tt1, tt2, tt3 := op(qui)
member(vas, tt1, ou1) ; member(vas, subsop(ou1 = KKK, tt1), ou2)
lesdex := [seq](Catenate(tut, j), j = 1..nops(tt1))
lesdey := (op@subsop)(ou1 = NULL, ou2 = NULL, lesdex)
lesdim := (op@subsop)(ou1 = NULL, ou2 = NULL, ArrDim(tt3))
lesdex := (op@subsop)(ou1 = vas, ou2 = vas, lesdex)
lestt1 := subsop(ou1 = NULL, ou2 = NULL, tt1)
lestt2 := subsop(ou1 = NULL, ou2 = NULL, tt2)
lemu := ArrDim(tt3)[ou1] ; lesdeyy ↦ add(tt3[lesdexx], vas = lemuu)
laproc := subs(lesdexx = lesdex, lesdeyy = lesdey, lemuu = lemu, %)
if nops(lestt2) = 0 then return eval(laproc)()
return [lestt1, lestt2, rtable(lesdim, eval(laproc))]
```

LISTING 21.5: The rtens procedure (R means Reduce)

Example 21.7.10. $x_{,\mu}^{m}\,\mathrm{d}u^{\mu}$ means $\left[\left(\frac{\partial x_j}{\partial u}\,\mathrm{d}u + \frac{\partial x_j}{\partial v}\,\mathrm{d}v\right), j = 1..3\right]$. In our formalism, a sequence DCR is used to construct this 1-index tensor. From a complexity point of view, this is far from being optimal. But, in what we are doing, the underlying arrays are not sufficiently large to require a careful optimization.

Maple 21.7.11. Adding two tensors require that indices lists are exactly the same (and the up/down list also). Reordererering of indices is done by 21.6, while the addition itself is done by 21.7.

```
XTENS := proc(qui, org) ; global tt3_
local tt1, tt2, ss2, frum, vers, lafun_, udpo, dims
tt1, tt2, tt3_ := op(qui)
if convert(org, set) <> convert(tt1, set) then Error("wrong set of indices")
frum := cat(`@`(op, map2)(sprintf, "%a,", org))[1.. − 2]
vers := cat(`@`(op, map2)(sprintf, "%a,", tt1))[1.. − 2]
cat("(", frum, ") − > tt3_(", vers, ")") ; lafun_ := parse(Ditto1())
udpo := table([seq](tt1[j] = tt2[j], j = 1..nops(tt1))) ; ss2 := [seq](udpo[j], j = org)
dims := table([seq](tt1[j] = ArrDim(tt3_)[j], j = 1..nops(tt1)))
return [org, ss2, eval(rtable(seq(dims[j], j = org), lafun_))]
```

Listing 21.6: The xtens procedure (X means Xcross)

```
ADDTENS2 := proc(trr, tss, laproc := factor, {mul1 := 1, mul2 := 1})
local tt1, tt2, tt3, ss1, ss2, ss3
tt1, tt2, tt3 := op(trr) ; ss1, ss2, ss3 := op(tss)
if tt1 <> ss1 or tt2 <> ss2  then error ("indices doesn't match")
return [tt1, tt2, map(`@`(eval, laproc), mul1 ∗ tt3 + mul2 ∗ ss3)]
```

Listing 21.7: The addtens2 procedure

21.8 The sphere: dealing with an example

21.8.1 External and internal coordinates

Fact 21.8.1. *When describing a "surface", the simplest way to proceed is embedding the surface into some larger space, leading to something like*

$$(x, y, z) \in (E) \text{ means } x^2 + y^2 + z^2 - 1 = 0$$

The first step towards a more intrinsic way of doing is to describe these external coordinates from "the real world" i.e. from the surface itself, as if there was nothing outside.

Example 21.8.2. One can check that equations $[u, v, w] = W$ and $[x, y, z] = \widehat{X}$ where :

$$W^a \underset{x}{=} \left[[a], [1], \left[\frac{x}{1+z}, \frac{y}{1+z}, \frac{x^2+y^2+z^2-1}{1+z} \right] \right]$$

$$\widehat{X}^m \underset{w}{=} \left[[m], [1], \left[\frac{u\,(w+2)}{1+u^2+v^2}, \frac{v\,(w+2)}{1+u^2+v^2}, \frac{1+w-u^2-v^2}{1+u^2+v^2} \right] \right]$$

describes a pair of bijections $(x, y, z) \mapsto (u, v, w) \mapsto (x, y, z)$. They are differentiable quite everywhere. When making $w = 0$, we are selecting the points of the sphere $x^2 + y^2 + z^2 = 1$.

Let us define U and X by:

$$U^\mu \underset{x}{=} \left[[\mu], [1], \left[\frac{x}{1+z}, \frac{y}{1+z} \right] \right]$$

$$X^m \underset{u}{=} \left[[m], [1], \left[\frac{2\,u}{u^2+v^2+1}, \frac{2\,v}{u^2+v^2+1}, \frac{1-u^2-v^2}{1+u^2+v^2} \right] \right]$$

Then the pairs $U = (u, w)$ are a system of internal coordinates for the sphere, while the triples (x, y, z) are a system of external coordinates for the same surface, bound by the so called implicit equation of the sphere.

We will also introduce another set of internal coordinates $T^\sigma = [t, s]$ by equations

$$T^\sigma \underset{u}{=} \left[[\sigma], [1], \left[\arcsin\left(\frac{1-u^2-v^2}{1+u^2+v^2}\right), \arctan\left(\frac{v}{1+u^2+v^2}, \frac{u}{1+u^2+v^2}\right)\right]\right]$$
$$\underset{x}{=} [[\sigma], [1], [\arcsin(z), \arctan(y, x)]]$$

and their converses:

$$U^\mu \underset{s}{=} \left[[\mu], [1], \left[\frac{\cos(s)\cos(t)}{1+\sin(t)}, \frac{\sin(s)\cos(t)}{1+\sin(t)}\right]\right]$$

$$X^m = [[m], [1], [\cos s\, \cos t,\, \sin s\, \cos t,\, \sin t]]$$

21.8.2 Jacobians

Notation 21.8.3. More than ever, notations of Table 21.1 are used.

Fact 21.8.4. *The external description of the tangent plane at M is*

$$\frac{\partial\left(x^2+y^2+z^2-1\right)}{\partial(x; y; z)} \cdot \overrightarrow{[dx, dy, dz]} = 0$$

At a regular point, i.e. almost everywhere, the $[\,\mathrm{d}x,\ \mathrm{d}y,\ \mathrm{d}z]$ *variations of the external coordinates* $[x, y, z]$ *are linearly correlated with the variations of any other set of coordinates. In fact, this is the very existence of such an invertible transform which decides if changing from a set of coordinates to another set is allowed or not. The matrices expressing these transformations are called the Jacobians and noted* $\mathfrak{N}^\mu_{,m}$ *(where* $\mathfrak{N}$ *indicates Jacobian,* μ *indicate new=*u, v *and* m *indicates old=*x, y, z*. In other words:* $\mathrm{d}^\mu = \mathfrak{N}^\mu_{,m}\, \mathrm{d}^m$.

21.8.2.1 Internal versus another internal

Proposition 21.8.5. *The* 2×2 *tensors* $\mathfrak{N}^\mu_{,\sigma} \mathbf{1}^\nu_\mu \mathfrak{N}^\tau_{,\nu}$ *and* $\mathfrak{N}^\mu_{,\sigma} \mathbf{1}^\sigma_\tau \mathfrak{N}^\tau_{,\nu}$ *are respectively equal to and* $\mathbf{1}^\tau_\sigma$ *and* $\mathbf{1}^\mu_\nu$.

Proof. Both tensors describe respectively $[\,\mathrm{d}s,\ \mathrm{d}t]$ and $[\,\mathrm{d}u,\ \mathrm{d}v]$ wrt themselves, so that obvious is obvious. But, in order to check the procedures given above, let us compute explicitly the Jacobians of (u, v) versus (s, t) and conversely. And then rewrite them using the other set of variables. This leads to: □

$$\frac{\partial(u, v)}{\partial(s, t)} \equiv \mathfrak{N}^\mu_{,\sigma} \underset{t}{=} \left[[\mu, \sigma], [1, 0], \begin{bmatrix} \dfrac{-\cos(s)}{1+\sin(t)} & \dfrac{-\cos(t)\sin(s)}{1+\sin(t)} \\ \dfrac{-\sin(s)}{1+\sin(t)} & \dfrac{\cos(t)\cos(s)}{1+\sin(t)} \end{bmatrix}\right]$$
$$\underset{u}{=} \left[[\mu, \sigma], [1, 0], \begin{bmatrix} \dfrac{-u\left(1+u^2+v^2\right)}{2\sqrt{u^2+v^2}} & -v \\ \dfrac{-v\left(1+u^2+v^2\right)}{2\sqrt{u^2+v^2}} & +u \end{bmatrix}\right]$$
$$\underset{x}{=} \left[[\mu, \sigma], [1, 0], \begin{bmatrix} \dfrac{-x}{(1+z)\sqrt{1-z^2}} & \dfrac{-y}{1+z} \\ \dfrac{-y}{(1+z)\sqrt{1-z^2}} & \dfrac{+x}{1+z} \end{bmatrix}\right]$$

$$\frac{\partial(s,t)}{\partial(u,v)} \equiv \mathfrak{N}^{\sigma}_{,\mu} \underset{u}{=} \left[[\sigma,\mu],[1,0],\left[\begin{array}{cc} \frac{-2\,u}{(1+u^2+v^2)\sqrt{u^2+v^2}} & \frac{-2\,v}{(1+u^2+v^2)\sqrt{u^2+v^2}} \\ \frac{-v}{u^2+v^2} & \frac{+u}{u^2+v^2} \end{array}\right]\right]$$

$$\underset{t}{=} \left[[\tau,\nu],[1,0],\left[\begin{array}{cc} -\cos s\,(1+\sin t) & -\sin s\,(1+\sin t) \\ \frac{-\sin s\,(1+\sin t)}{\cos t} & \frac{\cos s\,(1+\sin t)}{\cos t} \end{array}\right]\right]$$

$$\underset{x}{=} \left[[\sigma,\mu],[1,0],\left[\begin{array}{cc} \frac{-x\,(1+z)}{\sqrt{1-z^2}} & \frac{-y\,(1+z)}{\sqrt{1-z^2}} \\ \frac{-y}{1-z} & \frac{x}{1-z} \end{array}\right]\right]$$

Therefore, $\mathfrak{N}^{\mu}_{,\sigma}\mathbf{1}^{\nu}_{\mu}\mathfrak{N}^{\tau}_{,\nu}$ can be computed using any of u,t,x as algebraic basis, leading to

```
seq(rtens(ctens(
   Jac(sigma,mu,X),Jac(nu,tau,X),kron(mu,nu)), mu, nu), X=[u,t,x]);
```

The same remarks apply to

```
seq(rtens(ctens(
   Jac(sigma,mu,X),Jac(nu,tau,X),kron(tau,sigma)), sigma,tau), X=[u,t,x]);
```

21.8.2.2 internal versus external

Exercise 21.8.6. Show that the (u,v) versus (x,y,z) Jacobians are respectively:

$$\mathfrak{N}^{\mu}_{,m} \underset{x}{=} \left[[\mu,m],[1,0],\left[\begin{array}{ccc} \frac{1}{1+z} & 0 & \frac{-x}{(1+z)^2} \\ 0 & \frac{1}{1+z} & \frac{-y}{(1+z)^2} \end{array}\right]\right]$$

$$\underset{u}{=} \left[[\mu,m],[1,0],\frac{1}{2}\left(1+u^2+v^2\right)\left[\begin{array}{ccc} 1 & 0 & -u \\ 0 & 1 & -v \end{array}\right]\right]$$

$$\mathfrak{N}^{m}_{,\mu} \underset{u}{=} \left[[m,\mu],[1,0],\frac{2}{(1+u^2+v^2)^2}\left[\begin{array}{cc} 1-u^2+v^2 & -2\,uv \\ -2\,uv & 1+u^2-v^2 \\ -2\,u & -2v \end{array}\right]\right]$$

$$\underset{x}{=} \left[[n,\nu],[1,0],\left[\begin{array}{cc} 1-x^2+z & -yx \\ -yx & 1-y^2+z \\ -x\,(1+z) & -y\,(1+z) \end{array}\right]\right]$$

Moreover, compute $\mathfrak{N}^{\mu}_{,m}\mathbf{1}^{m}_{n}\mathfrak{N}^{n}_{,\nu}$ and $\mathfrak{N}^{\mu}_{,m}\mathbf{1}^{\nu}_{\mu}\mathfrak{N}^{n}_{,\nu}$.

Exercise 21.8.7. Show that the (t,s) versus (x,y,z) Jacobians are respectively:

$$\mathfrak{N}^{\sigma}_{,m} \underset{x}{=} \left[[\sigma,m],[1,0],\left[\begin{array}{ccc} 0 & 0 & \frac{1}{\sqrt{1-z^2}} \\ \frac{-y}{1-z^2} & \frac{+x}{1-z^2} & 0 \end{array}\right]\right]$$

$$\underset{t}{=} \left[[\sigma,m],[1,0],\left[\begin{array}{ccc} 0 & 0 & \frac{1}{\cos t} \\ -2\,\sin s & +2\,\cos s & -2\,\cos s \end{array}\right]\right]$$

$$\underset{u}{=} \left[[\sigma,m],[1,0],\frac{1+u^2+v^2}{2\,(u^2+v^2)}\left[\begin{array}{ccc} 0 & 0 & \sqrt{u^2+v^2} \\ -v & +u & 0 \end{array}\right]\right]$$

$$\mathfrak{N}^m_{,\sigma} \underset{t}{=} \left[[m,\sigma], [1,0], \left[\begin{array}{cc} -\cos s \sin t & -\sin s \cos t \\ -\sin s \sin t & \cos s \cos t \\ \cos t & 0 \end{array} \right] \right]$$

$$\underset{x}{=} \left[[m,\sigma], [1,0], \left[\begin{array}{cc} \dfrac{-xz}{\sqrt{1-z^2}} & -y \\ \dfrac{-yz}{\sqrt{1-z^2}} & +x \\ \sqrt{1-z^2} & 0 \end{array} \right] \right]$$

$$\underset{u}{=} \left[[m,\sigma], [1,0], \left[\begin{array}{cc} \dfrac{-u\left(1-u^2-v^2\right)}{\sqrt{u^2+v^2}\,(1+u^2+v^2)} & \dfrac{-2\,v}{1+u^2+v^2} \\ \dfrac{-v\left(1-u^2-v^2\right)}{\sqrt{u^2+v^2}\,(u^2+v^2+1)} & \dfrac{+2\,u}{1+u^2+v^2} \\ \dfrac{2\sqrt{u^2+v^2}}{1+u^2+v^2} & 0 \end{array} \right] \right]$$

Moreover, compute $\mathfrak{N}^\sigma_{,m}\mathbf{1}^m_n\mathfrak{N}^n_{,\tau}$ and $\mathfrak{N}^\sigma_{,m}\mathbf{1}^\tau_\sigma\mathfrak{N}^n_{,\tau}$.

21.8.3 More about the projectors

Proposition 21.8.8. *The 2×2 tensor $\mathfrak{N}^\sigma_{,m}\mathbf{1}^m_n\mathfrak{N}^n_{,\tau}$ equals $\mathbf{1}^\sigma_\tau$, while the 3×3 tensor $\mathfrak{N}^\sigma_{,m}\mathbf{1}^\tau_\sigma\mathfrak{N}^n_{,\tau}$ describes a projector onto the tangent plane.*

Proof. The first tensor describes $[\,\mathrm{d}t,\, \mathrm{d}s]$ wrt itself and therefore is the Kronecker tensor. The second assertion comes from the fact that the normal vector necessarily belongs to orthogonal of colspan $\left(\mathfrak{N}^n_{,\sigma}\right)$. This can be checked using

```
pt_x, pt_u, pt_t:= seq(rtens(ctens(
   Jac(m,sigma,X),Jac(tau,n,X),kron(sigma,tau)), sigma,tau)[3], X=[x,u,t]):
```

which produces a matrix having $X\,(X-1)^2$ as characteristic polynomial:

$$U_t \underset{x}{=} \left[[m,n], [1,0], \frac{1}{x^2+y^2} \left[\begin{array}{ccc} +y^2 & -yx & -xz \\ -yx & +x^2 & -yz \\ 0 & 0 & x^2+y^2 \end{array} \right] \right] \underset{u}{=} \frac{1}{u^2+v^2} \left[\begin{array}{ccc} v^2 & -uv & \frac{u}{2}\left(u^2+v^2-1\right) \\ -uv & u^2 & \frac{v}{2}\left(u^2+v^2-1\right) \\ 0 & 0 & 1 \end{array} \right]$$

And now, we can consider matrices P, Q where:

$$P = \left[\begin{array}{c|c} x & \\ y & \mathfrak{N}^m_{,\sigma} \\ 0 & \end{array} \right] = \left[\begin{array}{ccc} x & -zx & -y \\ y & -yz & +x \\ 0 & 1-z^2 & 0 \end{array} \right] \cdot \operatorname{diag}\left(1, \frac{1}{W}, 1\right)$$

$$Q = \left[\begin{array}{c} \dfrac{1}{1-z^2} \left[\begin{array}{ccc} x & y & z \end{array} \right] \\ \hline \mathfrak{N}^\sigma_{,m} \end{array} \right] = \frac{1}{1-z^2} \left[\begin{array}{ccc} x & y & z \\ 0 & 0 & W \\ -y & +x & 0 \end{array} \right]$$

Then $P.Q = \mathbf{1}$ while $Q\cdot U_t\cdot P = \operatorname{diag}(0,1,1)$. □

21.8.4 The metric tensor

Proposition 21.8.9. *For each set of variables, the* ***metric tensor*** *$\mathfrak{G}_{mn}$ is defined by*

$$\mathrm{d}s^2 = \mathrm{d}x^m\, \mathfrak{G}_{mn}\, \mathrm{d}x^n$$

This is indeed a tensor, since a change of variable results into

$$\mathfrak{G}_{\mu\nu} = \mathfrak{N}^m_\mu \mathfrak{N}^n_\nu \mathfrak{G}_{mn} \tag{21.7}$$

Proof. Substitute $\mathrm{d}x^n = \mathfrak{N}^n_\nu\, \mathrm{d}u^\nu$, etc and obtain an equality which must hold for all the $\overrightarrow{\mathrm{d}u}$ □

Example 21.8.10. Here are the values of the two metric tensors $\mathfrak{G}_{\mu\nu}$ and $\mathfrak{G}_{\sigma\tau}$.

$$\begin{aligned}\mathfrak{G}_{\mu\nu} &\underset{u}{=} \left[[\mu,\nu],[0,0],\left(\frac{2}{1+u^2+v^2}\right)^2\begin{bmatrix}1&0\\0&1\end{bmatrix}\right]\\ &\underset{x}{=} \left[[\mu,\nu],[0,0],\frac{1}{(1+z)^2}\begin{bmatrix}1&0\\0&1\end{bmatrix}\right]\end{aligned}$$

$$\begin{aligned}\mathfrak{G}_{\sigma\tau} &\underset{t}{=} \left[[\sigma,\tau],[0,0],\begin{bmatrix}1&0\\0&\cos t^2\end{bmatrix}\right]\\ &\underset{u}{=} \left[[\sigma,\tau],[0,0],\begin{bmatrix}1&0\\0&\left(\frac{2\left(u^2+v^2\right)}{1+u^2+v^2}\right)^2\end{bmatrix}\right]\\ &\underset{x}{=} \left[[\sigma,\tau],[0,0],\begin{bmatrix}1&0\\0&1-z^2\end{bmatrix}\right]\end{aligned}$$

21.9 Christoffel symbols

Notation 21.9.1. More than ever, notations of Table 21.1 are in use.

21.9.1 Covariance

Definition 21.9.2. God said to Abraham: "down is covariant, up is contravariant. Don't discuss any more about that".

Remark 21.9.3. Before the divine decree, some theologians were arguing that

$$\overrightarrow{v} \doteq [\overrightarrow{e_1},\overrightarrow{e_2}]\cdot\left(\mathrm{d}x^1;\,\mathrm{d}x^2\right) = \mathrm{d}x^1\overrightarrow{e_1}+\mathrm{d}x^2\overrightarrow{e_2}$$

provides a model for covariant/contravariant. If $\overrightarrow{v}$ is supposed to be a real, observable thing then the $\mathrm{d}x^j$ have to be divided by a factor 2 when the $\overrightarrow{e_j}$ are multiplied by the same factor 2. Indeed, observable doesn't means that all observers obtain the same figures for their measurements, but that each observer can compute and predict which figures another specified observer will obtain for her measurements.

But the devil is in the details. The symbol $\mathrm{d}x^1$ means the first element in the set of all the $\left(\mathrm{d}^u\right)_{u\in J}$, while d^j means all of them... i.e. another copy of $\mathrm{d}^k = \left(\mathrm{d}^u\right)_{u\in J}$.

Remark 21.9.4. In linear algebra, $\underset{\text{old}}{x} = P\cdot\underset{\text{new}}{x}$ implies $\underset{\text{new}}{x} = P^{-1}\cdot\underset{\text{old}}{x}$ where P is the change of basis matrix. And this is because $\underset{\text{new}}{\text{basis}} = P\cdot\underset{\text{old}}{\text{basis}}$. Here, in tensor calculus, you only have to write:

$$\mathfrak{G}_{\mu\nu} = \mathfrak{G}_{mn}\mathfrak{N}^m_{,\mu}\mathfrak{N}^n_{,\nu}\ ;\ \mathfrak{G}_{mn} = \mathfrak{G}_{\mu\nu}\mathfrak{N}^\mu_{,m}\mathfrak{N}^n_{,\nu}$$

and the pairing of indices tells you which is the one to use from $\mathfrak{N}^m_{,\mu}$or $\mathfrak{N}^\mu_{,m}$.

21.9.2 Taking the steepest line as an example

Definition 21.9.5. Vector $\overrightarrow{k} \doteq \overrightarrow{k_1,k_2,k_3} \doteq {}^t[k_1,k_2,k_3]$ is a "true" vector in the $\mathbb{C}^3$ space. Its tensor representation is $[[m],[1],[k_1,k_2,k_3]]$. Then the tensor:

$$\left(\wedge\overrightarrow{k}\right)^m_n \doteq \left[[m,n],[1,0],\begin{bmatrix}0&k_3&-k_2\\-k_3&0&k_1\\k_2&-k_1&0\end{bmatrix}\right]$$

implements the rule $\left(\wedge\overrightarrow{k}\right)^m_n\left(\overrightarrow{v}\right)^n = \left(\overrightarrow{v}\wedge\overrightarrow{k}\right)$.

Fact 21.9.6. *When seeing the usual sphere (E) as embedded into $\mathbb{C}^3$, a normal vector to (E) at $M = (x, y, z)$ is $\overrightarrow{\mathbf{n}} = {}^t\,(x,\ y,\ z)$ while the vertical vector is $\overrightarrow{e_z} = {}^t\,(0,\ 0,\ 1)$. Then an horizontal vector at M and a* ***North pointing vector*** *at M are respectively B^m and A^m where*

$$\begin{array}{lll} B^m & \doteq \left(\wedge \overrightarrow{k}\right)^m_n (\overrightarrow{e_z})^n & \underset{x}{=} [[m],[1],[-y,x,0]] \\ A^m & \doteq \left(\wedge \overrightarrow{k}\right)^m_n A^n & \underset{x}{=} [[m],[1],[xz,yz,-x^2-y^2]] \end{array}$$

On the other hand, the horizontal line and the ***steepest line*** *at M are respectively $\mathfrak{G}_{mn}B^n$ and $\mathfrak{G}_{mn}A^n$.*

Fact 21.9.7. *This requires that A^m and B^m aren't null, i.e. that $(x, y, z) \neq (0, 0, \pm 1)$. Moreover, these vectors belong to the tangent plane. Therefore, it makes sense to compute the "local coordinates" of these vectors using the formula already in use for the* d, *namely $\mathrm{d}^\mu \doteq \mathrm{d}^m \mathfrak{N}^\mu_{,m}$. We have:*

$$\begin{array}{llllll} B^\mu & \doteq B^m\mathfrak{N}^\mu_{,m} & = [[\mu],[1],[-v,+u]] & ;\ A^\mu & \doteq A^m\mathfrak{N}^\mu_{,m} & = [[\mu],[1],[+u,+v]] \\ B^\sigma & \doteq B^m\mathfrak{N}^\sigma_{,m} & = [[\sigma],[1],[0,1]] & ;\ A^\sigma & \doteq A^m\mathfrak{N}^\sigma_{,m} & = [[\sigma],[1],[-\cos t,0]] \end{array}$$

And we can check that $B_\sigma = B_\mu \mathfrak{N}^\mu_{,\sigma}$, etc*, as required for a covariant tensor while we can check that $A^\mu = A^\sigma \mathfrak{N}^\mu_{,\sigma}$,* etc*, as required for a contravariant tensor.*[1]

21.9.3 Computing the Christoffels

A theoretical definition of the Christoffel symbols is delayed to a later subsection. Indeed, we want to discuss how to correctly define these objects, and indicate some wrong ways of doing. Therefore, we will introduce these symbols by taking their most important property... as a provisional definition, which can be used to check every equalities given in this document.

Definition 21.9.8. Quantities $\Gamma_{\psi|\mu\nu}$ and $\Gamma^\phi_{\mu\nu}$ (where μ, ν, ϕ, ψ are internal indices) are respectively called the Christoffels of first kind and second kind. They are not tensors, but will be used to form other tensors by contraction. They are computed from the metric, using

$$\begin{array}{rcl} \Gamma_{\psi|\mu\nu} & = & \dfrac{1}{2}\left(\mathfrak{G}_{\mu\psi,\nu} + \mathfrak{G}_{\psi\nu,\mu} - \mathfrak{G}_{\nu\mu,\psi}\right) \\ \Gamma^\phi_{\mu\nu} & = & \Gamma_{\psi|\mu\nu}\mathfrak{G}^{\phi\psi} \iff \Gamma_{\psi|\mu\nu} = \mathfrak{G}_{\psi\phi}\Gamma^\phi_{\mu\nu} \end{array}$$

Maple 21.9.9. The ALG. 21.8 algorithm implements the computation of the Christoffel symbols.

Require: $\sigma, \tau, \upsilon, t$ must be compatible with ω
CHDOTENS := **proc** $(\omega,\ \sigma,\ \tau,\ t)$; **global** *glodex*, *part1_*
local *setglodex*, *lesvars*, *part1*, fu_1, gu_2, *qqq*, *qq*, *tmp1*, *tmp2*
lesvars := *setvars*(ω) ; *setglodex* := *convert*(*glodex*, *set*)
if $\{\omega, \sigma, \tau\}$ '*minus*' *setglodex* $\neq \{\}$ **then** *Error*("wrong indexes") **end if**
if *not member*(*t*, *lesvars*) **then** *Error*("*wrong variable*") **end if**
qqq:= *setglodex minus* $\{\omega, \sigma, \tau\}$; *qq* := *qqq*[*1*] – un test a lieu
*part1_* := *dtens*(*GG*(σ, τ, t), *qq*) ; fu_1 := *mkfu*(*part1_*)
$gu_2 := \omega, \sigma, \tau \mapsto eval((fu_1(\omega,\sigma,\tau) + fu_1(\omega,\tau,\sigma) - fu_1(\tau,\sigma,\omega))/2)$
tmp1 := $[[\omega,\sigma,\tau],[0,0,0], rtable(1..2, 1..2, 1..2, gu2)]$
tmp2 := *ctens*(*uGG*(qq, ω, t), *tmp1*)
return *tmp1*, *rtens*(*tmp2*, ω)
Ensure: in $\Gamma_{\omega|\sigma\tau}$ and $\Gamma^\upsilon_{\sigma\tau}$ the first index is the "derivative" index

LISTING 21.8: The chdotens procedure

[1]If you are affected by a dyslexia disorder more severe than 5%, then never ever use "contravariant" and "like" in the same sentence. Better use: A is contravariant, while dx is also contravariant. Don't create a mess like "the north end of a compass is attracted to the south magnetic pole of the earth, which lies close to the geographic north pole".

Exercise 21.9.10. Consider the usual unit sphere $x^2 + y^2 + z^2 = 1$ and introduce the usual geographic coordinates (t, s) by the equations

$$[x, y, z] = [\cos t \cos s, \cos t \sin s, \sin t],$$

where the latitude t ranges from $-\pi/2$ (aka 90° South) to $+\pi/2$ (aka 90° North). Everybody acts like that, except from a small minority of snobs. Therefore, $\cos t \geq 0$ is ever assumed. Show that the corresponding Christoffels are:

$[t, s]$	111 121	112 122	211 221	212 222
$\Gamma_{\omega\|\sigma\tau}$	0 0	0 $\cos t \sin t$	0 $-\cos t \sin t$	$-\cos t \sin t$ 0
$\Gamma^{\epsilon}_{\sigma\tau}$	0 0	0 $\cos t \sin t$	0 $-\tan t$	$-\tan t$ 0

Exercise 21.9.11. Consider now the usual stereographic coordinates (u, v) defined by:

$$[x, y, z] = \left[\frac{2\,u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{1 - u^2 - v^2}{1 + u^2 + v^2}\right]$$

and show that the corresponding Christoffels are:

$[u, v]$	K	111 121	112 122	211 221	212 222
$\Gamma_{\psi\|\mu\nu}$	$\dfrac{8}{(u^2 + v^2 + 1)^3}$	$-u$ $-v$	$-v$ $+u$	$+v$ $-u$	$-u$ $-v$
$\Gamma^{\phi}_{\mu\nu}$	$\dfrac{2}{u^2 + v^2 + 1}$	$-u$ $-v$	$-v$ $+u$	v $-u$	$-u$ $-v$

21.9.4 Defining the Christoffels

Definition 21.9.12. Consider two tensors X^m_Z, Y^m_Z where Z is a set of internal indices in the same upper or lower places for both tensors and m is an external index (here $m \in 1..3$ is assumed). We will say that X and Y are equal up to their normal components, and note

$$X^m_Z = Y^m_Z + \mathcal{O}\left(\overrightarrow{\mathfrak{n}}\right)$$

when the property

$$\overrightarrow{X^1_z, X^2_z, X^3_z} - \overrightarrow{Y^1_z, Y^2_z, Y^3_z} \in \mathbb{C}\,\overrightarrow{\mathfrak{n}} \subset \mathbb{C}^3.$$

holds for all the instanciations z of Z.

Proposition 21.9.13. *Equivalently, tensors X^m_Z, Y^m_Z are equal up to their normal components when*

$$(X^m_Z\text{-}Y^m_Z)\,\mathfrak{G}_{mn}\mathfrak{N}^n_{,\nu} = (0)_{Z\nu} \qquad \text{where } \nu \notin Z$$

Proof. When $\mathrm{d}\nu$ varies, the 3D-vectors $\overrightarrow{\mathfrak{N}^1_{,\nu}\,\mathrm{d}^\nu, \mathfrak{N}^2_{,\nu}\,\mathrm{d}^\nu, \mathfrak{N}^3_{,\nu}\,\mathrm{d}^\nu}$ span the whole tangent plane. Thus an $\mathcal{O}\left(\overrightarrow{\mathfrak{n}}\right)$ vector is orthogonal to all of them, and conversely. □

Definition 21.9.14. The Christoffel symbols are defined by

$$\frac{\partial^2 x^m}{\partial u^\mu\,\partial u^\nu} = \Gamma^\phi_{\mu\nu}\,\frac{\partial x^m}{\partial u^\phi} + \mathcal{O}\left(\overrightarrow{\mathfrak{n}}\right)\ ;\ \Gamma_{\psi|\mu\nu} = \mathfrak{G}_{\psi\phi}\Gamma^\phi_{\mu\nu}$$

Equivalently, we have:

$$\mathfrak{G}_{mn}\left(\frac{\partial^2 x^m}{\partial u^\mu\,\partial u^\nu} - \Gamma^\phi_{\mu\nu}\,\frac{\partial x^m}{\partial u^\phi}\right)\frac{\partial x^n}{\partial u^\psi} = (0)_{\psi\mu\nu} \tag{21.8}$$

$$\Gamma_{\psi|\mu\nu} = \mathfrak{G}_{\phi\psi}\Gamma^{\phi}_{\mu\nu} = \mathfrak{G}_{mn}\mathfrak{N}^{m}_{,\mu\nu}\mathfrak{N}^{n}_{,\psi}$$

Remark 21.9.15. In cyril@ERE (2016), $\Gamma^{\phi}_{\mu\nu}$ is defined by $\Gamma^{\phi}_{\mu\nu} = \dfrac{\partial u^{\phi}}{\partial x^{m}} \dfrac{\partial^{2} x^{m}}{\partial u^{\mu}\partial u^{\nu}}$... but this formula is wrong when we simply replace each quantity by its (correct) definition. On the contrary, we have to define the $\Gamma^{\phi}_{\mu\nu}$ by the already given formula

$$x^{m}_{,\mu\nu} = x^{m}_{,\phi}\,\Gamma^{\phi}_{\mu\nu} + \mathcal{O}\left(\overrightarrow{\mathbf{n}}\right)$$

Theorem 21.9.16. *When the Christoffels are defined by (21.8), then the provisional definition*

$$\begin{aligned} \Gamma_{\psi|\mu\nu} &= \frac{1}{2}\left(\mathfrak{G}_{\mu\psi,\nu} + \mathfrak{G}_{\psi\nu,\mu} - \mathfrak{G}_{\nu\mu,\psi}\right) \\ \Gamma^{\phi}_{\mu\nu} &= \Gamma_{\psi|\mu\nu}\mathfrak{G}^{\phi\psi} \iff \Gamma_{\psi|\mu\nu} = \mathfrak{G}_{\psi\phi}\Gamma^{\phi}_{\mu\nu} \end{aligned} \tag{21.9}$$

becomes a theorem.

Proof. Start from $g_{\mu\nu} \doteq \eta_{mn} \dfrac{\partial x^{m}}{\partial u^{\mu}} \dfrac{\partial x^{n}}{\partial u^{\nu}}$ and obtain the following relations:

$$\begin{aligned} \frac{\partial g_{\mu\nu}}{\partial u^{\phi}} &= \frac{\partial}{\partial u^{\phi}}\left(\eta_{mn}\frac{\partial x^{m}}{\partial u^{\mu}}\frac{\partial x^{n}}{\partial u^{\nu}}\right) \\ \frac{\partial g_{\mu\nu}}{\partial u^{\phi}} &= \eta_{mn}\left(\frac{\partial^{2} x^{m}}{\partial u^{\phi}\,\partial u^{\mu}}\right)\frac{\partial x^{n}}{\partial u^{\nu}} + \eta_{mn}\left(\frac{\partial^{2} x^{n}}{\partial u^{\phi}\,\partial u^{\nu}}\right)\frac{\partial x^{m}}{\partial u^{\mu}} \\ &= \eta_{mn}\left(\frac{\partial x^{m}}{\partial u^{\psi}}\,\Gamma^{\psi}_{\mu\phi}\right)\frac{\partial x^{n}}{\partial u^{\nu}} + \eta_{mn}\left(\frac{\partial x^{n}}{\partial u^{\psi}}\,\Gamma^{\psi}_{\nu\phi}\right)\frac{\partial x^{m}}{\partial u^{\mu}} \\ &= \left(\eta_{mn}\frac{\partial x^{m}}{\partial u^{\psi}}\frac{\partial x^{n}}{\partial u^{\nu}}\right)\Gamma^{\psi}_{\mu\phi} + \left(\eta_{mn}\frac{\partial x^{m}}{\partial u^{\mu}}\frac{\partial x^{n}}{\partial u^{\psi}}\right)\Gamma^{\psi}_{\nu\phi} \\ \frac{\partial g_{\mu\nu}}{\partial u^{\phi}} &= g_{\psi\nu}\,\Gamma^{\psi}_{\mu\phi} + g_{\mu\psi}\,\Gamma^{\psi}_{\nu\phi} = g_{\psi\nu}\,\Gamma^{\psi}_{\mu\phi} + g_{\psi\mu}\,\Gamma^{\psi}_{\phi\nu} \end{aligned}$$

Then call $\mathcal{F}(\mu,\nu,\phi)$ this formula, and compute $\mathcal{F}(\mu,\psi,\nu) + \mathcal{F}(\psi,\nu,\mu) - \mathcal{F}(\nu,\mu,\psi)$. □

Proposition 21.9.17. *The Christoffels transform according to the rules:*

$$\begin{aligned} \Gamma^{\epsilon}_{\sigma\tau} &= \Gamma^{\phi}_{\mu\nu}\,\mathfrak{N}^{\mu}_{,\sigma}\,\mathfrak{N}^{\nu}_{,\tau}\,\mathfrak{N}^{\epsilon}_{,\phi} + \mathfrak{N}^{\nu}_{,\sigma,\tau}\,\mathfrak{N}^{\epsilon}_{,\nu} \\ \Gamma_{\omega|\sigma\tau} &= \Gamma_{\phi|\nu\mu}\,\mathfrak{N}^{\phi}_{,\omega}\,\mathfrak{N}^{\nu}_{,\sigma}\,\mathfrak{N}^{\mu}_{,\tau} + \mathfrak{N}^{\nu}_{,\sigma\tau}\,\mathfrak{G}_{\nu\phi}\,\mathfrak{N}^{\phi}_{,\omega} \end{aligned} \tag{21.10}$$

In other words, the "linear tensorial" term which depends on the up/down nature of the indices is completed by a second term containing some second order derivatives.

Proof. The formula $\Gamma_{\omega|\sigma\tau} = \mathfrak{G}_{mn}x^{m}_{,\sigma\tau}\,x^{n}_{,\omega}$ can be extended into:

$$\begin{aligned} \Gamma_{\omega|\sigma\tau} &= \mathfrak{G}_{mn}\frac{\partial}{\partial t^{\tau}}\left(\frac{\partial x^{m}}{\partial u^{\nu}}\frac{\partial u^{\nu}}{\partial t^{\sigma}}\right)\left(\frac{\partial x^{n}}{\partial u^{\psi}}\frac{\partial u^{\psi}}{\partial t^{\omega}}\right) \\ &= \frac{\partial}{\partial t^{\tau}}\left(\frac{\partial x^{m}}{\partial u^{\nu}}\right)\frac{\partial x^{n}}{\partial u^{\psi}}\frac{\partial u^{\psi}}{\partial t^{\omega}}\frac{\partial u^{\nu}}{\partial t^{\sigma}} + \frac{\partial}{\partial t^{\tau}}\left(\frac{\partial u^{\nu}}{\partial t^{\sigma}}\right)\left(\mathfrak{G}_{mn}\frac{\partial x^{m}}{\partial u^{\nu}}\frac{\partial x^{n}}{\partial u^{\psi}}\right)\frac{\partial u^{\psi}}{\partial t^{\omega}} \\ &= \mathfrak{G}_{mn}\left(\frac{\partial^{2} x^{m}}{\partial u^{\nu}\partial u^{\mu}}\frac{\partial u^{\mu}}{\partial t^{\tau}}\right)\frac{\partial x^{n}}{\partial u^{\psi}}\frac{\partial u^{\psi}}{\partial t^{\omega}}\frac{\partial u^{\nu}}{\partial t^{\sigma}} + \left(\frac{\partial^{2} u^{\nu}}{\partial t^{\sigma}\partial t^{\tau}}\right)\mathfrak{G}_{\nu\psi}\frac{\partial u^{\psi}}{\partial t^{\omega}} \\ &= \left(\mathfrak{G}_{mn}\frac{\partial^{2} x^{m}}{\partial u^{\nu}\partial u^{\mu}}\frac{\partial x^{n}}{\partial u^{\psi}}\right)\frac{\partial u^{\mu}}{\partial t^{\tau}}\frac{\partial u^{\psi}}{\partial t^{\omega}}\frac{\partial u^{\nu}}{\partial t^{\sigma}} + \left(\frac{\partial^{2} u^{\nu}}{\partial t^{\sigma}\partial t^{\tau}}\right)\mathfrak{G}_{\nu\psi}\frac{\partial u^{\psi}}{\partial t^{\omega}} \\ &= \Gamma_{\psi|\nu\mu}\,\mathfrak{N}^{\psi}_{,\omega}\,\mathfrak{N}^{\nu}_{,\sigma}\,\mathfrak{N}^{\mu}_{,\tau} + \mathfrak{N}^{\nu}_{,\sigma\tau}\,\mathfrak{G}_{\nu\psi}\,\mathfrak{N}^{\psi}_{,\omega} \end{aligned}$$

□

21.9.5 Moving

From the external point of view, the most probable result when moving, even a little bit, from a point $M \in (E)$ is to left the surface, evading into the external world. Therefore we need a more elaborated concept of "moving".

Definition 21.9.18. We distinguish three different notions of "variation".

1. Δx denotes a non elaborated difference, according to the model $\Delta x = x_2 - x_1$;
2. δx denotes an "infinitesimal variation", obeying to the informal rules $\delta x \neq 0$ together with $(\delta x)^2 = 0$... and to the formal rules stated by Newton, Leibniz and Landau.
3. $\mathrm{d}x$ denotes a "long range variable", not submitted to the $(\mathrm{d}x)^2 = 0$ rule, but bound to some specific tangent plane.

Remark 21.9.19. Let B be a covariant vector, depicted at a point M by the pair of tensor equations: $B_\mu = x^m_{,\mu} B_m$ and $B_m = u^\mu_{,m} B_\mu$. Then we move it from the map in use at point M to the map in use at point $M + \delta M$. In an ordinary cartesian frame, we simply have $\delta B_m = 0$. In another frame, we have:

$$\begin{aligned} \delta B_\mu &= \delta\left(x^m_{,\mu} B_m\right) = \delta\left(x^m_{,\mu}\right) B_m = x^m_{,\mu\nu}\, \mathrm{d}u^\nu\, B_m = \left(x^m_{,\mu\nu}\, u^\psi_{,m}\right) B_\psi\, \mathrm{d}u^\nu \\ &= \Gamma^\psi_{\mu\nu}\, B_\psi\, \mathrm{d}u^\nu \end{aligned}$$

Definition 21.9.20. The variation due to a parallel transport of quantity B_μ is defined by

$$\begin{aligned} \delta B_\mu &\doteq +B_\psi\, \Gamma^\psi_{\mu\nu}\, \mathrm{d}u^\nu \\ \delta A^\mu &\doteq -A^\psi\, \Gamma^\mu_{\psi\nu}\, \mathrm{d}u^\nu \end{aligned}$$

and this is extended to any tensor (a corrective term per index, with the right sign). Therefore this variation doesn't depend on the embedding chosen to introduce the surface, but depends only on its (intrinsic) metric $\mathfrak{G}$.

Proof. For a contravariant vector A^σ, the constraint $\delta\left(B_\sigma A^\sigma\right) = 0$ leads to

$$\begin{aligned} \delta\left(B_\mu A^\mu\right) &= A^\mu \delta\left(B_\mu\right) + B_\mu \delta\left(A^\mu\right) \\ &= A^\mu B_\psi\, \Gamma^\psi_{\mu\nu}\, \mathrm{d}u^\nu - A^\psi B_\mu\, \Gamma^\mu_{\psi\nu}\, \mathrm{d}u^\nu \quad \square \end{aligned}$$

Definition 21.9.21. The variation $\mathrm{d}B_\sigma = B_{\sigma,\tau}\, \mathrm{d}u^\tau$ of a vector is seen as the sum of two terms. One of them is due to the parallel transport, i.e. the change of the local map due to the displacement. And the difference can be seen as "the true variation" of the vector. This term is called the **covariant derivative** of B_σ and noted with a semi-colon. In other words:

$$B_{\sigma;\tau} \doteq B_{\sigma,\tau} - \Gamma^\epsilon_{\sigma\tau} B_\epsilon \;;\; A^\sigma_{;\tau} \doteq A^\sigma_{,\tau} + \Gamma^\epsilon_{\sigma\tau} B_\epsilon$$

Proposition 21.9.22. *The covariant derivative of a tensor is another tensor.*

Proof. Starting from the definition, we have:

$$\begin{aligned} B_{\sigma;\tau} &= B_{\sigma,\tau} - \Gamma^\epsilon_{\sigma\tau} B_\epsilon = \left(t^\mu_{,\sigma}\, B_\mu\right)_{,\tau} - \left(\Gamma^\phi_{\mu\nu}\, t^\mu_{,\sigma}\, t^\nu_{,\tau}\, u^\epsilon_{,\phi} + t^\nu_{,\sigma\tau}\, u^\epsilon_{,\nu}\right) t^\psi_{,\epsilon}\, B_\psi \\ &= t^\mu_{,\sigma\tau}\, B_\mu + t^\mu_{,\sigma}\, B_{\mu,\nu}\, t^\nu_{,\tau} - \Gamma^\phi_{\mu\nu}\, t^\mu_{,\sigma}\, t^\nu_{,\tau}\, B_\psi \left(u^\epsilon_{,\phi}\, t^\psi_{,\epsilon}\right) - t^\nu_{,\sigma\tau}\, B_\psi \left(u^\epsilon_{,\nu}\, t^\psi_{,\epsilon}\right) \\ &= \left(t^\mu_{,\sigma\tau}\, B_\mu - t^\nu_{,\sigma\tau}\, B_\psi \left(u^\epsilon_{,\nu}\, t^\psi_{,\epsilon}\right)\right) + t^\mu_{,\sigma}\, t^\nu_{,\tau}\, B_{\mu,\nu} - \Gamma^\phi_{\mu\nu}\, t^\mu_{,\tau}\, t^\nu_{,\sigma}\, B_\psi \left(u^\epsilon_{,\phi}\, t^\psi_{,\epsilon}\right) \\ &= \left(t^\mu_{,\sigma\tau}\, B_\mu - t^\nu_{,\sigma\tau}\, B_\nu\right) + t^\mu_{,\sigma}\, t^\nu_{,\tau} \left(B_{\mu,\nu} - \Gamma^\phi_{\mu\nu}\, B_\phi\right) = t^\mu_{,\sigma}\, t^\nu_{,\tau}\, B_{\mu;\nu} \end{aligned} \quad \square$$

21.10 Curvature

Definition 21.10.1. Explicit coordinates. Consider the surface $(E) = \{(x, y, z) \,|\, z = F(x, y)\}$, where the $[x, y, z]$ are living in an euclidean 3D space. Then we can use $[u, v] \doteq [x, y]$ as internal coordinates.

Notation 21.10.2. Using indices $\mu, \nu, \phi, \psi, \rho$ in association with variables u, v and indices m, n, p, q, r in association with variables x, y, z, we have:

$$\mathfrak{N}^k_{,\mu} = \left[[k,\mu], [1,0], \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ F_u & F_v \end{bmatrix} \right] \; ; \; \mathfrak{G}_{\mu\nu} = \left[[\mu,\nu], [0,0], \begin{bmatrix} F_u^2+1 & F_uF_v \\ F_uF_v & F_v^2+1 \end{bmatrix} \right]$$

where the F_u, F_v are the Frenet symbols (in any case, we will never use i, j, u, v as tensorial indices).

Fact 21.10.3. *Let $\overrightarrow{\mathbf{n}}$ be a normal vector to the tangent plane and N^q an associated tensor. We have:*

$$N^q = [[q], [1], [-F_u, -F_v, 1]]$$

Remark 21.10.4. We have the matrix product:

$$\begin{bmatrix} 0 & 0 & F_{uu} \\ 0 & 0 & F_{uv} \\ 0 & 0 & F_{vv} \end{bmatrix} = \frac{1}{1+F_u^2+F_v^2} \begin{bmatrix} -F_uF_{uu} & -F_vF_{uu} & F_{uu} \\ -F_uF_{uv} & -F_vF_{uv} & F_{uv} \\ -F_uF_{vv} & -F_vF_{vv} & F_{vv} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -F_u \\ 0 & 1 & -F_v \\ F_u & F_v & 1 \end{bmatrix}$$

where the coefficients are $\begin{bmatrix} -\Gamma_{11}^1 & -\Gamma_{11}^2 & L \\ -\Gamma_{12}^1 & -\Gamma_{12}^2 & M \\ -\Gamma_{22}^1 & -\Gamma_{22}^2 & N \end{bmatrix}$

Proposition 21.10.5. *When cutting (E) by a moving plane P_ϑ containing $\mathbf{n}$, we obtain a curve γ_ϑ and we examine its curvature κ_ϑ. This quantity is obtained by dividing twice the normal increment by the squared tangential increment. When the moving point $\mathbf{r}$ on the curve depends on a single parameter, we have the formula :*

$$\kappa \doteq \left| \ddot{\mathbf{r}} \wedge \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} \right| \div |\dot{\mathbf{r}}|^2 = \frac{|\ddot{\mathbf{r}} \wedge \dot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3}$$

where the fluxions are taken wrt the parameter of the curve.

Proof. Well known formula. □

Proposition 21.10.6. *The curvature is given by the quotient of two quadratic forms:*

$$\kappa_\vartheta = \frac{1}{\sqrt{\det \mathfrak{G}_{\mu\nu}}} \frac{\mathcal{II}(dx,dy)}{\mathcal{I}(dx,dy)}$$

where $\mathcal{I}(dx,dy)$ is the metric form (noted $\mathfrak{G}_{\mu\nu}$ here) and gives the squared tangential increment, while the normal increment is measured using:

$$\mathcal{II}(dx,dy) = 2(\delta\mathbf{r}) \cdot \mathbf{n} = (\,\mathrm{d}x,\,\mathrm{d}y) \begin{pmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}$$

Proof. We have $|\mathbf{n}|^2 = 1+F_u^2+F_v^2 = \det \mathfrak{G}_{\mu\nu}$, while $\mathbf{x} \wedge_P \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \mathbf{x} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}$ holds for any vector $\mathbf{x}$ in the tangent plane. □

Exercise 21.10.7. Use $F(u,v) = \sqrt{R^2-u^2-v^2}$ and obtain

$$\overrightarrow{n} = \left[\frac{u}{z}, \frac{v}{z}, 1\right] \; ; \; |\overrightarrow{n}| = \frac{R^2}{z^2} \; ; \; \mathcal{I} = \frac{1}{z^2} \begin{bmatrix} R^2-v^2 & uv \\ uv & R^2-u^2 \end{bmatrix} \; ; \; \mathcal{II} = -z\mathcal{I}$$

Finally, $\kappa_\vartheta = -1/R$ ($\overrightarrow{n}$ is the outer normal... and the center is "inside").

Lemma 21.10.8. *When $\mathcal{II}$ and $\mathcal{I}$ are two quadratic forms (and $\mathcal{I}$ is definite), then*

$$\max\left(\frac{\mathcal{II}(x,y)}{\mathcal{I}(x,y)}\right) * \min\left(\frac{\mathcal{II}(x,y)}{\mathcal{I}(x,y)}\right) = \frac{\det \mathcal{II}}{\det \mathcal{I}}$$

Proof. Let $\mathcal{II} = \left[\begin{smallmatrix} L & M \\ M & N \end{smallmatrix}\right]$ and $\mathcal{I} = \left[\begin{smallmatrix} E & F \\ F & G \end{smallmatrix}\right]$.
(1) brute force method: differentiate $\kappa \doteq \dfrac{Lu^2 + 2\,Muv + Nv^2}{Eu^2 + 2\,Fuv + Gv^2}$ and solve in u and v. Then substitute and simplify $\kappa_{\min} * \kappa_{\max}$
(2) educated method: diagonalize the matrix $\mathcal{II}/\mathcal{I}$. This allows a simultaneous reduction of the quadratic forms, so that $\kappa_{\min}$ and $\kappa_{\max}$ are the eigenvalues of matrix $\mathcal{II}/\mathcal{I}$. □

Lemma 21.10.9. *The following quantity is a tensor, named the Riemann-Christoffel curvature tensor.*

$$B^i_{jkl} \quad = \quad \Gamma^i_{rk}\,\Gamma^r_{jl} - \Gamma^i_{rl}\,\Gamma^r_{jk} + \Gamma^i_{jl,k} - \Gamma^i_{jk,l}$$

This tensor is anti-symmetric wrt k, l, and doesn't contains any third order derivative of F.

Proof. Tensorial property comes from the Christoffel formula. Morover, we have:

$$B^i_{jkl} = \frac{\left(F_{uu}F_{vv} - F_{vu}^2\right)}{\left(1 + F_u^2 + F_v^2\right)^2} \left(\begin{bmatrix} 1121 & -F_uF_v \\ 2221 & +F_uF_v \\ 1112 & +F_uF_v \\ 2212 & -F_uF_v \end{bmatrix} ; \begin{bmatrix} 2121 & +\left(F_u^2+1\right) \\ 1221 & -\left(F_v^2+1\right) \\ 2112 & -\left(F_u^2+1\right) \\ 1212 & -\left(F_v^2+1\right) \end{bmatrix} \right)$$

$$B_{ijkl} = \left(\frac{F_{uu}F_{vv} - F_{vu}^2}{1 + F_u^2 + F_v^2}\right) \left(\begin{bmatrix} 2121 & -1 \\ 1221 & +1 \\ 2112 & +1 \\ 1212 & -1 \end{bmatrix} \right)$$

(other components being nul). □

Theorem 21.10.10. *The product of the extremal curvatures, i.e.*

$$K = \kappa_{\min}\kappa_{\max} = \frac{F_{uu}F_{vv} - F_{uv}^2}{\left(1 + F_u^2 + F_v^2\right)^2}$$

doesn't depends on the chosen explicit parametrization.

Proof. Contract B^i_{jkl} on i, l and obtain the Ricci tensor :

$$R_{jk} = B^i_{jki} = \left[[\mu, \nu], [0,0], \ \frac{\left(F_{uu}F_{vv} - F_{vu}^2\right)}{\left(F_u^2 + F_v^2 + 1\right)^2} \begin{bmatrix} -\left(F_u^2+1\right) & -F_uF_v \\ -F_uF_v & -\left(F_v^2+1\right) \end{bmatrix} \right]$$

Contract again and obtain the scalar curvature

$$R \doteq g^{jk}\,R_{jk} = (-2)\,\frac{F_{uu}F_{vv} - F_{uv}^2}{\left(1 + F_u^2 + F_v^2\right)^2} = -2K$$

Being tensorial, this quantity is therefore an invariant of the surface (E). □

Remark 21.10.11. Formula B^i_{jkl} is top and formemost about general Riemann spaces, with dimensions greater then 2. When dealing with surfaces, due the skew-symmetry of B_{ijkl} wrt pairs of indices (i, j) et (k, l), B_{1212} is the only independent component of the Riemann-Christoffel tensor. And then, the curvature formula simplifies into:

$$K = B_{1212}/\det g_{ij}$$

21.11 Back to Poincaré and Klein

$$\begin{aligned}
\text{Klein}_\mu &= \left[[\mu],[1],\left[\frac{2\,z}{1+\zeta\,z},\frac{2\,\zeta}{1+\zeta\,z}\right]\right]\\
\text{Poincare}_\sigma &= \left[[\sigma],[1],\left[\frac{1-W}{\kappa},\frac{1-W}{k}\right]\right]\\
W &= \frac{1-\zeta\,z}{1+\zeta\,z}=\sqrt{1-k\kappa}
\end{aligned}$$

As a mnemonic, k,κ are related to Klein, while z,ζ are related to Poincaré.

Exercise 21.11.1. Compute the Jacobians and obtain:

$$\begin{aligned}
\mathfrak{N}^\sigma_{,\mu} &= \left[[\sigma,\mu],[1,0],\ \frac{1+z\zeta}{2\,(1-z\zeta)}\begin{bmatrix}1 & z^2\\ \zeta^2 & 1\end{bmatrix}\right]=\\
\mathfrak{N}^\mu_{,\sigma} &= \left[[\mu,\sigma],[1,0],\ \frac{2}{(1+z\zeta)^2}\begin{bmatrix}1 & -z^2\\ -\zeta^2 & 1\end{bmatrix}\right]=\left[[\mu,\sigma],[1,0],\ \frac{1}{2}\begin{bmatrix}(1+W)^2 & -k^2\\ -\kappa^2 & (1+W)^2\end{bmatrix}\right]
\end{aligned}$$

Check they are inverse of each other.

Exercise 21.11.2. Formulate the metrics in tensor form, and obtain:

$$\begin{aligned}
\mathfrak{G}_{\sigma\tau} &\underset{z}{=} \left[[\sigma,\tau],[0,0],\ \frac{2}{(1-z\zeta)^2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right]\\
&\underset{k}{=} \left[[\sigma,\tau],[0,0],\ \frac{(1+W)^2}{2\,(1-k\kappa)}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right]
\end{aligned}$$

$$\begin{aligned}
\mathfrak{G}_{\mu\nu} &\underset{z}{=} \left[[\mu,\nu],[0,0],\ \frac{(1+z\zeta)^2}{2\,(1-z\zeta)^4}\begin{bmatrix}2\zeta^2 & 1+z^2\zeta^2\\ 1+z^2\zeta^2 & 2z^2\end{bmatrix}\right]\\
&\underset{k}{=} \left[[\mu,\nu],[0,0],\ \frac{1}{4\,(1-k\kappa)^2}\begin{bmatrix}\kappa^2 & 2-k\kappa\\ 2-k\kappa & k^2\end{bmatrix}\right]
\end{aligned}$$

Check the validity of the transformation formula (21.7).

Exercise 21.11.3. Compute the Christoffels and obtain:

	111 121	112 122	211 221	212 222
$\Gamma^{\epsilon}_{\sigma\tau}$	$\frac{2\,\zeta}{1-z\zeta}$ 0	0 0	0 0	0 $\frac{2\,z}{1-z\zeta}$
$\Gamma^{\phi}_{\mu\nu}$	$\frac{\kappa}{1-k\kappa}$ $\frac{k}{2\,(1-k\kappa)}$	$\frac{k}{2\,(1-k\kappa)}$ 0	0 $\frac{\kappa}{2\,(1-k\kappa)}$	$\frac{\kappa}{2\,(1-k\kappa)}$ $\frac{k}{1-k\kappa}$

Check the validity of the transformation formula (21.10)

Exercise 21.11.4. Compute the Riemann-Christoffel tensors and obtain:

$$\frac{1}{(1-z\zeta)^2}\begin{bmatrix}1121 & +2\\ 2221 & -2\\ 1112 & -2\\ 2212 & +2\end{bmatrix}\ ;\ \frac{1}{4\,(1-k\kappa)^2}\left[\begin{array}{cc|cc}2121 & -\kappa^2 & 1121 & -k\kappa+2\\ 1221 & +k^2 & 2221 & +k\kappa-2\\ 2112 & +\kappa^2 & 1112 & +k\kappa-2\\ 1212 & -k^2 & 2212 & -k\kappa+2\end{array}\right]$$

Check the validity of the tensor transformation formula.

Proposition 21.11.5. *The Gauss curvature of both the Poincaré and the Klein hyperbolic planes is* -1.

Proof. Obvious from the former results. □

Chapter 22

About cubics

For a catalog with sketches, visit Gibert (2004-2024), especially Ehrmann and Gibert (2005). Some notations used there :

I	G	O	H	N	K	L			
X(1)	X(2)	X(3)	X(4)	X(5)	X(6)	X(20)			

22.1 Characterisation of a cubic

Definition 22.1.1. A cubic is a curve defined by an homogeneous polynomial of degree 3. Using barycentrics, or trilinears or Morley affixes is irrelevant, the degree is the same. Notation : $\mathcal{K}$.

Proposition 22.1.2. *A cubic is defined by nine general points.*

Proof. There are ten coefficients, defined up to a global proportionality factor. We found them by computing

$$\det_{j=10}\left[x^3, x^2y, xy^2, y^3, y^2z, yz^2, z^3, z^2x, zx^2, xyz\right] \tag{22.1}$$

applied to the nine given points and the generic point. □

Example 22.1.3. ***Pivotal isocubics.*** Let F, U be two points not on the sidelines. Then $p\mathcal{K}(\#F, U)$ is the cubic that goes through the nine points: ABC, $A_UB_UC_U$ (the cevians of U) and $F_AF_BF_C$ (the anticevians of F). This important class will be studied in details at Section 22.4.

Proposition 22.1.4 (Cayley Bacharach)**.** *When two cubics $\mathcal{K}_1, \mathcal{K}_2$ haven't a line or a conic in common, they cut into exactly nine points. But, even when they are distincts, these nine points aren't "general points" with respect to the previous proposition. More precisely the family $\mathcal{F}$ of the cubics that are going through eight of these points is exactly $\lambda\mathcal{K}_1 + \mu\mathcal{K}_2$, and all of these cubics are going through the last point.*

Proof. The first part is Bezout theorem. The second one is obvious: the nine points are not characterizing a cubic, since $\mathcal{F}$ contains at least two cubics. The last part, i.e. that $\mathcal{F}$ doesn't contain any other cubics is proven in great details in Eisenbud et al. (1996). □

22.1.1 More about the folium

Some properties of the Descartes curve have already be given at Section 12.2.

Exercise 22.1.5. Using the parametrization $M_p \simeq 6p : 6p^2 : 1 + p^3$, give the condition for three points of the folium be aligned. Deduce the parameter of the tangential, i.e. the point where the tangent at M_p cuts again the curve. And then, determine the 'conjugate' of M_p, i.e. the point having the same tangential.

Exercise 22.1.6. Using the same parametrization, find the ninth point of eight distincts points on the Folium (i.e. illustrate the Cayley Bacharach property).

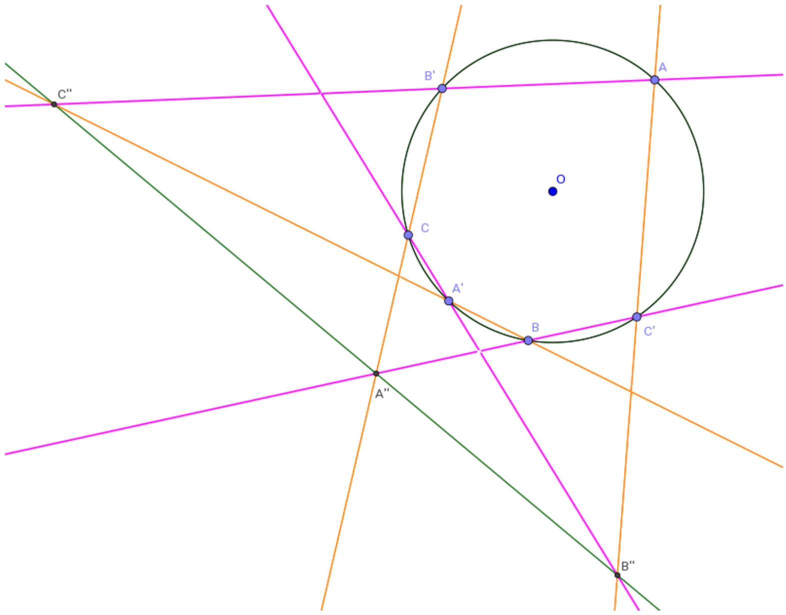


Figure 22.1: Pascal's theorem: A",B",C" are aligned

22.1.2 Pascal's theorem

Theorem 22.1.7 (Pascal's theorem)**.** *Let A, C', B, A', C, B' be six points on a conic. Define $A'' \doteq BC' \cap CB'$, etc. Then A'', B'', C'' are on a same line.*

Proof. Draw the magenta, orange and green cubics of Figure 22.1, i.e. the cubics by A, B, C, A', B', C', A'', B'' and M_j where $M_1 = (A + B')/2$, $M_2 = (A + C')/2$, $M_3 = (A'' + B'')/2$ and conclude by Cayley Bacharach. □

22.2 Group structure of a cubic

Definition 22.2.1. Suppose that the cubic $\mathcal{K}$ is not singular (no cusp, no nodes). When $A \neq B$ are on the cubic, notation $A@B$ will be used to design the third point where the line (AB) cuts again the cubic. In the same vein, $A@A$ will denote the **tangential** of A, i.e. the point where the tangent at A cuts again $\mathcal{K}$.

Proposition 22.2.2. *Operation @ is commutative, but not associative. By convention, $A@B@C$ is to be understood as $(A@B)@C$. And we have*
1. $P@Q@P = Q$
2. $P@Q = R@Q$ if and only if $P = R$
3. $P@Q = R$ if and only if $R@Q = P$

Definition 22.2.3. Chose a special point $O \in \mathcal{K}$ and note + the operation

$$(A, B) \mapsto A + B \doteq A@B@O$$

Proposition 22.2.4. *(1) Operation + is commutative*
(2) O is the neutral point, i.e. $P + O = P$;
(3) Defining N by $N \doteq O@O = O_t$, then $-N = N_t$;

Proof. (1) Obvious. (2) $P + O = (P@O)@O = P$ since $P, O, Q \doteq P@O$ are the three intersections of some line with the cubic, while $Q@O$ is "not Q nor O" on this line
(3) One has $N + N_t \doteq N@N_t@O = N@O = O_t@O = O$. □

Theorem 22.2.5. *Operation + is associative, and therefore $(\mathcal{K}, +)$ is a group.*

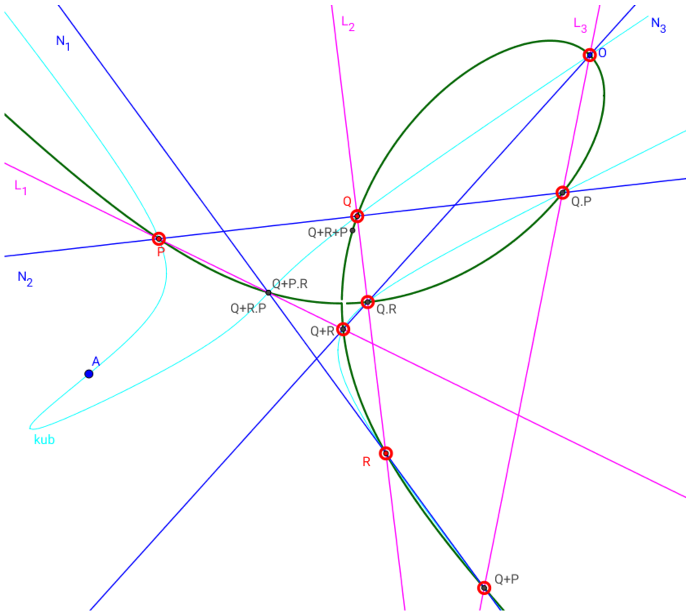


Figure 22.2: Using Cayley-Bacharach to prove the cubic associativity

Proof. See Figure 22.2 and Durège (1871, p. 135). Since the operation is commutative, what is to be proved can be written as:

$$(\forall P, Q, R \in \mathcal{K})\,(E = F) \qquad \text{where } E \doteq (Q+R)\,@P \;;\; F \doteq (Q+P)\,@R$$

1. Let us introduce two degenerate cubics: $\mathcal{K}_2$ as the union of the three magenta lines

$$L_1 = [P, Q+R, E]\,, L_2 = [Q, R, Q@R]\,, L_3 = [O, Q@P, Q+P]$$

   and $\mathcal{K}_3$ as the union of the three blue lines

$$N_1 = [R, Q+P, F]\,, N_2 = [Q, P, Q@P]\,, N_3 = [O, Q@R, Q+R]$$

2. So we see that cubic $\mathcal{K}$ intersects the other two at

$$\begin{aligned} \mathcal{K} \cap \mathcal{K}_2 &= O, P, Q, R, Q@R, Q@P, Q+R, Q+P, E \\ \mathcal{K} \cap \mathcal{K}_3 &= O, P, Q, R, Q@R, Q@P, Q+R, Q+P, F \end{aligned}$$

   Since $\mathcal{K}_3$ goes by eight of the nine $\mathcal{K} \cap \mathcal{K}_2$ points, the third cubic must go also through E.

3. Suppose that E lies on $N_3 = [O, Q@R, Q+R]$. Since L_1 already goes through $Q+R$, this would induce $L_1 = N_3$ and therefore $P \in (O, Q+R)$. Suppose that E lies on $N_2 = [Q, P, Q@P]$. Since L_1 already goes through P, this would induce $L_1 = N_2$ and therefore $P \in (Q, Q+R)$.

4. It remains $E \in N_1$, enforcing either $E = R$ or $E = Q+P$ or $E = F$. The first two possibilities couldn't be the general case, since $E \doteq P@\,(Q+R)$ depends on the three points. It remains only $E = F$ as the general case, implying $O@E = O@F$ as required.

5. Remark: the cyan cubic goes through the eight points and the random point A. One can see that it goes also through the ninth point $Q+R@P$

□

Theorem 22.2.6. *$3k$ points $P_i \in \mathcal{K}$ are on a curve of order k if and only if $\sum P_i = kN$.*

Proof. When $k = 1$, $P@Q@O@R@O = N = O@O$ leads to $P@Q@O@R = O$, and to $P@Q@O = R@O$, implying $P@Q = R$.
When $k = 2$, define $X \doteq P@Q, Y \doteq R@S, Z \doteq T@U$ and prove the equivalence: the P_j are on a conic with XYZ aligned. The property results since $P+Q+X = N$, etc. □

Remark 22.2.7. We have to be careful to the existence of torsion points i.e points with the property k P = O. Indeed the equation k X = Q has k^2 solutions (complex) for the point X.

Some complements are given at Subsection 22.4.2 (concerning only the $p\mathcal{K}$ cubics).

22.3 Isocubics

Proposition 22.3.1. *Define an isocubic $\mathcal{K}$ with pole P as a cubic which is invariant wrt the P isoconjugacy. Then $\mathcal{K}$ is either a "pivotal isocubic" (see Section 22.4 for more details) with equation:*

$$p\mathcal{K}\left(P,U\right) \doteq p\mathcal{K}\left(\#F,U\right) \doteq \left(h^2y^2-g^2z^2\right)ux+\left(f^2z^2-h^2x^2\right)vy+\left(g^2x^2-f^2y^2\right)wz \tag{22.2}$$

or a "non pivotal isocubic" (see Section 22.5 for more details) with equation :

$$n\mathcal{K}\left(P,U,k\right) \doteq ux\left(r\,y^2+qz^2\right)+vy\left(p\,z^2+r\,x^2\right)+wz\left(q\,x^2+p\,y^2\right)+k\,xyz \tag{22.3}$$

Proof. Direct inspection from $\mathcal{K}\left(X_P^*\right)=\lambda\mathcal{K}\left(X\right)$. It can be seen that terms like xy^2 and xz^2 are to be paired, and that terms like x^3 are to be avoided. As a corollary, such a cubic goes through the vertices A,B,C. □

Proposition 22.3.2. *When the pole P is fixed, and X_P^* is defined by (18.4), i.e. by $x^*=pyz$, the property "$\mathcal{K}$ is an isocubic" is characterized by the exact formula:*

$$\mathcal{K}\left(X_P^*\right)/\mathcal{K}\left(X\right)=\mp\left(pqr\,xyz\right)$$

Sign "-" characterizes a $p\mathcal{K}$ cubic, sign "+" characterizes a $n\mathcal{K}$ cubic. Each class form a projective space, whose dimensions are respectively 3 and 4.

Definition 22.3.3. The **triangular cubic** is the union of the three sidelines: $n\mathcal{K}_0=(BC)\cup(CA)\cup(AB)$. The standard equations of this cubic are **defined** by :

$$\begin{aligned} n\mathcal{K}_0\left(X\right) &\underset{b}{=} xyz \\ &\underset{z}{=} \left(\overline{\mathcal{Z}}\,\gamma\,\beta-\mathbf{T}\beta-\mathbf{T}\gamma+\mathbf{Z}\right)\left(\overline{\mathcal{Z}}\,\alpha\,\gamma-\mathbf{T}\alpha-\mathbf{T}\gamma+\mathbf{Z}\right)\left(\overline{\mathcal{Z}}\,\alpha\,\beta-\mathbf{T}\alpha-\mathbf{T}\beta+\mathbf{Z}\right)\div s_3 \end{aligned} \tag{22.4}$$

Proposition 22.3.4. *When substituting the isogonal formulas (18.5) into the equation of a #X(1) pivotal cubic and using (22.4), we have the following equalities :*

$$\begin{aligned} p\mathcal{K}\left(isog\left(X\right)\right) &= -p\mathcal{K}\left(X\right)\times n\mathcal{K}_0\left(X\right) \\ n\mathcal{K}\left(isog\left(X\right)\right) &= +n\mathcal{K}\left(X\right)\times n\mathcal{K}_0\left(X\right) \end{aligned}$$

that are exact identities, not up to a proportionality factor. Once again, the set of the X(6)-isocubics splits into two projective subspaces, the $p\mathcal{K}$ one (dimension 3) and the $n\mathcal{K}$ one (dimension 4).

Example 22.3.5. The cubic "circumcircle union infinity" is a $n\mathcal{K}$ cubic, characterized by:

$$\left[n\mathcal{K},\begin{pmatrix}a\\b\\c\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix},a^2+b^2+c^2\right]\underset{b}{=}\left(x+y+z\right)\left(a^2yz+b^2zx+c^2yx\right)\underset{z}{=}\mathbf{T}\left(\mathbf{Z}\overline{\mathcal{Z}}-\mathbf{T}^2\right)$$

22.4 Pivotal isocubics pK(P,U)

Definition 22.4.1. ***Pivotal isocubics***. Let F, U be two points not on the sidelines and define $p\mathcal{K}\left(\#F,U\right)$ as the cubic that goes through the nine points: ABC, $A_UB_UC_U$ (the cevians of U) and $F_AF_BF_C$ (the anticevians of F). Then U and $F_0\doteq F$ are also on the cubic. Its equation is :

$$p\mathcal{K}\left(P,U\right) \doteq p\mathcal{K}\left(\#F,U\right) \doteq \left(h^2y^2-g^2z^2\right)ux+\left(f^2z^2-h^2x^2\right)vy+\left(g^2x^2-f^2y^2\right)wz \tag{22.5}$$

Remark 22.4.2. Part of the time, isocubics are defined using the pole of the conjugacy, i.e. $P\simeq p:q:r\doteq f^2:g^2:h^2$.

$p\mathcal{K}$	name	F	U,U^*	E,E^*	D,D^*	some other points on the cubic
$p\mathcal{K}_A$	vertex 22.4.20	1	A			
ZU(1)	22.4.21	1	X(1)			
E_1, E_3	hidden 22.4.23	1	Ω			
K002	Thomson 22.4.29	1	2,6	3,4	9,57	223, 282, 1073, 1249
K003	McKay 22.4.24	1	3,4	1075,?	1745,3362	
K006		1	4,3	155,254	46,90	371, 372, 485, 486, 487, 488
K005		1	5,54	2120,2121	3460,3461	3, 4, 17, 18, 61, 62, 195, 627, 628
K004	Darboux 22.4.5	1	20,64	2130,2131	3182,3347	3, 4, 40, 84, 1490, 1498
K001	Neuberg 22.4.25	1	30,74	2132,2133	3464,?	3, 4, 13, 14, 15, 16, 399, 484, ...
		1	98,511	?,?	1756,?	1687, 1688, 2009, 2010
K035		1	99,512	39,83	1019,1018	1379, 1380
		1	100,513	1,1	513,100	1381, 1382
		1	110,523	5,54	?,?	1113, 1114
K020		1	384,695	?,?	?,?	3,4,32,39,76,83
K021		1	512,99	2142,2143	?,?	
K1155	shortest 22.4.28	1	523,110	39138	21381,39137	
K007	Lucas 22.4.5	2	X(69)			4, 7, 8, 20, 189, 253, 329... (15)
K170	22.4.9	2	X(4)			
K155	EAC2 22.4.55	$\sqrt{31}$	238			
K060	22.4.7	$\sqrt{1989}$	265			
$n\mathcal{K}$		F	root			
Δ	22.3.3	1				
circle	22.3.5	1				
K137		1	513	$Z^+(X_1X_6)$		1, 44, 88, 239, 241, 292, 294, 1931
???		1	649	$Z^+(X_1X_2)$		1, 238, 291, 899, 2107
K040	Pelletier	1	650	$Z^+(X_1X_3)$		1,105,243,296,518,1155,1156, 2651, 2652
K018	Brocar2 22.14	1	523	$Z^+(X_3X_6)$		2, 6, 13, 14, 15, 16, 111, 368, 524
K010	Simson 22.5.3	2		$c\mathcal{K}\,(\#X_2,\, X_{69})$		2, X(2394) upto X(2419)
K162	22.5.19	6		$c\mathcal{K}\,(\#X_6,\, X_3)$		6, X(2420) upto X(2445)

$F = \sqrt{P}$ (central fixed point), U, U_P^*, $E = cevadiv\,(U, U^*)$, E^*, $D = cevadiv(U, \sqrt{P})$

Table 22.1: Some well-known cubics

Definition 22.4.3. A Kimberling *ZU* cubic is a $p\mathcal{K}\,(X_6, U)$, giving a special place to isogonal conjugacy. Some examples are given in Table 22.1.

Remark 22.4.4. Only 8 *ZU* cubics have a reflection center : the Darboux cubic (center= X_3), the four degenerate cubics that are union of the three bisectors through an inexcenter, and three other (Maple length = 135712 using RootOf, [4948, 5345, 4215] using alias).

Theorem 22.4.5. pivotal isocubic property. *When point X is on a sideline of triangle ABC, then $X_F^{\#}$ is undefined (geometrically), while the formula gives the third vertex. Otherwise, $X_F^{\#}$ belongs to $p\mathcal{K}\,(\#F, U)$ if and only if X belongs to $p\mathcal{K}\,(\#F, U)$. And then U, X, $X_F^{\#}$ are collinear. For this reason, point U is called the* **pivot** *of the cubic. Alternate formulation: $p\mathcal{K}\,(\#F, U)$ is the locus of the X such that U, X, $X_F^{\#}$ are collinear (to be taken 'Cremona more', i.e. allowing indeterminacies)*

Proof. Compute the determinant of the 10 rows (22.1) relative to the nine points given in the definition and the variable point $X = x : y : z$. And remark that this quantity is proportional to

$\det\left(U,\,X,\,X_F^{\#}\right)$. □

Theorem 22.4.6. cevadiv property. *When X is on the sidelines of the cevian triangle of U, then $Y=\text{cevadiv}\,(U,X)$ is undefined geometrically, while the formula gives $0:0:0$. Otherwise, this Y belongs to $p\mathcal{K}\,(\#F,U)$ if and only if X belongs to $p\mathcal{K}\,(\#F,U)$. And then $U_F^{\#}$, X, cevadiv (U,X) are collinear. This fact is underlined by the name* **isopivot** *given to the point $U_F^{\#}$.*

Proof. Compute $p\mathcal{K}\,(\#F,U)\,(\text{cevadiv}\,(U,X))$ and obtain $p\mathcal{K}\,(\#F,U)\,(X)$ times the incidence relations, i.e. the rows of Adjoint $(cevian\,(U))\cdot X$. And remark that this quantity is proportional to $\det\left(U_F^{\#},X,\text{cevadiv}\,(U,X)\right)$. □

Proposition 22.4.7. ***The 22 points property.*** *The $p\mathcal{K}\,(\#F,U)$ cubic goes through*

1. *U,$A_U B_U C_U$ and their conjugates $U_F^{\#}$, ABC (8)*
2. *$F_0F_AF_BF_C$ (cf Theorem 22.4.5) (4)*
3. *cevadiv $\left(U,\,U_F^{\#}\right)$, the four cevadiv $(U,\,F_j)$ and their isoconjugates (10)*

Proof. Follows directly from the two theorems. □

Example 22.4.8. When U is one of the fixed points of the isoconjugacy (i.e. $P=U\underset{b}{*}U$), the $p\mathcal{K}\,(P,U)$ cubic degenerates into the lines through the remaining three fixed points.

Example 22.4.9. K170 is $p\mathcal{K}\,(X_2,X_4)$. Equation $\sum x\left(y^2-z^2\right)/S_a=0$. On Figure 22.3, one can see the following alignments (general properties, applicable to any $p\mathcal{K}$) :

1. Fixed points : F_0, F_a, A are collinear, and cyclically for the other fixed points and the other vertices ;
2. From U : U, X, $X_F^{\#}$ are collinear (e.g. E and E^* are aligned with U). Therefore, each line from U to a fixed point is tangent to the cubic at this fixed point ; in the same vein, point $U_b=UB\cap AC$ is on the cubic and viewing B as $(U_b)_F^{\#}$ makes sense, but not viewing U_b as $B_F^{\#}$ (this object would be "quite all points on line AC").
3. From $U_F^{\#}$: $U_F^{\#}$, X, *cevadiv* (U,X) are collinear (e.g. F and D are aligned with $U_F^{\#}$). Therefore, each line from $U_F^{\#}$ to a vertex or to U is tangent to the cubic at this point.

Definition 22.4.10. The $PK_F^{\#}(X)$ point is the intersection of the trilinear polars of points X and $X_F^{\#}$. Using barycentrics and $F=f:g:h$, we have :

$$PK_F^{\#}\,(X)\;=\;f^2x\left(g^2z^2-h^2y^2\right)\,:\,g^2y\left(h^2x^2-f^2z\right)\,:\,h^2z\left(f^2y^2-x^2g^2\right)$$

Proof. Direct computation. □

Remark 22.4.11. This transform was introduced by Ehrmann and Gibert (2005) as PK_P, i.e. putting forwards the pole of the conjugacy rather than its fixed points, and turned into Definition 25.4.1.

Example 22.4.12. Using F=X(1), i.e. P=X(6), we have PK(X(I)) = X(J) for these (I,J):

I	2	3	4	5	6	9	19	31	44	54	57	63
J	512	647	647	2081	512	663	810	2084	3251	2081	663	810

Proposition 22.4.13. *Points of indeterminacy of $PK_F^{\#}$ are the fixed points F_j and their diagonal vertices i.e. A,B,C. The exceptional curves are the six lines through two of the fixed points. Otherwise, each point $P=PK_F^{\#}\,(X)$ characterizes a pair $\left\{X,\,X_F^{\#}\right\}$ (so that $PK^{\#}$ is not a Cremona transform !).*

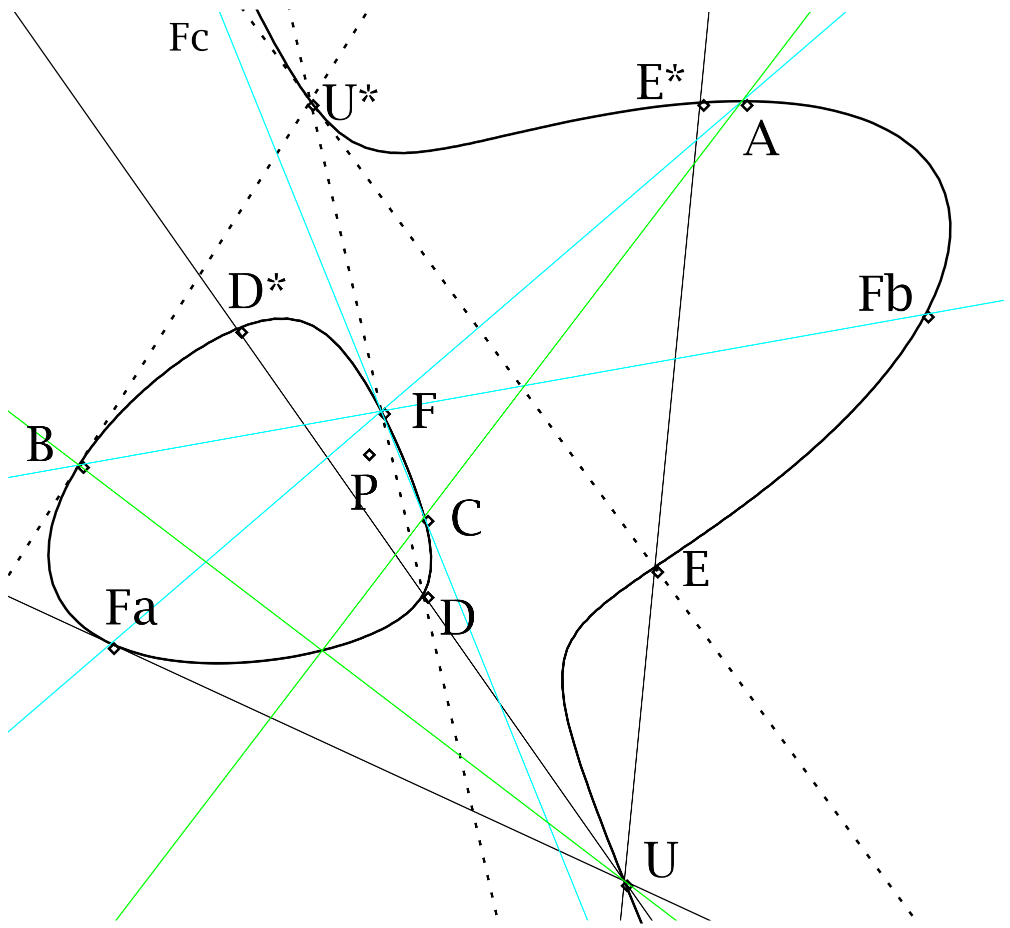


Figure 22.3: pK(2,4)

Proof. Solving $P = 0 : 0 : 0$ gives the first result, and solving the Jacobian gives the second. Otherwise, set $P \simeq p : q : r$ and obtain :

$$\begin{bmatrix} 2\,h^2qf^2prg^2 \\ -rg^2\left(+g^2h^2p^2 - f^2g^2r^2 + f^2h^2q^2\right) - rg^2W \\ -qh^2\left(+g^2h^2p^2 + f^2g^2r^2 - f^2h^2q^2\right) + qh^2W \end{bmatrix}$$

where $W = 4i\ f^2g^2h^2\ S\left(\frac{p}{f}, \frac{q}{g}, \frac{r}{h}\right)$ and S is the usual Heron formula for the area (7.8). □

Proposition 22.4.14. *When $PK_F^{\#}(X)$ belongs to the tripolar line of $U_F^{\#}$, then point X belongs to the cubic $p\mathcal{K}(P,U)$. This amounts to say that X is on $p\mathcal{K}$ if and only if the tripolars of X, $X_F^{\#}$, $U_F^{\#}$ are concurrent.*

Proof. Direct computation. In Kimberling (1998, p. 240) the corresponding cubic is noted $Z(U, Y)$ where $P = Y \underset{b}{*} X_6$. □

Proposition 22.4.15. *When point $U = u : v : w$ is at infinity, the $p\mathcal{K}\,(P,U)$ cubic can be rewritten as :*

$$\left(\frac{p}{x} + \frac{q}{y} + \frac{r}{z}\right)(x\rho + y\sigma + z\tau) - (x+y+z)\left(\frac{a^2\rho}{x} + \frac{b^2\sigma}{y} + \frac{c^2\tau}{z}\right) = 0$$

where $[\rho, \sigma, \tau]$ is any line whose direction is U. In other words $u = \sigma - \tau$, etc.

Proof. Direct examination. We can check that $p\mathcal{K}(P,U)$ contains the intersections of both conics —vertices A,B,C and U_P^*—, the points at infinity of $\mathcal{C}_{cir}(P)$, the intersections of line $[\rho,\sigma,\tau]$ with the associated conic, and U itself ! □

22.4.1 Another description

1. Consider $p\mathcal{K}(\#F,P)$ and use $F_j \simeq \pm f : \pm g : \pm h$ (isoconjugacy), $P \simeq p:q:r$ (pivot). The name $U \simeq u:v:w$ will be used for the generic point of the cubic. Let CP(F,P) be the diagonal conic that goes through P and the four F_j. In fact, this conic goes also through the 3 associates of P since

$$CP(F,P) \simeq \begin{bmatrix} g^2r^2-h^2q^2 & 0 & 0 \\ 0 & h^2p^2-f^2r^2 & 0 \\ 0 & 0 & f^2q^2-g^2p^2 \end{bmatrix}$$

2. Cut this conic by line PU and obtain point N. Acting that way, we obtain a symmetric expression. Cut rather by line $UU_F^\#$ and say that P is the other solution.. We obtain :

$$N \simeq \begin{bmatrix} -f^2pv^2+2\,f^2quv-g^2pu^2 \\ f^2qv^2-2\,g^2puv+g^2qu^2 \\ 2\,w\left(f^2qv-g^2pu\right)+r\left(g^2u^2-f^2v^2\right) \end{bmatrix}$$

Now, define $N(U)$ by this formula. Then $N(U)=N\left(U_F^\#\right)$ is equivalent to $U \in p\mathcal{K}$... except when P is one of the F_j.

3. Cut the tangent at N to CF(F,P) with line $PP_F^\#$ and obtain the point M

$$M \simeq \begin{bmatrix} \dfrac{qw-rv}{r^2g^2-q^2h^2} \\ \dfrac{ru-pw}{p^2h^2-r^2f^2} \\ \dfrac{pv-qu}{q^2f^2-p^2g^2} \end{bmatrix} \;;\; t=-\frac{(pw-ru)\,q}{(qw-rv)\,p}$$

4. For all U, point M is aligned with $P, P_F^\#$. When U is on the cubic,
   - $N = \text{cevadiv}(M,P)$.
   - M is aligned with $N, N_F^\#$.
   - $N_F^\#$ is the intersection of tangents at U and $U_F^\#$ to the cubic.
   - $M_F^\#$ is the intersection of the line $P,U,U_F^\#$ and the fixed circumconic through $P,P_F^\#$.

5. Line $\text{cevadiv}(P,U)\,;\text{cevadiv}\left(P,U_F^\#\right)$ goes through the fixed point $Q=\left(\text{cevadiv}\left(P,P_F^\#\right)\right)_F^\#$. Conversely, the locus of such U is the union of $p\mathcal{K}(\#F,P)$ et $n\mathcal{K}\left(\#F,Q,-1\div p^2q^2r^2\right)$.

22.4.2 Group structure (pivotal cubics)

Proposition 22.4.16. *When using the pivot U as the neutral point, then*

1. $N \doteq U_t = U^* = [f^2vw : g^2uw : h^2uv]$
2. X,Y,Z are collinear if and only if $X+Y+Z=N$. (generic property)
3. $X@X^*=U$; $X@U=X^*$; $X^*@U=X$; $X+X^*=N$ (isocubic property)
4. $X+Y=(X@Y)^*$; $X@Y=(X+Y)^*$
5. $-X=X@N=U/X$ (cevadiv property)
6. On the cubic, $A^*=A_U$ (the cevian of U) ; $\text{grad}(\text{A})=[0,-\text{h}^2\text{v},+\text{g}^2\text{w}]$; $A_t=N$

7. $A+A=B+B=C+C=U$; $A+B+C=U$

Proof. (7) $A+A=A@A@U=A_t@U=U_t@U=U$
$B+C=B@C@U=U_A@U=A$; □

Proposition 22.4.17. *Four points are said to form a tangential quadruple when they have the same tangential.*

1. The four F_j form such a quadruple, the tangential being U.
2. The four A,B,C,U form such a quadruple, , the tangential being N.
3. When (X_1,X_2,X_3,X_4) is a tangential quadruple, then $(X_1^*,X_2^*,X_3^*,X_4^*)$ is such a quadruple.
4. Every tangential quadruple is of the form $(X, X+A, X+B, X+C)$.

Proof. (3) From the tangential definition, and the isocubic property

$$\begin{aligned} N &= T+2P^*=S+2P=S+2Q \\ N &= P+P^*=Q+Q^* \quad \text{then} \end{aligned}$$

$$\begin{aligned} T+2Q^* &= N-2P^*+2Q^*=N-2(N-P)+2(N-Q)=N+2P-2Q \\ &= N+(N-S)-(N-S)=N \end{aligned}$$

so that T is the tangential of Q^* too.
$(4)(P+A)_t=N-2(P+A)=N-2P-2A=N-2P=S.$ □

Exercise 22.4.18. Find the tangential common to A_U, B_U, C_U, U^*.

22.4.3 $ABCIJKL$ cubics: the Lubin(2) point of view

Notation 22.4.19. All visible curves are normalised by $\mathcal{C}\div\operatorname{conj}\mathcal{C}=\pm 1$.

Proposition 22.4.20. ***Cubics PKA.*** *The set of all cubics that go through points $ABCIJKL$ is a projective space. Its dimension is 3. A generating family is given by the three cubics $p\mathcal{K}_A=(BC)\cup(AI)\cup(KL)$, etc. Pivot of $p\mathcal{K}_A$ is A. Its fully factored equation requires Lubin(2), but Lubin(1) is sufficient to use $\det(X,\,X^*,\,A)=0$ where X^* is given by the isogonal conjugacy formula. One has:*

$$\begin{aligned} p\mathcal{K}_A &\underset{2}{=} \frac{1}{s_{3^2}}\left(-\overline{\mathcal{Z}}\,\beta^2\gamma^2+\mathbf{T}\beta^2+\mathbf{T}\gamma^2-\mathbf{Z}\right)\left(-\overline{\mathcal{Z}}\,\alpha^2\beta\,\gamma+\mathbf{T}\alpha^2+\mathbf{T}\beta\,\gamma-\mathbf{Z}\right)\left(\overline{\mathcal{Z}}\,\alpha^2\beta\,\gamma+\mathbf{T}\alpha^2-\mathbf{T}\beta\,\gamma-\mathbf{Z}\right) \\ &\underset{1}{=} \frac{1}{s_3}\left(-\overline{\mathcal{Z}}\,\beta\,\gamma+\mathbf{T}\beta+\mathbf{T}\gamma-\mathbf{Z}\right)\left(-\alpha^2\beta\,\gamma\,\overline{\mathcal{Z}}^2+2\,\mathbf{T}\overline{\mathcal{Z}}\,\alpha\,\beta\,\gamma+\mathbf{T}^2\alpha^2-\mathbf{T}^2\beta\,\gamma-2\,\alpha\,\mathbf{Z}\mathbf{T}+\mathbf{Z}^2\right) \end{aligned}$$

Proof. Equation of $p\mathcal{K}(F,U)$ is $\det\left(X,\,X_F^{\#},\,U\right)=0$, leading to dimension 3, and allowing to check that pivot of $p\mathcal{K}_A$ is the vertex A. After that, we can go back to Lubin(1) since isogonal conjugacy doesn't require to identify which is the incenter among the inexcenters. □

Example 22.4.21. The Kimberling $Z(X(1))$ cubic, i.e. $(IJ)\cup(IK)\cup(IL)$, is obtained as :

$$Z(X(1))\underset{2}{=}\frac{\alpha\,(\beta+\gamma)}{(\alpha-\beta)\,(\alpha-\gamma)}\,p\mathcal{K}_A+\frac{\beta\,(\alpha+\gamma)}{(\beta-\gamma)\,(\beta-\alpha)}\,p\mathcal{K}_B+\frac{\gamma\,(\alpha+\beta)}{(\gamma-\alpha)\,(\gamma-\beta)}\,p\mathcal{K}_C$$

Proposition 22.4.22. *The Kiepert RH construction (see Proposition 13.22.2 for more details) can be summarized as follows. Let $K=\cot\phi$ be a fixed real and define circularly the points $A'B'C'$ by :*

$$\cot\left(\overbrace{BC,\,BA'}\right)-K=0\,;\ \cot\left(\overbrace{CB,\,CA'}\right)+K=0$$

These 3 points and the 7 $ABCIJKL$ are on a same cubic, together with X(3), X(4) and the $A'B'C'$ points related with the other orientation. Naming this cubic $p\mathcal{K}_{(K)}$, we have:

$$\begin{aligned} p\mathcal{K}_{(K)} &= p\mathcal{K}_{Darboux}+K^2\times p\mathcal{K}_{Thomson} \\ &= \left(K^2+1\right)\,p\mathcal{K}_{Neuberg}+\left(1-3K^2\right)\,p\mathcal{K}_{McKay} \end{aligned}$$

(See below for more details on these four cubics). The pivot of $p\mathcal{K}_{(K)}$ is $\left(3K^2zX\,(2)-zX\,(20)\right)\div\left(3K^2-1\right)$.

Proof. Let us compute the points A', B', C' and obtain:

$$A' \underset{1}{\simeq} \beta+\gamma+i\left(\gamma-\beta\right)/K \,:\, 2 \,:\, \left(\beta+\gamma+i\left(\gamma-\beta\right)/K\right) \div \beta\gamma, \text{ etc}$$

Then substitute A' and B' into $\sum_3 x_j\, p\mathcal{K}_j = 0$. This system has non zero solutions, proving the result. The circumcenter $X(3)$ (the $\mathbf{T}^3$ coefficient), the orthocenter (isogonality) and the $A'B'C'$ points related to the other orientation ($p\mathcal{K}_{(K)}$ depends only on K^2). □

22.4.4 Using a more handy basis

Proposition 22.4.23. The hidden IJKL cubics. *Let $p\mathcal{K}_{\Omega x}$ and $p\mathcal{K}_{\Omega y}$ be the isogonal cubics whose pivots are the umbilics $\Omega_x \simeq 0:0:1$ and $\Omega_y \simeq 1:0:0$. Then*

$$\begin{aligned} p\mathcal{K}_{\Omega x} &\underset{1}{=} \mathbf{Z}^2\overline{\mathcal{Z}} + \mathbf{T}\left(\sigma_3\,\overline{\mathcal{Z}}^2\right) - \mathbf{T}^2\left(2\,\mathbf{Z}+\sigma_2\,\overline{\mathcal{Z}}\right) + \mathbf{T}^3\sigma_1 \\ p\mathcal{K}_{\Omega y} &\underset{1}{=} \mathbf{Z}\overline{\mathcal{Z}}^2 + \mathbf{T}\left(\frac{1}{\sigma_3}\,\mathbf{Z}^2\right) - \mathbf{T}^2\left(\frac{\sigma_1}{\sigma_3}\,\mathbf{Z}+2\,\overline{\mathcal{Z}}\right) + \frac{\sigma_2}{\sigma_3}\,T^3 \end{aligned}$$

They are conjugate of each other, but not self conjugate (hidden curves). Their nine intersections are $ABCIJKL$ (stable by isogonal conjugacy) and the two umbilics (stable as a pair).

Proof. When $M \in \mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$, and $isog\,(M)$ is taken from (18.5), then

$$E \doteq \left(\mathbf{T}^2 - \mathbf{Z}\overline{\mathcal{Z}}\right)\, M - \mathbf{T}\, isog\,(M) = E_1 : 0 : E_3$$

The nine points property comes from (1) at an umbilic, column E vanishes since both multipliers are zero; (2) at a vertex, the circle vanishes and $isog\,(X)$ is $0:0:0$; (3) at a fixed point of the conjugacy, column E is proportional to M but E_2 is ever 0 and the M aren't at infinity. Obviously, this result can also be checked in Lubin(2) by substitution. Or even by factoring the $\mathbf{T}$ resultant of both equations. □

Proposition 22.4.24. ***K003, the McKay Cubic***, $p\mathcal{K}\,(6,3)$. *A visible cubic is described by the expression $\left(\overline{\mathcal{Z}}\, p\mathcal{K}_{\Omega x} - \mathbf{Z}\, p\mathcal{K}_{\Omega y}\right)/\mathbf{T}$. And then, pole=X(6), pivot=X(3) and equation:*

$$p\mathcal{K}_{McKay} \underset{1}{=} -\frac{1}{\sigma_3}\,\mathbf{Z}^3 + \sigma_3\,\overline{\mathcal{Z}}^3 + \mathbf{T}\left(\frac{\sigma_1}{\sigma_3}\,\mathbf{Z}^2 - \sigma_2\,\overline{\mathcal{Z}}^2\right) + \mathbf{T}^2\left(-\frac{\sigma_2}{\sigma_3}\,\mathbf{Z} + \sigma_1\,\overline{\mathcal{Z}}\right)$$

K003 goes through X(3) and X(4). Its points at infinity are $\Theta : 0 : 1/\Theta$ where $\Theta^3 = \sigma_3$. These points are the directions of the Morley triangles. The asymptotes are :

$$\left[\begin{array}{ccccc} 3\,\dfrac{\Theta^2}{\sigma_3} & ; & \dfrac{\sigma_2}{\Theta^2} - \dfrac{\sigma_1\,\Theta^2}{\sigma_3} & ; & -3\,\dfrac{\sigma_3}{\Theta^2} \end{array}\right]$$

where Θ ranges over the three cubic roots, using $j\,\Theta$ or $j^2\,\Theta$. They concur at X(2).

Proof. Coeff of T^3 is 0. Thus X(3) and then X(4). Asymptotes are obtained from the gradient. Pivot can be guessed as the intersection of two well chosen lines MM^*, for example for two of the points at infinity. □

Proposition 22.4.25. ***K001, the Neuberg cubic***, $p\mathcal{K}\,(6,30)$. *A visible cubic is described by expression $(\sigma_2/\sigma_3)\, p\mathcal{K}_{\Omega x} - \sigma_1 p\mathcal{K}_{\Omega y}$. And then pole=X(6), pivot=X(30) and equation:*

$$p\mathcal{K}_{Neuberg} = \frac{\sigma_2}{\sigma_3}\,\mathbf{Z}^2\overline{\mathcal{Z}} - \sigma_1\,\mathbf{Z}\overline{\mathcal{Z}}^2 + \mathbf{T}\left(-\frac{\sigma_1}{\sigma_3}\,\mathbf{Z}^2 + \sigma_2\,\overline{\mathcal{Z}}^2\right) + \mathbf{T}^2\left(\frac{\sigma_1^2 - 2\,\sigma_2}{\sigma_3}\,\mathbf{Z} - \frac{\sigma_2^2 - 2\,\sigma_1\,\sigma_3}{\sigma_3}\,\overline{\mathcal{Z}}\right)$$

This is a circular curve. The third point at infinity is $\sigma_1\sigma_3 : 0 : \sigma_2$, i.e. X(30), the direction of the Euler line. The asymptotes are :

$$\left[\begin{array}{ccc} 0 & -\sigma_1 & \sigma_2 \end{array}\right], \left[\begin{array}{ccc} -\sigma_1 & \sigma_2 & 0 \end{array}\right], \left[\begin{array}{ccccc} \sigma_2^2\sigma_1 & ; & -\sigma_1^3\sigma_3 + \sigma_2^3 & ; & -\sigma_2\,\sigma_1^2\,\sigma_3 \end{array}\right]$$

Intersecting the first two asymptotes, we obtain a singular focus at $\sigma_2^2 : \sigma_1\,\sigma_2 : \sigma_1^2$, i.e. point X(110). Moreover, the cubic goes through X(3) and therefore through X(4). Visible asymptote goes through X(74), the isogonal of X(30).

Proof. X(30) is the pivot because it is the only common point to the curve and line MM^* when M is an umbilic. An asymptote is the gradient evaluated at the corresponding point at infinity. □

Corollary 22.4.26. *The Neuberg cubic is the locus of points X such that the isogonal line $X\,X^*$ is parallel to the Euler line. See (Gibert, 2004-2024, 2005). Its (barycentric) equation is :*

$$\sum_{cyclic} x\left(y^2c^2-z^2b^2\right)\left(2\,a^4-\left(b^2+c^2\right)a^2-\left(b^2-c^2\right)^2\right)=0 \tag{22.6}$$

and can be rewritten as :

$$\left(\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}\right)(x\rho+y\sigma+z\tau)-(x+y+z)\left(\frac{a^2\rho}{x}+\frac{b^2\sigma}{y}+\frac{c^2\tau}{z}\right)=0$$

where $[\rho,\sigma,\tau]$ is the Euler line –remember: tripole = X(648). In other words, K001=circumcircle×Euler - infinity×Jerabek.

Proof. See Proposition 22.4.15. We can check that K001 contains the intersections of Γ and Jerabek —vertices A,B,C and X(74)—, the points at infinity of the circumcircle —the umbilics—, the intersections of Euler line and Jerabek hyperbola —X(3) and X(4)— and X(30) itself. □

Remark 22.4.27. The name "shortest cubic" was introduced here, in v42 (2012), as the circular $p\mathcal{K}$ cubic whose expression requires the shortest number of characters. Its main property was to provide an handy basis when used together with the McKay and the Neuberg cubics.

Proposition 22.4.28. ***Shortest cubic**, $p\mathcal{K}\,(6,523)$. The expression $(1/\sigma_1)\,p\mathcal{K}_{\Omega x}+(\sigma_3/\sigma_2)\,p\mathcal{K}_{\Omega y}$ gives a visible cubic, with pole=X(6), pivot=X(523) and equation:*

$$p\mathcal{K}_{shortest}=\frac{1}{\sigma_1}\,\mathbf{Z}^2\overline{\mathcal{Z}}+\frac{\sigma_3}{\sigma_2}\,\mathbf{Z}\overline{\mathcal{Z}}^2+\mathbf{T}\left(\frac{1}{\sigma_2}\,\mathbf{Z}^2+\frac{\sigma_3}{\sigma_1}\,\overline{\mathcal{Z}}^2\right)-\left(\frac{\sigma_1^2+2\,\sigma_2}{\sigma_2\,\sigma_1}\,\mathbf{Z}+\frac{\sigma_2^2+2\,\sigma_1\,\sigma_3}{\sigma_2\,\sigma_1}\,\overline{\mathcal{Z}}\right)+2\,T^3$$

The third point at infinity is $-\sigma_1\sigma_3:0:\sigma_2$, i.e. X(523), the orthodir of the Euler line. This is also the pivot. The three asymptotes concur at X(110), giving a singular focus. Barycentric equation of this curve is ;

$$\left(b^2-c^2\right)x\left(b^2z^2-c^2y^2\right)+\left(c^2-a^2\right)y\left(c^2x^2-a^2z^2\right)+\left(a^2-b^2\right)z\left(a^2y^2-b^2x^2\right)$$

The ETC databasis provides $U=523, U^*=110, D=39138, E=21381, E^*=39137$ and no other points. *Moreover the imaginary foci of the MacBeath inconic Example 12.11.2 are here. Thus we have 3 vertices, 4 inexcenters, 3 points at infinity, and 6 others, i.e. 16 points.*

Proposition 22.4.29. K002, the Thomson cubic, *$p\mathcal{K}\,(6,2)$. The expression*

$$\left(\left(\mathbf{T}s_2/s_3-3\,\overline{\mathcal{Z}}\right)p\mathcal{K}_{\Omega x}-\left(\mathbf{T}s_1-3\,\mathbf{Z}\right)p\mathcal{K}_{\Omega y}\right)/\mathbf{T}$$

gives a visible cubic, with pole=X(6), pivot=X(2) and equation:

$$p\mathcal{K}_{Thomson}=$$
$$\frac{3}{\sigma_3}\,\mathbf{Z}^3+\frac{\sigma_2}{\sigma_3}\,\mathbf{Z}^2\overline{\mathcal{Z}}-\sigma_1\,\mathbf{Z}\overline{\mathcal{Z}}^2-3\,\sigma_3\,\overline{\mathcal{Z}}^3+\mathbf{T}\left(-\frac{4\sigma_1}{\sigma_3}\,\mathbf{Z}^2+4\,\sigma_2\,\overline{\mathcal{Z}}^2\right)+\mathbf{T}^2\left(\frac{\sigma_2+\sigma_1^2}{\sigma_3}\,\mathbf{Z}-\frac{\sigma_2^2+\sigma_1\,\sigma_3}{\sigma_3}\,\overline{\mathcal{Z}}\right)$$

This cubic is the locus of points X whose trilinear polar is parallel to their polar line in the circumcircle. This is the $K=\infty$ cubic of the Kieper RH construct.

Proposition 22.4.30. K004, the Darboux cubic, *$p\mathcal{K}\,(6,20)$. The expression*

$$\left(\left(\mathbf{T}s_2/s_3+\overline{\mathcal{Z}}\right)p\mathcal{K}_{\Omega x}-\left(\mathbf{T}s_1+\mathbf{Z}\right)p\mathcal{K}_{\Omega y}\right)/T$$

gives a visible cubic, with pole=X(6), pivot=X(20) and equation:

$$p\mathcal{K}_{Darboux}=-\frac{\mathbf{Z}^3}{\sigma_3}+\frac{\sigma_2}{\sigma_3}\,\mathbf{Z}^2\overline{\mathcal{Z}}-\sigma_1\,\mathbf{Z}\overline{\mathcal{Z}}^2+\sigma_3\,\overline{\mathcal{Z}}^3+\mathbf{T}^2\left(\frac{\sigma_1^2-3\,\sigma_2}{\sigma_3}\,\mathbf{Z}-\frac{\sigma_2^2-3\,\sigma_1\,\sigma_3}{\sigma_3}\,\overline{\mathcal{Z}}\right)$$

This is the $K=0$ cubic of the Kieper RH construct. See the next section for more details.

Proposition 22.4.31. ***Resulting pencils***. *We have the following pencils :*

1. *The pencil generated by* $p\mathcal{K}_{Neuberg}$ *and* $p\mathcal{K}_{shortest}$ *is the set of all the circular* $p\mathcal{K}$ *cubics. Their pivots are at infinity.*
2. *The pencil generated by* $p\mathcal{K}_{McKay}$ *et* $p\mathcal{K}_{Neuberg}$ *is the set of the* $p\mathcal{K}$ *cubics that goes through the X(3), X(4) pair. Their pivots are on the Euler line.*
3. *The pencil generated by* $p\mathcal{K}_{McKay}$ *et* $p\mathcal{K}_{shortest}$ *is the set of* $p\mathcal{K}$*-cubics whose pivots are on the line X(3), X(523), i.e. the line through X(3) and perpendicular to the Euler line. The common points of these cubics are on* Δ *(horrible formula, with a 24th degree radicand).*

Exercise 22.4.32. Does it exist a cubic XXX such that :

1. XXX, K001, K003 provide a basis of the ZU cubics space
2. XXX contains "many" ETC points
3. Pencils (XXX,K001) and (XXX,K003) contain "many" known cubics
4. Lubin equation of XXX remains practicable

22.4.5 Darboux and Lucas cubics

In this section, $P \in Darboux$ and $U \in Lucas$ while pole and pivot are noted otherwise.

22.4.5.1 Presentation of K004 and K007

Definition 22.4.33. The Darboux cubic K004 is the locus of point P such that the pedal triangle of P is the Cevian triangle of some other point U, while the Lucas cubic K007 is the locus of point U such that the cevian triangle of U is the pedal triangle of some other point P.

Proposition 22.4.34. *Darboux cubic is a* $p\mathcal{K}$ *cubic, with X(6) as pole and X(20) as pivot. X(20) is the de Longchamps point. Lucas cubic is a* $p\mathcal{K}$ *cubic, with X(2) as pole and X(69) as pivot. X(69) is the anticomplement of X(6). Their equations are :*

$$\det\left(X_{20},\, P,\, isog\left(P\right)\right) = 0 \tag{22.7}$$
$$\det\left(X_{69},\, U,\, isot\left(U\right)\right) = 0 \tag{22.8}$$

Moreover, K004 has a reflection center at X(3), the circumcenter. Using Morley affixes and expanding, we have :

$$p\mathcal{K}_{Darboux} = -\frac{\mathbf{Z}^3}{\sigma_3} + \frac{\sigma_2}{\sigma_3}\,\mathbf{Z}^2\overline{\mathcal{Z}} - \sigma_1\,\mathbf{Z}\overline{\mathcal{Z}}^2 + \sigma_3\,\overline{\mathcal{Z}}^3 + \frac{\sigma_1^2 - 3\,\sigma_2}{\sigma_3}\,T^2\mathbf{Z} + \frac{3\,\sigma_1\,\sigma_3 - \sigma_2^2}{\sigma_3}\,T^2\overline{\mathcal{Z}}$$

$$p\mathcal{K}_{Lucas} = \left(\begin{array}{l} \dfrac{3}{s_3}\mathbf{Z}^3 + \dfrac{s_2}{s_3}\,\mathbf{Z}^2\,\overline{\mathcal{Z}} - s_1\,\mathbf{Z}\overline{\mathcal{Z}}^2 - 3\,s_3\,\overline{\mathcal{Z}}^3 - \left(\dfrac{s_1}{s_3} + \dfrac{s_2^2}{s_3^2}\right)\mathbf{Z}^2\mathbf{T} + \left(s_1^2 + s_2\right)\overline{\mathcal{Z}}^2\mathbf{T} \\ -\left(\dfrac{3\,s_1^2}{s_3} - \dfrac{s_1 s_2^2}{s_3^2} - \dfrac{4\,s_2}{s_3}\right)\mathbf{Z}\mathbf{T}^2 + \left(\dfrac{3\,s_2^2}{s_3} - \dfrac{s_1^2 s_2}{s_3} - 4\,s_1\right)\overline{\mathcal{Z}}\,\mathbf{T}^2 + \left(\dfrac{s_1^3}{s_3} - \dfrac{s_2^3}{s_3^2}\right)\mathbf{T}^3 \end{array}\right)$$

Their common points are A, B, C*,X(4),X(20) and four other points.*

Proof. Straightforward from (9.1) and (3.6). □

Fact 22.4.35. *The barycentric equations of these cubics can be rewritten as:*

$$Darboux = \sum_3 \left(2\,S^2 - S_b\,S_c\right)x\left(b^2z^2 - c^2y^2\right)\ ;\ Lucas = \sum_3 S_a\,x\left(y^2 - z^2\right)$$

while, as of 2019, the following points are known:

P	1	3	**4**	**20**	40	64	84	1490	1498	2130	2131	3182	3183
U	7	2	**4**	69	8	253	189	329	**20**	14362	?	5932	14361
P	3345	3346	3347	3348	3353	3354	3355	3472	3473	3637			
U	1034	1032	?	14365	?	?	?	?	?	?			

Asymptotes of the Darboux cubic are the perpendicular bissectors of the sidelines.

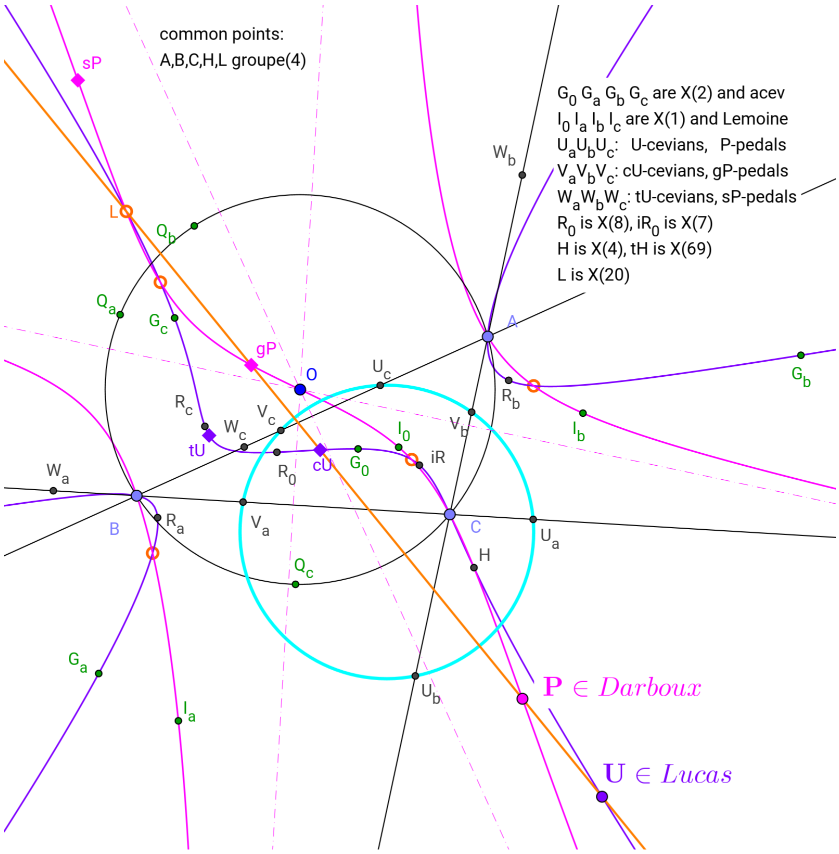


Figure 22.4: The Darboux and Lucas cubics

Definition 22.4.36. The ψ transform sends a point P onto the intersection of lines AA_P and BB_P where $A_PB_PC_P$ is the pedal triangle of P, while the ψ^{-1} transform sends a point U onto the intersection of the perpendicular to BC through U_A with the perpendicular to AC through U_B where $U_AU_BU_C$ is the cevian triangle of U.

Proposition 22.4.37. *When defined that way, i.e. using the same vertex of ABC to play the non-symmetric role, then ψ and ψ^{-1} are reciprocal Cremona transforms of the whole plane and satisfy:*

$$\psi\left(\begin{array}{c} p\\ q\\ r \end{array}\right)\simeq\left(\begin{array}{c} \left(ra^2+pS_b\right)\left(pb^2+qS_c\right)\\ \left(qa^2+pS_c\right)\left(rb^2+q\,S_a\right)\\ \left(rb^2+q\,S_a\right)\left(ra^2+pS_b\right) \end{array}\right) \tag{22.9}$$

$$\psi^{-1}\left(\begin{array}{c} u\\ v\\ w \end{array}\right)\simeq\left(\begin{array}{c} a^2\left(u\,w\,b^2+u\,v\,S_a-v\,w\,S_c\right)\\ b^2\left(v\,wa^2+u\,v\,S_b-u\,w\,S_c\right)\\ v\,w\,S_b\,S_c+u\,w\,S_a\,S_c-u\,v\,S_a\,S_b+4S^2w^2 \end{array}\right) \tag{22.10}$$

Points of indeterminacy of ψ are $a^2:-S_c:-S_b$, $-S_c:b^2:-S_a$ (the directions of AH and BH) and $a^2S_a:b^2S_b:-S_a\,S_b$(the antipode of C, i.e. the point $2O-C$) while points of indeterminacy of ψ^{-1} are A,B and $G_c=A+B-C$.

Proof. Direct computation shows that:

$$\begin{aligned} \mathcal{E}_{\psi} &= 8\,a^2b^2S^2\times\left(rb^2+qS_a\right)\left(a^2r+pS_b\right)(p+q+r)\\ \mathcal{E}_{\psi^{-1}} &= 32\,a^2b^2S^4\times(v+w)\,(u+w)\,(w) \end{aligned}$$

while we have an exact reciprocity on the whole plane when taking eclatements into account. □

Proposition 22.4.38. *When angle in C is not a straight one, then $P \in Darboux$ is equivalent to the alignment of $P, \psi(P)$ with X(20), while $U \in Lucas$ is equivalent to the alignment of $U, \psi^{-1}(U)$ with X(20).*

Proof. Both determinants are the product of S_c by the equation of the corresponding cubic and the corresponding $\mathcal{E}$. When $S_c = 0$, then X(20) comes at G_c and degeneracy is not a surprise. □

Remark 22.4.39. When restricted to both cubics, ψ and ψ^{-1} define two reciprocal central transforms and doesn't depend on the choice of the special vertex.

Proposition 22.4.40. *Darboux, i.e. $p\mathcal{K}(6,20)$, is invariant by isog and by sym3, the reflection about X(3), while Lucas, i.e. $p\mathcal{K}(2,69)$ is invariant by isot and cycl, the cyclopedal transform (as defined at Section 13.23). Moreover, $\psi^{-1} \circ isot \circ \psi = sym3$ holds over the whole plane, while $\psi^{-1} \circ cycl \circ \psi = isog$ holds when restricted to the cubics.*

Proof. When P is on *Darboux*, then $isog(P)$ is also on *Darboux*. By Section 9.3, they share the same pedal circle. Therefore, the corresponding $U \in Lucas$ are cyclocevian. The other formula is easily computed. □

Corollary 22.4.41. *The known points on Lucas cubic can be used to build the following chains. They were emphasized in Kimberling (2002a). When $P = X_3$, then $gP = X_4$, and the center of the cyclopedal circle is X_5; when $P = X_{20}$, this center is X_{5894} ; when $P = X_{40}$, this center is X_{1158}, ,* etc.

$$
\begin{array}{cccccccccccccc}
\boxed{X_1} & \xrightarrow{sym3} & X_{40} & \xrightarrow{isog} & X_{84} & \xrightarrow{sym3} & X_{1490} & \xrightarrow{isog} & X_{3345} & \xrightarrow{sym3} & X_{3182} & \xrightarrow{isog} & X_{3347} \\
1\ \downarrow\psi & & \downarrow\psi & 1158 & \downarrow\psi & & \downarrow\psi & ?? & \downarrow\psi & & \downarrow\psi & & \downarrow\psi \\
\boxed{X_7} & \xrightarrow{isot} & X_8 & \xrightarrow{cycl} & X_{189} & \xrightarrow{isot} & X_{329} & \xrightarrow{cycl} & X_{1034} & \xrightarrow{isot} & X_{5932} & \xrightarrow{cycl} & X_{???}
\end{array}
$$

$$
\begin{array}{cccccccccccccc}
\boxed{X_3} & \xrightarrow{isog} & X_4 & \xrightarrow{sym3} & X_{20} & \xrightarrow{isog} & X_{64} & \xrightarrow{sym3} & X_{1498} & \xrightarrow{isog} & X_{3346} & \xrightarrow{sym3} & X_{3183} \\
\downarrow\psi & 5 & \downarrow\psi & & \downarrow\psi & 5894 & \downarrow\psi & & \downarrow\psi & ?? & \downarrow\psi & & \downarrow\psi \\
\boxed{X_2} & \xrightarrow{cycl} & X_4 & \xrightarrow{isot} & X_{69} & \xrightarrow{cycl} & X_{253} & \xrightarrow{isot} & X_{20} & \xrightarrow{cycl} & X_{1032} & \xrightarrow{isot} & X_{14361}
\end{array}
$$

Since X_1 is fixed by *isog* and X_3 by *sym*3, these chains are unidirectional.

Claim 22.4.42. Let $Q = (sym3 \circ isog \circ sym3 \circ isog \circ sym3)\, P$. Then P is on darboux if and only if $cevamul(P, Q)$ =X(20). When P is on the branch of X_3, so is Q (obvious).

When P is not on *Darboux*, ??? In any case, a simple division by polynomial (22.7) isn't sufficient.

22.4.5.2 The Orion bundle

Lemma 22.4.43. *The nine intersections of K004 and K007 are: A, B, C, X(4)= H, X(20)= $2O - H$ and four other points called the Orion points. Their affixes are the solutions of:*

$$
64\,\mathbf{Z}^4 - 16\left(s_1 + \frac{s_2^2}{s_3}\right)\mathbf{T}\mathbf{Z}^3 + \left(-4\,s_1^4 + \frac{s_1^3 s_2^2}{s_3} + 18\,s_1^2 s_2 - 4\,\frac{s_1\,s_2^3}{s_3} - 75\,s_1\,s_3 + 16\,s_2^2\right)\mathbf{T}^4
$$
$$
+\left(16\,\frac{s_1\,s_2^2}{s_3} + 64\,s_2 - 48\,s_1^2\right)\mathbf{T}^2\mathbf{Z}^2 + \left(28\,s_1^3 - 7\,\frac{s_1^2 s_2^2}{s_3} - 78\,s_1\,s_2 + 12\,\frac{s_2^3}{s_3} + 125\,s_3\right)\mathbf{T}^3\mathbf{Z}
$$

Proof. Direct elimination from the equations. □

Definition 22.4.44. Reflect point $P \simeq p : q : r$ through the sidelines of its cevian triangle $A_P B_P C_P$. The obtained triangle is perspective with triangle ABC, and the perspector is called the **Orion transform** of P (Ehrmann, 2003). Notation and barycentrics are:

$$
\mathfrak{O}(M) \simeq \left[\begin{array}{c} \left(b^2r^2 + c^2q^2 + 2\,S_a\,rq\right)p^3 - a^2q^2r^2p \\ \left(a^2r^2 + c^2p^2 + 2\,S_b\,pr\right)q^3 - b^2p^2qr^2 \\ \left(a^2q^2 + b^2p^2 + 2\,S_c\,pq\right)r^3 - c^2p^2q^2r \end{array}\right]
$$

Remark 22.4.45. When M is on the sidelines, $\mathfrak{O}(M) = M$.

Example 22.4.46. One can identify the following pairs:

1	2	3	4	6	7	8	13	14	15	16	20
35	69	2055	24	2056	57	2057	11581	11582	2058	2059	2060
40	63	69	74	75	98	99	100	101	102	103	104
2061	2062	2063	10419	2064	2065	249	59	15378	15379	15380	15381
105	106	107	108	109	110	111	112	190	253	476	477
15382	15383	15384	15385	15386	250	15387	15388	4998	14572	15395	15396
523	651	675	1113	1114	1141	1292	1293	1294	1295	1296	1297
12064	7339	15397	15461	15460	15401	15402	15403	15404	15405	15406	15407

Definition 22.4.47. The A-Orion cubic K_A is the locus of points M such that $MM_a \perp M_bM_c$ where $M_aM_bM_c$ is the cevian triangle of M.

Proposition 22.4.48. *For M not on the sidelines, $M \in K_A$ is equivalent to the alignment of $A, M, \mathfrak{O}(M)$. Equation of K_A is*

$$K_A \simeq c^2xy^2 - b^2xz^2 + S_b\,y^2z - S_c\,yz^2$$

so that K_A goes once through B, C and twice through vertex A. Tangent at B, C are going through $T_a = 2O - A$ (on circumcircle and Darboux as well), while tangents at A are the bissectors of angle A. The 6th point on Γ is $a^2 : c^2 - b^2 : b^2 - c^2$, the 3rd one on BC is $0 : S_c : S_b$ (the foot of the A-altitude).

Proof. Direct computation, using the gradient (at B, C) and the hessian (at A). □

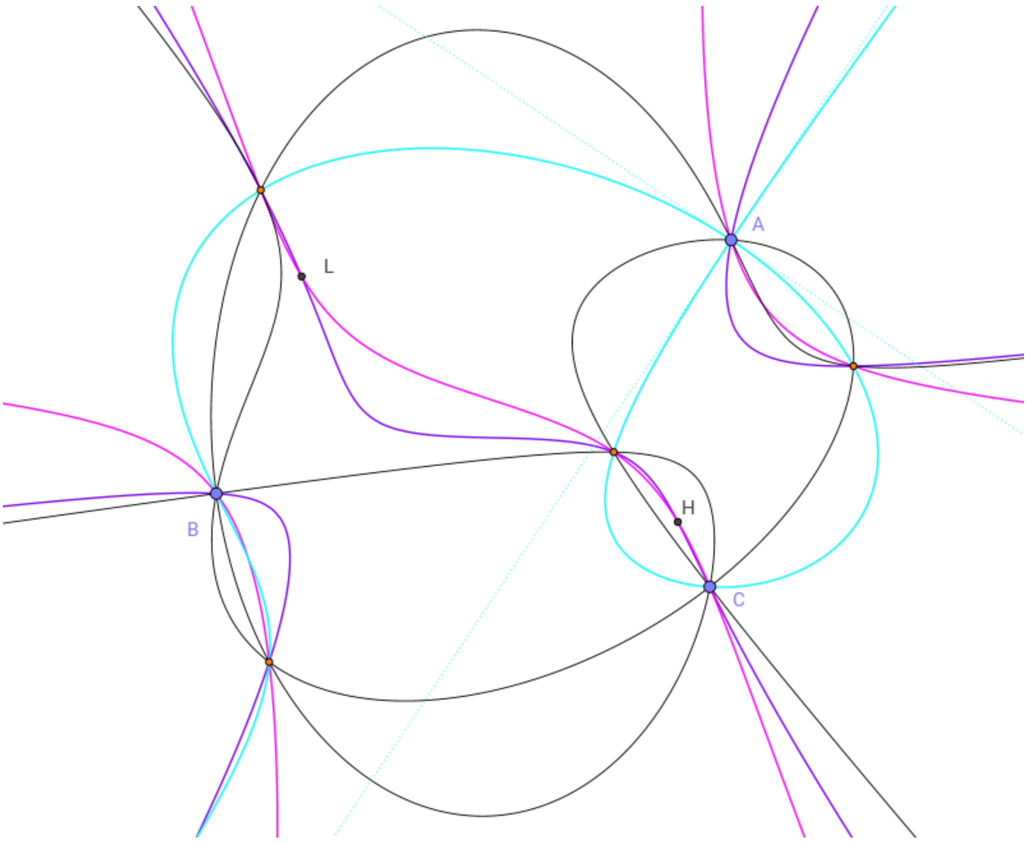


Figure 22.5: The Orion points

Proposition 22.4.49. *The four Orion points are the only ones which are the orthocenter of their cevian triangle. Moreover, the three Orion cubics K_A, K_B, K_C generate the bundle of all the cubics through the 3 vertices and the 4 Orion points.*

Proof. Since B counts twice on K_B, it counts also twice in $K_B \cap K_C$. The same with C, while A counts for one. Thus it remains four other common points. For each of them, M is on two altitudes of $M_a M_b M_c$ and therefore belongs to the third one, so that $M \in K_A$. Therefore the cevian and the pedal triangle of M are equal, and M is on both K004 and K007. Moreover, neither X(4) nor X(20) are the orthocenter of their cevian triangle. □

Proposition 22.4.50. *Let $P \simeq p : q : r$ be a fixed point. When M is not on the sidelines, the alignment of $P, M, \mathfrak{O}(M)$ is equivalent to $M \in K_P$ where K_P is defined by $K_P \doteq p\,K_A + q\,K_B + r\,K_C$.*

Proof. Determinant is linear wrt any column. Moreover, one can check that

$$\begin{array}{rcl} lucas & = K006 = & K_a + K_b + K_c \\ darboux & = K004 = & a^2 S_a K_a + b^2 S_b K_b + c^2 S_c K_c \end{array} \quad \square$$

Proposition 22.4.51. *When P is on the Thomson cubic, then K_p is a $p\mathcal{K}(F,U)$ cubic and we have*

$$F^2 \simeq \operatorname{cevadiv}(G,P) \simeq \begin{pmatrix} (q+r-p)\,p \\ (r+p-q)\,q \\ (p+q-r)\,r \end{pmatrix} \;;\; U \simeq \operatorname{anticomplem} P^* \simeq \begin{pmatrix} b^2rp + c^2pq - a^2qr \\ c^2pq + a^2qr - b^2rp \\ a^2qr + b^2rp - c^2pq \end{pmatrix}$$

Proof. $P \in K002$ comes from elimination. Then symmetric formula for F^2 and U can be checked modulo $K002$. □

Example 22.4.52. Here are some of the K_p cubics.

P	F^2	U	Kxxx	name
X(1)	X(9)	X(8)	K199	
X(2)	X(2)	X(69)	K007	Lucas
X(3)	X(6)	X(20)	K004	Darboux
X(6)	X(3)	X(2)	K168	

Proposition 22.4.53. *Consider the six conics*

$$\begin{array}{rcl} f_a(x,y,z) & = & b^2y^2 - c^2z^2 + x\,(S_c y - S_b z)\,,\ \text{etc} \\ g_a(x,y,z) & = & a^2\left(c^2y^2 - b^2z^2\right) + x\left(c^2 S_c y - b^2 S_b z\right)\,,\ \text{etc} \end{array}$$

Then the f_j are going through the isotomic conjugates of the Orion points while the g_j are going through their iogonal conjugates

Proof. Since A is double in K_A, then x^2yz comes in factor at both isotom K_A and isogon K_A. □

22.4.6 Equal areas (second) cevian cubic aka K155

Definition 22.4.54. Cubic shadow. Triangle centers on a cubic $\mathcal{K}$ yield non-central points on the cubic; e.g., if Q_1 and Q_2 are on $\mathcal{K}$, then the line Q_1Q_2 meets $\mathcal{K}$ in a "third" point, $L(Q_1,Q_2)$, possibly Q_1 or Q_2. If $A'B'C'$ is a central triangle (cf Section 2.2), R a triangle center, $A'' = L(R,A')$ and cyclically, then triangle $A''B''C''$ is a central triangle on $\mathcal{K}$.

Definition 22.4.55. Cubic EAC2, the equal areas (second) cevian cubic is $K155$ in Gibert (2004-2024). This cubic is $p\mathcal{K}(X31, X238)$, i.e self-isoconjugate wrt $P = X_{31} = a^3 : b^3 : c^3$ and pivotal wrt $U = X_{238} = a^3 - abc : b^3 - abc : c^3 - abc$.

Proposition 22.4.56. *It happens that $P \in EAC2$. When a point Q is on EAC2, its isoconjugate Q_P^* aka $X31 \underset{b}{\div} Q$ is on EAC2 too. In the following table, for each (I,J), the centers X(I) and X(J) are on EAC2 and are an isoconjugate pair. Each pair is collinear with the pivot X(238).*

R	A'	A''	B''
$a^{3/2}$	A	$-a^{3/2}$	$b^{3/2}$
$X(2)$	A	$-a^2$	bc
$X(238)$	$-a^2 : bc :$	$-abc$	b^3
$X(31)$	A	$-abc$	b^3
$X(1)$	A	$-a^2(a+b+c)$	$b(bc+ca+ab)$
$X(238)$	upright	$-a(bc+ca+ab)$	$b^2(a+b+c)$
$X(6)$	A	$-a(bc+ca+ab)$	$b^2(a+b+c)$
$X(6)$	$-a^2 : bc :$	$a^2(a+b+c)$	$b\left(a^2+b^2-c(a+b)\right)$
$X(1)$	$-abc : b^3 :$	$a^2(bc+ca+ab)$	$(c-b)a^2b+b^3(c-a)$
$X(31)$	$-a^2 : bc :$	$2a^2\left(b^2+ca\right)\left(c^2+ab\right)$	$b\left(b^2+ca\right)\left(a^3+b^3-c^3-abc\right)$
$X(1)$	$-a^2 : bc :$	$a(a+b)(c+a)\left(a^2+b^2+c^2+bc+ca+ab\right)$	$b^2(c+a)\left(a^2+b^2+ab-c(a+b+c)\right)$
$X(2)$	$-abc : b^3 :$	$2a^3bc\left(b^2+ca\right)\left(c^2+ab\right)$	$\left(b^2+ca\right)\left(b^4c^3+a^3bc^3-a^3b^4-b^3a^2c^2\right)$
$X(6)$	$-abc : b^3 :$	$a(a+b)(c+a)\left(\left(c^2+bc+b^2\right)a^2+bc(b+c)a+b^2c^2\right)$	$b^2(c+a)\left(\left(c^2-bc-b^2\right)a^2+abc(-b+c)+b^2c^2\right)$

Table 22.2: Some cubic shadows on EAC2

$$\begin{bmatrix} 1 & 2 & 105 & 238 & 365 & 1423 & 1931 \\ 6 & 31 & 672 & 292 & 365 & 2053 & 2054 \end{bmatrix}$$

$$\begin{bmatrix} 2106 & 2108 & 2110 & 2112 & 2114 & 2116 & 2118 & 2144 & 2146 \\ 2107 & 2109 & 2111 & 2113 & 2115 & 2117 & 2119 & 2145 & 2147 \end{bmatrix}$$

Table 22.2 gives some cubic shadows on EAC2. Column 1 gives the perspector $R \in \mathcal{K}$. Column 2 gives $A' \in \mathcal{K}$, the A vertex of the original triangle. Columns 3 and 4 give $A'' \in \mathcal{K}$, the A vertex of the shadow triangle. When expressions are growing, these coordinates are given in two rows. For example, in row 1, the perspector is the centroid, the original triangle is ABC itself and $A'' = -a^2 : bc : bc$, the third coordinate being obtained by swapping b and c. Points X_2, $A' = A$ and A'' are collinear. Obviously, $(A'')^*_P$, $(B'')^*_P$, $(C'')^*_P$, is another central triangle inscribed in $\mathcal{K}$.

22.4.7 The cubic K060

Proposition 22.4.57. *Let A', B', C' be the reflections of a point M into the sidelines BC, CA, AB. When triangle $A'B'C'$ is perspective with ABC, point M lies on the Neuberg cubic K001, while the resulting perspector N lies on another cubic (K060).*

Proof. The matrix of the reflection into the sideline BC is :

$$\boxed{\sigma_A} \simeq \begin{pmatrix} -a^2 & 0 & 0 \\ a^2+b^2-c^2 & a^2 & 0 \\ a^2-b^2+c^2 & 0 & a^2 \end{pmatrix}$$

Start from $M = p : q : r$. Compute $A' = \sigma_A(M)$, etc and obtain :

$$A'B'C' \simeq \begin{pmatrix} -pa^2 & \left(a^2+b^2-c^2\right)q+pb^2 & \left(a^2-b^2+c^2\right)r+pc^2 \\ \left(a^2+b^2-c^2\right)p+a^2q & -qb^2 & \left(b^2+c^2-a^2\right)r+qc^2 \\ \left(a^2-b^2+c^2\right)p+a^2r & \left(b^2+c^2-a^2\right)q+b^2r & -rc^2 \end{pmatrix}$$

Then $\det\left(AA', BB', CC'\right)$ is computed and identified with $K001$. Now, start from $N = u : v : w$. Compute $\delta_A = \sigma_A\left(AN\right) = \left(A \wedge N\right) \cdot \boxed{\sigma_A}^{-1}$, etc and obtain :

$$\left(\begin{array}{c} \delta_A \\ \delta_B \\ \delta_C \end{array}\right) \simeq \left(\begin{array}{ccc} 2\,vS_b - 2\,wS_c & -wa^2 & va^2 \\ wb^2 & 2\,wS_c - 2\,uS_a & -ub^2 \\ -vc^2 & uc^2 & 2\,uS_a - 2\,vS_b \end{array}\right)$$

Then $\det\left(\delta_A, \delta_B, \delta_C\right)$ is computed, and we obtain yet another $p\mathcal{K}$ cubic, defined as K060. □

Proposition 22.4.58. K060, the $p\mathcal{K}\,(1989, 265)$ cubic. *The pole P, the pivot U and the Morley equations of this cubic are respectively :*

$$P = \frac{1}{b^2c^2 - 4\,S_a^2} \text{ etc} \quad ; \quad z_P = \frac{\sigma_1^3\sigma_2^2 - \sigma_1\sigma_2^3 - \left(4\,\sigma_1^4 - 9\,\sigma_1^2\sigma_2 + 9\,\sigma_2^2\right)\sigma_3}{3\,\sigma_1^2\sigma_2^2 - 6\,\sigma_2^3 + \left(9\,\sigma_1\sigma_2 - 6\sigma_1^3\right)\sigma_3}$$

$$U = \frac{S_a}{b^2c^2 - 4\,S_a^2} : \frac{S_b}{a^2c^2 - 4\,S_b^2} : \frac{S_c}{a^2b^2 - 4\,S_c^2} \quad ; \quad z_U = \frac{\sigma_1^2 - \sigma_2}{\sigma_1}$$

$$\left(\begin{array}{l} \dfrac{s_2}{s_3}\,\mathbf{Z}^2\overline{\mathcal{Z}} - s_1\,\mathbf{Z}\overline{\mathcal{Z}}^2 + \left(\dfrac{2s_1}{s_3} - \dfrac{s_2^2}{s_3^2}\right)\mathbf{T}\mathbf{Z}^2 + \left(s_1^2 - 2\,s_2\right)\mathbf{T}\overline{\mathcal{Z}}^2 + \\ \quad \left(\dfrac{s_2 - 3\,s_1^2}{s_3} + \dfrac{s_2^2 s_1}{s_3^2}\right)\mathbf{T}^2\mathbf{Z} + \left(\dfrac{3\,s_2^2 - s_1^2 s_2}{s_3} - s_1\right)\mathbf{T}^2\overline{\mathcal{Z}} + \left(\dfrac{s_1^3}{s_3} - \dfrac{s_2^3}{s_3^2}\right)\mathbf{T}^3 \end{array}\right)$$

Both umbilics belong to the curve. The corresponding asymptotes intersect at X(3448). The real asymptote :

$$\left[\sigma_1\sigma_2^2,\, 2\,\sigma_3\sigma_1^3 - 2\,\sigma_2^3, -\sigma_1^2\sigma_2\sigma_3\right]$$

is parallel to the Euler line. The sixth intersection with the circumcircle is X(1141).

Proof. Direct inspection. □

D on K001, F=isogD on K001

$$\begin{array}{rcl} Nd & = & antig(D) = \left(isg \circ inv \circ isg\right)(D) \\ Nf & = & antig\,(F) = \left(isg \circ inv\right)(D) \end{array}$$

22.4.8 Eigentransform

Definition 22.4.59. The mapping $U \mapsto cevadiv\left(U, U_F^{\#}\right)$ is called eigentransform of U wrt pole $P = F^2$. In ETC, F =X(1), i.e. P =X(6), is assumed, and notation $ET\,(U)$ is used. Here, the same notation is used, but isogonal conjugacy isn't assumed.

Example 22.4.60. Assuming P =X(6), pairs (I,J) such that X(J) = ET(X(I)) include :

1	1	13	62	81	3293	174	266	664	2082	1156	1		
2	3	14	61	86	3294	190	1	673	1	1492	1		
3	1075	19	2128	88	1	512	2142	694	384	1821	1		
4	155	20	2130	92	47	648	185	771	1	1942	1941		
5	2120	30	2132	94	49	651	1	799	1				
6	194	37	2134	99	39	653	1	811	2083				
7	218	57	2136	100	1	655	1	823	1				
8	2122	63	1712	101	2140	658	1	897	1				
9	2124	69	2138	110	5	660	1	1113	3				
10	2126	75	2172	162	1	662	1	1114	3				

Proposition 22.4.61. *For any point U not on a sideline of triangle ABC, the following properties of eigentransform are easy to verify :*

1. *The barycentrics of $ET(U)$ are (cyclically) :*

$$vwf^2\left(u^2v^2h^2+u^2w^2g^2-v^2w^2f^2\right)$$

2. *$ET(U)$ is the eigencenter of the cevian triangle of U as well as the eigencenter of the anticevian triangle of U_P^*.*

3. *$ET(U)=F\doteq f:g:h$ (fixed point of the isoconjugacy) if and only if $U=F$ or U lies on the $CC(F)$ circumellipse. When P =X(6), then F =X(1) and this locus is the Steiner circumellipse: $yz+zx+xy=0$.*

4. *Points U, $ET(U)$ and $(ET(U))_F^{\#}$ are collinear points of the cubic $p\mathcal{K}(P,U)$.*

5. *Points F, $ET(U)$ and $(cevadiv(U,F))_F^{\#}$ are collinear. The last point is also on the cubic.*

6. *ET(U) is the tangential of U_F^*.*

22.5 Non pivotal isocubics nK(P,U,k) and nK0(P,U)

Definition 22.5.1. The non pivotal isocubic with pole P, root U and parameter k is defined by the equation :

$$n\mathcal{K}(P,U,k)\doteq ux\left(y^2r+qz^2\right)+vy\left(z^2p+rx^2\right)+wz\left(qx^2+py^2\right)+k\,xyz \tag{22.11}$$

When $k=0$, the cubic is noted $n\mathcal{K}0(P,U)$.

Remark 22.5.2. The more efficient method for specifying k is to indicate a point that belongs to the cubic. This is noted $n\mathcal{K}(P,U,X)$.

Proposition 22.5.3. *The "third intersections" of a $n\mathcal{K}(P,U,k)$ with the sidelines are the cocevians of the root U. Therefore, they are aligned. This is to be compared with the fact that, for a p$\mathcal{K}$ cubic, these points are the cevians of the pivot.*

Proof. Direct inspection. □

Remark 22.5.4. In the general case, a $n\mathcal{K}0(P,U)$ contains neither P nor U nor any of the four fixed points F of the conjugacy.

Definition 22.5.5. We define the F-**crosssum** of two points $U=u:v:w$ and $X=x:y:z$ that aren't lying on a sideline of ABC as :

$$\mathrm{crosssum}_F(U,X)=f^2(wy+vz):g^2(uz+wx):h^2(vx+uy)$$

Remark 22.5.6. In ETC, F =X(1) is assumed. Defined as above, the operation $(F,U,X)\mapsto \mathrm{crosssum}_F(U,X)$ is globally type-keeping and provides a point when the entries are points (F is any of the four fixed point of the conjugacy $X\mapsto X_P^*$).

Definition 22.5.7. The $NK_P(X)$ point is the pole of the line $X\,X_P^*$ with respect to the circumconic that passes through X and X_P^* (Bernard Gibert, 2003/10/1). Using barycentrics and $P=p:q:r=f^2:g^2:h^2$, we have :

$$\begin{aligned} NK_p(X) &= p\,x\left(r\,y^2+q\,z^2\right):q\,y\left(p\,z^2+r\,x^2\right):rz\left(q\,x^2+p\,y^2\right)\\ &= \mathrm{crosssum}_F\left(X,\,X_P^*\right)=\mathrm{crossmul}\left(X,\,X_P^*\right)=\left(\mathrm{cevamul}\left(X,\,X_P^*\right)\right)_P^* \end{aligned}$$

Proof. Direct computation. □

Remark 22.5.8. Here, X_P^* is the perspector of the cevian triangle of $NK_P(X)$ and the anticevian triangle of X.

Example 22.5.9. Using F=X(1), i.e. P=X(6), we have NK(X(I)) = X(J) for these (I,J):

I	1	2	3	4	6	9	19	31	57	63
J	1	39	185	185	39	2082	2083	2085	2082	2083

Proposition 22.5.10. *When $NK_P(X)$ belongs to the tripolar line of U_P^*, then point X belongs to the cubic $n\mathcal{K}0(P,U)$.*

Proof. Direct computation. In Kimberling (1998, p. 240), notation $Z+(XY)$ is used to denote the $n\mathcal{K}0\,(\#1,\,U)$ cubic where the pole U is the isogonal of the tripole of line XY. Therefore,

$$\begin{array}{rcl} Z^{+}(X_1X_6) &=& n\mathcal{K}0\,(\#1,\,513) \\ Z^{+}(X_3X_6) &=& n\mathcal{K}0\,(\#1,\,523) \\ Z^{+}(X_1X_2) &=& n\mathcal{K}0\,(\#1,\,649) \\ Z^{+}(X_1X_3) &=& n\mathcal{K}0\,(\#1,\,650) \end{array}$$

□

22.5.1 vanRees cubic

Remark 22.5.11. vanRees cubics are studied in detail at Section 28.11. Seen from triangle ABC, they are described by

$$\left[nkcub, \#F \simeq \begin{bmatrix} a \\ b \\ c \end{bmatrix}, U \simeq \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \widehat{X} \simeq \begin{bmatrix} q-r \\ r-p \\ p-q \end{bmatrix} \right]$$

Remember: $\#F$=X(1) means P=X(6), while $\widehat{X}$ is a point on the cubic, only intended to define the coefficient of xyz in the equation.

22.5.2 Conicopivotal isocubics cK(#F,U)

Definition 22.5.12. A conico-pivotal isocubic $c\mathcal{K}\,(\#F,U)$ (Ehrmann and Gibert, 2005) is a non pivotal isocubic $n\mathcal{K}\,(P,U,k)$ that contains one of the fixed points of the isoconjugacy ($F \neq U$ is assumed). When using $F = f : g : h$ instead of $P = p : q : r = f^2 : g^2 : h^2$, $k = -2\,(ghu + fhv + fgw)$, the equation becomes :

$$x(gz-hy)^2u + y(fz-hx)^2v + z(fy-gx)^2w = 0 \tag{22.12}$$

Proposition 22.5.13. *The pivotal conic is defined as the conic $\mathcal{C}$ tangent to the six lines F_BF_C, AA'_U and cyclically where $F_AF_BF_C$ are the anticevian of F and $A'_UB'_UC'_U$ the cocevian of U. Then the dual conic of $\mathcal{C}$ is conicev $(1/F, 1/U)$ and $\mathcal{C}$ itself has equation :*

$$\sum_{cyclic} (gw-hv)^2\,x^2 - 2\,\left(gu^2h + 3\,f\,(gw+vh)\,u + f^2vw\right)zy = 0$$

Proof. We have the equations :

$$\begin{array}{rcl} (F_BF_C) &=& F_B \wedge F_C = \begin{pmatrix} f \\ -g \\ h \end{pmatrix} \wedge \begin{pmatrix} f \\ g \\ -h \end{pmatrix} = [0,\,2fh,\,2fg] = \left[0,\,\dfrac{1}{g},\,\dfrac{1}{h}\right] \\ AA'_U &=& U_BU_C = \left[0,\,\dfrac{1}{v},\,\dfrac{1}{w}\right] \end{array}$$

The equation of $\mathcal{C}$ follows by duality. Barycentrics of the center are $2\,fu - (v+w)\,f - (g+h)\,u$, etc.

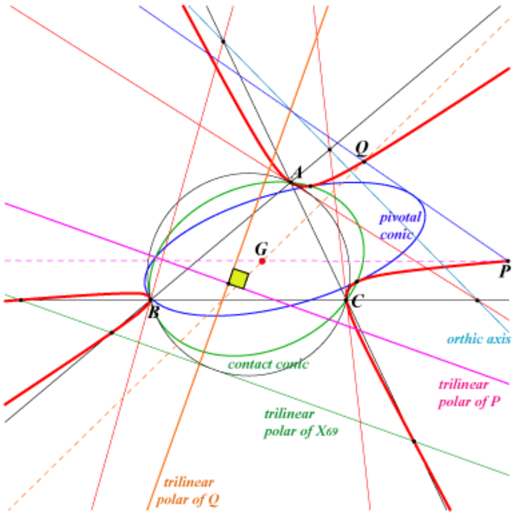
□

Proposition 22.5.14. *The contact conic (K) is defined as the circumconic whose perspector is*

$$K \simeq \left(2\,\frac{f}{u} + \frac{g}{v} + \frac{h}{w}\right) f, \text{ etc}$$

Assuming $F \neq U$, three of the intersections of the pivotal and contact conics are the three contacts of $c\mathcal{K}$ with $\mathcal{C}$, the fourth point being :

$$T_4 \simeq \left(2\,\frac{f}{u} + \frac{g}{v} + \frac{h}{w}\right) \div (gw - vh)\,, \text{ etc}$$



Centroid $G = X_2$ is isolated, but belongs nevertheless to the cubic.

Figure 22.6: The Simson cubic (as depicted in Gibert-CTP)

Proof. Eliminate z between (K) and $\mathcal{C}$. Obtain $P_1(x,y)$ $P_3(x,y)$, where degrees are respectively 1 and 3. Solving for P_1 gives directly T_4. Eliminate z between $c\mathcal{K}$ and $\mathcal{C}$. This leads again to P_3, proving that each common point is a contact and belongs also to (K). □

Example 22.5.15. A special case is obtained when $U = F$, i.e. when the root is a fixed point of the isoconjugacy. Then $\mathcal{C} = (K)$ is a circumconic. The Tucker cubic K015 is obtained with F =X(2), while K228 is obtained with F =X(1) and K229 is obtained with F =X(6).

22.5.3 Simson cubic, aka K010

Notation 22.5.16. In this section, the involved pole is the centroid, so that $X^* = isot(X)$. Due to the nature of the cubic, the key point is $F = X_2$ (involved as fixed point) rather than $P = X_2$ (involved as pole). Therefore, letter P has been used not to describe the pole, but the independent moving point of various parametrization.

Definition 22.5.17. The **Simson cubic** is the locus of the tripoles of the Simson lines. Depicted as K010 (cf. Figure 22.6) in Gibert (2004-2024). Founding paper is Ehrmann and Gibert (2001).

Proposition 22.5.18. *The Simson cubic (K010) is* $n\mathcal{K}(\#X_2,\, X_{69},\, X_2)$ *shortened into* $c\mathcal{K}(\#X_2,\, X_{69})$. *Centroid* $G = X_2$ *belongs to K010 (Simson line of* Q *is* $\mathcal{L}_b$ *when* $Q \in \mathcal{L}_b$*). Apart from this isolated point, a parametrization of K010 is given in (28.3), starting from* $\mathcal{L}_b$. *An other parametrization, using the barycentrics of the involved point on* Γ *is as follows :*

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \Gamma \mapsto \begin{pmatrix} b^2wu^2 - c^2vu^2 + \left(b^2 - c^2\right)wvu \\ c^2uv^2 - a^2v^2w + \left(c^2 - a^2\right)wuv \\ a^2vw^2 - w^2b^2u + \left(a^2 - b^2\right)wuv \end{pmatrix} \in K010$$

Definition 22.5.19. Cubic K162 is the isogonal transform of the Simson cubic (that can also be obtained by $Q \mapsto Q \underset{b}{\div} X_6$). Therefore, K162 is $c\mathcal{K}(\#X_6,\, X_3)$.

Definition 22.5.20. The **Gibert-Simson transform** is another parametrization of the Simson cubic that also uses an $U \in \Gamma$:

$$GS(U) = \text{cyclic}\left[\left(\frac{b^2\left(c^2 + a^2 - b^2\right)}{va^2} - \frac{c^2\left(a^2 + b^2 - c^2\right)}{a^2w}\right)u\right]$$

The lack of uniqueness is due to the binding relation $\sum a^2vw = 0$. Definition introduced in ETC on 2003/10/19, leading to points X(2394) - X(2419) on the Simson cubic and points X(2420)-X(2445) –their isoconjugates– on K162.

Remark 22.5.21. Regarding triangle centers that do not lie on the circumcircle, GS(X(I)) = X(J) for these (I,J): (32,669), (48,1459), (187,1649), (248,879), (485,850), (486,850). Of course, other realizations of $U \mapsto K$ give other results. Here again, only parametrization (28.3) ensures uniqueness.

Example 22.5.22. Use ${}^tP = {}^tX_{525}$ as entry point Figure 22.7 (arrow at the left of the bottom diagram). Obtain $X_{30} = Q_1 \in \mathcal{L}_b$ by (7.16), then $X_{74} = U_1 \in \Gamma$ by isogonal conjugacy. The Steiner line St_1 hasn't received any name, while the Simson line Si_1 of U_1 is ${}^tX_{247}$. The trilinear pole of this line, i.e. $X_{2394} = K_1 \in K010$, can be obtained by $isot \circ ({}^t)$ from Si_1, by gs from U_1 and also directly from P (the dotted line) using parametrization (28.3).

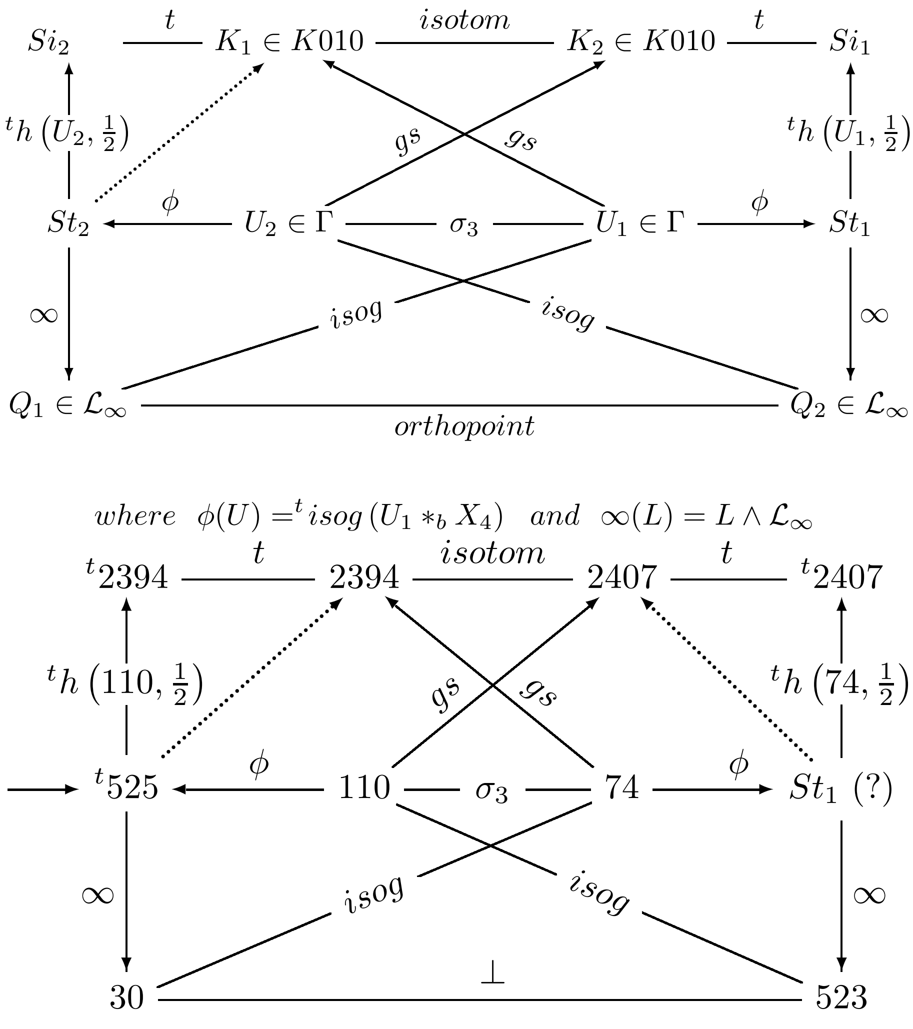


Figure 22.7: The Simson diagram

Proposition 22.5.23 (Fools' Day Theorem)**.** *Direct arrows $U_1 \mapsto K_2$ and $U_2 \mapsto K_1$ have a geometrical meaning : $(gs\,(U))^*$ is the eigencenter of the pedal triangle of U.*

Proof. Point K_2 has no other choice : he *is* the unary cofactor of the pedal triangle of U, and therefore the perspector of this triangle with anything else. ☐

Proposition 22.5.24. *The barycentric equation of the Simson cubic is*

$$\sum_{cyclic} x(z^2 + y^2)(b^2 + c^2 - a^2) - 2(a^2 + b^2 + c^2)x\,y\,z \qquad (22.13)$$

In other words, the Simson cubic is $n\mathcal{K}\,(X_2, X_{69}, \#X_{2394})$. Moreover, one of the fixed points (namely $G = X_2$) belongs to the cubic and the Simson cubic is in fact $c\mathcal{K}\,(\#X_2, X_{69})$.

Proof. Obtained from the parametric representation. The converse property is more easily obtained from next coming proposition. □

Definition 22.5.25. Special points wrt K010. Points on sidelines of triangle ABC or of antimedial triangle, together with the centroid are said to be special wrt K010 (because quite every "general" formula turns wrong when dealing with these points).

Proposition 22.5.26. *Among the special points, the following are the sole and only elements of K010 :*
(i) then centroid itself (fixed point under isoconjugacy)
(ii) the vertices of triangle ABC, the cocevians of X_{69} (the root) and points $b+c : b-c : c-b$ or $b-c : b+c : -b-c$ and cyclically.

Proof. Direct inspection. □

Proposition 22.5.27. *When point X is on the Simson cubic but not G, A, B, C then :*
(i) the trilinear polars of X and X^* are perpendicular
(ii) the trilinear polars of X and X^* are concurrent on the nine-point circle
Conversely, when X is not special and either property holds, then X is on the cubic.

Proof. When X is on the Simson cubic, $tripolar\,(X)$ is a Simson line and conclusion follows from $tripolar = {}^t\circ isot$. Conversely, if (i) then points ${}^tX^* \wedge \mathcal{L}_b$ and ${}^tX \wedge \mathcal{L}_b$ are the infinity points of both tripolars. They have to be the orthopoint of each other, and (22.13) is re-obtained by elimination. If (ii) then dividing $nineq\,(X \wedge X^*)$ by (22.13), leads to $\prod (y+z)\,/x^2$. When X is on a sideline of ABC, conjugacy is no more defined, and when $y+z=0$ (implying X on the sidelines of the antimedial triangle) then $X \wedge X^*$ is ever $0:1:1$. Outside of these six lines, both conditions are equivalent. □

22.5.4 Brocard second cubic aka K018

Definition 22.5.28. The Brocard second cubic is inventoried as K018 in (Gibert, 2004-2024). This cubic is $n\mathcal{K}0\,(X_6,\, X_{523})$. It is a circular isogonal focal nK cubic with root X(523) and singular focus X(111). The real asymptote is parallel to GK. It is also the orthopivotal cubic O(X6) and Z+(L) with L = X(3)X(6) in TCCT p.241. See also Z+(O) = CL025 and CL034.

Proposition 22.5.29. *The barycentric equation of K018 is :*

$$\sum_{cyclic} x(b^2z^2+y^2c^2)(b^2-c^2)=0 \tag{22.14}$$

Chapter 23

Tripolar curves

Definition 23.0.1. Given three fixed distinct points E, F, G, and three real numbers u, v, w, "this branch of a tripolar curve" $\mathcal{W}(u, v, w)$ is the locus of points M such that :

$$u\ |EM| + v\ |FM| + w\ |GM| = 0$$

On the contrary, the corresponding algebraic tripolar curve is the locus of points M such that

$$\pm\, u\ |EM| \pm v\ |FM| \pm w\ |GM| = 0 \tag{23.1}$$

Remark 23.0.2. When one of the u, v, w vanishes, a tripolar curve degenerates into an ordinary conic (this is excluded in what follows).

Notation 23.0.3. In this chapter, quantities $\alpha, \beta, \gamma, \delta$ are related to points E, F, G, H, while u, v, w are some multipliers. Quantities $\underset{Q}{S}, S_u, S_v, S_w$ are related to these multipliers u, v, w. They mimic the usual area and Conway symbols. In other words,

$$\underset{Q}{S}^2 = \frac{1}{16}\,(u+v+w)\,(-u+v+w)\,(u-v+w)\,(u+v-w)\,,\; S_u = \frac{1}{2}\left(v^2+w^2-u^2\right)$$

On the contrary, symmetric functions are related to $\alpha, \beta, \gamma, \delta$. In other words,

$$q_1 \doteq \sum_4 \alpha\ ;\ q_2 = \sum_6 \alpha\beta\ ;\ q_3 = \sum_4 \alpha\beta\gamma\ ;\ q_4 = \alpha\beta\gamma\delta \tag{23.2}$$

Remark 23.0.4. Section Section 23.1 provides some background by studying the whole space of the bicircular quartics. Then a section describes how to use Maple and Geogebra to draw the tripolar curves. A later section gives some further properties of these curves.

23.1 The bicircular space

Definition 23.1.1. A general quartic q requires 5+4+3+2+1=15 coefficients. When q is singular at each umbilic, its gradient has to vanish there, so that

$$q_{40} = q_{31} = q_{30} = q_{03} = q_{13} = q_{04} = 0$$

The set of all bicircular quartics is therefore a copy of $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^9\right)$. In this chapter, "quartic" is assumed everywhere, and this space will simply be depicted as "the bicircular space $\mathbb{P}(\mathcal{Q})$" (see Werner, 2012, p.86)).

Proposition 23.1.2. *The equation of any curve $q \in \mathbb{P}(\mathcal{Q})$ can be written in the following matrix form:*

$$q(M) \doteq {}^t\!\begin{pmatrix} \overline{\mathcal{Z}}^2 \\ \overline{\mathcal{Z}}\mathbf{T} \\ \mathbf{T}^2 \end{pmatrix} \cdot \begin{bmatrix} q_{22} & q_{21} & q_{20} \\ q_{12} & q_{11} & q_{10} \\ q_{02} & q_{01} & q_{00} \end{bmatrix} \cdot \begin{pmatrix} \mathbf{Z}^2 \\ \mathbf{ZT} \\ \mathbf{T}^2 \end{pmatrix} = 0$$

Therefore, any homography H acting over the points of $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ according to

$$(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}})\mapsto\left(\frac{\mathrm{a}\mathbf{Z}+\mathrm{b}\mathbf{T}}{\mathrm{c}\mathbf{Z}+\mathrm{d}\mathbf{T}}:1:\frac{\mathrm{a}'\overline{\mathcal{Z}}+\mathrm{b}'\mathbf{T}}{\mathrm{c}'\overline{\mathcal{Z}}+\mathrm{d}'\mathbf{T}}\right)$$

induces an action which is linear over the bicircular space $\mathbb{P}\left(\mathcal{Q}\right)$ according to

$$\boxed{q}\mapsto\left[\begin{array}{ccc}\mathrm{d}'^2 & -\mathrm{c}'\,\mathrm{d}' & \mathrm{c}'^2\\ -2\,\mathrm{b}'\,\mathrm{d}' & \mathrm{a}'\,\mathrm{d}'+\mathrm{b}'\,\mathrm{c}' & -2\,\mathrm{a}'\,\mathrm{c}'\\ \mathrm{b}'^2 & -\mathrm{a}'\,\mathrm{b}' & \mathrm{a}'^2\end{array}\right]\cdot\boxed{q}\cdot\left[\begin{array}{ccc}\mathrm{d}^2 & -2\,\mathrm{b}\,\mathrm{d} & \mathrm{b}^2\\ -\mathrm{c}\,\mathrm{d} & \mathrm{a}\,\mathrm{d}+\mathrm{b}\,\mathrm{c} & -\mathrm{a}\,\mathrm{b}\\ \mathrm{c}^2 & -2\,\mathrm{a}\,\mathrm{c} & \mathrm{a}^2\end{array}\right]$$

Proof. Obvious from the definitions. □

Proposition 23.1.3. *The bicircular quartic q admit four singular foci, namely*

$$\frac{-q_{21}\pm\sqrt{q_{21}^2-4\,q_{22}\,q_{20}}}{2\,q_{22}}:1:\frac{-q_{12}\pm\sqrt{q_{12}^2-4\,q_{22}\,q_{02}}}{2q_{22}}$$

Proof. Direct computation: cut by $M\Omega_x$, factor, substitute $\mathbf{T}=0,\mathbf{Z}=1$ and equate to 0. □

Proposition 23.1.4. *Among the elements of $\mathbb{P}\left(\mathcal{Q}\right)$, we have (1) the union (=product) of two cycles ; (2) the image of any conic by an homography.*

Proof. Direct computation □

Proposition 23.1.5. *Whatever could be meaning of* $\sqrt{ux}$, etc, *then the three terms relation (TTR)*

$$\sqrt{ux}+\sqrt{vy}+\sqrt{wz}=0 \tag{23.3}$$

can be rewritten as:

$$\begin{aligned}\mathrm{TTR}\quad&\iff\quad u^2x^2+v^2y^2+w^2z^2-2\left(uv\,xy+vw\,yz+wu\,zx\right)=0\\ &\iff\quad [x,y,z]\cdot\left[\begin{array}{ccc}u^2 & -uv & -uw\\ -uv & v^2 & -vw\\ -uw & -vw & w^2\end{array}\right]\cdot{}^t[x,y,z]=0\end{aligned}$$

1. *When x,y,z are the coordinates of a line wrt the reference trigone, the TTR describes a tangential conic$\mathfrak{C}^*$. The associated punctual conic is the circumscribed conic*

$$\mathfrak{C}\simeq\left[\begin{array}{ccc}0 & w & v\\ w & 0 & u\\ v & u & 0\end{array}\right]$$

2. *When x,y,z are circles' equations, one obtains the equation of a quartic $\boxed{\mathcal{Q}}$. And this quartic is bitangent to each of these circles.*

Proof. Direct computation. □

Proposition 23.1.6. *Assume that u,v,w be three coefficients and $\mathcal{C}_j, j=1..3$ be three circles, with centers z_j and radiuses r_j. This defines a common orthogonal cycle $\mathcal{C}_0$, and a z_j-circumscribed conic $\mathfrak{C}$. A moving point $P\in\mathfrak{C}$ defines a circle $\mathcal{C}_P$ orthogonal to $\mathcal{C}_0$. Then, the enveloppe of the circles $\mathcal{C}_P$ is the bicircular quartic $\boxed{\mathcal{Q}}$ of the just above proposition.*

Proof. The usual property of the enveloppes !(Casey, 1871, §5, p.459) □

Example 23.1.7. All the three sub-figures of Figure 23.1 have been drawn with $z_1=-2-3i$, $z_2=2+3i$, $z_3=3-i$ and $u=5,v=6,w=11$ Thus, they share the same focal conic. But using $1,2,1$, resp $3,2,1$ and $3,2,3$ as radiuses lead to different orthogonal circles. As a result, there are resp. four, none and two visible intersections of $\mathfrak{C}$ with $\mathcal{C}_0$. are visible on the top. A not visible point is alluded with its $\mathbf{Z}:\mathbf{T}$ part, noted F_j'. Then a "true" intersection is $F_{34}\simeq F_3':1:\overline{F_4'}$, etc (on the $\mathcal{C}_0$ circle), while F_3' and F_4' are inverse in the focal circle.

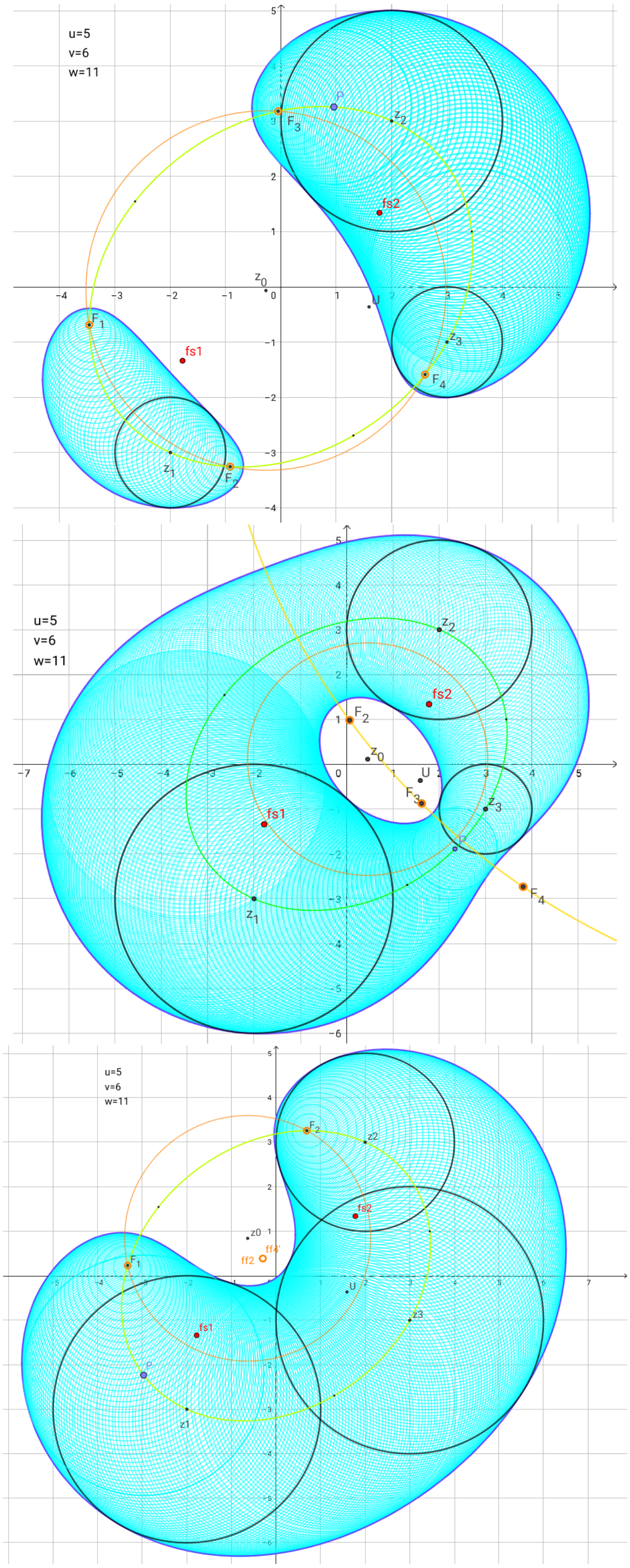


Figure 23.1: Does the focal conic intersect the orthogonal circle?

Proposition 23.1.8. *The four intersections of the focal conic and the orthogonal cycle, visible or not, are four of the 16 foci of the quartic.(Casey, 1871, §16, p.464) On the other hand, the foci of the focal conic are the singular foci of the quartic.*

Proof. (1) Each of these intersections define a null circle which is bitangent to the curve. But a null circle is the product of the isotropic lines through its center. (2) Direct computation. □

23.2 Define and draw

Maple 23.2.1. The Maple library `tcurv.m` deals with the following objects:

zptE1;zptF1;zptG1;zptH1 the four focuses

zptDe_1;zptDf_1;zptDg_1;les3dia_1 the diagonal triangle

VALdelta1 value of δ from $a, b, c, \alpha, \beta, \gamma$

VALeq11 raw $\mathbf{Z}, \mathbf{T}, \overline{\mathcal{Z}}$ equation from $a, \alpha, b, \beta, c, \gamma$

VALmeth2 new values of a, b, c when changing the poles

ledeb vector $a : b : c$

valabc_1;valabc_2 values of a, b, c from $\alpha, \beta, \gamma, \delta, \nu$

zptK0_2;zptKa_2;zptKb_2;zptKc_2;les4cir_2;VALnu the four points on the focal circle.

zptX60_1;zptX6e_1;zptX6f_1;zptX6g_1

rotEFG;rotEFG3;les4moines rotate the variables

Maple 23.2.2. contains the procedures

`geotcurv`

and `buildmeth` .

The procedure receives $u, v, w \in \mathbb{R}$ together with $E, F, G \in \mathbb{C}$ and produces $\alpha, \beta, \gamma, \delta \in \Gamma$ (the reduced coordinates of the four focuses). The seventh argument $\nu \in \Gamma$ is used to select a connected piece of the curve. When given, the procedure checks that ν in one of the four acceptable values. Otherwise, one of the four possibilities is returned.

The result is two Maple sequences (to be used in Lubin1 or Lubin2 context), together with a set of commands to be transmitted to geogebra.

Maple 23.2.3. The curve, already depicted using a set of three focuses, is depicted using the other triples of focuses. And then, one can check that all the four descriptions are leading to the same cartesian equation *eqq1*.

23.3 The generic case

Proposition 23.3.1. *The tripolar curve is globally invariant under the reflection μ into the EFG cycle.*

Proof. This is obvious if EFG are aligned. Otherwise, let U be the center of the reflection circle. Then, for generic P, Q, the triangles (U, P, Q) and $(U, \mu(Q), \mu(P))$ are similar, leading to

$$|\mu(P)\,\mu(Q)| = |PQ| \times \frac{R^2}{|UP|\,|UQ|} \text{ therefore } |E\mu(P)| = |EP| \times \frac{R}{|UP|}$$

□

Require: *VALeq11* (23.4), *VALnu* (23.5), *VALdelta1* (23.6) are already stored somewhere

```
GEOTCURV := proc u_, v_, w_, E_, F_, G_, nu_
global icidou1, icidou2, eqq1, cen_, rad2, rad1, VALeq11, VALnu, VALdelta1
local msg, H_, z0_, icinu
(op@map)(z2mor, [E_, F_, G_]) ; cen_, rad2 := (colu2mor@zcircle3)(%)
rad1 := sqrt(rad2) ; z0_ := mor2z(cen_)
icidou1 := α = (E_ − z0_)/rad1, β = (F_ − z0_)/rad1, γ = (G_ − z0_)/rad1
icidou1 := u = u_, v = v_, w = w_, icidou1
H_ := (factor@subs) (icidou1, VALdelta1)
icidou2 := α = sqrt((E_ − z0_)/rad1), β = sqrt((F_ − z0_)/rad1), γ = sqrt((G_ − z0_)/rad1)
icidou2 := u = u_, v = v_, w = w_, icidou2
if nargs = 7 then
    icinu := nu_ ;
    {seq} ((factor@subs) (j = −j, icidou2, VALnu), j = {u, v, w, qxq})
    print(%) ; ASSERT(member(icinu, %))
else
    icinu := (factor@subs)(icidou2, valnu)
end if
icidou1 := icidou1, delta = H_, nu = icinu
icidou2 := icidou2, delta = sqrt(H_), nu = icinu
eqq1 := collect( subs(icidou1, Z = x + I y, ZZ = x − I y, T = 1, VALeq11),
                 [x, y], real, factor, distributed )
convert(‘\n\nExecute
  [{ "I = ToComplex[(0, 1)]", "O = (0, 0)", "cir = Circle[O, 1]",
     "u = AAA", "v = BBB", "w = CCC",
     "α = EEE", "E = O + α", "β = FFF", "F = O + β", "γ = GGG", "G = O + γ",
     "δ = HHH", "H = O + δ", "tcur = ImplicitCurve(TTCCUR)", "nu = NUU" }]
\n\n‘, string)
SubstituteRec( %,
     AAA, convert(u_, string), EEE, convert(subs(icidou1, α), string),
     BBB, convert(v_, string), FFF, convert(subs(icidou1, β), string),
     CCC, convert(w_, string), GGG, convert(subs(icidou1, γ), string),
     HHH, convert(subs(icidou1, δ), string),
     TTCCUR, convert(eqq1, string), NUU, convert(icinu, string) )
return convert(%, symbol)
```

Listing 23.1: The geotcurv procedure

```
Require: geotcurv has already been executed
BUILDMETH := proc; global icimeth, VALmeth2, icidou2
local mynu, ddist, rr, test, k, mymeth
map(evalc, subs(icidou2, VALmeth2))
(combine@redurow@SubMatrix) (%, 1..4, 4..6)
icimeth := ⟨SubMatrix(%%, 1..4, 1..3), %, SubMatrix(%%, 1..4, 7..9))⟩
mynu := subs(icidou2, nu)
ddist := (α, β) ↦ √(abs ((α − β)²/α/β))
for rr to 4 do
  test := add(icimeth[rr, j] ∗ ddist(mynu, icimeth[rr, j + 3]), j = 4..6)
  if factor(test) = 0 then
    print(rr, OK, 0)
  else
    for k to 3 do
      mymeth := Copy(icimeth)
      mymeth[rr, 3 + k] := −mymeth[rr, 3 + k]
      test := add(mymeth[rr, j] ∗ ddist(mynu, mymeth[rr, j + 3]), j = 4..6)
      if factor(test) = 0 then
        print(rr, OK, k) ; icimeth := Copy(mymeth) ; Break()
      end if
    end for
  end if
end for
return icimeth
```

Listing 23.2: The buildmeth procedure

Notation 23.3.2. For this reason, we will use turns $\alpha, \beta, \gamma, \delta$ to represent the points E, F, G (and the next coming point H). Part of the time, Lubin-1 will be used, and results are written as $z_E \underset{1}{=} \alpha$, etc. Part of the time Lubin-2 will be used and results are written as $z_E \underset{2}{=} \alpha^2$. Increasing the degree allows to split some algebraic equations at the price of lengthening the expressions. Coexistence of both systems requires to tag any equal sign. This third option is the one we have chosen.

Proposition 23.3.3. *When* $\underset{Q}{S} \neq 0$*, the equation of the tripolar curve can be written as*

$$\begin{aligned}
\mathcal{W}(M) &\underset{2}{=} \left(\mathbf{Z}^2\overline{\mathcal{Z}}^2 + \mathbf{T}^4\right) + \left(\mathbf{Z}\overline{\mathcal{Z}} + \mathbf{T}^2\right)\left(W_{21}\mathbf{Z} + W_{12}\overline{\mathcal{Z}}\right)\mathbf{T} + \mathbf{T}^2\left(W_{20}\,\mathbf{Z}^2 + W_{11}\,\mathbf{Z}\overline{\mathcal{Z}} + W_{02}\overline{\mathcal{Z}}^2\right) \\
&= {}^t\!\begin{pmatrix} \overline{\mathcal{Z}}^2 \\ \overline{\mathcal{Z}}\mathbf{T} \\ \mathbf{T}^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & W_{12} & W_{02} \\ W_{21} & W_{11} & W_{12} \\ W_{20} & W_{21} & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{Z}^2 \\ \mathbf{Z}\mathbf{T} \\ \mathbf{T}^2 \end{pmatrix} \qquad \text{where}
\end{aligned} \tag{23.4}$$

$$W_{21} = -2\sum_3 \frac{S_u\,u^2}{8\,\underset{Q}{S}^2}\frac{1}{\alpha^2} \;;\; W_{12} = -2\sum_3 \frac{S_u\,u^2}{8\,\underset{Q}{S}^2}\alpha^2 \;;\; W_{20} = \frac{1}{16\underset{Q}{S}^2}\prod_4\left(\frac{u}{\alpha^2} \pm \frac{v}{\beta^2} \pm \frac{w}{\gamma^2}\right)$$

$$W_{11} = \frac{-1}{4\underset{Q}{S}^2}\sum u^4 + \frac{1}{8\underset{Q}{S}^2}\sum_3 v^2w^2\left(\frac{\beta}{\gamma} + \frac{\gamma}{\beta}\right)^2 \;;\; W_{02} = \prod_4\left(\frac{u\alpha \pm v\beta \pm w\gamma}{u \pm v \pm w}\right)$$

and the curve is a bicircular quartic. Coefficients in H_{21}, H_{12} *are the usual normalized barycentrics of the circumcenter (as of now, no interpretation has been given). When u±v±w=0 ,the curve degenerates into a simply-circular cubic. Otherwise, the visible part of* $\mathcal{W}$ *is bounded.*

Proof. This equation can be rationalized into :

$$\sum_3\left(u^4\left|EM\right|^4\right) - 2\sum_3\left(v^2w^2\left|FM\right|^2\left|GM\right|^2\right) = 0$$

leading to a not so huge expression and, when making $\mathbf{T} = 0$, it only remains :

$$(u+v+w)(-u+v+w)(+u-v+w)(+u+v-w) \times \alpha^2\beta^2\gamma^2\mathbf{Z}^2\overline{\mathcal{Z}}^2 \qquad \square$$

Proposition 23.3.4. *There are four intersections with the circumcircle. One has:*

$$K_0 \doteq \left(\begin{array}{c}\nu \\ 1 \\ 1/\nu\end{array}\right) \underset{2}{\simeq} \left[\begin{array}{c}\dfrac{u\,\alpha+v\,\beta+w\,\gamma}{u/\alpha+v/\beta+w/\gamma} \\ 1 \\ \dfrac{u/\alpha+v/\beta+w/\gamma}{u\,\alpha+v\,\beta+w\,\gamma}\end{array}\right] \tag{23.5}$$

together with $K_\alpha K_\beta K_\gamma$ obtained by replacing the corresponding α by its opposite.

Proof. The main occasion to use Lubin-2 here. It must be noted that the α,β,γ used in this Lubin-2 formula are only defined up to a change of sign, so that none of the four is the "true $u\,\alpha+v\,\beta+w\,\gamma$" □

Proposition 23.3.5. *The tripolar curve has four focuses, the already involved $E,F,G \underset{1}{=} \alpha,\beta,\gamma$ and a fourth point, H, also on the unit circle and given by:*

$$\delta \underset{1}{=} \frac{1}{\alpha\beta\gamma}\,\frac{\mathrm{id}}{\mathrm{conj}}\left(\sum_3 u^2\left(\alpha^2-\alpha\,\beta-\alpha\,\gamma+\beta\,\gamma\right)\right) \tag{23.6}$$

Proof. We cut by a line $P\Omega_y$. Since the curve is bi-circular, the degree falls and we only have to nullify the discriminant of a second degree equation. One can notice that:

$$\delta^2 \underset{2}{=} \frac{\mathrm{id}}{\mathrm{conj}}\left(\frac{1}{\alpha\beta\gamma}\left[\alpha^2,\beta^2,\gamma^2\right]\cdot\boxed{\mathcal{M}_b}\cdot{}^t\left[\alpha^2,\beta^2,\gamma^2\right]\right) \tag{23.7}$$

where $\boxed{\mathcal{M}_b}$ is the usual matrix (7.20) build on (u,v,w). □

Definition 23.3.6. There are three bi-transpositions of the set $\{E,F,G,H\}$. We note them σ_j by using who is paired with H. Thus σ_β notes $\alpha\leftrightarrow\gamma$, $\beta\leftrightarrow\delta$. This convention will also be used to describe the Cremona homography of the whole plane (see Definition 18.1.4) specified by $\alpha\leftrightarrow\gamma$, $\beta\mapsto\delta$. We have

$$\sigma_\beta\ :\ \left(\begin{array}{c}\mathbf{Z}\\ \mathbf{T}\\ \overline{\mathcal{Z}}\end{array}\right) \underset{1}{\mapsto} \left(\begin{array}{c}\dfrac{(\delta\,\beta-\alpha\,\gamma)\,\mathbf{Z}+((\beta+\delta)\,\alpha\,\gamma-(\alpha+\gamma)\,\beta\,\delta)\,\mathbf{T}}{(\beta+\delta-\alpha-\gamma)\,\mathbf{Z}-(\delta\,\beta-\alpha\,\gamma)\,\mathbf{T}}\\ 1\\ \dfrac{(\delta\,\beta-\alpha\,\gamma)\,\overline{\mathcal{Z}}+(\alpha+\gamma-\beta-\delta)\,\mathbf{T}}{((\beta+\delta)\,\alpha\,\gamma-(\alpha+\gamma)\,\beta\,\delta)\,\overline{\mathcal{Z}}-(\delta\,\beta-\alpha\,\gamma)\,\mathbf{T}}\end{array}\right)$$

Theorem 23.3.7. *The tripolar curve remains globally unchanged by the $2\times2\times2$ group $\mathfrak{Q}$ generated by μ and the σ_j.*

Proof. Invariance by μ has been proven at Proposition 23.3.1. Invariance by σ_α is direct computation. The type $2\times2\times2$ of the group $\mathfrak{Q}$ comes from the underlying action on $\{E,F,G,H\}$. □

Definition 23.3.8. Transformation $\pi_\beta \doteq \mu\circ\sigma_\beta$ is involutory and is necessarily a reflection into a circle $\mathcal{C}_\beta$.

$$\pi_\beta\ :\ \left(\begin{array}{c}\mathbf{Z}\\ \mathbf{T}\\ \overline{\mathcal{Z}}\end{array}\right) \underset{1}{\mapsto} \left(\begin{array}{c}\dfrac{((\beta+\delta)\,\alpha\,\gamma-(\alpha+\gamma)\,\beta\,\delta)\,\overline{\mathcal{Z}}-(\delta\,\beta-\alpha\,\gamma)\,\mathbf{T}}{(\delta\,\beta-\alpha\,\gamma)\,\overline{\mathcal{Z}}+(\alpha+\gamma-\beta-\delta)\,\mathbf{T}}\\ 1\\ \dfrac{(\beta+\delta-\alpha-\gamma)\,\mathbf{Z}-(\delta\,\beta-\alpha\,\gamma)\,\mathbf{T}}{(\delta\,\beta-\alpha\,\gamma)\,\mathbf{Z}+((\beta+\delta)\,\alpha\,\gamma-(\alpha+\gamma)\,\beta\,\delta)\,\mathbf{T}}\end{array}\right)$$

Let D_β and ρ_β be the center and radius of $\mathcal{C}_\beta$.

Proposition 23.3.9. *Triangle $D_\alpha D_\beta D_\gamma$ is the common diagonal triangle to both the quadrangle D,E,F,G and the quadrangle of the K_j, while the four circles Γ and $\mathcal{C}_j$ are orthogonal to each other.*

Proof. By construction, the set $\{E, F, G, H\}$ is invariant by σ_β. Since Γ and $\mathcal{W}$ are globally invariant, so is their intersection, the set of the four K_j. And the first conclusion follows, since object and image of a reflection in a circle are aligned with the center. Moreover, orthogonality is required for a given circle be invariant by reflection into another circle.

We can also take the representatives of all these circles

$$\left[\begin{array}{c}0\\0\\0\\1\end{array}\right]\left[\begin{array}{c}\beta+\gamma-\alpha-\delta\\2\,(\delta\,\alpha-\beta\,\gamma)\\(\alpha+\delta)\,\beta\,\gamma-(\beta+\gamma)\,\alpha\,\delta\\\delta\,\alpha-\beta\,\gamma\end{array}\right]\left[\begin{array}{c}\gamma+\alpha-\beta-\delta\\2\,(\delta\,\beta-\gamma\,\alpha)\\(\beta+\delta)\,\gamma\,\alpha\,-(\gamma+\alpha)\,\beta\,\delta\\\delta\,\beta-\gamma\,\alpha\end{array}\right]\left[\begin{array}{c}\alpha+\beta-\delta-\gamma\\2\,(\delta\,\gamma-\alpha\,\beta)\\(\gamma+\delta)\,\alpha\,\beta-(\alpha+\beta)\,\gamma\,\delta\\\delta\,\gamma-\alpha\,\beta\end{array}\right]$$

then use the generic formulas to obtain center and power

$$D_\beta \underset{1}{\simeq} \left(\begin{array}{c}\beta\delta\,(\alpha+\gamma)-\alpha\gamma\,(\beta+\delta)\\\beta\delta-\alpha\gamma\\\beta+\delta-\alpha-\gamma\end{array}\right)\;;\;\rho_\beta^2 \underset{1}{=} -\frac{(\delta-\alpha)\,(\delta-\gamma)\,(\beta-\alpha)\,(\beta-\gamma)}{(\delta\beta-\alpha\gamma)^2} \tag{23.8}$$

and finally conclude by taking the Gramm matrix of the four circles. □

Exercise 23.3.10. Prove that product $\rho_\alpha\rho_\beta\rho_\gamma$ is imaginary, so that one of the circles is imaginary.

Exercise 23.3.11. The 12 fixed points of the three homographies σ_j are the intersections of the four circles Γ and $\mathcal{C}_j$. To obtain the visible points among them, we have to discard the imaginary circle (see previous exercise).

Exercise 23.3.12. The action of π_A on the four intersections with the circumcircle amounts to change the sign of a in the formulas (23.5). Hint: use Lubin2, and substitute δ^2 from (23.7).

23.4 Cross-ratios

Lemma 23.4.1. *Equation (23.1) can be rewritten using cross-ratio, transforming $M \in \mathcal{W}$ into:*

$$\mathcal{W}(M) \underset{1}{=} \left(\frac{u}{(\gamma-\beta)\sqrt{\alpha}}\right)+\left(\frac{v}{(\alpha-\gamma)\sqrt{\beta}}\right)\left|\frac{(M-F)\,(G-E)}{(M-E)\,(G-F)}\right|+\left(\frac{w}{(\beta-\alpha)\sqrt{\gamma}}\right)\left|\frac{(M-G)\,(F-E)}{(M-E)\,(F-G)}\right| \tag{23.9}$$

Proof. One has $|GF| = |G-F| = \pm i\,(\beta-\gamma) \div \sqrt{\beta\gamma}$, etc. □

Proposition 23.4.2. *The defining equation (23.1) can be rewritten for each triple of focuses. And we have*

$$\left(\begin{array}{c}u\\v\\w\end{array}\right)_{E,F,G} \mapsto \left(\begin{array}{c}d\\e\\f\end{array}\right)_{E,H,G} \underset{1}{\simeq} \left[\begin{array}{c}+w\sqrt{\dfrac{\alpha}{\gamma}}\,(\gamma-\beta)\,(\gamma-\delta)\\-v\sqrt{\dfrac{\delta}{\beta}}\,(\beta-\alpha)\,(\beta-\gamma)\\+u\sqrt{\dfrac{\gamma}{\alpha}}\,(\alpha-\delta)\,(\alpha-\beta)\end{array}\right]$$

Iterating the process, we get the final table:

E	F	G	u	v	w	α	β	γ
F	G	H	$w\,K_e$	$v\,K_e\,\dfrac{\gamma\,(\beta-\delta)\,(\beta-\alpha)}{\beta\,(\alpha-\gamma)\,(\delta-\gamma)}$	$u\,K_e\,\dfrac{(\beta-\alpha)\sqrt{\delta}\sqrt{\gamma}}{(\delta-\gamma)\sqrt{\alpha}\sqrt{\beta}}$	β	γ	δ
E	G	H	$w\,K_f$	$u\,K_f\,\dfrac{\gamma\,(\alpha-\delta)\,(\alpha-\beta)}{\alpha\,(\beta-\gamma)\,(\delta-\gamma)}$	$v\,K_f\,\dfrac{(\alpha-\beta)\sqrt{\delta}\sqrt{\gamma}}{(\delta-\gamma)\sqrt{\alpha}\sqrt{\beta}}$	α	γ	δ
E	F	H	$v\,K_g$	$u\,K_g\,\dfrac{\beta\,(\alpha-\delta)\,(\alpha-\gamma)}{\alpha\,(\gamma-\beta)\,(\delta-\beta)}$	$w\,K_g\,\dfrac{(\alpha-\gamma)\sqrt{\delta}\sqrt{\beta}}{(\delta-\beta)\sqrt{\alpha}\sqrt{\gamma}}$	α	β	δ

Proof. On the one hand, substitute $[E = H, F = G, G = F]$ (and nothing else) into formula (23.9). On the other hand, substitute not only $[E = H, F = G, G = F]$ but also $[\alpha = \delta, \beta = \gamma, \gamma = \beta]$ and $[u = k_h, v = k_g, w = k_f]$. And equate both results. The second move is only a flat application of the formula, the first one is using the invariance of the curve under the cross-ratio preserving homography σ_α. □

23.5 Introducing the cut parameter

Definition 23.5.1. Choosing one of the four intersections of $\mathcal{W}$ with Γ as $K_0 \simeq \nu : 1 : 1/\nu$ breaks the symmetry of the problem. But this provides what is required to split the action of μ, σ, π.

Proposition 23.5.2. *Using the cut parameter, we obtain:* ${}^t(u, v, w) \underset{1}{=}$

$$\frac{1}{\alpha\beta\gamma\sqrt{\delta}} \left(\begin{array}{c} \sqrt{\alpha}\,(\beta-\gamma)\left((\alpha+\delta-\beta-\gamma)\,\nu - 2\,(\delta\,\alpha-\beta\,\gamma) + \dfrac{\delta\,\alpha\,(\beta+\gamma)-\beta\,\gamma\,(\delta+\alpha)}{\nu}\right) \\ \sqrt{\beta}\,(\gamma-\alpha)\left((\beta+\delta-\gamma-\alpha)\,\nu - 2\,(\delta\,\beta-\gamma\,\alpha) + \dfrac{\delta\,\beta\,(\gamma+\alpha)-\alpha\,\beta\,(\delta+\beta)}{\nu}\right) \\ \sqrt{\gamma}\,(\alpha-\beta)\left((\gamma+\delta-\alpha-\beta)\,\nu - 2\,(\delta\,\gamma-\alpha\,\beta) + \dfrac{\delta\,\gamma\,(\alpha+\beta)-\alpha\,\beta\,(\delta+\gamma)}{\nu}\right) \end{array} \right) \in \mathbb{R}^3$$

Proof. Compute the D_j from the K_0, K_j and compare with those obtained at (23.8). Then the simultaneous reality of the multipliers is proven by their invariance under conjugacy. □

Corollary 23.5.3. *Equation of $\mathcal{W}$ can be rewritten using the symmetric functions* (23.2) *The* $\mathbf{Z}^2\overline{\mathcal{Z}}\mathbf{T}$ *coefficients are a sum of terms like*

$$\frac{-2\,q_1^4 - 32\,q_1\,q_3 + 32\,q_2^2 - 128\,q_4}{q_4^2}\,\nu^4$$

i.e. a power ν^k (where $-4 \le k \le 4$) times a $\mathbb{Z}[q_1, q_2, q_3, q_4]$ polynomial divided by a power of q_4, leading to a zero global degree in $\alpha, \beta, \gamma, \delta, \nu$. It remains to say that the resulting expressions have a rather huge length:

H_{22}	H_{12}	H_{02}	H_{11}
705	800	773	876

Fact 23.5.4. *We can use a varying ν as in Figure 23.2 to see that focuses cut the circle into arcs that contains either 2 or 0 intersections with the curve. Starting from violet arcs $\overset{\frown}{EG}$ and $\overset{\frown}{FH}$, we get something like two bananas surrounding both pairs of focuses. Then they cross each other at circle* cinvG *and reduce back to the original arcs. Then we get arcs $\overset{\frown}{EF}$ and $\overset{\frown}{HG}$, etc.*

Example 23.5.5. Figure 23.2 has been constructed by choosing

$$u = 11\sqrt{5}, v = 5\sqrt{5}, w = 26, \alpha \underset{1}{=} -4/5 + 3/5\,i, \beta = 4/5 + 3/5\,i, \gamma = -i$$

Then the fourth focus is $\delta = 1$. The multipliers are given by columns 4,5,6 to obtain the branch containig $\nu = (-28 - 45i)/53$ and are given by columns 7,8,9 to obtain the branch containing $\nu' = (80 + 39i)/89$.

			ν			ν'		
E	F	G	$11\sqrt{5}$	$-5\sqrt{5}$	-26	$11\sqrt{5}$	$5\sqrt{5}$	-26
F	G	H	$13\sqrt{5}$	-5	$-22\sqrt{2}$	$13\sqrt{5}$	5	$-22\sqrt{2}$
E	G	H	$13\sqrt{5}$	-33	$-10\sqrt{2}$	$13\sqrt{5}$	-33	$+10\sqrt{2}$
E	F	H	$5\sqrt{5}$	$33\sqrt{5}$	$-52\sqrt{2}$	$5\sqrt{5}$	$33\sqrt{5}$	$-52\sqrt{2}$

23.6 Barycentrics wrt the diagonal triangle

Proposition 23.6.1. *When using the diagonal triangle $D_\alpha D_\beta D_\gamma$ as barycentric basis, we obtain:*

$$H_0 \doteq (f : g : h) \underset{b,1}{\simeq} \left(\frac{(\alpha\,\delta-\gamma\,\beta)\,(\alpha-\delta)}{2\delta\,(\alpha-\gamma)\,(\alpha-\beta)} : \frac{(\beta\,\delta-\alpha\,\gamma)\,(\beta-\delta)}{2\delta\,(\beta-\gamma)\,(\beta-\alpha)} : \frac{(\gamma\,\delta-\alpha\,\beta)\,(\gamma-\delta)}{2\delta\,(\gamma-\beta)\,(\gamma-\alpha)} \right)$$

where the f, g, h are real with $f + g + h = 1$, while the other three focuses $H_e H_f H_g$ are given by $\pm f : \pm g : \pm h$, leading to an anticevian configuration.

Proof. Direct computation. □

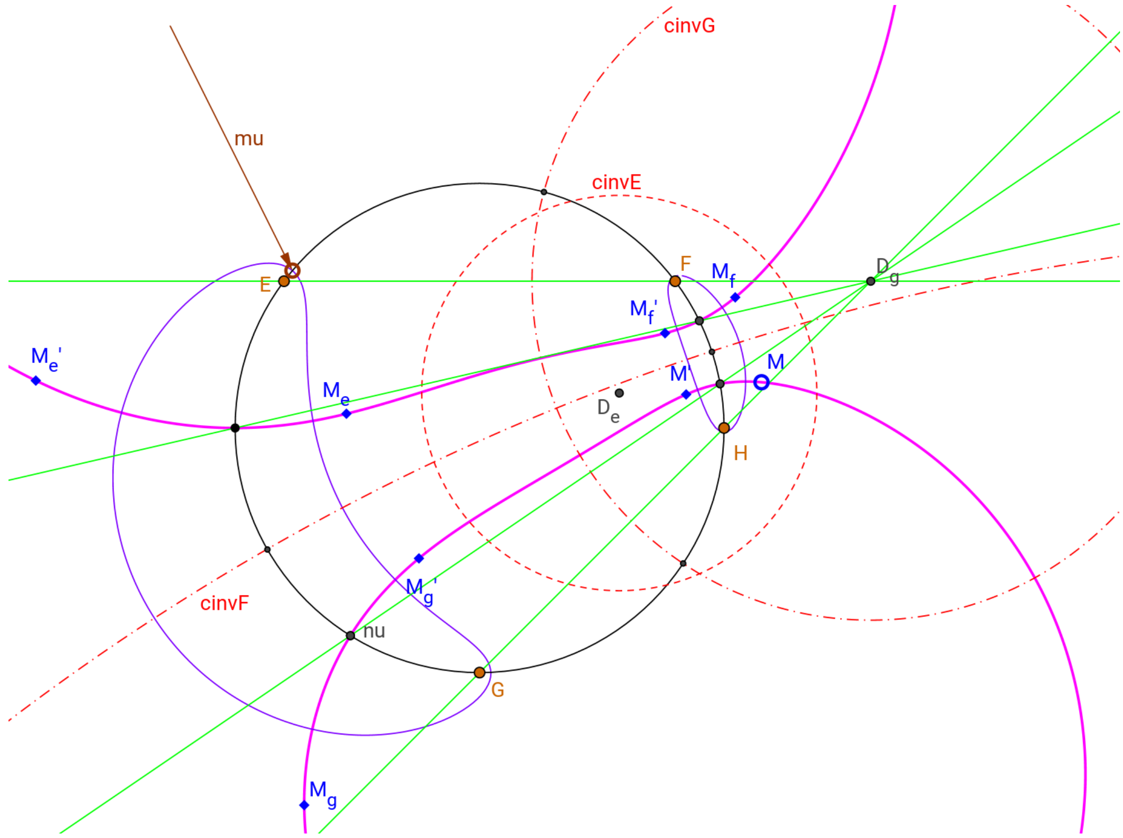


Figure 23.2: Tripolar curve

Proposition 23.6.2. *The point O, center of the focal circle, is the orthocenter of the diagonal triangle. Its matrix is*

$$\operatorname{diag}\left(\frac{1}{(\delta-\alpha)(\gamma-\beta)(\alpha\delta-\beta\gamma)^2},\frac{1}{(\delta-\beta)(\alpha-\gamma)(\beta\delta-\gamma\alpha)^2},\frac{1}{(\delta-\gamma)(\beta-\alpha)(\gamma\delta-\alpha\beta)^2}\right)$$

Proof. From the $\pm f:\pm g:\pm h$ property, the matrix of Γ has to be diagonal. But the orthocentroidal circle is the only one to share this property. □

Definition 23.6.3. For each triple of focuses, we introduce the Lemoine center L_h and the mass center M_h by:

$$\begin{aligned} L_h &\doteq \left(|FG|^2 E+|GE|^2 F+|EF|^2 G\right)/\left(|FG|^2+|GE|^2+|EF|^2\right)\\ M_h &\doteq \left(u^2E+v^2F+w^2G\right)/\left(u^2+v^2+w^2\right)\end{aligned}$$

The first time, we ponder by the squared lengths of the sidelines, the second time, we ponder by the square of the coefficients involved in (23.1).

Proposition 23.6.4. *Wrt the diagonal triangle, the barycentric coordinates of L_0 and K_0 are*

$$L_0 \underset{b,1}{\simeq} \begin{pmatrix}(\delta\alpha-\gamma\beta)(\beta-\gamma)^2(\delta-\alpha)^2\\(\delta\beta-\alpha\gamma)(\gamma-\alpha)^2(\delta-\beta)^2\\(\delta\gamma-\beta\alpha)(\alpha-\beta)^2(\delta-\gamma)^2\end{pmatrix}$$

$$K_0 \underset{b,1}{\simeq} \begin{pmatrix}(\delta-\alpha)(\beta-\gamma)(\delta\alpha-\beta\gamma)\left((\delta+\alpha-\beta-\gamma)\nu^2+2(\beta\gamma-\alpha\delta)\nu+(\beta+\gamma)\alpha\delta-(\delta+\alpha)\beta\gamma\right)\\(\delta-\beta)(\gamma-\alpha)(\delta\beta-\gamma\alpha)\left((\delta+\beta-\gamma-\alpha)\nu^2+2(\gamma\alpha-\beta\delta)\nu+(\gamma+\alpha)\beta\delta-(\delta+\beta)\gamma\alpha\right)\\(\delta-\gamma)(\alpha-\beta)(\delta\gamma-\alpha\beta)\left((\delta+\gamma-\alpha-\beta)\nu^2+2(\alpha\beta-\gamma\delta)\nu+(\alpha+\beta)\gamma\delta-(\delta+\gamma)\alpha\beta\right)\end{pmatrix}$$

and we have $M_j = K_j \underset{b}{*} K_j \underset{b}{*} L_j \underset{b}{\div} E_j \underset{b}{\div} E_j$.

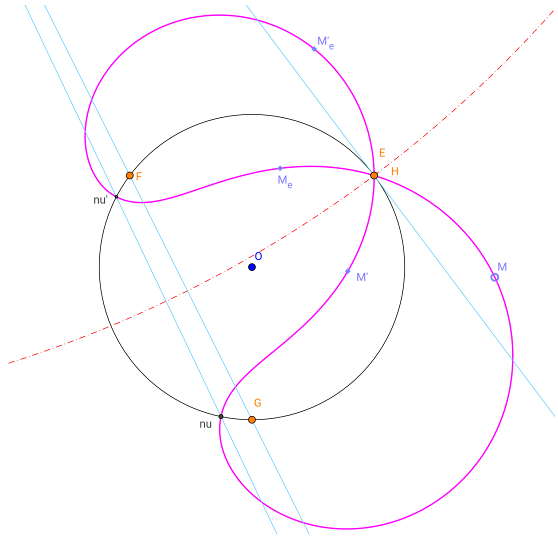


Figure 23.3: Tripolar curve, after the meltdown of two focuses

Proof. Everything remains in the Lubin1 domain, since we are using $K_0 \underset{1}{\simeq} \nu : 1 : \nu^{-1}$. A direct computation is easy. One can also use the fact that:

$$K_0 \simeq H_0 \underset{b}{*} \begin{pmatrix} u/FG \\ v/GE \\ w/EF \end{pmatrix} \;;\; L_0 \simeq H_0 \underset{b}{*} \begin{pmatrix} HE\,FG \\ HF\,GE \\ HG\,EF \end{pmatrix} \;;\; M_0 \simeq H_0 \underset{b}{*} \begin{bmatrix} u^2\,HE/FG \\ v^2 HF/GE \\ w^2\,HG/EF \end{bmatrix} \qquad \square$$

23.7 When E is on the curve

$u\,|EE| + v\,|EF| + w\,|EG| = 0$ implies $v : w \simeq |EG| : |EF|$.

Proposition 23.7.1. **** pompous *** Point D_e is on the curve when either $S = 0$ (this case will be studied in detail at Section 23.8) or one of the F, G is on the curve (focuses are supposed to be different).*

Proof. When substituting (23.8) into (23.4), we get:

$$S^2\,\frac{\alpha^4\gamma^4\beta^4\,|EF|^4\,|GE|^4\left(v^2\,|FG|^2 - u^2\,|GE|^2\right)^2\left(u^2\,|EF| - w^2\,|FG|\right)^2}{\left(\sqrt{\beta}\sqrt{\gamma}\,|GE|\,|EF|\,u^2 + \sqrt{\gamma}\sqrt{\alpha}\,|EF|\,|FG|\,v^2 + \sqrt{\alpha}\sqrt{\beta}\,|FG|\,|GE|\,w^2\right)^4}$$

$\square$

23.8 The tripolar circular cubics

Definition 23.8.1. In this section $\pm u \pm v \pm w = 0$ is assumed. Parametrization $u : v : w \simeq 1 : s : -1 - s$ will be overused. And notation $\mathcal{K}$ will be used when asserting properties that doesn't apply to the whole family of tripolar curves $\mathcal{W}$.

Example 23.8.2. Figure 23.5 has been drawn using the following data:

$$u = 1,\ v = 2,\ w = 3,\ E_0 = 12i,\ F_0 = 5,\ G_0 = 0$$

This results into

$$\alpha \underset{1}{=}= \frac{-5+12\,i}{13}, \beta \underset{1}{=} \frac{5-12\,i}{13}, \gamma \underset{1}{=} \frac{-5-12\,i}{13}, \nu = \frac{12-35\,i}{37}$$

$$\delta \underset{1}{=} \frac{-1555-48348\,i}{48373} \;;\; \delta \underset{2}{=} \frac{153\ -158i}{793}\,\sqrt{13}$$

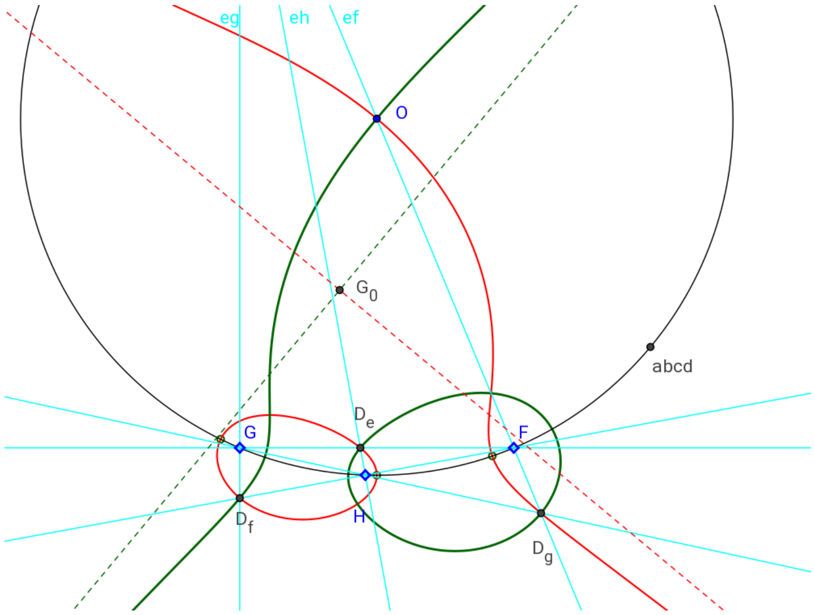


Figure 23.4: Tripolar cubic

while the coefficients describing the connected piece containing ν are

E	F	G	1	2	-3
F	G	H	33	28	-61
E	G	H	33	-155	122
E	F	H	28	155	-183

Exercise 23.8.3. Values $a = 1$, $b = 2$, $c = -3$, $\alpha \underset{1}{=} \dfrac{-12+5i}{13}$, $\beta = \dfrac{12-5i}{13}$, $\gamma = \dfrac{5-12i}{13}$ are given. Apply everything.

Proposition 23.8.4. ***Coefficients from focuses.*** *When $\pm u \pm v \pm w = 0$, but $uvw \neq 0$, the resulting curve is a circular cubic. It admits four concyclic focuses as any other $\mathcal{W}$. When parametrizing $u : v : w$ by $1 : s : -1-s$, formula (23.6) can be reverted into:*

$$s^{\|} \underset{2}{=} -\frac{\beta\left(\alpha^2-\gamma^2\right)\left(\alpha\,\gamma-\delta\,\beta\right)}{\alpha\left(\beta^2-\gamma^2\right)\left(\beta\,\gamma-\alpha\,\delta\right)}\;;\; s^{\perp} \underset{2}{=} -\frac{\beta\left(\alpha^2-\gamma^2\right)\left(\alpha\,\gamma+\beta\,\delta\right)}{\alpha\left(\beta^2-\gamma^2\right)\left(\beta\,\gamma+\alpha\,\delta\right)}$$

And then the coefficients can be re-obtained as:

$$\left(\begin{array}{c} a \\ b \\ c \end{array}\right)^{\|} \simeq \left(\begin{array}{c} 1 \\ s \\ -1-s \end{array}\right) \underset{2}{\simeq} \left(\begin{array}{c} \alpha\left(\beta^2-\gamma^2\right)\left(\alpha\,\delta-\beta\,\gamma\right) \\ \beta\left(\gamma^2-\alpha^2\right)\left(\delta\,\beta-\alpha\,\gamma\right) \\ \gamma\left(\alpha^2-\beta^2\right)\left(\delta\,\gamma-\beta\,\alpha\right) \end{array}\right) \simeq \left(\begin{array}{c} FG\,\left(\alpha\,\delta-\beta\,\gamma\right) \\ GE\left(\beta\,\delta-\gamma\,\alpha\right) \\ EF\,\left(\gamma\,\delta-\alpha\,\beta\right) \end{array}\right)$$

Proof. Obvious computation. Naming both values of s as "parallel" and "perpendicular" will be explained later. When s is known, the sign of δ is supposed to be chosen so that $s = s^{\|}(\delta)$, rather than the contrary. □

Proposition 23.8.5. ***Asymptotes.*** *Using $q_1 = \alpha+\beta+\gamma+\delta$, etc (the so called symmetric functions), the equation of the tripolar cubic becomes:*

$$\mathcal{K}\left(\alpha,\beta,\gamma,\delta\right) \underset{2}{=} 4\left(\mathbf{T}^2+\mathbf{Z}\overline{\mathcal{Z}}\right)\left(q_4\,\overline{\mathcal{Z}}-\mathbf{Z}\right)\left(q_1^2 q_4-q_3^2\right)+$$
$$\mathbf{T}\left(\begin{array}{c}\left(8\,q_1q_3q_4^2-4\,q_2q_3^2q_4+q_3^4\right)\overline{\mathcal{Z}}^2+\left(q_1^4-4\,q_1^2q_2+8\,q_1q_3\right)\mathbf{Z}^2 \\ +\left(4\,q_1^2q_2q_4-2\,q_1^2q_3^2-16\,q_1q_3q_4+4\,q_2q_3^2\right)\overline{\mathcal{Z}}\,\mathbf{Z}\end{array}\right)$$

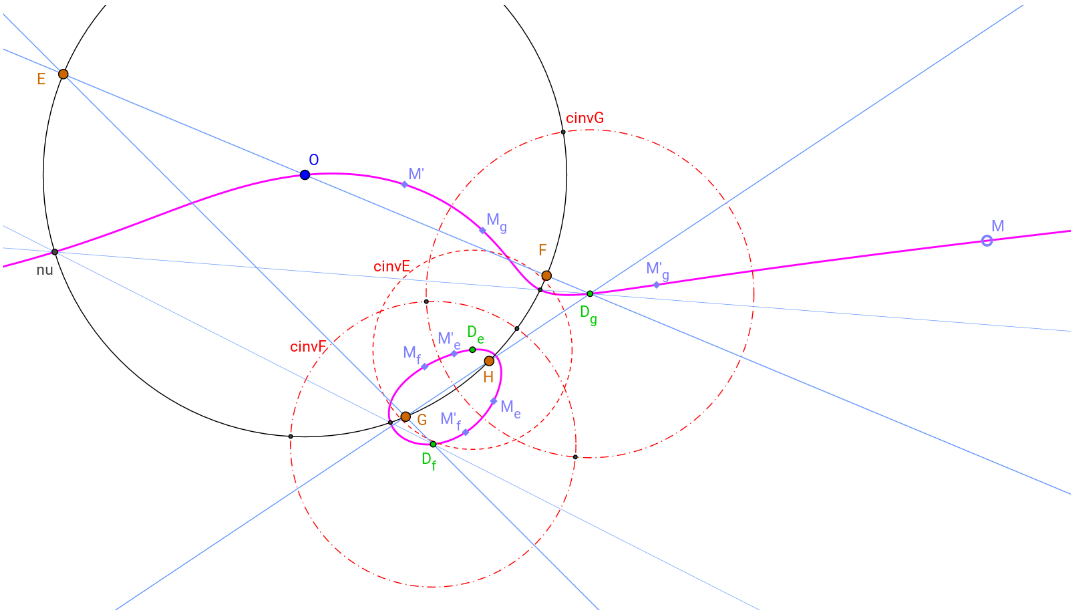


Figure 23.5: Tripolar cubic

The third point at infinity is $U(\delta) \doteq \alpha\beta\gamma\delta : 0 : 1$. Thus the asymptotes $\Delta^{\parallel}$ and $\Delta^{\perp}$ are orthogonal. Their intersection is the gravity center of $EFGH$. When s is known, sign of δ is supposed to be chosen so that $\mathcal{K} = \mathcal{K}^{\parallel}(\delta)$, rather than the contrary.

Proof. When substituting, one can see the symmetry of the expression, allowing to use the q_j. Then straightforward computations. Equations found for both asymptotes are:

$$\Delta^{\parallel} \underset{2}{\simeq} \left[4, -q_1^2 + q_3^2/q_4, -4q_4\right] \;;\; \Delta^{\perp} \underset{2}{\simeq} \left[4, -q_1^2 + q_2 - q_3^2/q_4, +4q_4\right]$$

□

Corollary 23.8.6. *The leading factor $\left(q_1^2 q_4 - q_3^2\right) = \prod_3 (\alpha\,\delta - \beta\,\gamma)$ forbids a possible degeneracy into a conic and the line at infinity.*

Proposition 23.8.7. ***Tangentials**. Each of the $\mathcal{K}^{\parallel}$ and $\mathcal{K}^{\perp}$ curves are invariant under the $\mathfrak{Q}$ group. Their $9 = 3 \times 3$ common points are Ω_x, Ω_y, O, the D_e and their inverses D'_e,* etc *in the unit circle. They form an orbit under the $\mathfrak{Q}$ group... when taking into account the indeterminacy at O for the inversion into Γ. Moreover:*

1. *$U^{\parallel}$ is the tangential of O, D_e, D_f, D_g wrt $\mathcal{K}^{\parallel}$.*

2. *The tangentials of $U^{\parallel}$ and $U^{\perp}$wrt their own curve are :*

$$U_t^{\parallel} \simeq \left(\begin{array}{c} 4\,q_2\,q_3 q_4 - 8\,q_1 q_3 q_4^2 - q_3^4 + 16\,q_4^3 \\ 4\,q_4\,\left(q_1^2 q_4 - 4\,q_2\,q + q_3^2\right) \\ q_4\left(4\,q_1^2 q_2 - 8\,q_1 q_3 - q_1^4 + 16\,q_4\right) \end{array}\right) \;;\; U_t^{\perp} \simeq \left[\begin{array}{c} -q_3\,\left(8\,q_1\,q_4^2 - 4\,q_2\,q_3\,q_4 + q_3^3\right) \\ 4\,q_4\,\left(q_1^2 q_4 - q_3^2\right) \\ q_1\,q_4\,\left(q_1^3 - 4\,q_1\,q_2 + 8\,q_3\right) \end{array}\right]$$

   Point U_t is also the tangential of D'_e, D'_f, D'_g. And it happens that $U_t(-\delta)$ is the singular focus of $\mathcal{K}(\delta)$.

3. *Apart from $\Omega^{\pm}$, the 8 intersections of both $\mathcal{K}$ and the unit circle are obtained from $-\sigma_3 \div \sigma_1$ by changing the signs of $\alpha, \beta, \gamma, \tau$ (odd changes move to $\mathcal{K}^{\perp}$, even ones remain on $\mathcal{K}^{\parallel}$). Each group of 4 forms an orbit under $\mathfrak{Q}$. Point O is the common tangential of all these points.*

4. *Both curves are orthogonal at each of their intersections.*

Proof. Due to the choice $z_A = \alpha^2$, everything factors and computations are easy. The six D_j, D'_j depend only on the squares α^2, etc: that is the reason why they can belong to both curves. □

Exercise 23.8.8. Consider the straight lines which are the bisectors of the angles $\left(\overrightarrow{OE}, \overrightarrow{OH}\right)$ and $\left(\overrightarrow{OF}, \overrightarrow{OG}\right)$. And then consider the gudule which bisects these two lines. How to help some constructions with this object ?

23.8.1 When the fourth focus is moving

Exercise 23.8.9. The locus of U_t, F_s is a circular quintic, and the directions of sidelines are the other points at infinity. The corresponding asymptotes are going through $(2A+B+C)/4$, etc while it exists a singular focus at X(143).

Exercise 23.8.10. The envelope of Δ is the deltoid :

$$M(\tau) = \left(\frac{1}{4}\,s_1^2 - \frac{1}{2}\,s_2\right) - \frac{1}{4}\,\tau^2 - \frac{s_3}{2\,\tau}$$

whose center is X(140)= $(A+B+C+O)/4$, its internal and external radiuses being $1/4$ and $3/4$. Cups are given by $\tau^3 = s_3$, i.e. the Morley's directions. Moreover $\Delta^{\parallel} \cap \Delta^{\perp -}$ describes the inner circle. Hint: use $\mathrm{d}\tau = ik\tau$ where k is evanescent.

Exercise 23.8.11. The reflection of $\mathcal{K}$ wrt a circle centered at a focus leads to Cartesian ovals whose focuses are the images of the remaining three original focuses (see Figure 23.7). Choosing $\rho = 1$ is a great choice, which leads to $O \mapsto O$).

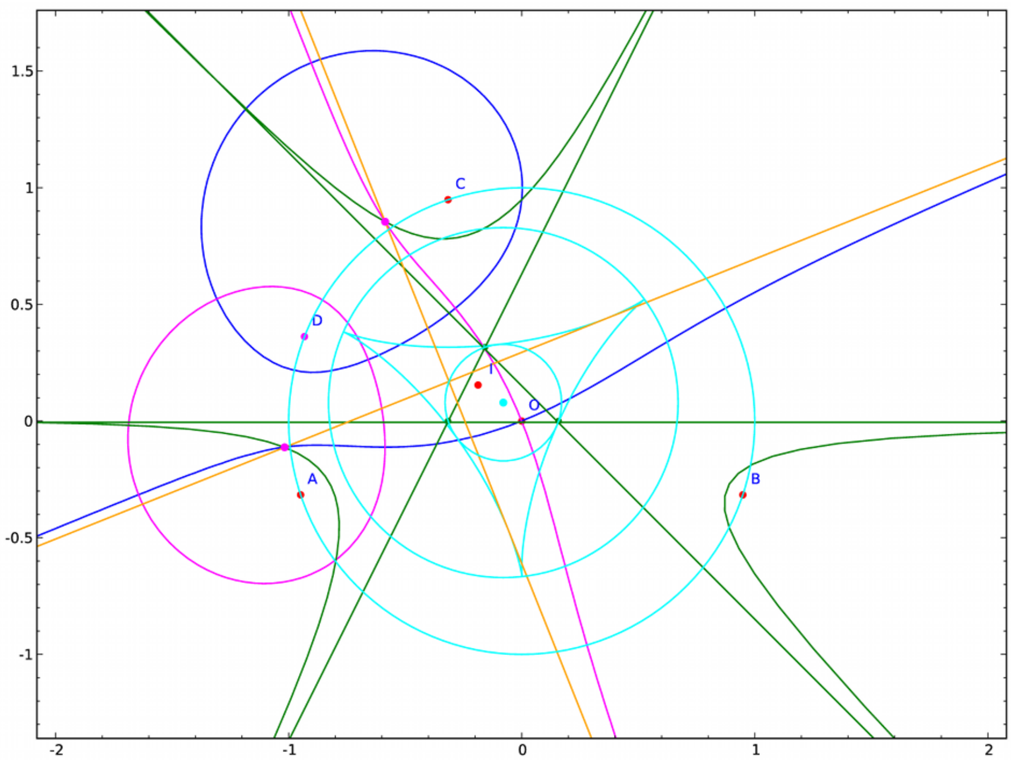


Figure 23.6: Angels

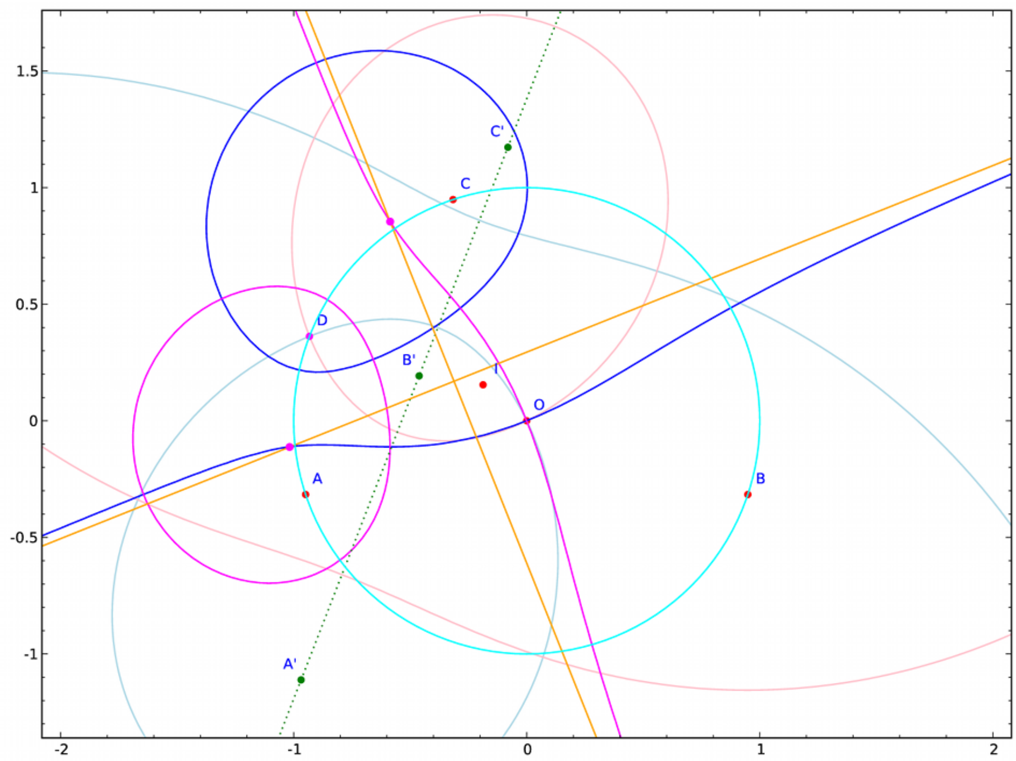


Figure 23.7: Cartesian ovals by inversion

Chapter 24

Special Triangles

Central triangles have been defined in Section 2.2.

24.1 Changing coordinates, functions and equations

Proposition 24.1.1. *Any triangle $\mathcal{T}$ can be used as a barycentric basis instead of triangle ABC. When columns of triangle $\mathcal{T}$ are synchronized, the old barycentrics $x : y : z$ (relative to ABC) can be obtained from the new ones $\xi : \eta : \zeta$ (relative to $\mathcal{T}$) by :*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boxed{\mathcal{T}} \cdot \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$

while the converse transformation can be done using the adjoint matrix.

Proof. A column of synchronized barycentrics acts on the matrix of rows containing the projective coordinates of the vertices of the reference triangle by the usual matrix multiplication. □

Remark 24.1.2. By definition of synchronized barycentrics, $\mathcal{L}_b \cdot \boxed{\mathcal{T}} \simeq \mathcal{L}_b$ and the line at infinity is (globally) invariant.

Proposition 24.1.3. *Let α, β, γ the side lengths of triangle $\mathcal{T}$ (computed using Theorem 7.4.4). Consider a central punctual transformation Φ that can be written as :*

$$p : q : r \mapsto u : v : w = \phi\,(a, b, c, p, q, r)$$

with all the required properties of symmetry and homogeneity. Consider now the corresponding punctual transformation Φ' with respect to triangle $\mathcal{T}$ (written in its normalized form) and define $\phi_{\mathcal{T}}$ as the action of Φ' on the old barycentrics (the ones related to ABC). Then :

$$\phi_{\mathcal{T}}\left(a, b, c, \begin{pmatrix} p \\ q \\ r \end{pmatrix}\right) = \mathcal{T} \cdot \phi\left(\alpha, \beta, \gamma, \mathcal{T}^{-1} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix}\right)$$

Example 24.1.4. Applied to the isogonal transform and some usual triangle, this leads to formulas given in Figure 24.1. The term "complementary conjugate" is a synonym for "medial isogonal conjugate", as is "anticomplementary conjugate" for "anticomplementary isogonal conjugate". Also, "excentral isogonal conjugate" is "X(188)-aleph conjugate" and "orthic isogonal conjugate" is "X(4)-Ceva conjugate".

Proposition 24.1.5. *Let α, β, γ the side lengths of triangle $\mathcal{T}$. Consider a conic Φ whose matrix can be written as $M\,(a, b, c)$ with the required properties of symmetry and homogeneity. Consider now the corresponding conic Φ' with respect to triangle $\mathcal{T}$ (written in its normalized form) and define $M_{\mathcal{T}}$ as the matrix defining Φ' wrt the old barycentrics. Then :*

$$M_{\mathcal{T}}(a, b, c) = {}^t\mathcal{T}^{-1} \cdot M\,(\alpha, \beta, \gamma) \cdot \mathcal{T}^{-1}$$

Example 24.1.6. Applied to the circumcircle and some usual triangles, this leads to formulas given in Figure 24.1.

24.2 Residual triangles

Definition 24.2.1. The residual triangles of a triangle $A'B'C'$ inscribed in a bigger one ABC are the triangles $AB'C', A'BC', A'B'C$. Mind the fact that the residuals are oriented counterclockwise wrt triangles ABC and $A'B'C'$.

Proposition 24.2.2. *Suppose that $A'B'C'$ is inside the convex hull ABC and note $\mathcal{A}, \mathcal{A}_0, \mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c$ the absolute values of the areas of the five triangles. Then $\mathcal{A} = \mathcal{A}_a + \mathcal{A}_b + \mathcal{A}_c + \mathcal{A}_0$. Moreover, $\mathcal{A}_0 \geq min\left(\mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c\right)$. More precisely(Bottema et al., 1969), $\mathcal{A}_0 \geq min\left(\sqrt{\mathcal{A}_a\mathcal{A}_b}, \sqrt{\mathcal{A}_b\mathcal{A}_c}, \sqrt{\mathcal{A}_c\mathcal{A}_a}\right)$. And equality occurs only when $A'B'C'$ are the mid-points.*

Proof. Write $A' \simeq 0 : x : x'$, $B' \simeq y' : 0 : y$, $C' \simeq z : z' : 0$ where $x' = 1 - x$, etc. Then

$$\mathcal{A}_0 = xyz + x'y'z' \geq 2\sqrt{xyzx'y'z'} = 2\sqrt{\mathcal{A}_a\mathcal{A}_b\mathcal{A}_c}$$

Suppose $\mathcal{A}_a \geq \mathcal{A}_b \geq \mathcal{A}_c$. If $\mathcal{A}_a \leq 1/4$, then $\mathcal{A}_0 \geq 1/4 \geq \sqrt{\mathcal{A}_j\mathcal{A}_k}$. If $\mathcal{A}_a \geq 1/4$ then $2\sqrt{\mathcal{A}_a} \geq 1$.
Equality occurs only if $xyz = x'y'z'$ and $\mathcal{A}_a = 1/4$. □

Definition 24.2.3. The residual cevian triangles associated with point P are the triangles AP_bP_c, P_aBP_c, P_aP_bC where $P_aP_bP_c$ is the cevian triangle of P (mind the order...).

Proposition 24.2.4. *The A-residual of the orthic (i.e. wrt X(4)) and the intouch (i.e. wrt X(7)) triangles have the following sidelengths :*

$$\begin{aligned} [\alpha,\, \beta,\, \gamma]_{orthic} &= \frac{S_a}{bc}\, [a,\, b,\, c] \\ \left[\alpha^2,\, \beta^2,\, \gamma^2\right]_{intouch} &= \frac{(b+c-a)^2}{4\,bc}\, \left[(a+b-c)\,(a-b+c)\,,\, bc,\, bc\right] \end{aligned}$$

The incenters of the orthic residuals are the orthocenters of the intouch residuals.

24.3 Incentral triangle

definition cevian triangle of the incenter X(1)

pythagoras (strong values)

$$\alpha^2 = \frac{abc\,\left(a^3 + a^2b + a^2c - ab^2 - ac^2 + 3\,abc - b^3 + b^2c + bc^2 - c^3\right)}{(a+b)^2\,(a+c)^2}$$

barycentrics (normalized)

$$\boxed{\mathcal{C}_1} = \begin{pmatrix} 0 & \frac{a}{a+c} & \frac{a}{a+b} \\ \frac{b}{b+c} & 0 & \frac{b}{a+b} \\ \frac{c}{b+c} & \frac{c}{a+c} & 0 \end{pmatrix}$$

(f, g) $(0;\ b)$

24.4 Excentral triangle

definition triangle of the excenters. And thus, anticevian triangle of the incenter X(1)

pythagoras (strong values)

$$\left[\alpha^2, \beta^2, \gamma^2\right] = \frac{2R}{\rho}\, \left[a\,(b+c-a)\,,\, b\,(c+a-b)\,,\, c\,(a+b-c)\right]$$

thus similar with the intouch triangle (center X(57), ratio $2R/\rho$)

barycentrics (normalized)

$$\boxed{\mathcal{A}_1} = \begin{pmatrix} \dfrac{-a}{b+c-a} & \dfrac{a}{c+a-b} & \dfrac{a}{a+b-c} \\ \dfrac{b}{b+c-a} & \dfrac{-b}{c+a-b} & \dfrac{b}{a+b-c} \\ \dfrac{c}{b+c-a} & \dfrac{c}{c+a-b} & \dfrac{-c}{a+b-c} \end{pmatrix}$$

(f, g) $(-a;\ b)$

24.5 Medial triangle

definition cevian triangle of the centroid X(2)

side_length (strong values)

$$[\alpha, \beta, \gamma] = \frac{1}{2}\ [a,\, b,\, c]$$

barycentrics (normalized)

$$\boxed{\mathcal{C}_2} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

24.6 Antimedial triangle

definition anticevian triangle of the centroid X(2)

side_length (strong values)

$$[\alpha, \beta, \gamma] = 2\ [a,\, b,\, c]$$

barycentrics (normalized)

$$\boxed{\mathcal{A}_2} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

24.7 Orthic triangle

definition cevian triangle of the orthocenter X(4)

side_length (strong values)

$$[\alpha, \beta, \gamma] = \frac{1}{abc}\ \left[a^2 S_a,\, b^2 S_b,\, c^2 S_c\right]$$

circumcircle center=X(5), $R_{orthic} = \frac{1}{2}\, R_{ABC}$

incircle center=X(4), $r_{orthic} = \dfrac{S_a S_b S_c}{2abc\, S}$

barycentrics (synchronized)

$$\boxed{\mathcal{C}_4} = \begin{pmatrix} 0 & S_c/b^2 & S_b/c^2 \\ S_c/a^2 & 0 & S_a/c^2 \\ S_b/a^2 & S_a/b^2 & 0 \end{pmatrix}$$

angles $\pi - 2A,\ \pi - 2B,\ \pi - 2C$

24.8 Tangential triangle

definition The sidelines are the tangents to the ABC-circumcircle at the vertices.

key_property Anticevian triangle of X(6).

side_length (strong values)

$$[\alpha, \beta, \gamma] = \frac{abc}{2\, S_a S_b S_c}\ \left[a^2 S_a,\, b^2 S_b,\, c^2 S_c\right]$$

Therefore, this triangle is similar to the orthic triangle.

barycentrics (synchronized)

$$\boxed{\mathcal{A}_6} = \begin{pmatrix} -\dfrac{a^2}{S_a} & \dfrac{a^2}{S_b} & \dfrac{a^2}{S_c} \\ \dfrac{b^2}{S_a} & -\dfrac{b^2}{S_b} & \dfrac{b^2}{S_c} \\ \dfrac{c^2}{S_a} & \dfrac{c^2}{S_b} & -\dfrac{c^2}{S_c} \end{pmatrix}$$

24.9 Brocard triangle (first)

Remember that Brocard points are defined by $\omega^+ = a^2b^2 : b^2c^2 : c^2a^2$ and $\omega^- = c^2a^2 : a^2b^2 : b^2c^2$ (see Proposition 7.11.1).

definition $A_1 \doteq B\omega^- \cap C\omega^+ \simeq a^2 : c^2 : b^2$ etc. This triangle is inscribed in the Brocard 3-6 circle, see , diameter X(3)X(6).

side_length (strong values)

$$[\alpha, \beta, \gamma] = \frac{W_2}{2\, S_\omega}\ [a,\, b,\, c]$$

and therefore $Broc_1$ is (anti-)homothetic to ABC. Perspector X(76), the third Brocard point. Moreover, anti-similar to ABC with center X(2).

barycentrics (synchronized)

$$\boxed{Broc_1} = \begin{pmatrix} a^2 & c^2 & b^2 \\ c^2 & b^2 & a^2 \\ b^2 & a^2 & c^2 \end{pmatrix}$$

24.10 Brocard triangle (second)

definition $U_a \simeq b^2 + c^2 - a^2 : b^2 : c^2$ is the projection of X(3) on the A-symmedian, etc. (and therefore belongs to the 3-6 Brocard circle).

key_property Triangle ABC and triangle $\boxed{Broc_2}$ share the same isodynamic centers X(15), X(16).

Exercise 24.10.1. The center E_a of circle $A, X(15), X(16)$ is the inverse in circumcircle of U_a, etc.

24.11 Brocard triangle (third)

definition $A_3 \doteq C\omega^- \cap A\omega^+ \simeq b^2c^2 : b^4 : c^4 = isog\,(A_1)$.

perspector with ABC: X(32)

24.12 Intouch triangle (contact triangle)

definition Cevian triangle of the Gergonne point X(7).

key_property contacts of the incircle and the sidelines.

pythagoras (strong equality)

$$\left[\alpha^2, \beta^2, \gamma^2\right] = \frac{\rho}{2R}\left[a\left(b+c-a\right), b\left(c+a-b\right), c\left(a+b-c\right)\right]$$

thus similar with the excentral triangle (ratio $\rho/2R$)

barycentrics (normalized)

$$\boxed{\mathcal{C}_7} = \begin{pmatrix} 0 & \dfrac{a+b-c}{b} & \dfrac{c+a-b}{c} \\ \dfrac{a+b-c}{a} & 0 & \dfrac{b-a+c}{c} \\ \dfrac{c+a-b}{a} & \dfrac{b-a+c}{b} & 0 \end{pmatrix}$$

24.13 Extouch triangle

definition cevian triangle of the Nagel point X(8).

side_length easy to compute, but nothing great !

barycentrics (normalized)

$$\boxed{\mathcal{C}_8} = \begin{pmatrix} 0 & \dfrac{b-a+c}{b} & \dfrac{b-a+c}{c} \\ \dfrac{c+a-b}{a} & 0 & \dfrac{c+a-b}{c} \\ \dfrac{a+b-c}{a} & \dfrac{a+b-c}{b} & 0 \end{pmatrix}$$

24.14 Hexyl triangle

definition symmetric of the excentral triangle $J_aJ_bJ_c$ wrt the circumcircle. Thus $H_aJ_bH_cJ_aH_bJ_c$ is an hexagon whose opposite sides are parallel.

key_property Vertex H_a is the point in which the perpendicular to AB through the excenter J_b meets the perpendicular to AC through the excenter J_c.

pythagoras (strong values)

$$\left[\alpha^2, \beta^2, \gamma^2\right] = \frac{2R}{\rho}\left[a\left(b+c-a\right), b\left(c+a-b\right), c\left(a+b-c\right)\right]$$

thus similar with the intouch triangle (center I, ratio $2R/\rho$).

barycentrics (normalized) $H_a \simeq a\left(aS_a+bS_b+cS_c+abc\right) : b\left(aS_a+bS_b-cS_c-abc\right) : c\left(aS_a-bS_b+cS_c-abc\right)$

circumcircle centered at X(1), radius $2R$.

24.15 Fuhrmann triangle

Definition 24.15.1. Fuhrmann triangle is $A''B''C''$ where $A'B'C'$ is the circumcevian triangle of X_1 and A'' is the reflection of A' in sideline BC (and cyclically for B'' and C'').

Proposition 24.15.2. *Side length of Fuhrmann triangle are :*

$$W_4 \times \left(\sqrt{\frac{a}{\left(a+b-c\right)\left(a-b+c\right)}}, \sqrt{\frac{b}{\left(b-c+a\right)\left(b+c-a\right)}}, \sqrt{\frac{c}{\left(c+a-b\right)\left(c-a+b\right)}}\right)$$

where $W_4 = \sqrt{a^3+b^3+c^3-\left(a^2b+a^2c+ab^2+ac^2+b^2c+bc^2\right)+3\,abc}$

Quantity W_4 is the Fuhrmann square root (13.14). *Circumcircle of the Fuhrmann triangle is the Fuhrmann circle of ABC (whose diameter is* $[X_4, X_8]$

Remark 24.15.3. As noticed in (Dekov, 2007), $OIFuNa$ and $OIHFu$ are parallelograms, whose respective centroids are the Spieker center and nine-point center, respectively, where $IFu = 2NI$ or $IFu = R - 2r$.

24.16 Star triangle

Definition 24.16.1. Star triangle. Consider the midpoints $A'B'C'$ of the sidelines of triangle ABC. Draw from each midpoint the perpendicular line to the corresponding bisector. These three lines determine a triangle $A^*B^*C^*$. This is our star.

Proposition 24.16.2. *The synchronized barycentrics and the sidelengths of the star triangle are :*

$$\boxed{\mathcal{T}_*} \simeq \begin{pmatrix} b+c & c-b & b-c \\ c-a & c+a & a-c \\ b-a & a-b & b+a \end{pmatrix} \cdot \begin{pmatrix} b+c-a & 0 & 0 \\ 0 & c+a-b & 0 \\ 0 & 0 & a+b-c \end{pmatrix}^{-1}$$

$$\left[\alpha^2, \beta^2, \gamma^2\right] = \left(\frac{R}{2\,\rho}\right)^2 \times \left[a\,(b+c-a)\,,\, b\,(c+a-b)\,,\, c\,(a+b-c)\right]$$

Similar to the intouch triangle ($k^2 = R^3 \div 2\rho^3$) and to the excentral triangle ($k^2 = R \div 8\rho$). We have the following central correspondences :

$\mathcal{T}_*$	$\mathcal{T}$	$\mathcal{T}_*$	$\mathcal{T}$	$\mathcal{T}_*$	$\mathcal{T}$	$\mathcal{T}_*$	$\mathcal{T}$
2	3817	133	121	542	2801	2393	527
3	946	134	122	647	3835	2501	4885
4	10	135	123	690	3887	2574	3308
5	5	136	124	804	926	2575	3307
6	142	137	125	924	522	2679	1566
20	4301	138	126	974	1387	2777	2802
25	3452	139	127	1112	3035	2781	528
30	517	143	140	1154	30	2782	2808
39	2140	184	226	1205	3254	2790	2810
51	2	185	1	1495	908	2794	2809
52	3	235	1329	1503	518	2797	2821
53	141	389	1125	1510	523	2799	2820
65	178	403	3814	1531	1512	2848	2832
113	119	418	2051	1562	4904	3258	3259
114	118	427	2886	1568	1532	3564	971
115	116	428	3740	1596	3820	3566	3900
125	11	511	516	1637	4928	3574	442
128	113	512	514	1824	2090	3575	960
129	114	520	3667	1843	9	3917	1699
130	115	523	513	1986	214		
131	117	525	3309	1990	3834		
132	120	526	900	2052	3840		

For example, orthocenter X(4, $\mathcal{T}_*$) is Spieker center X(10, $\mathcal{T}$).

Proof. Straightforward computations. In fact, $A'A^*$ and B^*C^* are orthogonal and the orthic triangle of $\mathcal{T}_*$ is the medial triangle of $\mathcal{T}$.

□

		$\phi_{\mathcal{T}}$ when ϕ is the isogonal conjugacy
medial	$\mathcal{C}_2$	$(v+w-u)\left((u+v-w)\,b^2+(u-v+w)\,c^2\right)$
antimedial	$\mathcal{A}_2$	$\dfrac{-a^2}{v+w}+\dfrac{b^2}{u+w}+\dfrac{c^2}{u+v}$
orthic	$\mathcal{C}_4$	$u\,(-S_a u+S_b v+S_c w)$
tangential	$\mathcal{A}_6$	$a^2\left(\dfrac{-a^2}{c^2v+wb^2}+\dfrac{b^2}{uc^2+wa^2}+\dfrac{c^2}{ub^2+a^2v}\right)$
excentral	$\mathcal{A}_1$	$a\left(\frac{-1}{(b+c-a)(cv+bw)}+\frac{1}{(a-b+c)(cu+aw)}+\frac{1}{(b+a-c)(bu+av)}\right)$

		circumcircle of $\mathcal{T}$
medial	$\mathcal{C}_2$	$\sum\left(b^2+c^2-a^2\right)x^2-2\sum a^2yz$
antimedial	$\mathcal{A}_2$	$\sum a^2x^2+\left(a^2+b^2+c^2\right)\sum yz$
orthic	$\mathcal{C}_4$	$\sum\left(b^2+c^2-a^2\right)x^2-2\sum a^2yz$
tangential	$\mathcal{A}_6$	$a^2b^2c^2\sum\left(b^2+c^2-a^2\right)x^2+\left(\sum_6 a^4b^2-\sum_3 a^6\right)\sum a^2yz$
excentral	$\mathcal{A}_1$	$\sum bcx^2+(a+b+c)\sum ayz$

Figure 24.1: Special Triangles

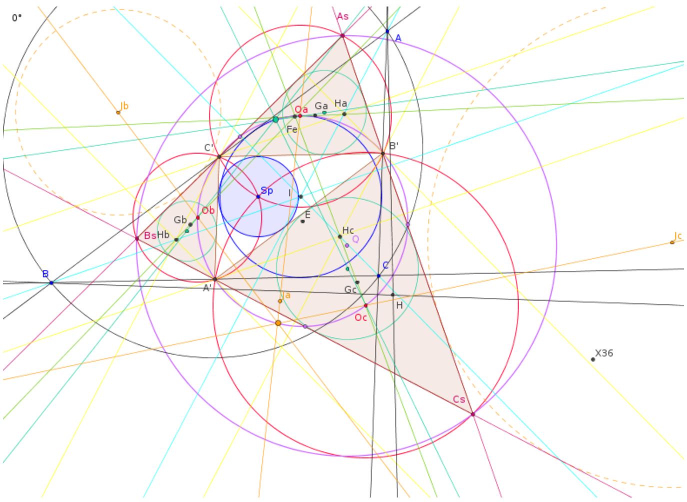


Figure 24.2: The star triangle

Chapter 25

Formal operations

Let us examine again some already defined operations, and consider in more details their formal properties

25.1 Unary operators

1. DP1, see 3.13
$$\mathrm{DP}_1\,(U) = u^2(v+w) : v^2(w+u) : w^2(u+v)$$

2. DP2, see 3.14
$$\mathrm{DP}_2\,(U) = u(v-w)^2 : v(w-u)^2 : w(u-v)^2$$

25.2 cevamul, cevadiv, crossmul, crossdiv

These operations were considered in Kimberling to formalize various properties related to cevian nests.

Remark 25.2.1. **cevadiv** has been defined in Section 3.11 by :

$$\begin{aligned} \mathrm{cevadiv}\,(P\ ;\ U) &\doteq persp\,(Cevian\,(dividend,\ ABC)\,,\ Anticevian\,(divisor,\ ABC)) \\ &\simeq u(-q\,r\,u + r\,p\,v + p\,q\,w) : v(q\,r\,u - r\,p\,v + p\,q\,w) : w(q\,r\,u + r\,p\,v - p\,q\,w) \end{aligned}$$

Property $X = \mathrm{cevadiv}\,(P, U)$ is sometimes stated as "X is the P-ceva conjugate of U". This operation is clearly a type-keeping (P, U)-map and an involutory U-map (when P is fixed).

Remark 25.2.2. The **cevamul** operation is the converse of the previous one, and is sometimes called cevapoint.

$$\mathrm{cevamul}\,(u:v:w,\ x:y:z) = (uz+wx)(uy+vx) : (vz+wy)\,(uy+vx) : (vz+wy)\,(uz+wx)$$

This operation is clearly commutative and type-keeping.

Remark 25.2.3. The **crossdiv** has been defined in Section 3.10 by :

$$\begin{aligned} \mathrm{crossdiv}\,(P\ ;\ U) &\doteq persp\,(ABC,\ Cevian\,(dividend,\ Cevian\,(divisor,\ ABC))) \\ &\simeq \frac{u}{quw+ruv-pvw} : \frac{v}{pvw+ruv-quw} : \frac{w}{pvw+quw-ruv} \end{aligned}$$

Property $X = \mathrm{crossdiv}\,(P, U)$ s sometimes stated as "X is the P-crossconjugate of U". This operation is clearly a type-keeping (P, U)-map and an involutory U-map (when P is fixed).

Remark 25.2.4. The **crossmul** operation is the converse of the previous one, and is sometimes called crosspoint.

$$\mathrm{crossmul}\,(u:v:w,\ x:y:z) = (v\,z + w\,y)\,u\,x : (u\,z + w\,x)\,v\,y : (u\,y + v\,x)\,w\,z$$

This operation is clearly commutative and type-keeping.

Proposition 25.2.5. ***'div' formula**. The isoconjugacy that exchanges P, U also exchanges*

$$P \leftrightarrow U \;;\; \text{crossdiv}\,(P,U) \leftrightarrow \text{cevadiv}\,(U,P) \;;\; \text{crossdiv}\,(U,P) \leftrightarrow \text{cevadiv}\,(P,U)$$

Proof. We have $\text{crossdiv}\,(P,U) \underset{b}{*} \text{cevadiv}\,(U,P) = P \underset{b}{*} U$. □

Proposition 25.2.6. ***'mul' formula**. For any isoconjugacy, we have :*

$$\begin{aligned} \text{cevamul}\,(U,X) &= \text{crossmul}\left(U_F^{\#},\; X_F^{\#}\right)_F^{\#} \\ \text{crossmul}\,(U,X) &= \text{cevamul}\left(U_F^{\#},\; X_F^{\#}\right)_F^{\#} \end{aligned}$$

Proof. Direct examination is shorter, using the "div" formula just above is more stratospheric. □

25.3 Formal operators and conics

Since they are all Cremona transforms of second degree, operators wrt a triangle can be transformed into operators wrt a well chosen conic.

Proposition 25.3.1. ***barymul**. Let ψ be the ABC-isoconjugacy that swaps points P, U (not on the sidelines). Then ψ swaps $\mathcal{C}_{PU} \doteq conic\,(A,B,C,P,U)$ and $\mathcal{L}_{PU} \doteq line\,(P,U)$. The pole F^2 of the conjugacy is the barymul of P, U and therefore the "barysquare" of the four fixed points of ψ:*

$$F^2 \simeq P \underset{b}{*} U \;;\; f^2 = kp\; u,\; g^2 = kq\; v,\; h^2 = kr\; w,\; k \neq 0$$

Proposition 25.3.2. *We have the various 'cross' formulas*

formula	name	usefulness	inverse
$\dfrac{1}{qw} - \dfrac{1}{rv}$		intersections of the tripolars, perspector of $\mathcal{C}_{PU}$	$X \in \mathcal{C}_{PU}$
$\dfrac{1}{rv} + \dfrac{1}{qw}$	crossmul	conipole of $\mathcal{L}_{PU}$ wrt $\mathcal{C}_{PU}$	crossdiv
$\dfrac{1}{qw - rv}$		tripole of $\mathcal{L}_{PU}$	$X \in \mathcal{L}_{PU}$
$\dfrac{1}{qw + rv}$	cevamul	$U = persp\,(Acev\,(P)\,,\; cev\,(X))$	cevadiv

Thus the 'odd ones'[1] aren't Cremona transforms.

25.4 crossdiff, crosssum, polarmul, polardiv

Definition 25.4.1. The F-**crossdiff** of two points $U = u : v : w$ and $X = x : y : z$ that aren't lying on a sideline of ABC is defined by :

$$\text{crossdiff}_F\,(U,X) = f^2\,(wy - vz) : g^2\,(uz - wx) : h^2\,(vx - uy)$$

Remark 25.4.2. In ETC, F =X(1) is assumed. Defined as above, the operation $(F,U,X) \mapsto \text{crossdiff}_F\,(U,X)$ is globally type-keeping and provides a point when the entries are points (F is any of the four fixed point of the conjugacy $X \mapsto X^*$).

Proposition 25.4.3. *The F-crosssum of U, X that was defined at Definition 22.5.5 is constructible using* crossmul, cevamul *and isoconjugacies. Therefore a better definition is the globally type-keeping function :*

$$\begin{aligned} \text{crosssum}_F\,(U,X) &= f^2\,(wy + vz)\;:\; g^2\,(uz + wx)\;: h^2\,(vx + uy) \\ &= (\text{cevamul}\,(U,X))_F^{\#} = \text{crossmul}\left(U_F^{\#},\; X_F^{\#}\right) \end{aligned}$$

where $F = f : g : h$ is any of the fixed points of the conjugacy.

[1] English joke. Can not be translated.

Remark 25.4.4. Defined that way, $\text{crosssum}_F(U, X)$ is really different from the polar line of X wrt the circumconic $CC(U)$: $u\,yz + v\,zx + w\,xy = 0$ since this line is the next coming polarmul.

Proposition 25.4.5. *Given U and $P \doteq \text{crosssum}_F(U, X)$ one can find X using*

$$X \simeq \text{cevadiv}\left(P_F^{\#},\ U\right)$$

Proof. Direct inspection (here P is generic, not the "square" of the fixed point F). □

Proposition 25.4.6. *The* ***polarmul of two points*** *U, X is a line Δ. When $U = u : v : w$ and $X = x : y : z$ are not on the sidelines, then line Δ is defined as the conipolar of the point X wrt the circumconic $CC(U)$. We have polarmul (U, U) = tripolar (U) and*

$$\begin{aligned} \textit{polarmul}(U, X) &= \textit{complem}\left(X \underset{b}{\div} U\right) \underset{b}{\div} U \\ &= \text{crossmul}(\text{tripolar}(U), \text{tripolar}(X)) = \text{tripolar}(\text{cevamul}(U,\ X)) \\ &= [wy + vz\ ;\ uz + wx\ ;\ vx + uy] \end{aligned}$$

Operation polarmul is commutative and type-crossing (i.e output is a line when entries are points).

Proposition 25.4.7. *The* ***polarmul of two lines*** *D, Δ is the point P obtained by applying "the same rules" as above, i.e.*

$$\textit{polarmul}([u, v, w],\ [x, y, z]) \doteq wy + vz\ :\ uz + wx\ :\ vx + uy$$

Then polarmul (Δ, Δ) *is* tripolar (Δ) *and polarmul* (D, Δ) *is the conipolar of line* Δ *wrt the inconic whose perspector is* tripolar (D).

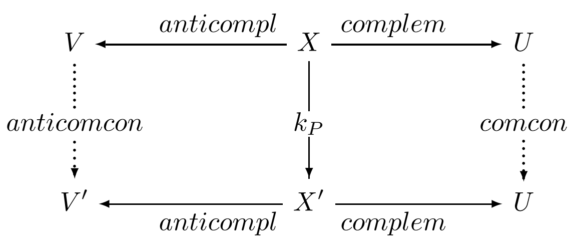
Proof. Direct inspection. Remember that the dual of an inconic "goes through" the sidelines. □

Definition 25.4.8. The **polardiv** of a line $\Delta \simeq [\rho; \sigma; \tau]$ and a point $U \simeq u : v : w$ is the reverse operation of the previous one. One has the formula :

$$\begin{aligned} \textit{polardiv}(\Delta,\ U) &= \text{cevadiv}(\text{tripolar}(\Delta),\ U) \\ &= (\sigma v + \tau w - \rho u)\,u : (\tau w + \rho u - \sigma v)\,v : (\rho u + \sigma v - \tau w)\,w \end{aligned}$$

that correctly defines a point when P is a line and U is a point.

25.5 Complementary and anticomplementary conjugates



$$x_{comcon} = \frac{g^2}{u - v + w} + \frac{h^2}{u + v - w}$$

$$x_{anticomcon} = -\frac{f^2}{v + w} + \frac{g^2}{u + w} + \frac{h^2}{u + v}$$

Figure 25.1: The complementary and anticomplementary conjugates

Definition 25.5.1. comcon, anticomcon. For points $P = f^2 : g^2 : rh^2$ and $U = u : v : w$, neither lying on a sideline of ABC, the P-complementary and P-anticomplementary conjugates of U are defined as in Figure 25.1, where k is the P-isoconjugacy. Most of the time, $P = X_1$ and the conjugacy reduces to isogonal conjugacy.

25.6 Hirstpoint aka Hirst inverse

Definition 25.6.1. Hirstpoint. Suppose $P = p : q : r$ and $U = u : v : w$ are distinct points, neither lying on a sideline of ABC. The hirstpoint X is the point of intersection of the line PU and the polar of U with respect to the circumconic $CC(P)$ conic :

$$p\,yz + q\,zx + r\,xy = 0.$$

Proposition 25.6.2. *We have the following properties :*
(i) H is a type-keeping operation as a "ramified product" of type-crossing transforms and :

$$\begin{aligned} hirstpoint\,(P,U) &= (P \wedge U) \wedge polarmul\,(P,U) \\ &= u^2\,q\,r - p^2\,v\,w : p\,v^2\,r - u\,q^2\,w : p\,q\,w^2 - u\,v\,r^2 \end{aligned} \tag{25.1}$$

(ii) H is commutative from the duality properties of polarization.
(iii) $H(P,U) = 0:0:0$ occurs only when $U = P$
(iv) $H(P,U) = P$ if and only if U lies on the polar line of P
(v) $H(P,\ H(P,\ U))$ is either $0:0:0$ or U. Indeterminate form is obtained (a) on the polar line of P and (b) on a conic containing P and having P has perspector ... i.e. a conic whose only real point is P.

Proof. Direct inspection for all properties. To be precise, these properties are valid only on the real part of the world. For example, (i) gives $U = p : jq : j^2 r$ where $j^3 = 1$, i.e. $U = P$ and two other "imagined" solutions. □

All these properties show that "Hirst inverse" is a poorly chosen term, since we aren't dividing, but multiplying. Concerning the designation "Hirst inverse," see the contribution Gunter Weiss: http://mathforum.org/kb/message.jspa?messageID=1178474.

25.7 Line conjugate

Suppose $P = p : q : r$ and $U = u : v : w$ are distinct points, neither equal to A, B, or C. The P-line conjugate of U is the point whose trilinears are given by :

$$p(v^2 + w^2) - u(qv + rw) : q(w^2 + u^2) - v(rw + pu) : r(u^2 + v^2) - w(pu + qv)$$

This is the point of intersection of line PU and the tripolar of the isogonal conjugate of U.

Using the same formula with barycentrics, another point is obtained, that is the intersection of PU and the dual of U. So what ?

25.8 Collings transform

Lemma 25.8.1. *Let M_i, $1 \leq i \leq 5$, be five (different) points, not four of them on the same line, such that $midpoint\,(M_1, M_2) = midpoint\,(M_3, M_4) = P$. They determine uniquely a conic whose center is P and contains $reflection(P, M_5)$.*

Proof. Take P as origin of the euclidian coordinate system and consider determinant γ whose lines are $\left[x_i^2, x_i y_i, y_i^2, x_i, y_i, 1\right]$, the last line ($i = 6$) referring to the generic point of the plane. Since $x_2 = -x_1$, $\cdots$ γ can be factored into $(x_1 y_3 - x_3 y_1)$ times an expression without terms of first degree in x_6, y_6. □

Lemma 25.8.2. *Let M_i, $1 \leq i \leq 4$, be four different points, not on the same line. The locus :*

$$\Theta = \{center\,(\gamma) \mid \gamma \text{ is a conic and } M_1, M_2, M_3, M_4 \in \gamma\}$$

is a conic. It contains the six $midpoint(M_i, M_j)$ and its center is $K = \sum M_i/4$.

Proof. Θ is a conic since degree is two. When $P = midpoint\,(M_1, M_2)$, the preceding lemma can be applied to $M_1, M_2, M_3, reflection\,(P, M_3)\,, M_4$, defining a conic whose center is P, so that $P \in \Theta$. Now, the lemma can be applied to Θ itself, since $K = midpoint\,(midpoint\,(M_1, M_2)\,, midpoint\,(M_3, M_4))$ –and cyclically. □

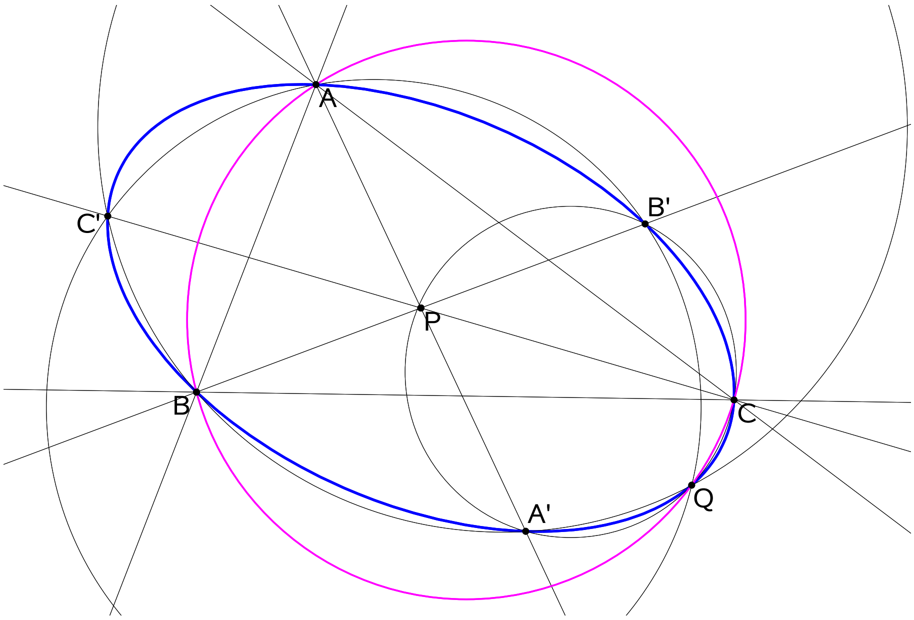


Figure 25.2: Collings configuration

Proposition 25.8.3. *Let P be a point not on a sideline of ABC, and A', B', C' the reflections of A, B, C in P.*
(i) It exists a conic γ through the six points A, B, C, A', B', C', its center is P and its perspector is $U = P \underset{b}{*} \operatorname{anticomplem}(P)$, so that $\gamma = CC(U)$. This conic intersects the circumcircle at point $Q = \operatorname{isotom}(X_6 \wedge U)$, i.e. the tripole of line UX_6. Moreover, the circumcircles of triangles $AB'C'$, $A'BC'$ and $A'B'C$ are also passing through point Q.
(ii) Conversely, when Q is given on the circumcircle, the locus of P is $\operatorname{conicev}(X_2, Q)$. This conic goes through the three $AB \cap CQ$ points, the three $\operatorname{midpoint}(A, Q)$ and through four fixed points : the vertices of medial triangle and through the circumcenter X_3. Moreover, this conic is a rectangular hyperbola, its center is $K = (A + B + C + Q)/4$ and belongs to the nine points circle of the medial triangle.
(iii) The anticomplement of this RH is the rectangular ABC-circumhyperbola whose center is the complement of Q.

Proof. For (i), only Q belongs to circumcircle of $AB'C'$ has to be proved. Barycentric computation. For (ii), point $AB \cap CQ$ lead to the degenerate conic $AB \cup CQ$ (and cyclically) while the six midpoints come from the lemma. When $P = X_3$, conic γ is the circumcircle... and passes through A, B, C, Q and $X_3 \in \Theta$. But X_3 of ABC is X_4 of the medial triangle, and Θ contains an orthic configuration, characteristic property of a rectangular hyperbola. □

For the sake of exhaustivity, if barycentrics of P are $p : q : r$ then barycentrics of Q are

$$Q_x : Q_y : Q_y \quad where \quad Q_x = \frac{1}{r(p+q-r)b^2 - q(p-q+r)c^2}$$

(and cyclically in a, b, c and p, q, r too). The transformation $P \mapsto Q$ was described by Collings (1974) and was further discussed by (Grinberg, 2003c).

Example 25.8.4. Examples are as follows :

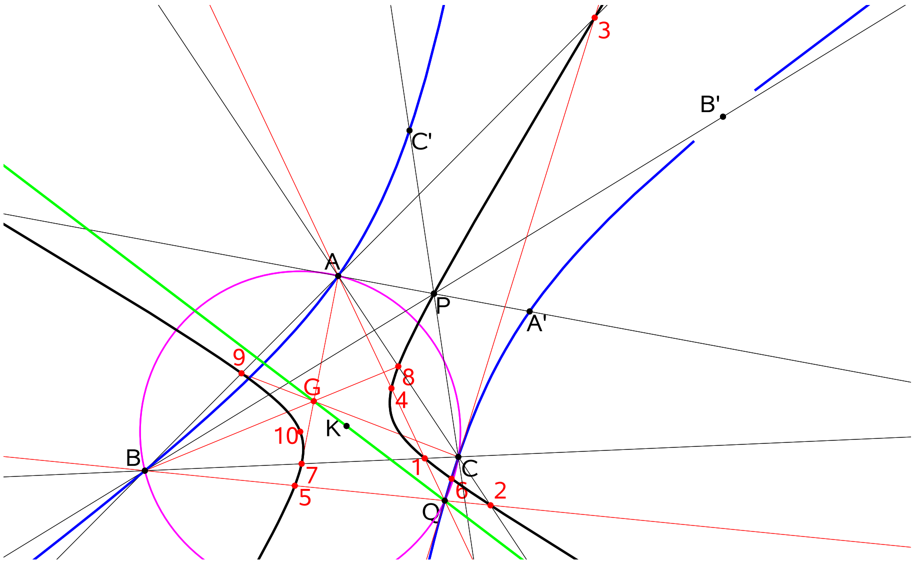


Figure 25.3: Collings locus is a ten points rectangular hyperbola

Q	points on the conic	wrt medial triangle
X_{74}	125	
X_{98}	115, 868	
X_{99}	2, 39, 114, 618, 619, 629, 630, 641, 642, 1125	Kiepert hyperbola
X_{100}	1, 9, 10, 119, 142, 214, 442, 1145	Feuerbach hyperbola
X_{107}	4, 133, 800, 1249	
X_{110}	5, 6, 113, 141, 206, 942, 960, 1147, 1209	Jerabek hyperbola
X_{476}	30	

Chapter 26

Sondat theorems

26.1 Perspective and directly similar

Remark 26.1.1. When triangle ABC is translated into $A'B'C'$, these triangles are in perspective and the perspector is the direction of the translation. The converse situation is not clear, so that translations will be excluded from what follows, implying the existence of a center. When using composition, we have to examine if nevertheless translations are reappearing.

Lemma 26.1.2. *When σ is a direct similarity (but not a translation) with center $S = z : 0 : \zeta$ and ratio $k\kappa$ (k is real while κ is unimodular, and $k\kappa \neq \pm 1$) then its matrix in the Morley space is :*

$$\boxed{\sigma} = \begin{pmatrix} k\kappa & \dfrac{z}{t}\,(1-k\kappa) & 0 \\ 0 & 1 & 0 \\ 0 & \dfrac{\zeta}{t}\left(1-\dfrac{k}{\kappa}\right) & \dfrac{k}{\kappa} \end{pmatrix}$$

Proof. One can check that umbilics are fixed points of this transform. The characteristic polynomial of matrix $\boxed{\sigma}$ is $\chi(\mu) = (\mu-1)(\mu-k\,\kappa)(\mu-k/\kappa)$. Excluding $k\kappa = \pm 1$ ensures the existence of a center. □

Proposition 26.1.3. *When a direct central similarity $\sigma(S,\, k\kappa)$ and a perspector $P \neq S$ are given, the locus $\mathcal{C}$ of points M such that P, M, $M' = \sigma(M)$ are collinear is the circle through S, P, $\sigma^{-1}P$.*

Proof. The locus contains certainly the five points such that $M = P$ or $M' = P$ or $M = M'$ i.e. S and both umbilics. The general case results from the fact that $\det[P,\, M,\, M']$ is a second degree polynomial in $\mathbf{Z}$, $\overline{\mathcal{Z}}$, $\mathbf{T}$ so that $\mathcal{C}$ is a conic. □

Proposition 26.1.4. *Suppose that triangles $\mathcal{T}_1$ and $\mathcal{T}_2$ are together in perspective (center P) and strictly similar (center S, ratio $k\kappa$, $\kappa \neq \pm 1$). Then P and S are the two intersections of their circumcircles ($P = S$ cannot occur).*

Proof. Use Lubin coordinates relative to $\mathcal{T}_1 = ABC$, and note $S \simeq z : t : \zeta$. Then $A'B'C'$ is obtained as :

$$\boxed{A'B'C'} \simeq \boxed{\sigma} \cdot \boxed{ABC}$$

The determinant of lines AA', BB', CC' factors as :

$$\frac{\sigma_4}{\sigma_3}\,\frac{k}{\kappa}\,\left(1-\kappa^2\right)\left(1-\frac{k}{\kappa}\right)(1-k\,\kappa) \times \left(\frac{z\,\zeta - t^2}{t^2}\right)$$

proving $S \in \Gamma$. Then perspector is also on this circle (from preceding proposition). We even have the more precise result :

$$P = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^{-1} \end{pmatrix} \cdot S \quad \text{where} \quad \omega = \frac{1-k\kappa}{1-\dfrac{k}{\kappa}}$$

□

Proposition 26.1.5. *Consider a fixed triangle ABC, and describe the plane using the Lubin frame. Consider points $P=\Phi:1:1/\Phi$ and $S=\Theta:1:1/\Theta$ on the unit circle (P as Phi, and S as Sigma. But Sigma is sum, use the next Greek letter). Assume $P\neq S$. Then all triangles $A'B'C'$ that are P-perspective and S-similar to triangle ABC are obtained as follows. Let point O' on the perpendicular bisector of (P,S) be defined by property $(SO,SO')=\kappa$ where κ^2 is a given turn. Draw circle γ centered at O' and going through P and S. Then $A'=\gamma\cap SA$,* etc.

Proof. Point O' can be written as $P+S+x\,(\Theta\Phi:0:1)$. This point is the $\sigma\,(S,\,k\kappa)$ image of O if and only if :

$$k=\kappa\,\frac{\Phi-\Theta}{\Phi-\kappa^2\Theta},\ \boxed{\sigma}=\left(\begin{array}{ccc}\kappa^2\,(\Phi-\Theta) & \left(1-\kappa^2\right)\Theta\,\Phi & 0\\ 0 & \Phi-\kappa^2\Theta & 0\\ 0 & 1-\kappa^2 & \Phi-\Theta\end{array}\right)$$

As it should be, σ depends only on κ^2, while the sign of k depends on the choice of κ among the square roots of κ^2. The matrix of circle γ is :

$$\boxed{\gamma}={}^t\boxed{\sigma}^{-1}\cdot\boxed{\Gamma}\cdot\boxed{\sigma}^{-1}\simeq\left(\begin{array}{ccc}0 & 1-\kappa^2 & \kappa^2\Theta-\Phi\\ 1-\kappa^2 & 2\,\left(\kappa^2\Phi-\Theta\right) & \left(1-\kappa^2\right)\Theta\,\Phi\\ \kappa^2\Theta-\Phi & \left(1-\kappa^2\right)\Theta\,\Phi & 0\end{array}\right)$$

And it can be checked that P, A, $A'=\sigma\,(A)$ are collinear. □

Proposition 26.1.6. *With same hypotheses, the perspectrix of triangles ABC and $A'B'C'$ is the line :*

$$XYZ=\left[\left(\Phi-\kappa^2\,\Theta\right)\Phi,\ -\Phi\Theta\,\left(\Phi-\kappa^2\,\sigma_1\right)+\kappa^2\,\left(\kappa^2\,\sigma_3-\sigma_2\Phi\right),\ \kappa^2\,\left(\Phi-\kappa^2\,\Theta\right)\sigma_3\right]$$

Points S,C,C',X,Y are concyclic (and circularly). When κ^2 reaches Φ/α, then A' moves to P and XYZ becomes the sideline BC. When S is not a vertex A,B,C, the envelope of line XYZ is the Steiner parabola of point S (focus at S, directrix the Steiner line of S). The tangential equation of this parabola is given by matrix :

$$\boxed{\mathcal{P}^*}\simeq\left(\begin{array}{ccc}2\,\Theta\,\sigma_3 & \sigma_3 & \sigma_2-\sigma_1\Theta\\ \sigma_3 & 0 & -\Theta\\ \sigma_2-\sigma_1\Theta & -\Theta & -2\end{array}\right)$$

Proof. Since XYZ is given by second degree polynomials, the envelope is a conic. It can be obtained by diff and wedge, then eliminate. Parabola comes from the central 0. Focus is obtained in the usual way, and one recognizes S. Directrix Δ is the locus of the reflections of the focus in the tangents. From the special cases, this is the Steiner line of S.

Last point, compute circle (S,X,Y). Special cases $\Theta=\alpha$, etc, and $\Phi=\kappa^2\Theta$ (O' at infinity) are appearing in factor. Otherwise, the equation is :

$$\alpha\,\left(\Phi-\kappa^2\,\Theta\right)\mathbf{Z}\overline{\mathcal{Z}}+\left(\kappa^2\,\alpha-\Phi\right)\left(\mathbf{Z}+\alpha\,\Theta\,\overline{\mathcal{Z}}\right)\mathbf{T}+\left(\Phi\,\Theta-\alpha^2\kappa^2\right)\mathbf{T}^2$$

and this circle goes through A and A'. □

Proposition 26.1.7. *When A,B,C,S,P are according the former hypotheses, let H, H' be the respective orthocenters of ABC and $A'B'C'$. Then midpoint of H,H' belongs to XYZ if and only if $\kappa^2=-1$ (so that κ is a quarter turn) or :*

$$\kappa^2=-\Phi^2\,\frac{(\Theta-\sigma_1)}{\sigma_2\,\Theta-\sigma_3},\quad i.e.\,\kappa=(BC,\,OP)+(SA,\,SH)$$

In the second case, XYZ is the perpendicular bisector of (H,H').

Proof. We have $H'=\sigma\,(H)$ and equation in κ^2 is straightforward. Then we have :

$$\omega^2\,(BC)=-\beta\gamma,\ \omega^2\,(OP)=\Phi^2,\ \omega^2\,(SA)=-\Theta\alpha,\ \omega^2\,(SH)=-\sigma_3\Theta\,\frac{\Theta-\sigma_1}{\sigma_2\Theta-\sigma_3}$$ □

Corollary 26.1.8. *Start from triangle ABC, and assume that* $XYZ = [\rho, \sigma, \tau]$ *while* κ *is a quarter turn. Then* X, Y, Z *are* $X = 0 : \tau : -\sigma$, etc. *Lines* $X\delta_A = \left[S_c\sigma + S_b\tau, a^2\sigma, a^2\tau\right]$, etc *are the perpendicular at* X *to* BC, etc. *Finally, point* A' *is* $Y\delta_B \cap Z\delta_C$, etc. *In other words :*

$$A' \simeq \left(\begin{array}{c} \rho\left(\rho S_b S_c + \sigma S_c S_a + \tau S_a S_b\right) - 4\,S^2\sigma\tau \\ b^2\rho\left(\tau c^2 - \sigma S_a - \rho S_b\right) \\ c^2\rho\left(\sigma b^2 - \tau S_a - \rho S_c\right) \end{array}\right), \text{ etc}$$

The perspector and the similicenter are :

$$P = isogon\left(\boxed{\mathcal{M}_b} \cdot {}^t\Delta\right)\ ;\ S = isogon\left(\left(\sigma-\tau\right)\rho : \sigma\left(\tau-\rho\right) : \tau\left(\rho-\sigma\right)\right)$$

while the ratio of the similarity is :

$$k = \frac{\left(a^2+b^2+c^2\right)\rho\sigma\tau - S_a\rho\left(\sigma^2+\tau^2\right) - S_b\sigma\left(\rho^2+\tau^2\right) - S_c\tau\left(\rho^2+\sigma^2\right)}{2\,S\left(\sigma-\tau\right)\left(\rho-\tau\right)\left(\rho-\sigma\right)}$$

Proof. Straightforward computation. □

26.2 Perspective and inversely similar

Lemma 26.2.1. *A circumscribed rectangular hyperbola goes through* A, B, C, H, Gu *where* H *is the orthocenter and* Gu *is the gudulic point, the intersection of the RH and the circumcircle. Directions of axes are given by the bisectors of, for example,* AGu *and* BC. *Then directions of asymptotes are obtained by a* $45°$ *rotation (or taking again the bisectors).*

Lemma 26.2.2. *When a rectangular hyperbola* $\mathcal{H}$ *is known by its implicit equation*

$$\kappa^2\overline{\mathcal{Z}}^2 - \frac{1}{\kappa^2}\,\mathbf{Z}^2 + \left(W\,\overline{\mathcal{Z}} + V\,\mathbf{Z}\right)\mathbf{T} + Q\,\mathbf{T}^2$$

then points $M \in \mathcal{H}$ *can be parametrized as :*

$$M = \frac{1}{2}\left(\begin{array}{c} +V\,\kappa^2 \\ 1 \\ -W\,\dfrac{1}{\kappa^2} \end{array}\right) + X\left(\begin{array}{c} \kappa \\ 0 \\ \dfrac{1}{\kappa} \end{array}\right) + Y\left(\begin{array}{c} +i\,\kappa \\ 0 \\ -i\,\dfrac{1}{\kappa} \end{array}\right) \tag{26.1}$$

where X, Y *are real quantities linked by :*

$$YX = \frac{-i}{16}\left(\kappa^2V^2 + 4\,Q - \frac{W^2}{\kappa^2}\right)$$

Proof. This way of writing may look weird but, most of the time, hyperbola equations are appearing that way. □

Lemma 26.2.3. *Consider four points* M_j *on a rectangular hyperbola, parametrized by* (26.1). *These points form an orthocentric quadrangle if and only if :*

$$256\,x_1x_2x_3x_4 = \left(\kappa^2V^2 + 4\,Q - \frac{W^2}{\kappa^2}\right)^2$$

Proof. We write that $(M_1 \wedge M_2)\cdot\boxed{\mathcal{M}_z}\cdot{}^t(M_3 \wedge M_4) = 0$, and obtain this condition. The conclusion follows from the symmetry of the the result. □

Lemma 26.2.4. *When* ψ *is a central inverse similarity (reflections in a line are allowed, but not the other isometries), then its matrix in the Morley space can be written as :*

$$\boxed{\psi} = \left(\begin{array}{ccc} 0 & \dfrac{z}{t} - k\,\kappa^2\,\dfrac{\zeta}{t} & k\kappa^2 \\ 0 & 1 & 0 \\ \dfrac{k}{\kappa^2} & \dfrac{\zeta}{t} - \dfrac{k}{\kappa^2}\,\dfrac{z}{t} & 0 \end{array}\right)$$

Point $S = z : 0 : \zeta$ *is the center, axes are directed by* $\pm\kappa^2$ *and ratio is* k *(*κ *is unimodular while* k *is real and* $k = \pm1$ *is a dubious case). One can check that umbilics are exchanged by this transform.*

Proof. Let $z_2 : t_2 : \zeta_2$ be the image of the origin $0 : 1 : 0$. The characteristic polynomial of matrix :

$$\left(\begin{array}{ccc} 0 & z_2/t_2 & k\,\kappa^2 \\ 0 & 1 & 0 \\ \dfrac{k}{\kappa^2} & \zeta_2/t_2 & 0 \end{array}\right)$$

is $\chi(\mu) = (\mu - 1)(\mu - k)(\mu + k)$. Excluding $k = \pm 1$ ensures the existence of a center. Consider the reflection δ about line through S and $\kappa^2 : 0 : 1$. We have :

$$\boxed{\delta} = subs\left(k = \pm 1, \boxed{\psi}\right) \quad ; \quad \boxed{\psi} \cdot \boxed{\delta} = \boxed{\delta} \cdot \boxed{\psi} = \left(\begin{array}{ccc} k & (1-k)\,\dfrac{z}{t} & 0 \\ 0 & & 0 \\ 0 & (1-k)\,\dfrac{\zeta}{t} & k \end{array}\right)$$

□

Remark 26.2.5. The unimodular κ was an intrinsic quantity when we were dealing with direct similarities. Now, κ^2 measure the angle between the real axis and one of the axes of the skew similarity.

Proposition 26.2.6. *When a central skew similarity $\psi(S,\, k\kappa)$ and a perspector $P \neq S$ are given, the locus $\mathcal{H}$ of points M such that $P,\, M,\, M' = \psi(M)$ are collinear is the conic through $S,\, P,\, \psi^{-1}P$ and directions of the ψ-axes (and this conic is a rectangular hyperbola).*

Proof. The locus contains certainly the five points such that $M = P$ or $M' = P$ or $M = M'$ i.e. S and both directions $\pm\kappa^2 : 0 : 1$. The general case results from the fact that $\det[P,\, M,\, M']$ is a second degree polynomial in $\mathbf{Z},\, \overline{\mathcal{Z}},\, \mathbf{T}$ so that $\mathcal{C}$ is a conic. □

Proposition 26.2.7. *Suppose that triangles $\mathcal{T}_1$ and $\mathcal{T}_2$ are together in perspective (center P) and strictly antisimilar (center S, ratio $k \neq \pm 1$, direction of axes $\pm\kappa^2 : 0 : 1$). Let X be one of the points $S,\, P,\, \pm\kappa^2 : 0 : 1$. Then the other three are obtained as the remaining intersections between conic $\mathcal{H}$ through A, B, C, H, X and conic $\mathcal{H}'$ through A', B', C', H', X.*

Proof. Use Lubin coordinates relative to $\mathcal{T}_1 = ABC$, and note $S \simeq z : t : \zeta$. Then $A'B'C'$ is obtained as :

$$\boxed{A'B'C'} \simeq \boxed{\psi} \cdot \boxed{ABC}$$

The determinant of lines AA', BB', CC' factors as :

$$\frac{\sigma_4}{t^2 \sigma_3}\, k\left(k^2 - 1\right) \times conic$$

proving that S belongs to the rectangular hyperbola whose implicit and parametric equations are :

$$\frac{-1}{\kappa^2}\,\mathbf{Z}^2 + \kappa^2 \overline{\mathcal{Z}}^2 + \left(\frac{\sigma_1}{\kappa^2} - \frac{\kappa^2}{\sigma_3}\right)\mathbf{TZ} + \left(\frac{\sigma_2\kappa^2}{\sigma_3} - \frac{\sigma_3}{\kappa^2}\right)\mathbf{T}\overline{\mathcal{Z}} + \left(\frac{\sigma_1\kappa^2}{\sigma_3} - \frac{\sigma_2}{\kappa^2}\right)\mathbf{T}^2$$

$$M(x) \simeq \left[\begin{array}{c} x\,\kappa - \dfrac{\kappa^4 - \sigma_1\sigma_3}{2\,\sigma_3} + \dfrac{1}{16\,x}\,\kappa\left(\kappa^2\left(\dfrac{\kappa^2}{\sigma_3} + \dfrac{\sigma_1}{\kappa^2}\right)^2 - \dfrac{1}{\kappa^2}\left(\dfrac{\sigma_3}{\kappa^2} + \dfrac{\sigma_2\kappa^2}{\sigma_3}\right)^2\right) \\ 1 \\ \dfrac{1}{\kappa}x + \dfrac{\sigma_2\kappa^4 - \sigma_3^2}{2\,\kappa^4\sigma_3} - \dfrac{1}{16\,x}\,\dfrac{1}{\kappa}\left(\kappa^2\left(\dfrac{\kappa^2}{\sigma_3} + \dfrac{\sigma_1}{\kappa^2}\right)^2 - \dfrac{1}{\kappa^2}\left(\dfrac{\sigma_3}{\kappa^2} + \dfrac{\sigma_2\kappa^2}{\sigma_3}\right)^2\right) \end{array}\right]$$

Then perspector is also on this hyperbola (from preceding proposition). We even have the more precise result :

$$k = \frac{x_S - x_P}{x_S + x_P}$$

□

Proposition 26.2.8. *Consider a triangle A, B, C, its orthocenter H and two points S, P such that the six points A, B, C, H, P, S are on the same conic $\mathcal{H}$. Then it exists exactly one triangle $A'B'C'$ that is together S-antisimilar and P perspective with ABC. Moreover A', B', C', H', P, S are on the same conic $\mathcal{H}'$, both conics are rectangular hyperbolas and share the same asymptotic directions. Finally, the gudulic point G_u of conic $\mathcal{H}$ sees triangle ABC at right angles with trigone $A'B'C'$, and the fourth intersection of conic $\mathcal{H}'$ with circle Γ' sees triangle $A'B'C'$ at right angles with trigone ABC.*

Proof. A conic through A, B, C, D is a rectangular hyperbola. Consider one of its asymptotes and draw a parallel Δ to this line through point S. Let A'', B'', C'', H'' be the reflections of A, B, C, H into Δ. Then we have A'=SA"∩PA, etc. Final result comes from □

Proposition 26.2.9. *When A, B, C, H, P are fixed, the direction of the perspectrix XYZ is also fixed.*

Proposition 26.2.10. *When A, B, C, H, S are fixed and P moves onto the A, B, C, H, S hyperbola, the envelope of the perspectrix XYZ is the parabola inscribed in triangle ABC whose directrix is line HS.*

26.3 Parallelogy

Definition 26.3.1. Two triangles $\mathcal{T}_1$ and $\mathcal{T}_2$ are **parallelogic** when a point M_1 exists that sees triangle $\mathcal{T}_1$ with rays parallel to the sidelines of $\mathcal{T}_2$. In other words, the lines drawn from the vertices of $\mathcal{T}_1$ and parallel to the corresponding sides of $\mathcal{T}_2$ are concurrent at a point M_1 (the ray source).

Proposition 26.3.2. *Parallelogy is a symmetric relation between triangles (defining a ray source M_1 that sees $\mathcal{T}_1$ and a ray source M_2 that sees $\mathcal{T}_2$).*

Proof. Using computer, symmetry is straightforward : condition of concurrence is the product of $\det \mathcal{T}_2$ by a polynomial that is invariant by exchange of the two triangles. □

Proposition 26.3.3. *When $\mathcal{T}_1, \mathcal{T}_2$ are parallelogic, the formula $\boxed{\phi} \doteq \boxed{\mathcal{T}_2} \cdot \boxed{\mathcal{T}_1}^{-1}$ (columns are supposed to be synchronized !) defines a collineation ϕ such that $\mathcal{L}_b \mapsto \mathcal{L}_b$ together with $A_1 \mapsto A_2$,* etc. *Then $\phi(M_1) = M_2$. Moreover, any other triangle is parallelogic with its image by ϕ (so that ϕ itself can be called a **parallelogy**).*

Conversely, a collineation ϕ is a parallelogy when (1) $\mathcal{L}_b \mapsto \mathcal{L}_b$ and (2) $\mathcal{L}_b \cdot \boxed{\phi} = \mathrm{trace}(\phi)\, \mathcal{L}_b$.

Proof. Direct computation, assuming that $\mathcal{T}_1, \mathcal{T}_2$ are parallelogic. □

Exercise 26.3.4. (spoiler). **Parallelogy in $\mathbb{P}_\mathbb{C}(\mathbb{C}^3)$.** Morley spaces: conditions are $\phi_{21} = \phi_{23} = 0$ to enforce $\mathcal{L}_z \mapsto \mathcal{L}_z$ and $\phi_{11} + \phi_{33} = 0$ to have the right trace.

26.4 Orthology

Definition 26.4.1. We say that point P sees triangle $A'B'C'$ at right angles to trigone ABC when P is different from A', B', C' and verifies $PA' \perp BC$ etc.

Remark 26.4.2. Should point P be at infinity, all sidelines of ABC would have the same direction, and triangle ABC would be degenerate (flat). Some problems are to be expected...

Lemma 26.4.3. *The orthodir of the BC sideline is :$\delta_A \doteq a^2 : -S_c : -S_b$. This is also the direction of line HA, where H is the orthocenter of the triangle.*

Proposition 26.4.4. *Consider two non degenerate finite triangles ABC, $A'B'C'$ and suppose that it exists a finite point P, different from A', B', C', that sees triangle $A'B'C'$ at right angles to trigone ABC. Then it exist a point U that sees triangle ABC at right angles to trigone $A'B'C'$.*

Proof. From hypothesis, A' belongs to line $P\delta_A$. Then it exists a real number $k_A \neq \infty$ such that $A' = k_A P + \delta_A$. And the same for B', C'. Now compute :

$$A'' \doteq \boxed{\mathcal{M}_b} \cdot {}^t (B' \wedge C') = -2S\,(p+q+r) \begin{pmatrix} -k_B - k_C \\ k_B \\ k_C \end{pmatrix}$$

This comes from $P \wedge P = 0$, together with $\boxed{\mathcal{M}_b} \cdot {}^t(\delta_B \wedge \delta_C) = \boxed{\mathcal{M}_b} \cdot {}^t\mathcal{L}_b = 0$. Since $p+q+r \neq 0$ and $k_j \neq 0$ is assumed, column A'' really defines a direction. We can therefore simplify and obtain :

$$A''B''C'' = \begin{pmatrix} -k_B - k_C & k_A & k_A \\ k_B & -k_A - k_C & k_B \\ k_C & k_C & -k_A - k_B \end{pmatrix}$$

This triangle is perspective to ABC, with perspector $U = k_A : k_B : k_C$. Therefore point U sees triangle ABC at right angles to trigone $A'B'C'$. □

Definition 26.4.5. We say that two triangles are **orthologic** to each other when it exists a point M_1 that sees triangle $\mathcal{T}_1$ with rays orthogonal to the sidelines of trigone $\mathcal{T}_2$ and a point M_2 that sees triangle $\mathcal{T}_2$ at right angles to the sidelines of trigone $\mathcal{T}_1$.

In other words, the lines drawn from the vertices of $\mathcal{T}_1$ and orthogonal to the corresponding sidelines of $\mathcal{T}_2$ are concurrent at a point M_1 (the ray source) , etc. In this definition, flat triangles and centers at infinity are allowed.

Remark 26.4.6. This property cannot be reworded in a shorter form (the so-called symmetry), since $P \notin \mathcal{L}_b$ is required to be sure of the existence of U, but this is not sufficient to be sure of $U \notin \mathcal{L}_b$.

Example 26.4.7. Cevian triangles of X(2) and X(7) are orthologic. Orthology centers are X(10) and X(1).

Proposition 26.4.8. *Let ABC be the reference triangle, P a point not on the sidelines, Q its isogonal conjugate and $P_A P_B P_C$, $Q_A Q_B Q_C$ their respective pedal triangles. Then ABC and $P_A P_B P_C$ are orthologic. The center which looks at $P_A P_B P_C$ is obviously P, while the center which. looks at A, B, C is Q. Let EF be the line through A and parallel to $Q_B Q_C$, etc. Then ABC is the pedal triangle of P wrt DEF, and ABC is orthologic with DEF. The center which looks at D, E, F is $K = 2O - P$, whose barycentrics wrt DEF are $p : q : r$. The triangle DEF is called the* ***anti-pedal triangle*** *of P and we have: $DEF \simeq$*

$$\begin{pmatrix} +qr\,(a^2q + S_c\,p)\,(a^2r + S_b\,p) & -pr\,(a^2q + S_c\,p)\,(b^2r + S_a\,q) & -pq\,(a^2r + S_b\,p)\,(c^2q + S_a\,r) \\ -qr\,(b^2p + S_c\,q)\,(a^2r + S_b\,p) & +pr\,(b^2p + S_c\,q)\,(b^2r + S_a\,q) & -pq\,(c^2p + S_b\,r)\,(b^2r + S_a\,q) \\ -qr\,(a^2q + S_c\,p)\,(c^2p + S_b\,r) & -pr\,(b^2p + S_c\,q)\,(c^2q + S_a\,r) & +pq\,(c^2p + S_b\,r)\,(c^2q + S_a\,r) \end{pmatrix}$$

Proof. Using the mkortho routine, one obtains the coordinates of Q, and $Q_A Q_B Q_C$. Then one computes the lines EF, etc and obtain the points D, E, F. Using again the mkortho routine, one obtains K. See Figure 26.1. The cyan circles are illustrating the cyclopedal property (Section 9.3). □

Proposition 26.4.9. *When $\mathcal{T}_1, \mathcal{T}_2$ are orthologic, the formula $\boxed{\phi} \doteq \boxed{\mathcal{T}_2} \cdot \boxed{\mathcal{T}_1}^{-1}$ defines a collineation ϕ such that $\mathcal{L}_b \mapsto \mathcal{L}_b$ together with $A_1 \mapsto A_2$, etc. Then $\phi(M_1) = M_2$. Moreover, any other triangle is orthologic with its image by ϕ (so that ϕ itself can be called an orthology).*
Conversely, a collineation ϕ is an orthology when (1) $\mathcal{L}_b \mapsto \mathcal{L}_b$ and (2) $\operatorname{trace}\left(\boxed{\phi} \cdot \boxed{\mathrm{OrtO}}\right) = 0$.

Proof. Direct computation, assuming that $\mathcal{T}_1, \mathcal{T}_2$ are orthologic. □

Exercise 26.4.10. (spoiler). **Orthology in** $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$. Morley spaces:. Conditions are $\phi_{21} = \phi_{23} = 0$ to enforce $\mathcal{L}_z \mapsto \mathcal{L}_z$ and $\phi_{11} = \phi_{33}$ to have the right trace.

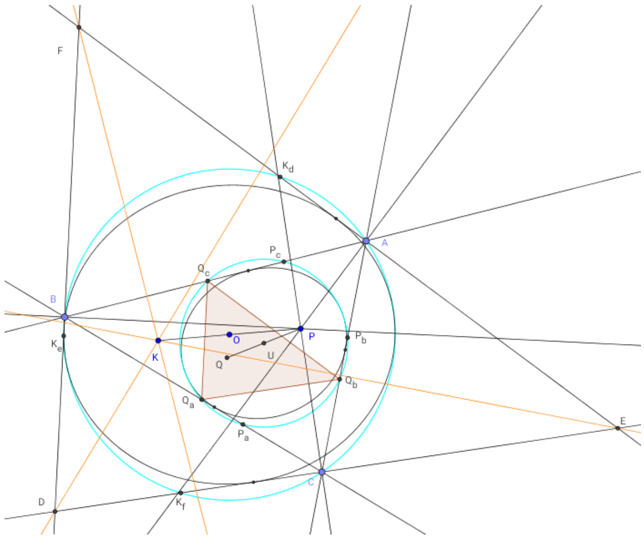


Figure 26.1: Antipedal triangle

Proposition 26.4.11. *Assume that triangle ABC is not degenerate and let the finite points P, U be described by their barycentrics $P = p : q : r$ and $U = u : v : w$ (here $P = U$ is allowed). Moreover, assume that U is not on the sidelines of ABC. Then all triangles $A'B'C'$ such that point P sees triangle $A'B'C'$ at right angles to trigone ABC and point U sees triangle ABC at right angles to trigone $A'B'C'$ are given by formula :*

$$\boxed{A'B'C'} \simeq \left(\begin{array}{ccc} pu + a^2\mathbf{k} & pv - S_c\mathbf{k} & pw - S_b\mathbf{k} \\ qu - S_c\mathbf{k} & qv + b^2\mathbf{k} & qw - S_a\mathbf{k} \\ ru - S_b\mathbf{k} & rv - S_a\mathbf{k} & rw + c^2\mathbf{k} \end{array} \right) = \left(P \cdot {}^tU\right) + 2S\mathbf{k}\boxed{\mathcal{M}_b} \tag{26.2}$$

where $\mathbf{k}$ *describes an homothety centered at* P*.*

Proof. Since $u \neq 0$, relation $A' \in P\,\delta_A$ can be written as $uP + \vartheta_A\delta_A$, and the same holds for the other points. Now, compute :

$$\det\left[uP + \vartheta_A\delta_A,\, vP + \vartheta_B\delta_B,\, \boxed{\mathcal{M}_b}^t (C \wedge U)\right] = 2uvS\,(p+q+r)\,(\vartheta_A - \vartheta_B)$$

Due to the hypotheses, all the ϑ must have the same value, leading to the formula. Converse is obvious, when a triangle is as described by the formulas, both orthologies are verified. □

Remark 26.4.12. If P were at infinity, PA', PB', PC' would have the same direction, and also the sidelines of ABC. In the formula, this would lead to A', B', C' at infinity. If coordinate u was 0 then $UB \parallel UC$ so that $A'B' \parallel A'C'$. In the formula, this would lead to A' at infinity.

26.5 Simply orthologic and perspective triangles

Definition 26.5.1. When triangles are orthologic with $P \neq U$, we say they are simply orthologic. When $P = U$, we say they are bilogic.

Notation 26.5.2. In this chapter, when triangles ABC and $A'B'C'$ are in perspective, their perspector will be noted Ω (and never P, nor U), while the perspectrix will be noted XYZ with $X = BC \cap B'C'$, etc.

Theorem 26.5.3 (First Sondat Theorem)**.** *Assume that triangle ABC is finite and non degenerate ; P is at finite distance ; U is different from P and is not on the sidelines. Then it exists exactly one triangle $A'B'C'$ such that (1) P sees $A'B'C'$ at right angles to trigone ABC (2) U sees ABC at right angle to trigone $A'B'C'$ (3) $A'B'C'$ is perspective to ABC. When $A'B'C'$ is chosen that way, the corresponding perspector Ω is collinear with points P, U and the perspectrix XYZ is orthogonal to PU.*

Proof. We have assumed that $P = p : q : w$ is different from $U = u : v : w$. Then ϑ is fixed by the condition of being perspective and we have :

$$\begin{aligned}
\vartheta &= -\frac{u\,(vS_b - wS_c)\,qr + v\,(wS_c - uS_a)\,pr + w\,(uS_a - vS_b)\,pq}{(vS_b - wS_c)\,pS_a + (wS_c - uS_a)\,qS_b + (uS_a - vS_b)\,rS_c} \\
\Omega &= \frac{S_cw - S_bv}{qw - rv} : \frac{S_cw - S_au}{pw - ru} : \frac{S_bv - S_au}{pv - qu} \\
tripolar\,(\Delta) &= \left(\begin{array}{c}
\dfrac{(S_cw - S_bv)\,p + \left(S_bu + a^2w\right)q - \left(S_cu + a^2v\right)r}{S_bq - S_cr} \\
\dfrac{(S_au - S_cw)\,q + \left(b^2u + S_cv\right)r - \left(S_av + b^2w\right)p}{S_cr - S_ap} \\
\dfrac{(S_bv - S_au)\,r + \left(c^2v + S_aw\right)p - \left(S_bw + c^2u\right)q}{S_ap - S_bq}
\end{array}\right)
\end{aligned}$$

First assertion is proved by $\Omega \simeq \alpha\,U - \beta\,P$ where :

$$\left(\begin{array}{c}\alpha \\ \beta\end{array}\right) \simeq \left[\begin{array}{c}(qw - rv)\,puS_a + (ru - pw)\,vqS_b + rw\,(pv - qu)\,rwS_c \\ (qw - rv)\,u^2S_a + (ru - pw)\,v^2S_b + (pv - qu)\,w^2S_c\end{array}\right]$$

Second assertion is proved by checking that

$$\Delta \cdot \boxed{\mathrm{OrtO}} \cdot (normalized\,(P) - normalized\,(U)) = 0$$

□

26.6 Bilogic triangles

Proposition 26.6.1. *Bilogic triangles are ever in perspective. When orthology center is $U = u : v : w$ is finite and not on the sidelines, formula (26.2) becomes*

$$\boxed{A'B'C'} \simeq \left(\begin{array}{ccc}
u^2 + a^2\mathbf{k} & uv - S_c\mathbf{k} & uw - S_b\mathbf{k} \\
uv - S_c\mathbf{k} & v^2 + b^2\mathbf{k} & vw - S_a\mathbf{k} \\
uw - S_b\mathbf{k} & vw - S_a\mathbf{k} & r^2 + c^2\mathbf{k}
\end{array}\right) = \left(U \cdot {}^tU\right) + 2S\mathbf{k}\boxed{\mathcal{M}_b} \qquad (26.3)$$

while the perspector and the tripole of the perspectrix are respectively :

$$\Omega \simeq (\mathbf{k}\,vw - S_a)^{-1} : (\mathbf{k}\,wu - S_b)^{-1} : (\mathbf{k}\,uv - S_c)^{-1}$$

$$tripolar\,(XYZ) \simeq \left(\begin{array}{c}
\left(vwa^2 + u\,(vS_b + wS_c - uS_a)\right)\mathbf{k} - 4\,S^2 \\
\left(uwb^2 + v\,(wS_c + uS_a - vS_b)\right)\mathbf{k} - 4\,S^2 \\
\left(uvc^2 + w\,(uS_a + vS_b - wS_c)\right)\mathbf{k} - 4\,S^2
\end{array}\right)$$

Proof. Straightforward computation. □

Proposition 26.6.2. *When triangle ABC is fixed and the bilogic center U is given, the locus of the perspector Ω of the bilogic triangles $A'B'C'$ is the rectangular hyperbola that goes through A, B, C, U, H =X(4). The perspector of this circumconic is :*

$$u\,(vS_b - wS_c) : v\,(wS_c - uS_a) : w\,(uS_a - vS_b)$$

while the envelope of the perspectrix is the inconic whose perspector is :

$$(vS_b - wS_c)^{-1} : (wS_c - uS_a)^{-1}, (uS_a - vS_b)^{-1}$$

Proof. Eliminate ϑ from Ω. Then eliminate ϑ from XYZ and take the adjoint matrix. □

Proposition 26.6.3. *Let ABC, $A'B'C'$ be two bilogic triangles, with orthology center U, perspector Ω and perspectrix (XYZ) where $X = BC \cap B'C'$, etc. Then lines $U\Omega$ and XYZ are orthogonal. Moreover we have $(UX) \perp (AA'\Omega)$, etc.*

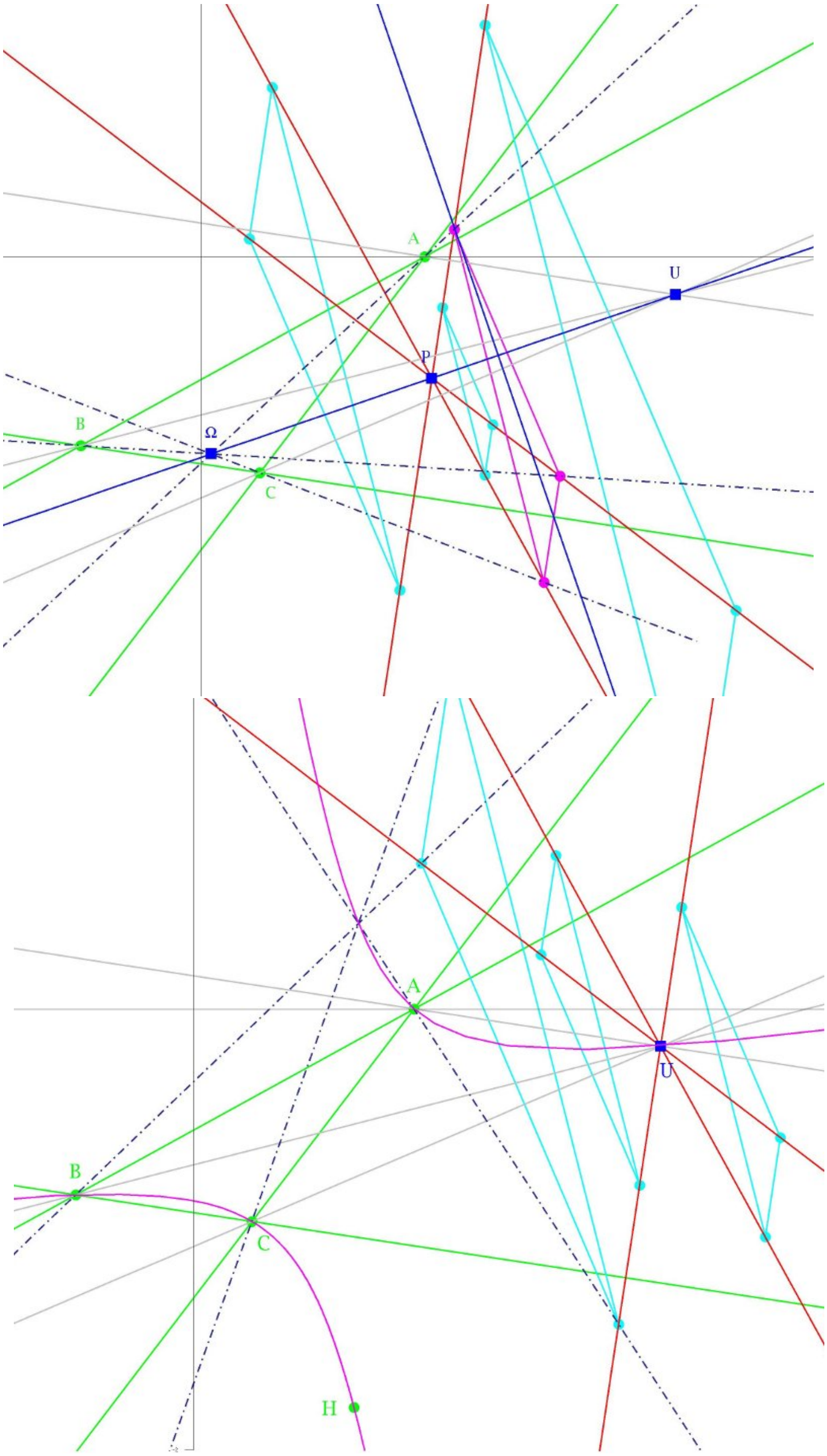


Figure 26.2: Orthology and perspective.

Proof. We compute the orthodir δ of line XYZ and find that :

$$\delta = \left[\begin{array}{c} \left(u^2\left(w+v\right)S_a - S_c u w^2 - S_b v^2 u\right)\theta + S_b S_c\left(v+w\right) - a^2 S_a u \\ \left(v^2\left(w+u\right)S_b - S_a v u^2 - S_c v w^2\right)\theta + S_c S_a\left(w+u\right) - b^2 S_b v \\ \left(w^2\left(v+u\right)S_c - S_a w u^2 - S_b w v^2\right)\theta + S_a S_b\left(u+v\right) - c^2 S_c w \end{array}\right]$$

It remains to check that $normalized\left(\Omega\right) - normalized\left(U\right)$ is proportional to δ. □

Chapter 27

Linear Families of Inscribed Triangles

This chapter is an extended version of Douillet (2014c) and summarises discussions with pappus, Poulbot and others at http://www.les-mathematiques.net

Notation 27.0.1. In all parts of this chapter, the relations

$$f+g+h=u+v+w=\rho+\sigma+\tau$$

will ever be assumed. These rules are related to the asymmetric parametrization described at Definition 27.2.18.

27.1 General linear families of triangles

Definition 27.1.1. Suppose that a_0, b_0, c_0 and a_1, b_1, c_1 are six points at finite distance, together with $a_0 \neq a_1, b_0 \neq b_1, c_0 \neq c_1$.Then we say that

$$\boxed{\mathcal{T}_t} = t\boxed{\mathcal{T}_1} + (1-t)\boxed{\mathcal{T}_0}$$

defines a (general) linear family of triangles.

Remark 27.1.2. Property $a_0 \notin \mathcal{L}_b$, etc is required to allow the projective definition:

$$a_t \doteq t\,\frac{a_1}{\mathcal{L}_b \cdot a_1} + (1-t)\,\frac{a_0}{\mathcal{L}_b \cdot a_0}$$

while property $a_0 \neq a_1$, etc is required to ensure the existence of $\mathcal{L}_a \doteq a_0 \wedge a_1$, etc.

Remark 27.1.3. Such a family can also be defined as

$$\boxed{\mathcal{T}_t} \simeq \left[\begin{array}{ccc} 1-t\left(v_{12}+v_{13}\right)-p_{12}-p_{13} & tv_{21}+p_{21} & tv_{31}+p_{31} \\ tv_{12}+p_{12} & 1-t\left(v_{21}+v_{23}\right)-p_{21}-p_{23} & tv_{32}+p_{32} \\ tv_{13}+p_{13} & tv_{23}+p_{23} & 1-t\left(v_{31}+v_{32}\right)-p_{31}-p_{32} \end{array}\right]$$

where triangles $\boxed{\mathcal{T}_t}$ are in a normalized form, while $\left(\boxed{\mathcal{T}_t} - \boxed{\mathcal{T}_s}\right) / (t-s)$ is a set of three non-zero constant vectors.

Definition 27.1.4. An **equicenter** $\mathcal{E}$ is a fixed point which has the same barycentrics wrt all the triangles $\mathcal{T}_t$ of the family. The column $F = f_a : f_b : f_c$ of these barycentrics is called the Neuberg column of the family (and doesn't depends on the barycentric frame used to compute them).

Definition 27.1.5. An **areal center** $\mathcal{S}$ is a fixed point which verifies:

$$\forall s,t:\ \operatorname{area}\left(\mathcal{S}, a_t, a_s\right) = \operatorname{area}\left(\mathcal{S}, b_t, b_s\right) = \operatorname{area}\left(\mathcal{S}, c_t, c_s\right)$$

For reasons given later, this point is also called the **slowness center** of the family.

Construction 27.1.6. *The Neuberg's (1921) construction of $\mathcal{E}, \mathcal{S}$ is as follows (see Figure 27.1). From an auxiliary point O, draw points $O_a \doteq O+a_1-a_0$, etc and compute $f_a = \text{area}\,(O, O_b, O_c)$, etc. Then O is the barycenter of the O_j with coefficients f_j, and this implies that $\boxed{\mathcal{T}_t} \cdot F$ is ever equal to $f_a a_0 + f_b b_0 + f_c c_0$.*

And now, draw the lines $\mathcal{L}_A \doteq a_0 \wedge a_1$, etc, obtain the points $A \doteq \mathcal{L}_b \wedge \mathcal{L}_c$, etc. and take their barycenter, i.e. compute $\mathcal{S} \doteq \boxed{ABC} \cdot F$.,

Discussion. If O is chosen as $1-y-z : y : z$ then

$$f_a = \det \begin{bmatrix} 1-y-z & 1-y-z-v_{21}1 & -y-z-v_{31} \\ y & y+v_{21}+v_{23} & y-v_{32} \\ z & z-v_{23} & z+v_{31}+v_{32} \end{bmatrix}$$

and, therefore:

$$F = \begin{pmatrix} v_{21}v_{31} + v_{21}v_{32} + v_{23}v_{31} \\ v_{31}v_{12} + v_{12}v_{32} + v_{32}v_{13} \\ v_{12}v_{23} + v_{21}v_{13} + v_{23}v_{13} \end{pmatrix}$$

When $f_a + f_b + f_c = 0$, the magenta triangle of Figure 27.1 if flat and point $\mathcal{E}$ is at infinity. When $f_a = f_b = f_c = 0$, we have the special-special case, discussed at Proposition 27.13.3

The trigone $(\mathcal{L}_j)$ degenerates when quantity $\det(\mathcal{L}_a, \mathcal{L}_b, \mathcal{L}_c) =$

$$(p_{12}v_{13} - p_{13}v_{12})\, f_a + (-p_{21}v_{23} + p_{23}v_{21})\, f_b + (p_{31}v_{32} - p_{32}v_{31})\, f_c + v_{12}v_{23}v_{31} - v_{13}v_{21}v_{32}$$

vanishes. Otherwise... everything becomes simpler when seen from ABC, the fixed circimscribed triangle. See next coming section for a dicussion of what happens when $\mathcal{S}$ is at infinity. □

27.2 Slowness- and equi-center of a LFIT

27.2.1 Slowness center

Definition 27.2.1. We say that triangle abc is **inscribed** in triangle ABC when $a \in BC$, etc. Conversely, we say that ABC is **circumcribed** to triangle abc.

Definition 27.2.2. LFIT. When the triangles $\mathcal{T}_t$ of the former section are all inscribed into a fixed (non degenerate) triangle ABC, this situation is described as a Linear Family of Inscribed Triangles. Using the values relative to $t = 0$ and $t = 1$, this can be written as:

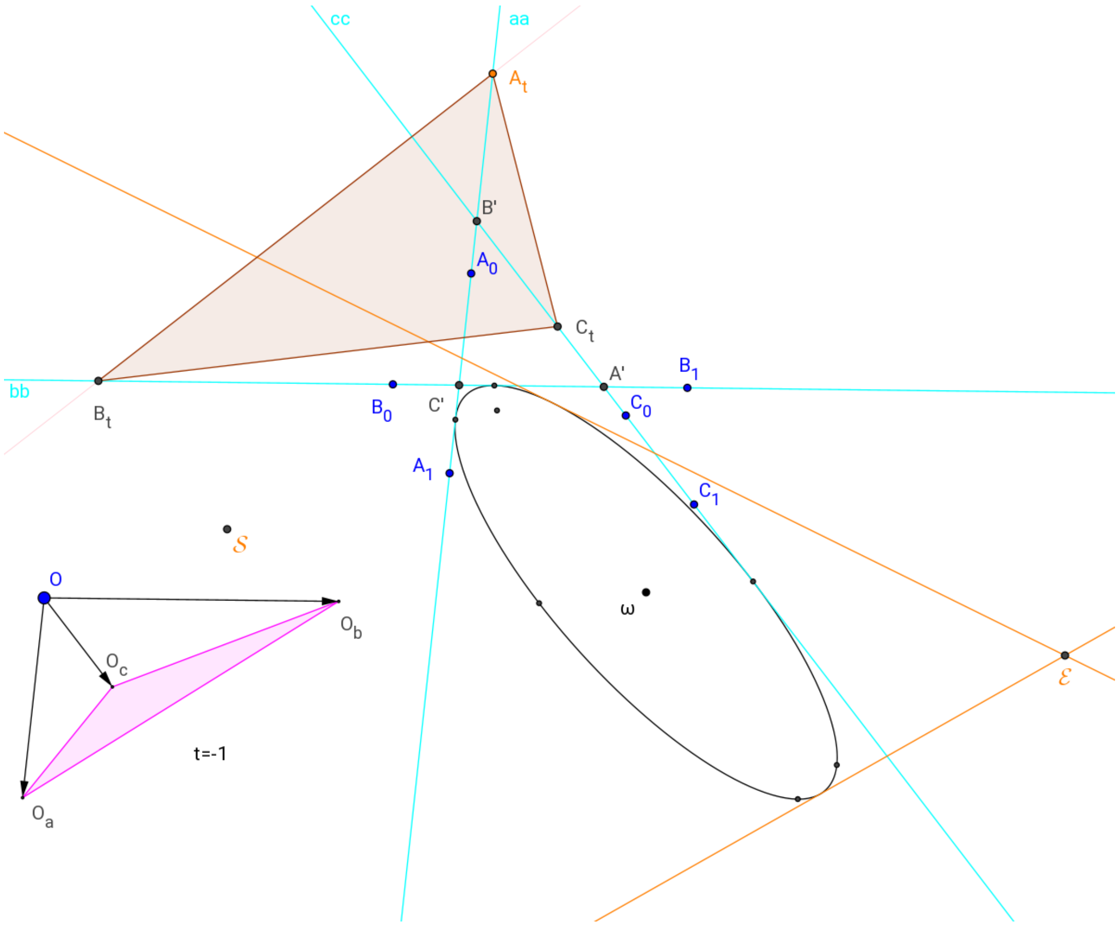
$$\begin{aligned} \boxed{\mathcal{T}_t} &\doteq \boxed{a_t b_t c_t} \doteq (1-t)\boxed{a_0 b_0 c_0} + (t)\boxed{a_1 b_1 c_1} \\ \boxed{\mathcal{T}_t} &= (1-t)\begin{pmatrix} 0 & 1-q_0 & r_0 \\ p_0 & 0 & 1-r_0 \\ 1-p_0 & q_0 & 0 \end{pmatrix} + (t)\begin{pmatrix} 0 & 1-q_1 & r_1 \\ p_1 & 0 & 1-r_1 \\ 1-p_1 & q_1 & 0 \end{pmatrix} \end{aligned} \quad (27.1)$$

Construction 27.2.3. *Geogebra: given $a_t \in BC$, the temporal parameter t is obtained as*

```
t= real ( (a_t - a_0)/(a_1 - a_0) )  ; real is required
b_t= (t)*b_1+(1-t)*b_0 + 0*I ; not a vector
```

Definition 27.2.4. As in Figure 27.2, the velocities $\overrightarrow{v_a}$, etc are defined by $\overrightarrow{v_a} \doteq a_1 - a_0 = [0,\, p_1 - p_0,\, p_0 - p_1]$, while the so called "speed vector" is defined by $\overrightarrow{V} \doteq [p_1 - p_0,\, q_1 - q_{0.},\, r_1 - r_0]$.

Remark 27.2.5. Many things are to be computed from the speed vector, with the following strange property: everything becomes simpler when introducing the reciprocal of these quantities (remember that $t \mapsto a$, etc are non constant). This leads to the following:



Spoiler: $\omega = (\mathcal{S} + \mathcal{E})/2$ is the center of the pilar conic $\mathfrak{C}$.

Figure 27.1: The Neuberg construction

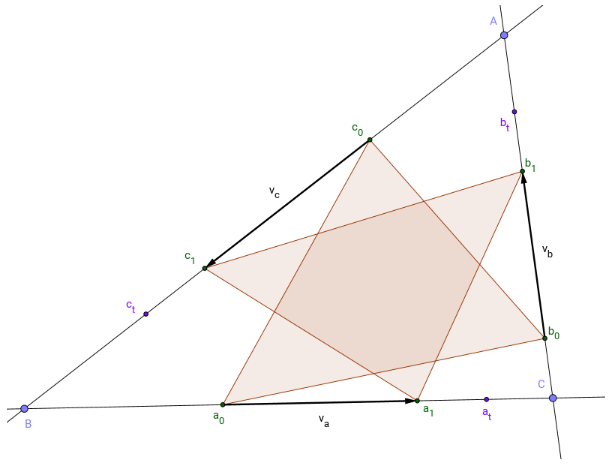


Figure 27.2: Variable triangle *abc* is inscribed into fixed triangle *ABC*

Definition 27.2.6. The **slowness center** of a linear family of triangles is:

$$\mathcal{S} \doteq f : g : h \simeq \frac{1}{p_1 - p_0} : \frac{1}{q_1 - q_0} : \frac{1}{r_1 - r_0}$$

leading to the following description (where indices 0 have been omitted) :

$$\boxed{abc}\,(t) = \begin{pmatrix} 0 & 1 - q - \dfrac{t}{g} & r + \dfrac{t}{h} \\ p + \dfrac{t}{f} & 0 & 1 - r - \dfrac{t}{h} \\ 1 - p - \dfrac{t}{f} & q + \dfrac{t}{g} & 0 \end{pmatrix} \tag{27.2}$$

Remark 27.2.7. Considering $\mathcal{S}$ as a projective quantity is only recognizing that choosing a time unit or another is unessential. Caveat: when used together with the later introduced $\mathcal{E} \simeq u : v : w$, the projective object is $f : g : h : u : v : w$, so that neither $f : g : h$ nor $u : v : w$ can be changed "at will", i.e. independently of the other.

Theorem 27.2.8 (Main result)**.** *The slowness center is the* ***areal center*** *of the family. In other words, the areas swept out from $\mathcal{S}$ by the $\mathcal{T}$-vertices are equivalent for any pair t_1, t_2.*

Proof. Suppose that $\mathcal{S}$ is at finite distance, compute the area, obtain:

$$\text{area}\,(\mathcal{S},\, a\,(t_1)\,,\, a\,(t_2)) = (t_2 - t_1)\, S \div (f + g + h)$$

and conclude from symmetry. Suppose now that $\mathcal{S}$ is at infinity (i.e. $h = -f - g$), compute the width of the strip created by the parallel lines $\Delta_1 \doteq \mathcal{S} \wedge a\,(t_1)$ and $\Delta_2 \doteq \mathcal{S} \wedge a\,(t_2)$, obtain:

$$\Delta_2 - \Delta_1 = [t_1 - t_2 : t_1 - t_2 : t_1 - t_2]$$

and here again conclude from symmetry. □

27.2.2 Equicenter

Theorem 27.2.9 (Neuberg)**.** *If we apply to abc the barycentrics of $\mathcal{S}$ wrt ABC, we obtain a fixed point, the so called equicenter $\mathcal{E}$ (except in the special-special case where $\mathcal{E}$ degenerates into $0 : 0 : 0$, see Proposition 27.13.3)*

Proof. From equation (27.2), it is obvious that $\mathcal{E} \doteq \boxed{abc} \cdot \mathcal{S}$ doesn't depend on t. And we have:

$$\mathcal{E} \simeq u : v : w = (1 - q)\, g + r\, h : (1 - r)\, h + p\, f : (1 - p)\, f + q\, g$$

This formula complies with the later requirement that: $u + v + w = f + g + h$. □

Remark 27.2.10. When one of the centers $\mathcal{S}$, $\mathcal{E}$ is at infinity, the other is also at infinity.

27.2.3 Pilar point

Definition 27.2.11. The pilar point $\Omega \simeq \rho : \sigma : \tau$ is defined by the hard equalities:

$$\begin{aligned} f+g+h &= u+v+w=\rho+\sigma+\tau \\ \mathcal{S}+\mathcal{E}+\Omega &= (f+g+h)\,(A+B+C) \end{aligned} \tag{27.3}$$

27.2.4 Parametrization of a LFIT

Parametrization (27.2) has many interesting properties, among them its genericity and its symmetry. Nevertheless, there are many situations where this parametrization induces too burdensome computations. There are several other choices, each of them depending on some assumption about the areas of the variable triangles.

Proposition 27.2.12. *The area of triangle $a_t b_t c_t$ equals the area of ABC times the quantity:*

$$\begin{aligned} \mathcal{A} &= \left(\frac{(f+g+h)}{fgh}\,t^2 + \frac{fg\,(p+q-1)+gh\,(q+r-1)+hf\,(r+p-1)}{fgh}\,t \right. \\ &\qquad \left. +pqr+(1-p)\,(1-q)\,(1-r) \right) \\ &= \frac{(f+g+h)\,t^2}{fgh} - \frac{(fu+2\,fv+gv-hw)\,t}{fgh} + \frac{v\,(f+g-w)}{gh} \end{aligned}$$

Proof. First formula is the characteristic property of the determinant. The second one doesn't apply when $\mathcal{S} \simeq \mathcal{E} \in \mathcal{L}_b$: in this case, the area is constant, but has to be evaluated using $\mathcal{E} = K\mathcal{S}$, and the area is $-K\,(K+1)$. More details at Section 27.13. ☐

Theorem 27.2.13. *When $\mathcal{S},\mathcal{E}$ are at finite distance, the area is a second degree quantity and therefore presents an extremum, characterizing the so-called critical triangle. Taking this event as the origin of a shifted time T, we obtain: $abc\,(T) =$*

$$\left(\begin{array}{ccc} 0 & -\dfrac{T}{g}+\dfrac{gv+2\,gu-hw+fu}{2\,(u+v+w)\,g} & +\dfrac{T}{h}+\dfrac{hw+2\,hu+fu-gv}{2\,(u+v+w)\,h} \\ +\dfrac{T}{f}+\dfrac{fu+2\,fv+gv-hw}{2\,(u+v+w)\,f} & 0 & -\dfrac{T}{h}+\dfrac{hw+2\,hv-fu+gv}{2\,(u+v+w)\,h} \\ -\dfrac{T}{f}+\dfrac{fu+2\,fw-gv+hw}{2\,(u+v+w)\,f} & +\dfrac{T}{g}+\dfrac{gv+2\,gw+hw-fu}{2\,(u+v+w)\,g} & 0 \end{array} \right) \tag{27.4}$$

And then we have:

$$\frac{area\,(T)}{S} = \frac{(f+g+h)}{fgh}\,T^2 - \frac{f+h+h}{(u+v+w)^2}\,\frac{f^2u^2+g^2v^2+h^2w^2-2\,fguv-2\,ghvw-2\,hfwu}{4\,fgh}$$

Proof. Only the apparent denominators were used to obtain this formula. The $\mathcal{A}$ formula becomes indeterminate if one tries to use it when $\mathcal{S} \in \mathcal{L}_b$. ☐

Remark 27.2.14. This parametrization is hard to use, due to its too huge coefficients. Except when the critical triangle is already given. See for more details at Proposition 27.10.14

Theorem 27.2.15. *When $\mathcal{S},\mathcal{E}$ are at infinity but aren't equal, the area is a first degree quantity. We can choose the origin of time at $\mathcal{A}=0$, leading to:*

$$\boxed{abc} = \left(\begin{array}{ccc} 0 & -\dfrac{T}{g}+\dfrac{u\,(h+w)}{gw-vh} & \dfrac{T}{h}-\dfrac{u\,(g+v)}{gw-vh} \\ \dfrac{T}{f}-\dfrac{(h+w)\,v}{gw-vh} & 0 & -\dfrac{T}{h}+\dfrac{v\,(f+u)}{gw-vh} \\ -\dfrac{T}{f}+\dfrac{(g+v)\,w}{gw-vh} & \dfrac{T}{g}-\dfrac{w\,(f+u)}{gw-vh} & 0 \end{array} \right)$$

Here, we have $gw-hv=hu-fw=fv-gu$ and the area is $T\,(hv-gw) \div fgh$ (caveat: replacing $\mathcal{E} \mapsto k\mathcal{E}$ changes the family and multiplies the area by k !)

Proof. $\mathcal{S} \wedge \mathcal{E} \simeq \mathcal{L}_b$, hence the equality. No hidden denominators are implied. ☐

Remark 27.2.16. CAVEAT. The remaining case, i.e. $\mathcal{S} \simeq \mathcal{E} \in \mathcal{L}_b$, is really special, worth of a specific section (see Section 27.13).

27.2.5 Asymetric parametrization

Remark 27.2.17. As a matter of experiment, the following parametrization is more computation-friendly as any other one, largely enough to justify a lack of symmetry, and the burden of enforcing a binding rule.

Definition 27.2.18. The asymmetric parametrization of a LFIT is defined as:

$$\boxed{\mathcal{T}_t} = \left[\begin{array}{ccc} 0 & 1 - \dfrac{t}{g} - \dfrac{w-f}{g} & \dfrac{t}{h} + \dfrac{h-v}{h} \\ \dfrac{t}{f} & 0 & 1 - \dfrac{t}{h} + \dfrac{v-h}{h} \\ 1 - \dfrac{t}{f} & \dfrac{t}{g} + \dfrac{w-f}{g} & 0 \end{array} \right] \tag{27.5}$$

where $f:g:h$ is the slowness center $\mathcal{S}$ and $u:v:w$ s the equicenter $\mathcal{E}$. This amounts to enforce $a_0 = C$ together with the rule

$$\boxed{f+g+h = u+v+w = \rho+\sigma+\tau = 0} \tag{27.6}$$

Remark 27.2.19. In fact, we have to consider $\mathcal{S}:\mathcal{E}:\Omega$ or $f:g:h:u:v:w:\rho:\sigma:\tau$ as an unique projective object. When $\mathcal{S} \notin \mathcal{L}_b$, the binding relation suffices to synchronize the three columns $\mathcal{S},\mathcal{E},\Omega$. But when $\mathcal{S},\mathcal{E} \in \mathcal{L}_b$, we already have $f+g+h = u+v+w = 0$ and this relation is no more sufficient for the purpose of synchronization. Caveat: when $\mathcal{S} \in \mathcal{L}_b$, replacing $\mathcal{E}$ by $k\mathcal{E}$ changes the family (and changes the area of the inscribed triangles).

27.2.6 Gravity centers

Proposition 27.2.20. *The locus of the gravity centers of the moving triangles of a given LFIT is a straight line, parallel to the tripolar of* $\mathcal{S}$.

Proof. The alignment property is obvious from $g_t = (1-t)\, g_0 + (t)\; g_1$, while premultiplying g_t by tripolar $\mathcal{S}$ leads obviously to a constant. One can also check directly that

$$\operatorname{locus}(g) = (3)\,[gh, hf, fg] - (f\rho + \sigma\, g + h\tau)\,\mathcal{L}_b$$

□

27.3 Hexagonal graphs

27.3.1 Constructions

Definition 27.3.1. The hexagonal points $\alpha_t, \beta_t, \gamma_t$ are defined by $\alpha_t = b_t + c_t - A$, etc, so that:

$$\boxed{abc} = \left(\begin{array}{ccc} 0 & 1-q & r \\ p & 0 & 1-r \\ 1-p & q & 0 \end{array} \right) \Longrightarrow \boxed{\alpha\beta\gamma} = \left(\begin{array}{ccc} r-q & r & 1-q \\ 1-r & p-r & p \\ q & 1-p & q-p \end{array} \right) \tag{27.7}$$

The locus of α_t is obviously a straight line, noted $\mathcal{G}_a$ and called "the A hexagonal graph". And circularly for the other two.

Remark 27.3.2. Since β_t is at intersection of lines $\left(a_t, \overrightarrow{AB}\right)$ and $\left(c_t, \overrightarrow{CB}\right)$, the locus of β_t is nothing but the graph of the correspondence $BC \rightleftharpoons AB \;:\; a(t) \longleftrightarrow c(t)$, drawn using the directions of the sidelines.

Construction 27.3.3. *When the LFIT is know by triangles* $\mathcal{T}_1$ *and* $\mathcal{T}_0$*, then* $\mathcal{G}_b$ *is defined by* β_1 *and* β_0*. We draw* $\left(a_t, \overrightarrow{AB}\right)$ *which cuts* $\mathcal{G}_b$ *at* β_t*. And then draw* $\left(\beta_t, \overrightarrow{BC}\right)$ *which cuts* AB *at* c_t*. Similarly,* b_t *is constructed. And we can check the figure by closing the hexagon and going back to* a_t*. See Figure 27.3 When* $\mathcal{T}_0$ *and* $\mathcal{S}$ *are known, draw the graphs using Proposition 27.3.5. When* $\mathcal{S},\mathcal{E}$, etc *are known, draw the graphs using 27.8.*

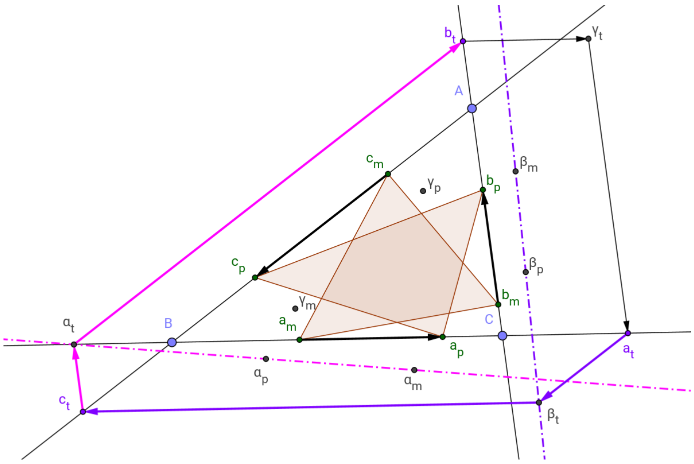


Figure 27.3: Hexagonal construction of the inscribed triangle

Proposition 27.3.4. *When $\mathcal{S}, \mathcal{E}, \Omega$ are known then, assuming the synchronization rule* (27.6), *the* ***hexagonal graphs****, locus of the $\alpha_t = b_t + c_t - A$, etc, are (by rows)*

$$\begin{pmatrix} \mathcal{G}_a \\ \mathcal{G}_c \\ \mathcal{G}_c \end{pmatrix} = \begin{bmatrix} g+h-u & g-u & h-u \\ f-v & h+f-v & h-v \\ f-w & g-w & f+g-w \end{bmatrix} = \begin{bmatrix} -\rho & h-\rho & g-\rho \\ h-\sigma & -\sigma & f-\sigma \\ g-\tau & f-\tau & -\tau \end{bmatrix} \tag{27.8}$$

Proof. Direct computation. □

Proposition 27.3.5. *Directions of the hexagonal graphs are those of the sidelines of the anticevian triangle of the slowness center $\mathcal{S}$.*

Proof. Obvious from the preceding proposition. One can also remark that:

$$\alpha\beta\gamma = \begin{bmatrix} r-q & r & 1-q \\ 1-r & p-r & p \\ q & 1-p & q-p \end{bmatrix} + t \begin{bmatrix} h^{-1}-g^{-1} & h^{-1} & -g^{-1} \\ -h^{-1} & f^{-1}-h^{-1} & f^{-1} \\ g^{-1} & -f^{-1} & g^{-1}-f^{-1} \end{bmatrix}$$ □

27.3.2 Hexagonal conics

Proposition 27.3.6. ***Hexagonal conic****. A conic $\mathcal{H}_t$ goes through the six points $a\gamma b\alpha c\beta$ of* (27.7). *Moreover the matrix*

$$\boxed{\mathcal{H}_t^H} \doteq {}^t\boxed{\mathcal{T}_t} \cdot \boxed{\mathcal{H}_t} \cdot \boxed{\mathcal{T}_t} \simeq \begin{pmatrix} 0 & p+q-1 & p+r-1 \\ p+q-1 & 0 & q+r-1 \\ p+r-1 & q+r-1 & 0 \end{pmatrix}$$

can be seen as a description of $\mathcal{H}_t$ from the abc frame, or as the description, from the ABC frame, of another conic, $\mathcal{H}_t^H$, image of $\mathcal{H}_t$ by the collineation $\phi_t : a, b, c \mapsto A, B, C$, $\mathcal{L}_b \mapsto \mathcal{L}_b$. In any case, the $\mathcal{H}_t^H$ conics form a linear family.

Proof. Computations are obvious, with huge simplifications. Linearity follows from the linearity of the p_t. One can also argue that opposite sides of the hexagon are parallel. □

Corollary 27.3.7. *The hexagonal conic is a parabola when area$(a, b, c) =$ area$(A, B, C) \div 4$, an ellipse when area(abc) is greater and an hyperbola otherwise.*

Proof. Since ϕ_t is an affinity, conics $\mathcal{H}_t$ and $\mathcal{H}_t^H$ have the same number of points at infinity. □

Proposition 27.3.8. *The $\mathcal{H}_t^H$ have a fourth fixed point in common, X, whose isotomic is given by:*

$$\text{isotom}\, X \simeq \begin{pmatrix} (q-r)\,gh + (p+q-1)\,fg - (r+p-1)\,fh \\ (r-p)\,hf + (q+r-1)\,gh - (p+q-1)\,gf \\ (p-q)\,fg + (r+p-1)\,hf - (q+r-1)\,hg \end{pmatrix} \simeq \begin{pmatrix} gh+fu-gv-hw \\ hf-fu+gv-hw \\ fg-fu-gv+hw \end{pmatrix}$$

Proof. Consequence of the linearity of the family. □

Corollary 27.3.9. *The hexagonal conic degenerates five times: (1) when $a_t b_t c_t$ is flat, two occurences, visible or not and (2) when one of the $p+q+1$ vanishes, three occurences. In the second case, $\beta_t\gamma_t = b_t c_t \parallel BC$ while $\mathcal{H}_t^H$ degenerates into $AX \cup BC$*, etc.

Proof. After simplification, one has $\det \boxed{\mathcal{H}_t} = \det \mathcal{T}_t \det \boxed{\mathcal{H}_t^H}$. □

Fact 27.3.10. *(spoiler) At area's peak, $\mathcal{H}_t$ goes through $\mathcal{E}$, while $\mathcal{H}_t^H$ goes through $\mathcal{S}$.*

Remark 27.3.11. Both triangles abc and $\alpha\beta\gamma$ have the same area, equal to $s \times S$ where

$$s \doteq pqr + (1-p)(1-q)(1-r) = \det \boxed{abc} = \det \boxed{\alpha\beta\gamma}$$

27.3.3 Some collineations

Executive summary: use some collineation to transform $\mathcal{H}_t^H$ into $\mathcal{H}_t$, and then go back using some other collineation.

Exercise 27.3.12. Let ϕ_a be the collineation defined by $A, B, C \mapsto a, b, c$, $\mathcal{L}_b \mapsto \mathcal{L}_b$. Using ABC as basis, the matrix of ϕ_a is $\boxed{abc}$ and its characteristic polynomial is:

$$(\mu-1)\left(\mu^2+\mu+s\right) = (\mu-1)\left(\mu - \frac{-1+W}{2}\right)\left(\mu - \frac{-1-W}{2}\right)$$

where W is defined by: $s = \left(1-W^2\right)/4$. With respect to ABC, the coordinates of the fixed point at finite distance are:

$$K_a \simeq \left(\begin{array}{c} qr-q+1 \\ rp-r+1 \\ pq-p+1 \end{array}\right) \tag{27.9}$$

Exercise 27.3.13. Let ϕ_α be the collineation:$ABC \mapsto \alpha\beta\gamma$, $\mathcal{L}_b \mapsto \mathcal{L}_b$. Using again ABC as basis, the matrix of ϕ_α is $\boxed{\alpha\beta\gamma}$, the characteristic polynomial is the same and, wrt ABC, the coordinates of the fixed point at finite distance are:

$$K_\alpha \simeq \left(\begin{array}{c} -qr+rp+pq-p-q+r+1 \\ +qr-rp+pq+p-q-r+1 \\ +qr+rp-pq-p+q-r+1 \end{array}\right)$$

Exercise 27.3.14. Let ψ be defined by $\boxed{\psi} \doteq \boxed{abc}^{-1} \cdot \boxed{\alpha\beta\gamma}$... and not by $\boxed{\alpha\beta\gamma} \cdot \boxed{abc}^{-1}$. The normalized matrix of ψ is:

$$\begin{aligned} \boxed{\psi} &= \frac{1}{s}\left(\begin{array}{ccc} (1-r)\,q & (p-1)\,(q+r-1) & (q+r-1)\,p \\ (p+r-1)\,q & (1-p)\,r & (q-1)\,(p+r-1) \\ (r-1)\,(p+q-1) & (p+q-1)\,r & (1-q)\,p \end{array}\right) \\ \chi_\psi(X) &= (\mu-1)\left(\mu - \frac{1+W}{1-W}\right)\left(\mu - \frac{1-W}{1+W}\right) \qquad \text{where } s = \frac{1}{4}\left(1-W^2\right) \end{aligned}$$

and its fixed points are:

$$K_1 = \frac{-1}{W^2}\left(\begin{array}{c} (2\,p-1)\,(q+r-1) \\ (2\,q-1)\,(r+p-1) \\ (2\,r-1)\,(p+q-1) \end{array}\right) \;;\; K_2, K_3 \simeq \left(\begin{array}{c} 2\,q+2\,r-2 \\ -2\,r+1-W \\ -2\,q+1+W \end{array}\right), \left(\begin{array}{c} 2\,q+2\,r-2 \\ -2\,r+1+W \\ -2\,q+1-W \end{array}\right) \tag{27.10}$$

So what ?

27.3.4 Graphs, the general case

Definition 27.3.15. A "delta -triple" is what is obtained by choosing three directions

$$\delta \doteq \begin{bmatrix} p & -1-q & 1 \\ 1 & q & -1-r \\ -1-p & 1 & r \end{bmatrix}$$

and using them to draw the graph of $b_t \mapsto c_t$, etc, i.e. the locus Δ_a of the $\alpha_t = b_t\delta_c \cap c_t\delta_b$, etc. Quite obviously, these graphs are straight lines.

Proposition 27.3.16. *The directions of these graphs (coded the same way as the original ones) are:*

$$[p', q', r'] = \left[\frac{(r+1)(q+1)h - g}{(r+1)(g - qh)}, \frac{(r+1)(p+1)f - h}{(p+1)(h - fr)}, \frac{g(q+1)(p+1) - f}{(q+1)(f - pg)}\right]$$

Only the directions of cevian $(\mathcal{S})$ *lead to a 1-sized orbit. The 2-sized orbits are characterized by*

$$(uvw + uw + u)\, gh + (uvw + uv + v)\, fh + (uvw + vw + w)\, fg = 0$$

For a given triple $[u', v', w']$ *there are at most two triples* $[u, v, w]$.

Proof. Computations are straightforward. Nevertheless, the general formula giving the other antecedent is rather huge. □

Example. Starting from $[0, 0, 0]$ (hexagonal conics), we obtain the directions of *anticev* $(\mathcal{S})$, and conversely. The other antecedent of $[0, 0, 0]$ is given by the directions of SA, SB, SC (Catalan conics).

Exercise 27.3.17. Study the cevian and anticevian conics.

27.4 Temporal graphs

27.4.1 Pilar point, pilar conic

In the previous section, we have constructed the hexagonal graphs $\mathcal{G}_a, \mathcal{G}_b, \mathcal{G}_c$ by drawing lines $b\alpha$ through b, parallel to AB together with drawing lines $c\alpha$ through c, parallel to AC. As a result, triangle $\alpha\beta\gamma(t)$ is the crosstri (see 4.6) of triangles $abc(t)$ and triangle dir_A, dir_B, dir_C. This method can be repeated, using other auxiliary directions than those of the sidelines BC, CA, AB and obtaining other graphs than the $\mathcal{G}_j$.

Definition 27.4.1. The **pilar point** $\Omega \simeq \rho : \sigma : \tau$ of a LFIT is defined by the hard equality

$$\Omega + \mathcal{S} + \mathcal{E} = (f + g + h)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \textit{i.e. } \rho \doteq g + h - u, \text{ etc}$$

Therefore $\rho + \sigma + \tau = f + g + h$, so that $f : g : h : u : v : w : \rho : \sigma : \tau$ is a projective object involving only one factor of proportionality. When $\mathcal{S} \notin \mathcal{L}_b$, G =X(2) is the ordinary barycenter of points $\mathcal{S}, \mathcal{E}, \Omega$. Moreover, the **pilar conic** $\mathfrak{C}$ is defined as the inscribed conic whose perspector is the isotomic conjugate of Ω, and the center is

$$\omega \doteq \frac{1}{2}(\mathcal{S} + \mathcal{E}) = \frac{1}{2}\begin{pmatrix} \sigma + \tau \\ \tau + \rho \\ \rho + \sigma \end{pmatrix} = \frac{1}{2}\begin{pmatrix} f + u \\ g + v \\ h + w \end{pmatrix}$$

Once again the same caveat: $\mathcal{S}, \mathcal{E}, \Omega$ have to be synchronized by (27.6).

Remark 27.4.2. The name can be tracked back to a joke describing a temporal conic $\mathcal{C}_t^s$ as a major general (conique divisionnaire), distinguished among all the hexagonal conics (conique brigadière). In this context, the pilar conic would be something like a lieutenant general in command.

Exercise 27.4.3. Consider a LFIT. The lines drawn by A and parallel to $b_t c_t$, etc determine a triangle $A'B'C'$.

1. Determine the locus of A', B', C' ;
2. Compare area $(A'B'C')$ and area $(a_t b_t c_t)$;
3. Consider the collineation $f : \mathcal{L}_b \mapsto \mathcal{L}_b$, $ABC \mapsto A'B'C'$ and determine a point X such that $f(X)$ is constant ;

27.4.2 Temporal graphs

Definition 27.4.4. The graphs Δ_A^s, etc obtained when using the directions of trigone $a_s b_s c_s$ as auxiliary directions, are called the temporal graphs of the LFIT. Obviously, they are straight lines, while the conics through $a_t \beta_t^s c_t \alpha_t^s b_t \gamma_t^s$ are called the temporal conics $\mathcal{C}_t^s$. This process amounts to observe the LFIT from the fixed triangle that coincides with abc at time $t = s$.

Proposition 27.4.5. *The temporal graph Δ_A^s is the tangent to the pilar conic $\mathfrak{C}$ issued from a_s (other than the obvious BC).*

Proof. The auxiliary directions are given by the directions of the sidelines of $\boxed{abc}(s)$, i.e.

$$\mathcal{D}_s \simeq \left[\begin{array}{ccc} (g+h)\,s+(hw-fh-gv) & -f\,s+(fv-f\,h) & f\,s-(f+g-w)\,f \\ -g\,s+(gv) & (f+h)\,s+(-fv) & g\,s \\ -h\,s+(fh-hw) & -hs+f\,h & (-f-g)\,s+(f+g-w)\,f \end{array}\right]$$

Thus points $\alpha(t), \beta(t), \gamma(t)$ are obtained as crosstri of $\boxed{abc}(t)$, given at (27.2), and $\mathcal{D}_s$. Eliminating t, we obtain the ABC-equation of the A-temporal graph:

$$\Delta_A^s \simeq \left[+\rho\,s^2-f\rho\,s\ ;\ -(\sigma+\tau)\,s^2+f\,(\sigma+2\,\tau)\,s-f^2\tau\ ;\ -(\sigma+\tau)\,s^2+f\tau\,s\right]$$

Property $a_s \in \Delta_A^s$ was obvious... and is easy to check.

For the contact: points α, β, γ are obtained as crosstri of $\boxed{abc}(t)$

which gives the auxiliary directions. Use locusconi, and obtain a line-conic circumscribed to trigone ABC, and see that its perspector is ${}^t\Omega$. □

Proposition 27.4.6. *Consider the temporal graph Δ_A^s. When point α_t^s moves along Δ_A^s, then line $\alpha_t^s a_t$ keeps a constant direction (depending on s, but not on t).*

Proof. For any kind of graphs, directions of $\alpha_t^s b_t$ and $\alpha_t^s c_t$ are constant by the very definition. The $\alpha_t^s a_t$ property is special even if easily checked. □

Exercise 27.4.7. Determine the graphs that share this property with the temporal graphs.

27.4.3 Temporal conics

Theorem 27.4.8. *The temporal conic $\mathcal{C}_t^s$ related to $a_t b_t c_t$ doesn't depends on the s chosen to construct the graphs (and now will be noted by $\mathcal{C}_t$). The centers of the $\mathcal{C}_t$ belong to line*

$$[(gh-vw)\,\rho\ ;\ (fh-uw)\,\sigma\ ;\ (fg-uv)\,\tau]$$

that goes through $\omega = \mathcal{S} + \mathcal{E}$. Moreover, the matrix

$$\boxed{\mathcal{C}^H} \doteq {}^t\boxed{\mathcal{T}_t} \cdot \boxed{\mathcal{C}_t} \cdot \boxed{\mathcal{T}_t} \simeq \left(\begin{array}{ccc} 0 & h\tau & g\sigma \\ h\tau & 0 & f\rho \\ g\sigma & f\rho & 0 \end{array}\right) \tag{27.11}$$

can be seen as a description of $\mathcal{C}_t$ from the abc frame, or as the description, from the ABC frame, of another conic, $\mathcal{C}^H$, image of $\mathcal{C}_t$ by the collineation $\phi_t : a, b, c \mapsto A, B, C$, $\mathcal{L}_b \mapsto \mathcal{L}_b$. Using the second point of view, we have a fixed circumconic, with perspector the point $\mathcal{S} \underset{b}{*} \Omega$.

Proof. Use the first point of view. Compute $\boxed{a_t b_t c_t}^{-1} \cdot \boxed{\alpha_t^s \beta_t^s \gamma_t^s}$ and then, according to Proposition 12.7.9, take the wedge of the tripolars of two columns. The result doesn't contain s. Moreover, being equal to $\mathcal{S} \underset{b}{*} \Omega$, this result doesn't depends on t either. Now use the formula giving center from perspector, go back to ABC- barycentrics and conclude. □

Theorem 27.4.9. *The temporal conic $\mathcal{C}_t$ is bi-tangent to the pilar conic $\mathfrak{C}$ and the chord through the contact points (that are not necessarily visible) is parallel to line $\mathcal{SE}$.*

Proof. A point on $\mathfrak{C}$ is $P(s) \simeq 1 \div \rho : s^2 \div \sigma : (1+s)^2 \div \tau$. Thus, using $SS = s_1 + s_2,\ PP = s_1 s_2$, an equation of the chord $P(s_1)\,P(s_2)$ is

$$chord \simeq [(g+h-u)(2\,PP+SS), (h+f-v)(SS+2), -(f+g-w)\ SS]$$

Then compute ${}^tP \cdot {}^t\boxed{a_t b_t c_t}^{-1} \cdot conicir\left(\mathcal{S} \underset{b}{*} \Omega\right) \cdot \boxed{a_t b_t c_t}^{-1} \cdot P$, and substitute $u = f+g+h-v-w$. The result factorizes into $\det \boxed{\mathcal{T}_t}$ times a perfect square. Use the usual formulas for sum and product, and conclude by obtaining:

$$chord(t) = [f+v-w,\ f-h+v,\ f+g-w] - 2t\,\mathcal{L}_b$$

□

Proposition 27.4.10. *When $\mathcal{S} \notin \mathcal{L}_b$, the area of $a_t b_t c_t$ is extremal exactly when the center of $\mathcal{C}_t$ is ω (and the chord of contacts is $\mathcal{SE}$).*

Proof. Substitute $t = \dfrac{fu + 2\,fv + gv - hw + T}{2\,(f+g+h)}$ in the previous results. □

Definition 27.4.11. When using the directions of $\mathcal{S}A, \mathcal{S}B, \mathcal{S}C$ as auxiliary directions, the results are called Catalan conics and Catalan graphs (when doing that, we assume that point $\mathcal{S}$ is at finite distance). Directions $\boxed{\mathcal{D}}$ and point α are:

$$\boxed{\mathcal{D}} \simeq \left(\begin{array}{ccc} -g-h & f & f \\ g & -h-f & g \\ h & h & -f-g \end{array}\right) \ ;\ \alpha \simeq \left(\begin{array}{c} f+u \\ g-u-v+\dfrac{hw-gv}{f}+t\,\dfrac{f+g+h}{f} \\ h+v+\dfrac{gv-hw}{f}-t\,\dfrac{f+g+h}{f} \end{array}\right)$$

Proposition 27.4.12. *The Catalan graphs Δ_a,* etc *are the symmetrics of the sidelines wrt ω, the midpoint of $\mathcal{S}$, $\mathcal{E}$. In fact, the Catalan graphs are nothing else than the temporal graphs related to $s = \infty$. When area of abc is extremal, the Catalan conic is centered at ω.*

Proof. Make $s \to \infty$ in $\mathcal{D}_s$. □

27.4.4 Tucker associate LFIT

Definition 27.4.13. Starting from K =X(6)$\simeq a^2 : b^2 : c^2$ and $t \in \mathbb{R}$, we define $A' = K + t\,\overrightarrow{KA}$, etc. And then define the 6 points: A_b, A_c on BC (intersections with $A'B'$ and $A'C'$), B_c, B_a on CA (intersections with $B'C'$ and $B'A'$) and C_a, C_b on AB (intersections with $C'A'$ and $C'B'$). Then

1. The polygon $A_b A_c B_c B_a C_a B_b$ is called the Lemoine's hexagon.
2. $[A_b, B_c, C_a]$ and $[A_c, B_a, C_b]$ are called the first and the second Brocard LFIT's.

Proposition 27.4.14. *Lemoine's hexagon is inscribed in the so called Tucker circle. Its center lies on line OD. Equicentre and Slowness center of both LFIT are the Brocard's points, and the common pilar conic is the Lemoine's inconic.*

Proof. We have the following coordinates: $LFIT_1, LFIT_2 =$

$$\left[\begin{array}{ccc} 0 & -ta^2+a^2 & tb^2+a^2+c^2 \\ tc^2+a^2+b^2 & 0 & -tb^2+b^2 \\ -tc^2+c^2 & ta^2+b^2+c^2 & 0 \end{array}\right], \left[\begin{array}{ccc} 0 & tc^2+a^2+b^2 & -ta^2+a^2 \\ -tb^2+b^2 & 0 & ta^2+b^2+c^2 \\ tb^2+a^2+c^2 & -tc^2+c^2 & 0 \end{array}\right]$$

So that $\mathcal{S}_1 = a^2b^2 : b^2c^2 : c^2a^2$, etc. □

Fact 27.4.15. *The flat triangles which belong to the Tucker LFIT's are the isotropes of the Brocard points.*

Definition 27.4.16. The points $\widehat{a_t}, \widehat{b_t}, \widehat{c_t}$ where the temporal conic $\mathcal{C}_t$ cuts again the sidelines of ABC form themselves a LFIT, called the **Tucker associate** of the original one.

Proposition 27.4.17. *The slowness center and the equicenter are exchanged when passing from the original LFIT to its Tucker associate. Therefore both LFITs have the same pilar conic.*

Proof. The coordinates of the Tucker associate LFITs are:

$$\boxed{\mathcal{T}_t} = \begin{bmatrix} 0 & 1 - \frac{t}{g} - \frac{w-f}{g} & 1 + \frac{t}{h} - \frac{v}{h} \\ \frac{t}{f} & 0 & -\frac{t}{h} + \frac{v}{h} \\ 1 - \frac{t}{f} & \frac{t}{g} + \frac{w-f}{g} & 0 \end{bmatrix} ; \boxed{\widehat{\mathcal{T}_t}} = \begin{bmatrix} 0 & \frac{t}{v} & -\frac{t}{w} + \frac{f}{w} \\ 1 - \frac{t}{u} - \frac{h-v}{u} & 0 & 1 + \frac{t}{w} - \frac{f}{w} \\ \frac{t}{u} + \frac{h-v}{u} & 1 - \frac{t}{v} & 0 \end{bmatrix}$$

so that $\widehat{\mathcal{S}} = \mathcal{E}$ is obvious. This implies $\widehat{\mathcal{E}} = \mathcal{S}$. □

Corollary 27.4.18. *When all temporal conics are circles, then $\mathcal{S}$ and $\mathcal{E}$ are isogonal conjugates.*

Proof. Use the parametrization given just above and write that $\widehat{a_t}$, etc belong to circle $a_t b_t c_t$. Take the t leading term (4th degree) of these equations, solve in u, v, w (don't forget $f+g+h = u+v+w$) and obtain

$$\mathcal{E} = \frac{f+g+h}{a^2gh + b^2hf + c^2fg} \left(a^2gh : b^2hf : c^2fg\right)$$

proving the necessity. Sufficiency is straightforward. □

27.5 HH and temporal point of view

27.5.1 Temporal embedding

Definition 27.5.1. Since we are dealing with moving points M_t, it makes sense to define a ***temporal embedding*** of such points by the following projective map:

$$\mathfrak{G}\left(\left(\begin{array}{c} x(t) \\ y(t) \\ z(t) \end{array}\right), t\right) \mapsto \left(\begin{array}{c} x \\ y \\ z \\ t(x+y+z) \end{array}\right)$$

Proposition 27.5.2. *Given a LFIT, the temporal embeddings of the variable vertices belong to a quadric $\mathfrak{Q}$, whose matrix is:*

$$\boxed{\mathfrak{Q}} \simeq \begin{bmatrix} 2v(f-w) & -fh+2fv+hw-vw & v\tau & -f-v+w \\ -fh+2fv+hw-vw & 2f(v-h) & f\tau & -f-v+h \\ v\tau & f\tau & 0 & -\tau \\ -f-v+w & -f+h-v & -\tau & 2 \end{bmatrix}$$

$$\boxed{\mathfrak{Q}^*} \simeq \begin{bmatrix} 0 & \tau & \sigma & \tau f \\ \tau & 0 & \rho & \tau v \\ \sigma & \rho & 0 & fh-hw+vw \\ \tau f & \tau v & fh-hw+vw & 2f\tau v \end{bmatrix}$$

And, for these instanciations, we have $\det\boxed{\mathfrak{Q}} = \det\boxed{\mathfrak{Q}^*} = (\rho\sigma\tau)^2$ *(squared coordinates of Ω in the triangle plane).*

Proof. Well known property of hyperbolic paraboloids. □

Signature is +2;-2. Using the completesquare algorithm, one obtains

$$\frac{(t_{yz})^2}{(fh-hw+vw)^2} - (t_{xyz})^2 + \left(1 - \frac{(t_z)^2}{(fh-hw+vw)^2}\right) K$$

where t_{xyz}, t_{yz}, t_z are linear expressions and K a constant.

Remark 27.5.3. Matrix $\boxed{\mathfrak{Q}^*}$ is symmetric, while matrix $\boxed{\mathfrak{Q}}$ is not. One could enforce symmetry by choosing $t=0$ when the area is extremal. But the price to pay would be a huge size for the coefficients.

Remark 27.5.4. The directions of sidelines of ABC are embedded at

$$0:1:-1:f \ ; \ -1:0:1:g \ ; \ 1:-1:0:h$$

Proposition 27.5.5. *The temporal embeddings of the s-temporal graphs, i.e. the lines*

$$\mathfrak{G}(\Delta_A^s) \doteq \left\{\mathfrak{G}(\alpha_t^s, t) \mid t \in \overline{\mathbb{R}}\right\}, \text{ etc}$$

are drawn on the $\mathfrak{Q}$ *quartic. Moreover,* $\mathfrak{G}(\alpha_t^s, t)_{t=s} = \mathfrak{G}(a_s, s)$. *Therefore* $\mathfrak{G}(\Delta_A^s)$ *is nothing but the other line drawn on* $\mathfrak{Q}$ *through* $\mathfrak{G}(a_s, s)$.

Proof. The first assertion is easily computed. The second is Proposition 27.4.5. And the conclusion follows. □

Proposition 27.5.6. *A point* $x:y:z$ *on the triangle plane has two temporal embeddings on the quartic* $\mathfrak{Q}$ *(counting multiplicity and complex values). The critical points of this double coating of the plane are the points of the pillar conic (and the points at infinity).*

Proof. First assertion is about the degree in t. Second assertion is about the discriminant, which is $(x+y+z)^2$ times the equation of the key conic.

Proposition 27.5.7. *The embedding* $\mathfrak{G}(\mathfrak{C})$ *of the pillar conic lies in the plane :*

$$[w-f-v \ ; \ h-f-v \ ; \ -\tau \ ; \ 2]$$

The horizontal plane $[s,s,s,-1]$ *cuts this curve in two points that are the embeddings of the contact points of the pillar conic with the temporal conic* $\mathcal{C}(s)$. *One of the horizontal lines of the polar plane is drawn through points* $\mathfrak{G}(\mathcal{S}, t^*)$ *and* $\mathfrak{G}(\mathcal{E}, t^*)$ *where* t^* *is the parameter related to the extremal area of triangle* $a_t b_t c_t$.

Using a parameter k, a point on the pillar conic can be described as: $1 \div \rho : k^2 \div \sigma : (1+k)^2 \div \tau$. It's only date of embedding is

$$t \quad = \quad \tau \, \frac{f\,\rho\,k^2 + \sigma\,\rho\,k + v\,\sigma}{(f+u)\,\rho\,k^2 + 2\,\sigma\,\rho\,k + (g+v)\,\sigma}$$

and the conclusion follows. One should remark that the plane equation is the last line of $\boxed{\mathfrak{Q}}$. □

Definition 27.5.8. A linear motion $M(t)$ on a given line Δ is said to be an ***incident motion*** to a given LFIT when the temporal parameters at cutting points are the same on the sidelines and on the transversal.

Proposition 27.5.9. *A given line* $\Delta \simeq [l,m,n]$ *is the support of an incident motion if and only if the line is tangent to the pillar conic. And then, the incident motion is embedded along a line drawn on* $\mathfrak{Q}$.

Proof. Solving in t the equation $[l,m,n] \cdot {}^t[0, p+t/f, (1-p)-t/f] = 0$ tells us when point $\Delta \cap BC$ is reached. Substituting into a_t and doing the same for the other two sidelines leads to the three embeddings:

$$\begin{bmatrix} 0 & -n & -m \\ -n & 0 & l \\ m & l & 0 \\ -nf & (l-n)(f-w)+lg & (l-m)v-lh \end{bmatrix}$$

Asking for the degeneracy of the plane drawn by these three points gives four equations whose *gcd* is the tangential equation of $\mathfrak{C}$. □

27.5.2 Menelaüs HH (parallelogy)

Caveat: in this subsection, vectors $P = {}^t[p, q, r, 1]$, etc aren't projective vectors, and $P[4] \doteq 1$ is a hard coded quantity.

Definition 27.5.10. The Menelaüs encoding of the inscribed triangle a, b, c is defined by:

$$\mathcal{T} \doteq (abc) \simeq \begin{pmatrix} 0 & 1-q & r \\ p & 0 & 1-r \\ 1-p & q & 0 \end{pmatrix} \mapsto P \doteq \begin{pmatrix} p \\ q \\ r \\ 1 \end{pmatrix}$$

while the Menelaüs quadric is defined by the symmetric matrix:

$$\boxed{\mathfrak{H}} = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ -1 & -1 & -1 & 2 \end{bmatrix}$$

Proposition 27.5.11. *(1) the area of an inscribed triangle is given by:* $\operatorname{area}(\mathcal{T}) = S \times {}^tP \cdot \boxed{\mathfrak{H}} \cdot P$; *(2) two non flat inscribed triangles are parallelogic if and only if* ${}^tQ \cdot \boxed{\mathfrak{H}} \cdot P = 0$.

Proof. Direct examination. □

Proposition 27.5.12. *When* $\mathcal{S} \notin \mathcal{L}_b$*, the triangles* $\mathcal{T}$ *of a* LFIT *are parallelogic by pairs. This relation is involutive and is described by the homography:*

$$s = \frac{(fu + 2\,fv + gv - hw)\,t - 2\,f\tau\,v}{2\,(h + g + f)\,t - (fu + 2\,fv + gv - hw)}$$

Proof. Immediate consequence of ${}^tP_t \cdot \boxed{\mathfrak{H}} \cdot P_s = 0$. □

Proposition 27.5.13. *Any flat inscribed triangle verifies* $(p + q - 1)\,(p + r - 1) = p\,(p - 1)$ *and determines two* LFIT *of flat triangles:*

1. Defining $\mu \doteq \dfrac{p + q - 1}{p} = \dfrac{p - 1}{p + r - 1}$ leads to

$$\boxed{\mathcal{T}_1}(p) = \begin{bmatrix} 0 & (1-\mu)\,p & \left(\mu^{-1} - 1\right)p + 1 - \mu^{-1} \\ p & 0 & \left(1 - \mu^{-1}\right)p + \mu^{-1} \\ 1 - p & (\mu - 1)\,p + 1 & 0 \end{bmatrix}$$

   Then we have $\mathcal{S} = \mu - 1 : 1 : -\mu$; $\mathcal{E} = -\mathcal{S}$; $\Omega = \overrightarrow{0}$. All the lines are parallel and directed by $\mathcal{S}$

2. Defining $\lambda \doteq \dfrac{p + q - 1}{p - 1} = \dfrac{p}{p + r - 1}$ leads to

$$\boxed{\mathcal{T}_2}(p) = \begin{bmatrix} 0 & \lambda + (1-\lambda)\,p & 1 + \left(\lambda^{-1} - 1\right)p \\ p & 0 & \left(1 - \lambda^{-1}\right)p \\ 1 - p & 1 - \lambda + (\lambda - 1)\,p & 0 \end{bmatrix}$$

   Then we have $\mathcal{S} = \lambda - 1 : 1 : -\lambda$; $\mathcal{E} = \overrightarrow{0}$; $\Omega = -\mathcal{S}$. All the lines are tangent to the inscribed parabola through $\mathcal{S}$ –whose focus is isogon (S):

$$\boxed{\mathcal{C}^*} \simeq \begin{bmatrix} 0 & -\lambda & 1 \\ -\lambda & 0 & \lambda - 1 \\ 1 & \lambda - 1 & 0 \end{bmatrix}$$

Proof. When $\mathcal{E} = K\mathcal{S}$ then $\operatorname{area}(\mathcal{T}_t) = -K\,(K + 1)\,S$. And so we have either $K = -1$ or $K = 0$. The fact that the $\mathfrak{H}$ contains two lines throught $P \in \mathfrak{H}$ is a general property of a HH. □

27.6 Miquel circles

Proposition 27.6.1. *When triangle A', B', C' is inscribed into trigone ABC (and none of the A', B', C' is a vertex), then circles $\mathrm{miq}_A \doteq AB'C'$, $\mathrm{miq}_B \doteq A'BC'$, $\mathrm{miq}_C \doteq A'B'C$ concur into some point M_{q}.*

Proof. Let $A' \simeq pB + (1-p)\,C$, etc. Then $\mathrm{miq}_A \simeq \left[0, c^2 r, -b^2 (q-1), 1\right]$. And one can see that the radius of $\bigwedge_3 (\mathrm{miq}_j)$ vanishes. Obviously, using directed angles is another possibility (Johnson, 1929, p. 131). □

Proposition 27.6.2 (Miquel LFIT)**.** *Conversely, we can fix $M_{\mathrm{q}} \simeq P$ and move $A' \in BC$. This leads to the* LFIT

$$\left[\begin{array}{ccc} 0 & 1+\dfrac{a^2 qt}{b^2 p}-\dfrac{powP}{b^2 p} & 1+\dfrac{(1-t)\,a^2 r}{c^2 p}-\dfrac{powP}{c^2 p} \\ 1-t & 0 & -\dfrac{(1-t)\,a^2 r}{c^2 p}+\dfrac{powP}{c^2 p} \\ t & -\dfrac{a^2 qt}{b^2 p}+\dfrac{powP}{b^2 p} & 0 \end{array}\right]$$

where $powP\,(p+q+r) \doteq a^2 qr + b^2 rp + c^2 qp$. The Neuberg center $\mathcal{E}$ is P itself, while the slowness center $\mathcal{S}$ is $\mathrm{isogon}\,(P)$*.*

Proof. Direct computation. □

Corollary 27.6.3. *In the context of a* LFIT*, circles (A, b_t, c_t)*, etc *are called the Miquel circles of the family. Their Veronese columns are:*

$$\mathrm{miq}_A, \mathrm{miq}_B, \mathrm{miq}_C \simeq \left[\begin{array}{c} 0 \\ c^2\, g\,(h-v+t) \\ b^2\, h\,(u+v-h-t) \\ gh \end{array}\right] \left[\begin{array}{c} c^2\, f\,(v-t) \\ 0 \\ a^2\, ht \\ fh \end{array}\right] \left[\begin{array}{c} b^2\, f\,(w-f+t) \\ a^2\, g\,(f-t) \\ 0 \\ fg \end{array}\right]$$

The μ_t coordinates themselves are not really handy, except from the second degree terms:

$$\mu_t \simeq \left(\begin{array}{c} a^2 gh \\ b^2 hf \\ c^2 fg \end{array}\right) t^2 + \mathbf{O}\,(t)$$

Proposition 27.6.4. *The $A-$Miquel circles of a* LFIT *are going through A and another fixed point O_A,* etc. *And we have $O_A, O_B, O_C \simeq$*

$$\left[\begin{array}{c} a^2 gh - b^2 hu - c^2 gu \\ b^2\, h\,(u-g-h) \\ c^2\, g\,(u-g-h) \end{array}\right] \left[\begin{array}{c} a^2 h\,(v-h-f) \\ -a^2 hv + b^2 fh - c^2 fv \\ c^2\, f\,(-h-f+v) \end{array}\right] \left[\begin{array}{c} a^2\, g\,(w-f-g) \\ b^2\, f\,(w-f-g) \\ -a^2 gw - b^2 fw + c^2 fg \end{array}\right] \qquad (27.12)$$

Proof. The existence is obvious from the linear nature of the Veronese's (and the computation is straightforward). □

Proposition 27.6.5. ***Circle of similarity****. For a given t, the three Miquel circles concur at a point μ_t, called the **Miquel point** of $\mathcal{T}_t$. The locus of this point is the circle Γ_σ* through *O_A, O_B, O_C. Moreover, $\mu_{(t=\infty)}$ is the isogonal conjugate of $\mathcal{S}$, i.e. $\mu_{(t=\infty)} = \mathcal{S}^* \doteq a^2 gh : b^2 hf : c^2 fg$. Therefore, $\mathcal{S}^*$ is the perspector of triangles ABC and $O_A O_B O_C$.*

Proof. Compute μ_t as above, and then use locusconi to obtain Γ_σ. Check that we have a circle, and obtain its Veronese, namely

$$\Gamma_\sigma \simeq \left(\begin{array}{c} b^2 c^2\,(g+h-u)\,f \\ c^2 a^2\,(h+f-v)\,g \\ a^2 b^2\,(f+g-w)\,h \\ a^2 gh + b^2 fh + c^2 fg \end{array}\right) = \left(\begin{array}{c} b^2 c^2\, f\rho \\ c^2 a^2\, g\sigma \\ a^2 b^2\, h\tau \\ a^2 gh + b^2 fh + c^2 fg \end{array}\right)$$

What remains is straightforward. The name "circle of similarity" is explained at Section 27.9. □

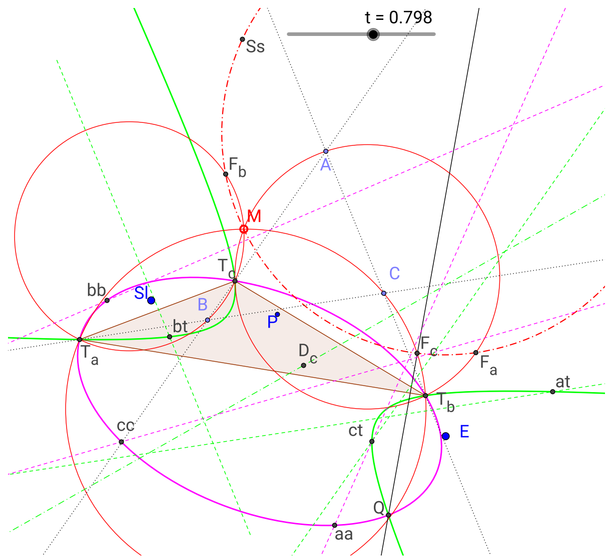


Figure 27.4: Miquel circles

Proposition 27.6.6. ***Miquel of a flat triangle.*** *When $\mathcal{T}_t = a_t b_t c_t$ is flat, then μ_t belongs to the circumcircle Γ and conversely*

Proof. When $\mathcal{T}_t$ is a single line, circles miq_A, etc are three of the ordinary Miquel circles of the quadrilateral $ABC, \mathcal{T}_t$. So that μ_t belongs also to the fourth one, the circumscribed circle of ABC. □

Proposition 27.6.7. ***Fixed point.*** *All the lines $a_t M_t$ are going through a fixed point P_a whose barycentrics are:*

$$P_a, P_b, P_c \simeq \left[\begin{array}{c} a^2gh \\ b^2fh - b^2hw + c^2gv \\ b^2hw + c^2fg - c^2gv \end{array}\right] \left[\begin{array}{c} a^2gh - a^2hw + c^2fu \\ b^2fh \\ a^2hw + c^2fg - c^2fu \end{array}\right] \left[\begin{array}{c} a^2gh - a^2gv + b^2fu \\ a^2gv + b^2fh - b^2fu \\ c^2fg \end{array}\right] \tag{27.13}$$

Points $\mathcal{E}, O_a, P_a$ are aligned and O_a, P_a belong to the circle of similarity. Moreover $\mathcal{S}^* P_A \parallel BC$.

Proof. Equation of line $a_t M_t$ is

$$\left[\left(b^2h + c^2g\right) ft - f\left(b^2fh - b^2hw + c^2gv\right),\, a^2ghf - a^2ght,\, -a^2ght\right]$$

This leads to P_a. One can see directly that $P_a = fgh\mathcal{S}^* + \left(b^2hw - c^2gv\right)\delta_{BC}$ and that $O_a - P_a$ (as written here) is $\simeq \mathcal{E}$. □

27.7 Constructions of the inscribed triangles

Construction 27.7.1. *Suppose now that $\mathcal{S} \simeq f : g : h$ and $\mathcal{E} \simeq u : v : w$ are given (with $f + g + h = u + v + w$, as ever). We have, inter alia, the following constructions.*

1. **Miquel**. Compute the O_j from $\mathcal{S}$, $\mathcal{E}$. When they are different, chose a point μ_t on circle $\Gamma_\sigma = (O_A, O_B, O_C)$. Draw the circles $\mathrm{miq}_A = (\mu_t, A, O_A)$, etc. Then $\mathrm{miq}_B, \mathrm{miq}_C$ concur on BC to give point a_t, etc. Condition $O_B = O_C$ induces $\mathcal{E} = \mathcal{S}^* = O_a = O_b = O_c$.
2. **Alt-Miquel**. Compute the P_j from $\mathcal{S}$, $\mathcal{E}$. When they are different, chose a point μ_t on circle $\Gamma_\sigma = (P_A, P_B, P_C)$. Draw the lines (μ_t, P_A) , etc. They cut the sidelines BC, etc at the required points a_t, etc. Condition $P_B = P_C$ induces $\mathcal{E} = \mathcal{S}^* = O_a = O_b = O_c$ (as for the Miquel construction.

3. **Catalan**. When $\mathcal{S}$, $\mathcal{E}$ are at finite distance, the Catalan graphs are obtained from the sidelines by a symmetry around $(\mathcal{S}+\mathcal{E})/2$.

4. **Hexagonal graphs**. The A-graph $\mathcal{G}_a$ is $[-\rho,\, h-\rho,\, g-\rho]$.

27.8 The three similarities theorem

This section has been inspired by Rouché and de Comberousse (1922b, Note III, p. 632-5)

Definition 27.8.1. Let be given three similarities $\sigma_a, \sigma_b, \sigma_c$, intended to act on three figures Φ_j according to

$$\Phi_a \overset{\sigma_c}{\longmapsto} \Phi_b \overset{\sigma_a}{\longmapsto} \Phi_c \overset{\sigma_b}{\longmapsto} \Phi_a \qquad \text{where } \sigma_c \cdot \sigma_b \cdot \sigma_a = \mathrm{Id}$$

We can consider the Φ_j as the images of a main figure Φ by three independent similarities $\sigma_1, \sigma_2, \sigma_3$. One can also introduce three other figures Φ'_j according to Figure 27.5. In what follows, similarity ψ will be forgotten most of the time.

Remark 27.8.2. In this section, the centers (fixed points) O_a, O_b, O_c are the main objects, and will be parametrized by ρ, σ, τ on the unit circle (called the circle of similarity and noted Γ_σ), while the ω^2 of the angles of the similarities are noted $\beta/\gamma, \gamma/\alpha, \alpha/\beta$. Thus, triangle ABC is only defined up to a rotation. In the same vein, the ubiquitous quantity $\delta \in \Gamma$

$$\delta \doteq -\frac{(\sigma-\tau)\,\alpha+(\tau-\rho)\,\beta+(\rho-\sigma)\,\gamma}{\rho\,(\sigma-\tau)\,\beta\,\gamma+\sigma\,(\tau-\rho)\,\alpha\,\gamma+\tau\,(\rho-\sigma)\,\alpha\,\beta} \tag{27.14}$$

is defined up a counter-wise rotation, so that only quantities like $\alpha\delta$ are well-defined.

Remark 27.8.3. Among the figures, we will take later $\Phi_a = BC$, $\Phi_b = CA$, $\Phi_c = AB$, so that σ_a sends CA onto AB and the angle of σ_a is $(AC, AB) = -\widehat{A}$. And one can check that $(id/conj)\,((\beta-\alpha)\,/\,(\gamma-\alpha)) = \beta/\gamma$.

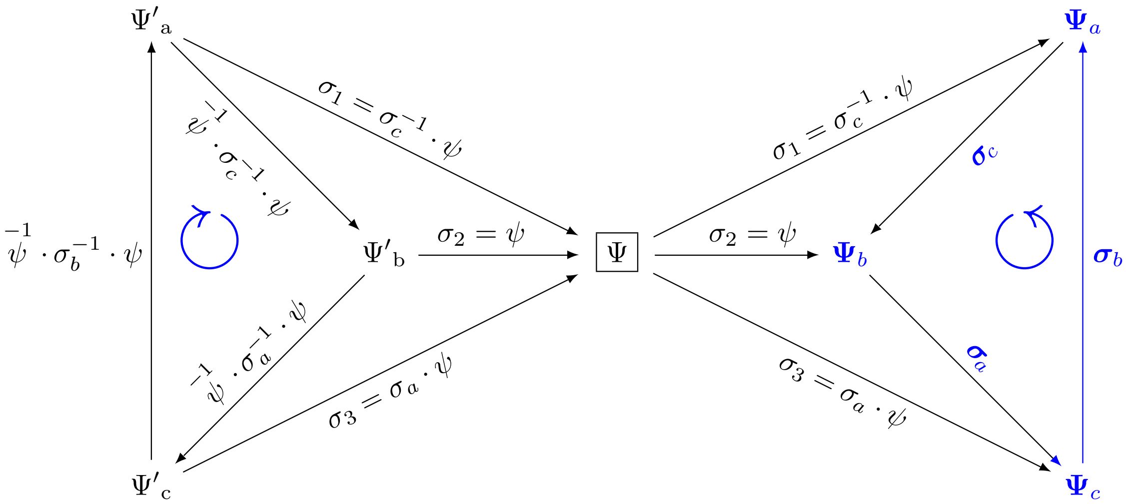


Figure 27.5: The "three" similarities configuration

Proposition 27.8.4. *The matrices of the three similarities are:*

$$\boxed{\sigma_a} \simeq \begin{bmatrix} \dfrac{\beta\,(\alpha\tau-\gamma\rho)\,(\rho-\sigma)}{\gamma\,(\alpha\sigma-\beta\rho)\,(\rho-\tau)} & \rho-\rho\,\dfrac{\beta\,(\alpha\tau-\gamma\rho)\,(\rho-\sigma)}{\gamma\,(\alpha\sigma-\beta\rho)\,(\rho-\tau)} & 0 \\ 0 & 1 & 0 \\ 0 & \dfrac{1}{\rho}-\dfrac{1}{\rho}\,\dfrac{(\alpha\tau-\gamma\rho)\,(\rho-\sigma)}{(\alpha\sigma-\beta\rho)\,(\rho-\tau)} & \dfrac{(\alpha\tau-\gamma\rho)\,(\rho-\sigma)}{(\alpha\sigma-\beta\rho)\,(\rho-\tau)} \end{bmatrix}, \text{ etc}$$

so that equalities $\alpha/\beta = \rho/\sigma$, etc are to be excluded. The Neuberg relation:

$$\forall M \in \Phi : (f\sigma_1 + g\sigma_2 + h\sigma_3)\,(M) \simeq \mathcal{E}$$

defining the equicenter $\mathcal{E}$ is satisfied by:

$$\left(\begin{array}{c} f\\ g\\ h \end{array}\right)\simeq i\left[\begin{array}{c} \dfrac{(\beta\tau-\gamma\sigma)(\beta-\gamma)}{\beta\gamma(\sigma-\tau)}\\ \dfrac{(\gamma\rho-\alpha\tau)(\gamma-\alpha)}{\gamma\alpha(\tau-\rho)}\\ \dfrac{(\alpha\sigma-\beta\rho)(\alpha-\beta)}{\alpha\beta(\rho-\sigma)} \end{array}\right]\in\mathbb{R}^3\;;\;\mathcal{E}\simeq\left[\begin{array}{c} \dfrac{\rho(\sigma-\tau)\alpha+\sigma(\tau-\rho)\beta+\tau(\rho-\sigma)\gamma}{(\sigma-\tau)\alpha+(\tau-\rho)\beta+(\rho-\sigma)\gamma}\\ 1\\ \dfrac{(\sigma-\tau)\beta\gamma+(\tau-\rho)\gamma\alpha+(\rho-\sigma)\alpha\beta}{\rho(\sigma-\tau)\beta\gamma+\sigma(\tau-\rho)\gamma\alpha+\tau(\rho-\sigma)\alpha\beta} \end{array}\right]$$

Proof. Write $\sigma_a = simil\,(O_a, k, \mu)$, $\sigma_b = simil\,(O_b, K, \lambda)$, $\mu^2 = \beta/\gamma$, $\lambda^2 = \gamma/\alpha$ and assume that O_c is the fixed point of σ_c. This gives the matrices. For the Neuberg relation, one can check that $\left(f\boxed{\sigma_c^{-1}} + g + h\boxed{\sigma_a}\right) = \mathcal{E}\cdot\mathcal{L}_z$. □

Proposition 27.8.5. *Exceptional cases are of two kinds*

1. *Except 1: $\rho:\sigma:\tau \simeq \alpha:\beta:\gamma$ occurs when $\mathcal{E}\in\mathcal{L}_z$ (and δ is undetermined)*
2. *Except 2: $\rho=\sigma=\tau$ occurs when centers O_a, O_b, O_c are collinear*

Proof. Equation $f+g+h=0$ is equivalent to $numer\,(\delta)\times denom\,(\delta)$ (see (27.14)). These two quantities are conjugate of each other, so they vanish together. Solving gives the property: ABC equal to $O_aO_bO_c$, up to a rotation. □

Proposition 27.8.6. *The R-C hodograph. Choose a point E in the plane (the handle of the hodograph) and M, N in figure Φ. The first modular triangle E_j is defined by $E_a = E + \phi_1\overrightarrow{MN}$, $E_b = E+\phi_2\overrightarrow{MN}$, $E_c = E+\phi_3\overrightarrow{MN}$. And the second modular triangle F_j is defined by $EE_a \perp F_bF_c$,* etc. *Then:*

1. *When E is translated to E', the whole R-C hodograph is translated by $\overrightarrow{EE'}$.*
2. *We have $f:g:h\simeq \det(E,E_b,E_c):\det(E_a,E,E_c):\det(E_a,E_b,E)$ while the ω^2 of $E_aE_bE_c$ are $\sigma\tau\,\alpha:\tau\rho\,\beta:\rho\sigma\,\gamma$. When $\mathcal{E}\in\mathcal{L}_b$, triangle $E_aE_bE_c$ is flat.*
3. *Triangle $F_aF_bF_c$ is similar to triangle ABC. Their common ω^2 are $\beta\,\gamma:\gamma\,\alpha:\alpha\,\beta$. When the O_j are collinear, then $F_a=F_b=F_c$.*
4. *From E, you see the F_j at $\omega^2=\rho:\sigma:\tau$, i.e. we have: $\angle(EF_b, EF_c) = \angle(O_aO_b, O_aO_c)$*

Proof. (1) is obvious ; (2) is easy to compute, and more powerful than only $(f+g+h)\,E = f\,E_a + g\,E_b + hE_c$; (3) describes why triangle $F_aF_bF_c$ is useful. Incantation for computing the ω^2: `U -> reduce(norztri( U.mWW)[1]);` □

Proposition 27.8.7. *Consider the triples $\sigma_1(M), \sigma_2(M), \sigma_3(M)$ where two elements are equal. We have three cases (in column):*

	to_next_point				fixed	adjunct
A	σ_c	O'_a	O_b	O_c	P_a	η_a
B	σ_a	O_a	O'_b	O_c	P_b	η_b
C	σ_b	O_a	O_b	O'_c	P_c	η_c

Pairs $O_jO'_j$ are aligned with $\mathcal{E}$ due to the Neuberg relation. Triangles $O'_aO_bO_c$ are similar with each other and with triangle $O_aO_bO_c$.

$$O'_a = \left[\begin{array}{c} \dfrac{\tau\,\sigma\,(\beta-\gamma)}{\tau\,\beta-\sigma\,\gamma} - \dfrac{(\sigma-\tau)\,\beta\,\gamma}{(\tau\,\beta-\sigma\,\gamma)\,\alpha}\,\rho\\ 1\\ \dfrac{\beta-\gamma}{\tau\,\beta-\sigma\,\gamma} - \dfrac{(\sigma-\tau)\,\alpha}{(\tau\,\beta-\sigma\,\gamma)}\,\dfrac{1}{\rho} \end{array}\right]$$

Proof. Straightforward computation. R-C are using angles not so clearly defined, remarking that σ_c maps $[O_c, O'_a]$ onto $[O_c, O_a]$, so that triangle $[O_c, O'_a, O_a]$ is similar to triangle $[E, E_a, E_b]$. □

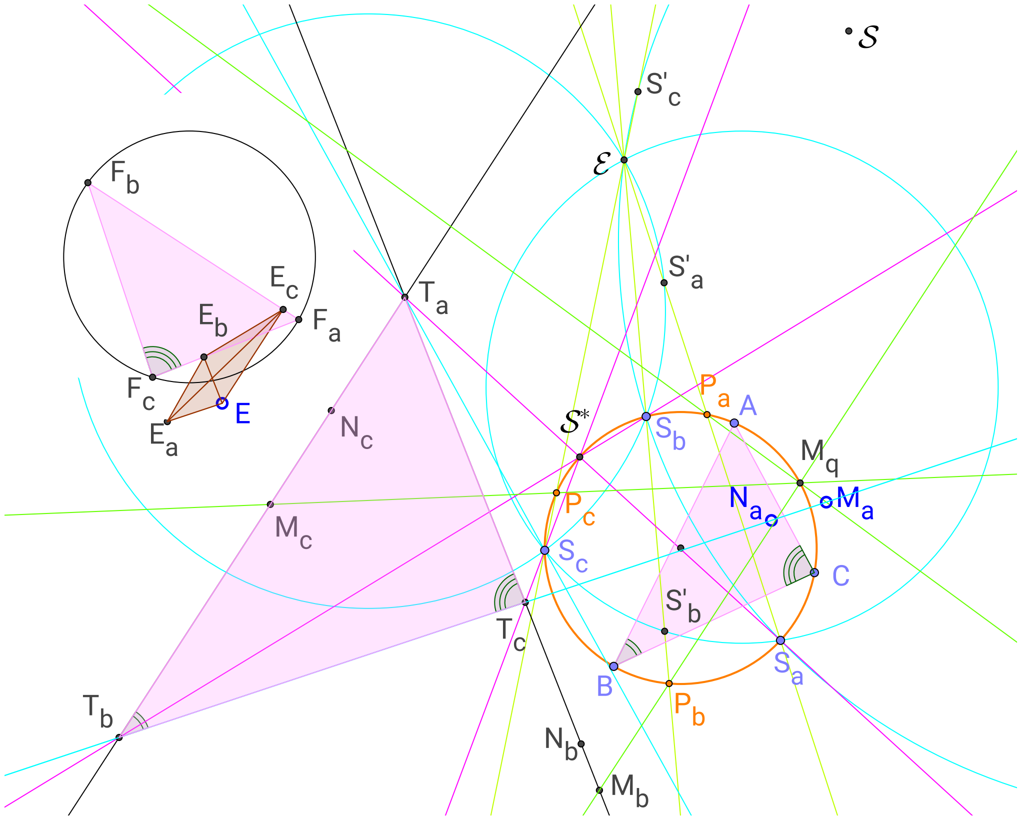


Figure 27.6: The RC hodograph

Definition 27.8.8. Points O'_j are called the adjunct points. The circle $\gamma_a \doteq (O'_a, O_b, O_c)$ is called the adjunct circle. It goes through the equicenter $\mathcal{E}$. Its center is called η_a and we have $\sigma_c(\eta_a) = \eta_b$ where

$$\gamma_a \simeq \left[\begin{array}{c} \gamma-\beta \\ (\sigma+\tau)(\beta-\gamma) \\ (\gamma-\beta)\,\sigma\,\tau \\ \beta\,\tau-\gamma\,\sigma \end{array}\right] \;;\; \eta_a \simeq \left[\begin{array}{c} (\beta-\gamma)\,\sigma\,\tau \\ \beta\,\tau-\gamma\,\sigma \\ \beta-\gamma \end{array}\right] \tag{27.15}$$

Corollary 27.8.9. *Circle $(\mathcal{E}, O_b, O_c)$ is the A-adjunct circle, while its second intersection with line $\mathcal{E}O_a$ is O'_a, the A-adjunct point.*

Proposition 27.8.10. *The trigone formed by three corresponding lines L_j is ever perspective with triangle $O_aO_bO_c$ and their perspector L_q (to be celebrated as $\mathcal{S}^*$) belongs to circle Γ.*

Proof. Writing $L_a \simeq [f \;;\; g \;;\; h]$, we obtain $z(L_q) = \alpha\delta\,(h/f)$, see (27.14). □

Proposition 27.8.11. *When three corresponding lines L_a, L_b, L_c are concurrent, their common point L_q lies on the circle Γ, while each line L_j goes through the second intersection P_j of $\mathcal{E}O_j$ and Γ. And we have: $z(P_a) = \dfrac{1}{\alpha\delta}$ (see (27.14)) so that triangle $P_aP_bP_c$ is skew similar with ABC.*

Proof. Obviously, $z(L_q) = \alpha\delta\,(h/f)$. □

Proposition 27.8.12. *The triangle formed by three corresponding points M_j is ever perspective with the fixed triangle $P_aP_bP_c$, and the perspector μ (Miquel) belongs to circle Γ.*

Proof. One obtains: $z(M_q) = (\alpha\,\delta\,z_1 - t_1) \div (\alpha\,\delta\,t_1 - \zeta_1)$. □

Proposition 27.8.13. *When three corresponding points M_j are aligned then M_a belongs to adjunct circle $\gamma_a = (O'_aO_bO_c)$, etc. And conversely. Moreover, the M_j are aligned with $\mathcal{E}$ (Neuberg property).*

Proof. Consider $\det(M_aM_bM_c)$ and obtain the locus. Then identification is easy from (27.15). □

27.9 Similarities and Cremona transforms

Now, we describe points A, B, C using the Lubin's parametrization, i.e. $A \simeq \alpha : 1 : 1/\alpha$ in the $\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}$ complex projective plane. We are giving both equations using f, g, h, p, q, r and using f, g, h, u, v, w (the equicenter).

Proposition 27.9.1. ***Three similarities theorem.*** *Let O_a, O_b, O_c be three generic points, not on the sidelines. Define the similarity σ_A by its center O_a together with $CA \mapsto AB$, etc. Note $\mu = \sigma_C \cdot \sigma_B \cdot \sigma_A$. Then*

1. μ is an homothety translation that maps CA onto CA.
2. μ is a translation if and only if lines AO_A, BO_B, CO_C concur at some point $\mathcal{S}^*$.
3. μ is the identity if and only if, moreover, $\mathcal{S}^*$ belongs to circle $O_aO_bO_c$ (the so-called circle of similarity).

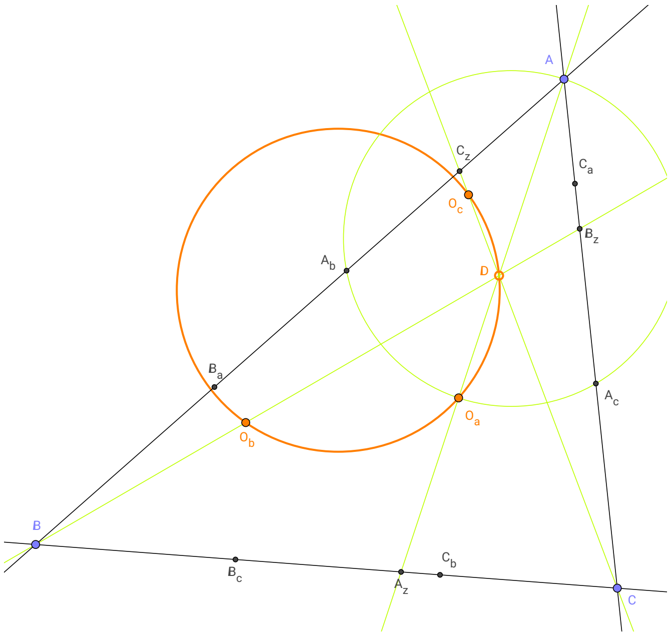


Figure 27.7: Circle of similarity

Proof. Let A_b, A_c be the orthogonal projections of O_a on AB and AC. We compute σ_A as the pointwise collineation $\Omega_x \mapsto \Omega_x$, $\Omega_y \mapsto \Omega_y$, $O_A \mapsto O_A$, $A_c \mapsto A_b$. Or as the inverse of the linewise collineation $\mathcal{L}_z \mapsto \mathcal{L}_z$, $\Omega_x O_a \mapsto \Omega_x O_a$, $\Omega_y O_a \mapsto \Omega_y O_a$, $AC \mapsto AB$. We get:

$$\boxed{\sigma_A} \simeq \left[\begin{array}{ccc} \dfrac{\alpha\,\beta\,\zeta_1 - (\alpha+\beta)\,t_1 + z_1}{\alpha\,\gamma\,\zeta_1 - (\alpha+\gamma)\,t_1 + z_1} & \dfrac{z_1\,(\gamma-\beta)\,(\alpha\,\zeta_1 - t_1)}{t_1\,(\alpha\,\gamma\,\zeta_1 - (\alpha+\gamma)\,t_1 + z_1)} & 0 \\ 0 & 1 & 0 \\ 0 & \dfrac{\zeta_1\,(\gamma-\beta)\,(\alpha\,t_1 - z_1)}{t_1\,(\alpha\,\gamma\,\zeta_1 - (\alpha+\gamma)\,t_1 + z_1)\,\beta} & \dfrac{\gamma\,(\alpha\,\beta\,\zeta_1 - (\alpha-\beta)\,t_1 + z_1)}{(\alpha\,\gamma\,\zeta_1 - (\alpha+\gamma)\,t_1 + z_1)\,\beta} \end{array}\right]$$

1. Obvious since μ fixes Ω_x, Ω_y and $\delta\,(AC)$. The factor of homothety is given by λ, the product of the three $(1,1)$ elements or by the product of the three $(3,3)$ elements of the matrices.
2. Let $A_z = BC \cap AO_A$. Then $\overline{A_zB} \div \overline{A_zC} = birap\,(B, C, A_z, \infty) = birap\,(\delta CA,\, \delta AB,\, \delta BC,\, \delta AO_a) =$

$$\frac{\gamma\,(\alpha-\beta)\,(\alpha\,\beta\,\zeta_1 - (\alpha+\beta)\,t_1 + z_1)}{\beta\,(\alpha-\gamma)\,(\alpha\,\gamma\,\zeta_1 - (\alpha+\gamma)\,t_1 + z_1)}$$

Clearly, $-\lambda$ is the product of these three quantities. And we conclude by the Ceva theorem.
3. Define $\mathcal{S}^* \doteq AO_A \cap BO_B$, $O_C = x\,\mathcal{S}^* + (1-x)\,C$, obtain $\mu = 1$ as a first degree equation in x and conclude. □

Proposition 27.9.2. ***LFIT Similarities***. *Given a LFIT, the correspondence $CA \longmapsto AB : b \mapsto c$ induces a similarity σ_A of the whole plane. Center is O_A, the fixed point of the Miquel circles (27.12), while angle and ratio are, respectively, $-A$ and $-cg/bh$. In the Morley frame, the matrix of σ_A is:*

$$\boxed{\sigma_A} = \begin{pmatrix} -\dfrac{g(\alpha-\beta)}{h(\alpha-\gamma)} & \alpha r+(1-r)\beta+\dfrac{(\alpha-\beta)(q\gamma+(1-q)\alpha)g}{h(\alpha-\gamma)} & 0 \\ 0 & 1 & 0 \\ 0 & \dfrac{r}{\alpha}+\dfrac{1-r}{\beta}+\dfrac{(\alpha-\beta)(q\alpha+(1-q)\gamma)g}{(\alpha-\gamma)\alpha\beta h} & -\dfrac{(\alpha-\beta)\gamma g}{(\alpha-\gamma)\beta h} \end{pmatrix}$$
$$= \begin{pmatrix} -\dfrac{g(\alpha-\beta)}{h(\alpha-\gamma)} & \beta+\dfrac{(u(\alpha-\gamma)+g\gamma)(\alpha-\beta)}{h(\alpha-\gamma)} & 0 \\ 0 & 1 & 0 \\ 0 & \dfrac{1}{\beta}+\dfrac{(u(\gamma-\alpha)+g\alpha)(\alpha-\beta)}{h(\alpha-\gamma)\alpha\beta} & -\dfrac{g(\alpha-\beta)\gamma}{h(\alpha-\gamma)\beta} \end{pmatrix}$$

The equicentric property holds for the whole plane, since:

$$f\boxed{1}+g\boxed{\sigma_C}+h\boxed{\sigma_B^{-1}}=\mathcal{E}_z\cdot\mathcal{L}_z$$

Proof. The quicker is to describe σ_A as the collineation: $\Omega_x \mapsto \Omega_x$, $\Omega_y \mapsto \Omega_y$, $b_0 \mapsto c_0$, $b \mapsto c$ and apply the general formulas. The obvious result $\boxed{\sigma_C}\boxed{\sigma_B}\boxed{\sigma_A}=1$ can be used to check the obtained matrices. Remember $O_A \simeq a^2gh-b^2hu-c^2gu : b^2h\rho : c^2g\rho$ □

Proposition 27.9.3. *Skew-similarities. The correspondence $CA \longmapsto AB : b \mapsto c$ induces a skew similarity σ_A^- of the whole plane. Center is*

$$Q_a \simeq \begin{bmatrix} h(qg-hr)b^2-g(qg-hr-g+h)c^2 \\ -(qg-hr+h)hb^2 \\ (qg-hr+h)gc^2 \end{bmatrix} \tag{27.16}$$

while $\boxed{\sigma_C^-}\boxed{\sigma_B^-}\boxed{\sigma_A^-}$ is the involutory affinity that fixes AC and reverts the direction of BH.

Proposition 27.9.4. ***Miquel homography***. *Assume that $\mathcal{S} \notin \mathcal{L}_b$ and define σ as the homography: $A \mapsto O_A$, $B \mapsto O_B$, $C \mapsto O_C$. The matrix of σ in the upper spherical map $(\mathbf{Z},\mathbf{T})$ is:*

$$\boxed{\sigma} = \begin{pmatrix} z_{\mathcal{E}} & \sigma_3\,\overline{z_{\mathcal{S}}+z_{\mathcal{E}}-z_H} \\ 1 & -z_{\mathcal{S}} \end{pmatrix} \; ; \; \boxed{\sigma^{-1}} = \begin{pmatrix} z_{\mathcal{S}} & \sigma_3\,\overline{z_{\mathcal{S}}+z_{\mathcal{E}}-z_H} \\ 1 & -z_{\mathcal{E}} \end{pmatrix}$$

where $z_{\mathcal{E}}=\dfrac{u\alpha+v\beta+w\gamma}{u+v+w}$, $z_{\mathcal{S}}=\dfrac{f\alpha+g\beta+h\gamma}{f+g+h}$, $z_H=\sigma_1$

This defines a Cremona transform $\widehat{\sigma}$ of the whole projective plane, whose indeterminacy points are the umbilics and $\mathcal{S}$ while exceptional lines are $\mathcal{L}_b$ and the isotropic lines of $\mathcal{S}$. By σ^{-1}, the center ω of circle $O_AO_BO_C$ is mapped to $1/\zeta_{\mathcal{S}}$ (symmetric of $\mathcal{S}$ in the circumcircle).

The four fixed points of $\widehat{\sigma}$ are the foci of the pillar conic and therefore verify:

$$\frac{\tau}{\mathbf{Z}-\mathbf{T}\gamma}+\frac{\sigma}{\mathbf{Z}-\mathbf{T}\beta}+\frac{\rho}{\mathbf{Z}-\mathbf{T}\alpha}=0$$

Proof. Direct computation. □

Proposition 27.9.5. *Homography σ degenerates when $\mathcal{S}$ and $\mathcal{E}$ are isoconjugates (thus none of them at infinity). In this case, the Miquel circle is reduced to point $\mathcal{E}=\mathcal{S}^*=O_a=O_b=O_c$.*

Proof. Obvious from (18.7) (when assuming $\mathcal{S} \notin \mathcal{L}_b$). Assuming $\mathcal{S} \in \mathcal{L}_b$, this would require $(u\,\alpha+v\,\beta+w\,\gamma)(f\,\alpha+g\,\beta+h\,\gamma)=0$, i.e. $\mathcal{E}$ at origin, instead of $\mathcal{E} \in \mathcal{L}_b$. □

27.10 Describing a LFIT from its degenerate triangles

27.10.1 General results

Fact 27.10.1. *In the general case, a LFIT contains two degenerate triangles $\mathcal{T}_j$ defining two lines $\Delta_j \simeq [p_j, q_j, r_j]$. Using an adapted timeline, we can enforce $t = \pm 1$ for these triangles and then:*

$$\mathcal{T}_t = \frac{1+t}{2}\left[\begin{array}{ccc} 0 & \dfrac{r_1}{r_1-p_1} & \dfrac{-q_1}{p_1-q_1} \\ \dfrac{-r_1}{q_1-r_1} & 0 & \dfrac{p_1}{p_1-q_1} \\ \dfrac{q_1}{q_1-r_1} & \dfrac{-p_1}{r_1-p_1} & 0 \end{array}\right] + \frac{1-t}{2}\left[\begin{array}{ccc} 0 & \dfrac{r_0}{r_0-p_0} & \dfrac{-q_0}{p_0-q_0} \\ \dfrac{-r_0}{q_0-r_0} & 0 & \dfrac{p_0}{p_0-q_0} \\ \dfrac{q_0}{q_0-r_0} & \dfrac{-p_0}{r_0-p_0} & 0 \end{array}\right]$$

Using (27.1), we obtain the following synchronized values:

$$\mathcal{S} = \left(\begin{array}{c} (p_0\, q_1 - q_0\, p_1)\,(r_0\, p_1 - p_0\, r_1)\,(q_1 - r_1)\,(q_0 - r_0) \\ (q_0\, r_1 - r_0\, q_1)\,(p_0\, q_1 - q_0\, p_1)\,(r_1 - p_1)\,(r_0 - p_0) \\ (r_0\, p_1 - p_0\, r_1)\,(q_0\, r_1 - r_0\, q_1)\,(p_1 - q_1)\,(p_0 - q_0) \end{array}\right)$$

$$\mathcal{E} = \Theta_2 \left(\begin{array}{c} q_0\, r_1 - r_0\, q_1 \\ r_0\, p_1 - p_0\, r_1 \\ p_0\, q_1 - q_0\, p_1 \end{array}\right) \; ; \; \Omega = \Theta_1 \left(\begin{array}{c} p_0\, p_1\, (q_0\, r_1 - r_0\, q_1) \\ q_0\, q_1\, (r_0\, p_1 - p_0\, r_1) \\ r_0\, r_1\, (p_0\, q_1 - q_0\, p_1) \end{array}\right)$$

$$\boxed{\mathfrak{C}^*} \simeq \left[\begin{array}{ccc} 0 & r_0\, r_1\, (p_0\, q_1 - p_1\, q_0) & q_0\, q_1\, (r_0\, p_1 - r_1\, p_0) \\ r_0\, r_1\, (p_0\, q_1 - p_1\, q_0) & 0 & p_0\, p_1\, (q_0\, r_1 - q_1\, r_0) \\ q_0\, q_1\, (r_0\, p_1 - r_1\, p_0) & p_0\, p_1\, (q_0\, r_1 - q_1\, r_0) & 0 \end{array}\right]$$

$$\text{where } \begin{array}{rcl} \Theta_1 & \doteq & (p_0\,(q_1 - r_1) + q_0\,(r_1 - p_1) + r_0\,(p_1 - q_1)) \\ \Theta_2 & \doteq & (p_0\, q_0\, r_1\,(p_1 - q_1) + q_0\, r_0\, p_1\,(q_1 - r_1) + r_0\, p_0\, q_1\,(r_1 - p_1)) \end{array}$$

Remark 27.10.2. One has $\mathcal{L}_b \cdot \mathcal{S} = \mathcal{L}_b \cdot \mathcal{E} = \mathcal{L}_b \cdot \Omega = \Theta_1 \times \Theta_2$. Then Θ_1 vanishes when the lines are parallel, while Θ_2 vanishes when the reciprocal of the lines are parallel.

Exercise 27.10.3. Point $\omega = (\mathcal{S} + \mathcal{E})/2$ is the intersection of lines *cevadiv* $(\mathcal{T}_j, \mathcal{L}_b)$. Spoiler: these lines are the Newton axes of quadrilaterals $(ABC, \mathcal{T}_j)$.

Proposition 27.10.4. *The temporal conic relative to a flat triangle $\mathcal{T}_j$ is the reunion of two parallels, each of them tangent to the pillar conic $\mathfrak{C}$: the line $\mathcal{T}_j$ itself (which goes through $\mathcal{E}$) and its symmetric wrt $\omega = (\mathcal{S} + \mathcal{E})/2$ (which goes through $\mathcal{S}$).*

Proof. The only safe proof is a direct computation, which is not difficult. Formula (27.11) cannot be used since $\det \mathcal{T}_j = 0$. While describing what happens when $t \to 0$ is enlightening... but doesn't prove anything. □

Theorem 27.10.5. *The two flat triangles (visible or not) that belongs to a LFIT are the tangents from equicenter $\mathcal{E}$ to the pillar conic $\mathfrak{C}$.*

Proof. This is only a paraphrase of the former proposition. Nevertheless, this is a key result. One can also note that $\mathcal{E}$ satisfies $(u+v+w)\,\mathcal{E} = f\,a_j + g\,b_j + h\,c_j$ so that $\mathcal{E} \in \mathcal{T}_j$ is an obvious requirement. □

Proposition 27.10.6. *About temporal conics. Applying (27.11), we have*

$$\boxed{\mathcal{C}^H} \simeq \left(\begin{array}{ccc} 0 & h\tau & g\sigma \\ h\tau & 0 & f\rho \\ g\sigma & f\rho & 0 \end{array}\right) =$$

$$\left[\begin{array}{ccc} 0 & (p_1 - q_1)\,(p_0 - q_0)\, r_0\, r_1 & (p_1 - r_1)\,(p_0 - r_0)\, q_0\, q_1 \\ (p_1 - q_1)\,(p_0 - q_0)\, r_0\, r_1 & 0 & p_0\, p_1\,(q_1 - r_1)\,(q_0 - r_0) \\ (p_1 - r_1)\,(p_0 - r_0)\, q_0\, q_1 & p_0\, p_1\,(q_1 - r_1)\,(q_0 - r_0) & 0 \end{array}\right]$$

whose points at infinity are

$$\delta_0^H,\, \delta^H \doteq \left(\begin{array}{c} p_0\,(q_0-r_0) \\ q_0\,(r_0-p_0) \\ r_0\,(p_0-q_0) \end{array}\right),\ \left(\begin{array}{c} p_1\,(q_1-r_1) \\ q_1\,(r_1-p_1) \\ r_1\,(p_1-q_1) \end{array}\right)$$

Except from the two flat triangles (where, in fact, $\boxed{\mathcal{C}_j^H}=0$*), these points generate the points at infinity of the temporal conic* $\boxed{\mathcal{C}_t}$ *itself:*

$$\delta_0=\boxed{\mathcal{T}_t}\cdot\delta_1^H=\operatorname{dir}\mathcal{T}_0=\left(\begin{array}{c} q_0-r_0 \\ r_0-p_0 \\ p_0-q_0 \end{array}\right)\ ;\ \delta_1=\boxed{\mathcal{T}_t}\cdot\delta_0^H=\operatorname{dir}\mathcal{T}_1=\left(\begin{array}{c} q_1-r_1 \\ r_1-p_1 \\ p_1-q_1 \end{array}\right)$$

Proof. Line at infinity is invariant by the collineation described by matrix $\boxed{\mathcal{T}_t}$. And thus we can avoid the direct computation of the points at infinity of the temporal conics. Once again: this doesn't apply to the flat triangles. □

Proposition 27.10.7. *Assume that $\mathcal{E}$ (and thus $\mathcal{S}$) remains at finite distance. Then all the following assertions are equivalent:*

1. *the LFIT contains only one degenerate triangle (and it happens that line $\mathcal{T}_0$ is tangent to $\mathfrak{C}$ at $\mathcal{E}$) ;*
2. *circumscribed Γ and similarity Γ_σ circles are tangent (the contact occurs at μ_0) ;*
3. *relation $f^2u^2+g^2v^2+h^2w^2-2\,fguv-2\,ghvw-2\,hfwu=0$ holds ;*
4. *point $\mathcal{E}$ belongs to $\mathfrak{C}$ (and therefore $\mathcal{S}\in\mathfrak{C}$ also) ;*
5. *point $\mathcal{S}$ belongs to the ABC-inconic centered at $(A+B+C-\mathcal{E})/2$ (and conversely).*

Proof. (1-2) is from Proposition 27.6.6; (3) is from the discriminant of $\det\mathcal{T}_t$; (4-5) are straightforward. □

27.10.2 Miscellany

Exercise 27.10.8. All the previous results were obtained "without circles". But we can identify the adjunct circle γ_a as the circle through $\mathcal{E}, a_0, a_1$, etc, point O_a as the other intersection of γ_b and γ_c, etc, point O_a' as the other intersection of line $\mathcal{E}O_a$ and circle γ_a. The similarity circle Γ_σ is the circle $O_aO_bO_c$, while fixed points P_a, etc are the other intersection of line $\mathcal{E}O_a$ and circle Γ_σ. And now, the moving part of the system: Miquel circles $\operatorname{miq}_A=(A,b_t,c_t)$, etc concur at $\mu_t\in\Gamma_\sigma$, as well as lines P_aa_t, etc.

27.10.3 Critical triangle

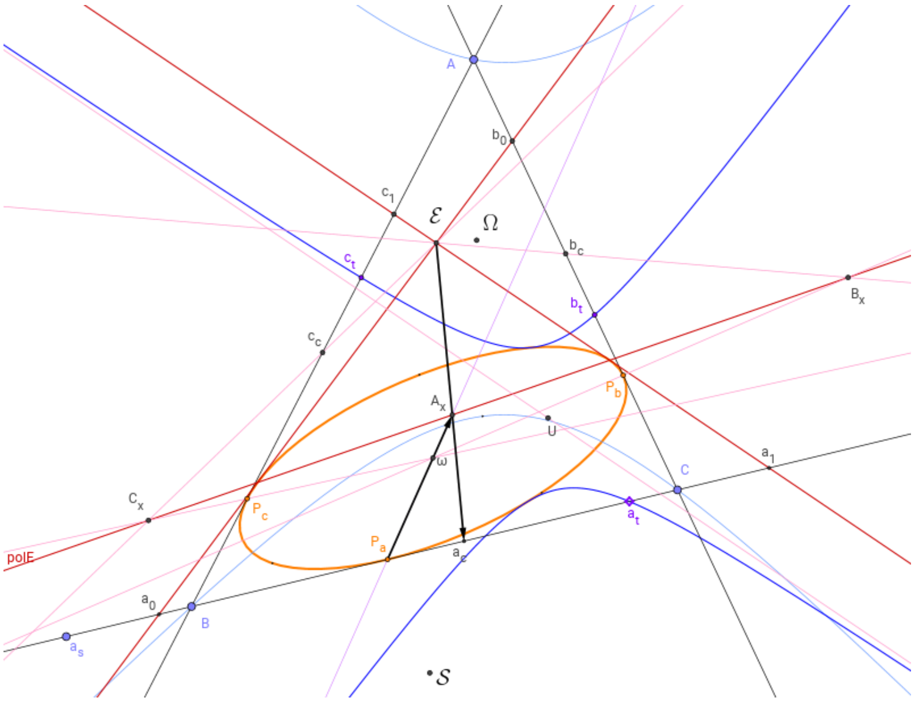
Definition 27.10.9. When $\mathcal{S}$ is not at infinity, *area* $(\mathcal{T}_t)$ is second degree in t, and presents an extremum. The corresponding triangle is called the critical triangle of the LFIT, and noted $\mathcal{T}_c$.

Proposition 27.10.10. *Consider $\mathcal{T}_0,\mathcal{T}_1$ the tangents issued from $\mathcal{E}\notin\mathcal{L}_b$ to the pillar conic $\mathfrak{C}$ (see Figure 27.8). Let* a_0, a_1, etc *be the intersections (visible or not) of these tangents with the sideline BC. Then critical triangle $\mathcal{T}_c$ is given by $a_c\doteq(a_0+a_1)/2$, etc.*

Proof. In the real domain, a second degree polynomial attains its extremum at the middle of its roots. □

Corollary 27.10.11. *Applied to the pedal triangles of the points P_t of a line Δ, the critical triangle is obtained at the projection P_0 of the circumcenter O onto the given line.*

Proof. Line OP_0 is the axis of symmetry of the figure formed by Γ and Δ. □



From P_a to A_x through ω. And then from $A_x$to a_c through $\mathcal{E}$.

Figure 27.8: Constructing the critical triangle

Construction 27.10.12. ***Construct $\mathcal{T}_{crit}$ when $\mathcal{S}, \mathcal{E}$ are known, but not at infinity.*** *It suffices to construct the middle of a subtangent as described at Construction 18.1.3. Obtain the contacts P_a,* etc *of the pillar conic from its perspector which is* isotom Ω. *Draw also Δ, the conipolar of $\mathcal{E}$ (see Figure 27.8). Cut this line by line ωP_a and obtain A_x,* etc. *Then we have $a_c = \mathcal{E}A_x \cap BC$,* etc.

Proof. This works even if $\mathcal{E}$ is inside $\mathfrak{C}$, i.e. when the flat triangles of the family are not visible. We have the following coordinates:

$$P_a \simeq \left(\begin{array}{c} 0 \\ (f+u)\,(u\rho-(v+w)\,\tau)-(g+v)\,u\,(\sigma+\tau) \\ (f+u)\,(u\rho-(v+w)\,\sigma)-(h+w)\,u\,(\sigma+\tau) \end{array} \right)$$

$$A_x \simeq \left(\begin{array}{c} u\,(\sigma+\tau) \\ u\,\rho-(v+w)\,\tau \\ u\,\rho-(v+w)\,\sigma \end{array} \right) \;;\; a_c \simeq \left(\begin{array}{c} 0 \\ u\,\rho-(v+w)\,\tau-v\,(\sigma+\tau) \\ u\,\rho-(v+w)\,\sigma-w\,(\sigma+\tau) \end{array} \right)$$

Moreover, a detailed proof has already been given at Construction 18.1.3. Obviously, we re-obtain (27.4). □

Remark 27.10.13. The point A_x is the conipole of the line $\left(\mathcal{E}, \overrightarrow{BC}\right)$.

Proposition 27.10.14. *When the critical triangle is known as*

$$\mathcal{T}_{crit} = \left[\begin{array}{ccc} 0 & 1-q & r \\ p & 0 & 1-r \\ 1-p & q & 0 \end{array} \right]$$

then the slowness center is constrained to the circumconic $\mathfrak{K}_S$ whose perspector and center are:

$$persp \simeq \left(\begin{array}{c} 1-q-r \\ 1-r-p \\ 1-p-q \end{array} \right) \;;\; cent \simeq \left(\begin{array}{c} (1-q-r)\,(1-2\,p) \\ (1-r-p)\,(1-2\,q) \\ (1-p-q)\,(1-2\,r) \end{array} \right)$$

One can parametrize $\mathcal{S},\mathcal{E}$ as

$$\mathcal{S}\simeq\left(\begin{array}{c} K\left(1+K\right)\left(1-q-r\right)\\ \left(1+K\right)\left(1-r-p\right)\\ -K\left(1-p-q\right) \end{array}\right)\;;\;\mathcal{E}=\mathcal{T}_{crit}\cdot\mathcal{S}$$

and $\mathcal{E}$ is constrained to $\mathfrak{K}_{\mathcal{S}}={}^{t}\mathcal{T}_{crit}^{-1}\cdot\mathfrak{K}_{S}\cdot\mathcal{T}_{crit}^{-1}$. This conic is nothing else than the hexagonal conic of $\mathcal{T}_{crit}$ that goes through the a_m, etc *and the $\alpha_m\doteq b_m+c_m-A$,* etc.

Proof. One must ensure that the area is a polynomial whose first degree coefficient is null. Caveat: the list $[p,q,r]$ is not to be treated up to a projective multiplier. □

27.10.4 Orthologic families

Proposition 27.10.15. *The lines that support the flat triangles are orthogonal if and only if $\mathcal{S}$ and $\mathcal{E}$ are conjugate with respect to the polar circle. In such a case, all temporal conics are rectangular hyperbolas... except from the degenerate ones*

Proof. Use parametrization (27.1) where $\mathcal{T}_0,\mathcal{T}_1$ are the flat triangles. Then we have

$${}^{t}S\cdot cir_{H}\cdot\mathcal{E}=\mathcal{L}_{0}\cdot\boxed{\mathcal{M}_{b}}\cdot{}^{t}\mathcal{L}_{1}\times\Theta_{2}\times\prod_{3}\left(p_{0}q_{1}-q_{0}p_{1}\right)$$

where Θ_2 was defined at Remark 27.10.2.

□

Proposition 27.10.16. *When two non flat triangles of a LFIT are orthologic, then all triangles of the LFIT are orthologic with each other. Moreover, $\mathcal{S}$ and $\mathcal{E}$ are conjugate wrt the polar circle (and the flat triangles orthogonal to each other).*

Proof. By Proposition 26.4.9, orthology between non degenerate triangles is characterized by trace $\left(\boxed{\mathcal{T}_s}\cdot\boxed{\mathcal{T}_t}^{-1}\cdot\boxed{\mathrm{OrtO}}\right)=0$. When using the asymmetric parametrization, this gives:

$$\frac{t-s}{f\,g\,h\,S}\left({}^{t}\mathcal{S}\cdot\left(\begin{array}{ccc} S_{a} & 0 & 0\\ 0 & S_{b} & 0\\ 0 & 0 & S_{c} \end{array}\right)\cdot\mathcal{E}\right)=0$$ □

Proposition 27.10.17. *Assume that preceding condition is fulfilled and note $P\left(t,s\right)$ the point which sees triangle $\mathcal{T}_t$ at right angle to trigone $\mathcal{T}_s$. The barycentrics of $P\left(t,s\right)$ have degrees in t and s that are, respectively, 1 and 2. The locus $t\mapsto P\left(t,s\right)$ is one of the two lines tangent to the pillar conic through the orthocenter of $a_sb_sc_s$, while the locus $s\mapsto P\left(t,s\right)$ is the temporal conic $\mathcal{C}_t$.*

Proof. The linear motion $t\mapsto P\left(t,s\right)$ is incident to the linear motions of $a_tb_tc_t$ (see Proposition 27.5.9). And therefore the locus is a line tangent to the pillar conic, while $P\left(s,s\right)$ is obviously the orthocenter of $a_sb_sc_s$. On the other hand, $P\left(t,t\right)\in\mathcal{C}_t$ was granted since this conic is a RH. □

27.11 Envelopes of the sidelines (parabolas)

Proposition 27.11.1. *The envelope of the line b_tc_t is a parabola $\mathfrak{P}_A$. Its point at infinity is $g+h:-g:-h$, i.e. the direction of line $\mathcal{S}A$. Its tangential equation is*

$$\boxed{\mathfrak{P}_{A}^{*}}=\left(\begin{array}{ccc} 2\left(q-1\right)g-2\,hr & \left(1-q\right)g-\left(1-r\right)h & hr-gq\\ \left(1-q\right)g-\left(1-r\right)h & 0 & \left(1-r\right)h+gq\\ hr-gq & \left(1-r\right)h+gq & 0 \end{array}\right)$$

$$\simeq\left(\begin{array}{ccc} 2\,u & h-u & g-u\\ h-u & 0 & -g-h+u\\ g-u & -g-h+u & 0 \end{array}\right)$$

This parabola is tangent to AB and AC at their intersections with locus (α). When using metric properties, the focus is the already encountered point O_A, while its directrix is

$$\Delta_A \simeq [S_a\,(g+h), -S_b\,g, -S_c\,h] - uS_a\,\mathcal{L}_b \qquad \text{where } u = \mathcal{E}_1 = hr - gq + g$$

Proof. The coefficients of line bc have degree 2 in t. Therefore, the envelope is a conic. Matrix $\boxed{\mathfrak{P}_A^*}$ is obtained by locusconi. Then $\mathcal{L}_b \cdot \boxed{\mathfrak{P}_A^*} = -g-h : g : h$ and we have a parabola. The focus comes from the Plucker method, and the directrix is the polar line of the focus. □

And circularly for the other two parabolas.

Proposition 27.11.2. *Tangents from $\mathcal{E}$ to the three parabolas are the same.*

Proof. The contact points themselves have a terrific expression, involving

$$W = \sqrt{f^2u^2 + g^2v^2 + h^2w^2 - 2\,fguv - 2\,ghvw - 2\,fhuw}$$

Nevertheless, one can see that the three expressions

$$(\mathcal{E} \cdot \mathfrak{P}_j \cdot \mathcal{E})\,(X \cdot \mathfrak{P}_j \cdot X) - (\mathcal{E} \cdot \mathfrak{P}_j \cdot X)^2$$

giving the pair of tangents issued from $\mathcal{E}$ are the same. □

27.12 Special shapes of the inscribed triangles

27.12.1 LFIT of equilateral triangles

Proposition 27.12.1. *For a given triangle ABC, there are two LFIT of equilateral triangles. They are described by: $a, b, c \simeq$*

$$(S_a + 2\,\Sigma\,)\begin{pmatrix} 0 \\ x \\ 1-x \end{pmatrix}, \begin{pmatrix} +4\,\Sigma\, - (S_b + 2\,\Sigma)\,x \\ 0 \\ S_a - 2\,\Sigma\, + (S_b + 2\,\Sigma)\,x \end{pmatrix}, \begin{pmatrix} S_a - b^2 + 2\,\Sigma + (S_c + 2\,\Sigma)\,x \\ +b^2 - (S_c + 2\,\Sigma)\,x \\ 0 \end{pmatrix}$$

where Σ stands for $\Sigma = S/\sqrt{3}$.

Proof. Chose $a \simeq 0 : x : 1-x$ on BC and consider the rotation $\rho\,(a, +60°)$, so that:

$$\boxed{\rho} = \begin{pmatrix} -a^2x + S_c + 2\,\Sigma & -a^2x + a^2 & -a^2x \\ (S_c + 2\,\Sigma\,)\,x - b^2 & (S_c + 2\,\Sigma)\,x - S_c + 2\,\Sigma & (S_c + 2\,\Sigma)\,x \\ (S_b - 2\,\Sigma\,)\,x + S_a + 2\,\Sigma & (S_b - 2\,\Sigma)\,x - S_b + 2\,\Sigma & (S_b - 2\,\Sigma)\,x + 4\,\Sigma \end{pmatrix}$$

Define c as $AB \cap \rho\,(CA)$ and b as $\rho^{-1}\,(c)$. Then abc is equilateral direct. It remains to synchronize the normalization of the three columns. □

Proposition 27.12.2. *For these two families, the pair $(\mathcal{S}, \mathcal{E})$ is either X(13), X(15) or X(14),X(16), i.e a Fermat center and the corresponding isodynamic center.*

Proof. From the values of a_t, b_t, c_t, one can read the values of f, g, h. And then apply these masses to the variable triangle. This leads to:

$$\begin{aligned} \mathcal{S} &\simeq \frac{1}{S_a + 2\,\Sigma} : \frac{1}{S_b + 2\,\Sigma} : \frac{1}{S_c + 2\,\Sigma} \\ \mathcal{E} &\simeq (S_a + 2\,\Sigma)\,a^2 : (S_b + 2\,\Sigma)\,b^2 : (S_c + 2\,\Sigma)\,c^2 \end{aligned}$$

Caveat: these two sets of coordinates are to be synchronized in order to enforce $f+g+h = u+v+w$ (leading to huge expressions !). □

Proposition 27.12.3. *The pedal triangle of X(15) is equilateral and belongs to the corresponding family.*

Proof. Since triangles abc are "turning around" point $\mathcal{E}$, the triangle of minimal area is obtained by orthogonal projection. The center of this triangle is the middle of $[\mathcal{S}, \mathcal{E}]$. Moreover, the locus of $g = (a+b+c)/3$ is directed by:

$$S_c - S_b : S_a - S_c : S_b - S_a \simeq b^2 - c^2 : c^2 - a^2 : a^2 - b^2$$

One recognizes X(531), the orthopoint of X(30): the locus of g is the perpendicular bisector of segment $[\mathcal{S}, \mathcal{E}]$, and therefore orthogonal to the Euler line. □

27.12.2 LFIT of similar triangles

Proposition 27.12.4. *The triangles of a linear family are similar to each other if and only if $\mathcal{S}$ and $\mathcal{E}$ form an isogonal pair. And then $\mathcal{E}$ is the center of similitude. Moreover, $O_A = O_B = O_C = \mathcal{E}$.*

Proof. We already know that $\mathcal{E}$ is a fixed point of $\boxed{\mathcal{T}_t} \cdot \boxed{\mathcal{T}_s}^{-1}$. In order to have a similitude, the other two must be the umbilics of the plane. This leads to:

$$\left(\begin{array}{c} u \\ v \\ w \end{array}\right) = \frac{h+g+f}{a^2gh + b^2fh + c^2fg} \left(\begin{array}{c} a^2gh \\ b^2fh \\ c^2fg \end{array}\right)$$

□

Proposition 27.12.5. *The pedal triangle of $\mathcal{E}$ belongs to the LFIT, and provides the extremal area of the family*

$$\frac{4fgh\,S^3\,(f+g+h)}{(a^2gh + b^2fh + c^2fg)^2} = \frac{4S^3\left(a^2vw + b^2uw + c^2uv\right)}{a^2b^2c^2\,(u+v+w)^2}$$

Proof. Direct computation. □

27.12.3 LFIT of pedal triangles

Proposition 27.12.6. *When a LFIT contains the pedal triangles of two different points P_0, P_1 (not at infinity), then each of the other inscribed triangles can be written as $\mathcal{T}_t \doteq (1-t)\,\mathcal{T}_0 + t\,\mathcal{T}_1$ and is the pedal triangle of $P_t = (1-t)\,P_0 + t\,P_1$.*

Spoiler: for such a LFIT, the slowness center $\mathcal{S}$ is on the circumcenter, its isogonal $\mathcal{S}^*$ is the orthopoint of $\overrightarrow{P_0P_1}$, while $\mathcal{E}$ belongs to the Simson line of $2O - \mathcal{S}$. This configuration is the core of the theory of the orthopoles, which is explored in detail at Section 28.8.

Proof of the first part. Consider the LFIT generated by $\mathcal{T}_0, \mathcal{T}_1$ and apply (9.2). Then check that P_t is the pedal center of $\mathcal{T}_t$. The slownesses are easy to obtain, and $\mathcal{S} \in \Gamma$ follows. □

Proposition 27.12.7. *Any LFIT whose slowness center $\mathcal{S}$ is not on the circumcircle contains exactly one pedal triangle.*

Proof. Apply (9.2) to the LFIT. This gives a first degree polynomial in t whose leading coefficient is $a^2gh + b^2hf + c^2fg$. □

Construction 27.12.8. *Embedded pedal triangle* *($\mathcal{S} \notin \Gamma$). Start form a triangle $\mathcal{T}_t$ and cut the perpendicular to AB issued from c_t by the perpendicular to AC issued from b_t. This gives a point BC_t. The locus Δ_a of these points BC_t is a straight line. Obtain another point by using another triangle $\mathcal{T}_s$. When $\mathcal{S} \notin \Gamma$, the three lines Δ_a,* etc *are concurrent, as proven in the previous proposition. This gives the central point of the embedded pedal triangle.*

27.12.4 Cevian triangles in a LFIT

Proposition 27.12.9. *A LFIT contains exactly three cevian triangles (up to visibility and multiplicity).*

Proof. The determinant of lines a_tA, b_tB, c_tC is a polynomial of degree 3 in t (its leading coefficient doesn't vanish). □

Construction 27.12.10. ***Embedded cevian triangles***. *Draw the conic $\mathcal{C}_A$ through $B, C, G_a = B + C - A$ and the two intersections of the hexagonal graph $\mathcal{G}_a$ with lines AB and AC. Draw the other two. The three conics concur at the three required centers (in the complex plane). Maybe only one of them is visible.*

Proof. Determine point BC_t as the intersection of lines $b_t B$, $c_t C$. The result is in t^2. Thus the locus of BC_t is a conic. Use locusconi to obtain its equation, and check for the five given points. Existence is given by the previous proposition. □

Exercise 27.12.11. Consider the LFIT generated by the cevians of the two points P_0 (fixed) and P_2 (mobile). Let P_3 be the center of the third cevian triangle of this family. Then, for quite all P_0, the transformation $P_2 \mapsto P_3$ is a Cremona involution. Determine the exceptional locus and the points of indetermination. Study how these points are blown out.

27.13 Families with constant area

27.13.1 Three non concurrent lines (rewritten)

Proposition 27.13.1. *Assume that* area$(\mathcal{T}_t)$ *remains constant when t changes. Then $\mathcal{S}$ is at infinity while $\mathcal{E} \simeq \mathcal{S}$, so that we can use the hard equalities $\mathcal{S} = [\tau - 1 : 1 : -\tau]$ and $\mathcal{E} = K\,\mathcal{S}$ for some values τ and K. The converse holds and leads to:*

$$\boxed{\mathcal{T}_t} = \left[\begin{array}{ccc} 0 & K\tau + \tau - \dfrac{t}{1} & 1 + \dfrac{K}{\tau} - \dfrac{t}{\tau} \\ \dfrac{t}{\tau - 1} & 0 & -\dfrac{K}{\tau} + \dfrac{t}{\tau} \\ 1 - \dfrac{t}{\tau - 1} & 1 - K\tau - \tau + \dfrac{t}{1} & 0 \end{array}\right] \;;\; \operatorname{area}(\mathcal{T}_t) = -K\,(K+1)\,S$$

Proof. The coefficient of t^2 in area$(\mathcal{T}_t)$ contains $f + g + h$, so that $\mathcal{S} \in \mathcal{L}_b$. And then we have $u + v + w = f + g + h = 0$, together with $\tau v + w = 0$ from the coefficient of t. □

Corollary 27.13.2. *Special case: when $\mathcal{S} \in \mathcal{L}_b$ and $K = -1$, the LFIT is the set of flat inscribed triangles whose "sideline" is $[1, -\tau + 1, 0] + t\mathcal{L}_b$ i.e. is directed by $\mathcal{S}$.*

Corollary 27.13.3. *Special-special case: when $\mathcal{S} \in \mathcal{L}_b$ and $K = 0$, the LFIT is the set of flat inscribed triangles whose "sideline" is tangent to the inscribed parabola having $\mathcal{S}$ as point at infinity. In this case, $\boxed{\mathcal{T}_t} \cdot \mathcal{S} = 0 : 0 : 0$ and the equicenter $\mathcal{E}$ is not defined.*

Proof. Apply locusconi to line $[t\,(-\tau + 1 + t)\,, (t - \tau)\,(-\tau + 1 + t)\,, (t - \tau)\,t]$. □

___***

Let us assume that $S = K_2$ (choosing the sign of W). One obtains:

$$K_3 \simeq W \begin{pmatrix} g - h \\ h - f \\ f - g \end{pmatrix} - \frac{2\,(fg + gh + hf)}{fgh} \begin{pmatrix} f \\ g \\ h \end{pmatrix} t$$

while the locus $\mathcal{P}$ of the K_1 becomes a conic since the $t-$degrees of the barycentrics of that point are now 2, due to $f + g + h = 0$. After computing $\mathcal{P}$ using the procedure *locusconi*, one can check that $K_2 \in \mathcal{P}$ together with $\mathcal{L}_b \in \mathcal{P}^*$: the conic is a parabola.

Moreover, the envelope of line $K_1 K_3$ is also a conic, since the $t-$degree of $K_1 \wedge K_3$ is two. Using again *locusconi*, we obtain the matrix of the tangential equation:

$$\boxed{Q^*} \simeq \left[\begin{array}{ccc} (1 - W)\,f & -Wh & -Wg \\ -Wh & (1 - W)\,g & -Wf \\ -Wg & -Wf & (1 - W)\,h \end{array}\right]$$

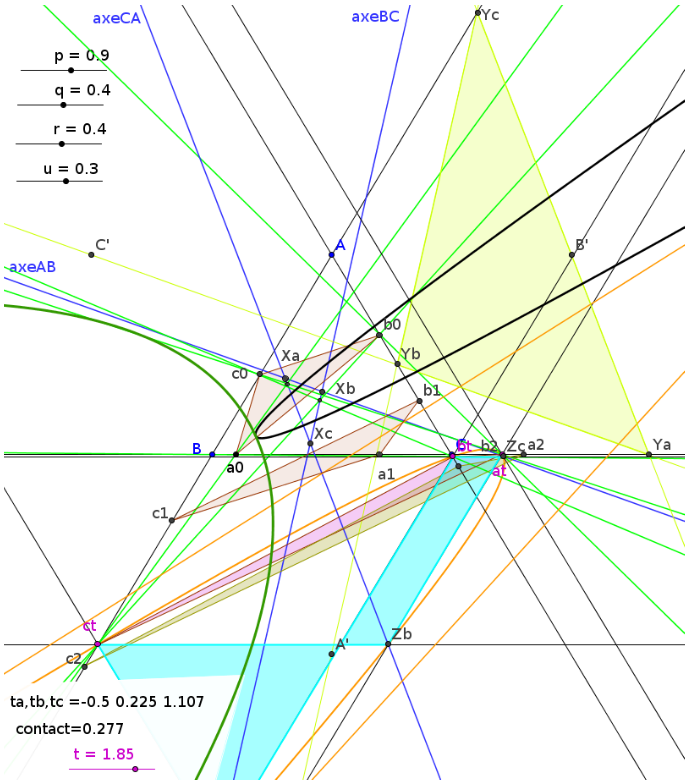


Figure 27.9: Two parabola

When searching for the common points of Q and $\mathcal{P}$, one obtains that these curves are bitangent. A first contact occurs in K_2, so that Q is a parabola with the same direction as $\mathcal{P}$. The second contact is the center of the hexagonal conic relative to

$$t_0 = \frac{(g-h)(f-h)(f-g)}{6(fg+fh+gh)} W$$

when using the former given parametrization. This value is the arithmetical mean of the dates of the degeneracies.

27.13.2 Three concurrent lines

Suppose now that points a, b, c are moving on concurrent lines. We only have to consider this case as the limit of what happens when $K \to 0$ to the figure obtained by applying an homothety of center G and ratio K to the previous results (and replacing X by X/K for $X = p, q, r, W, t$, while slownesses f, g, h are unchanged).

Then the inscribed triangle and the equicenter become:

$$\boxed{abc} \simeq \left(\begin{array}{ccc} \frac{1}{3} & \frac{1}{3} - q - \frac{t}{g} & \frac{1}{3} + r + \frac{t}{h} \\ \frac{1}{3} + p + \frac{t}{f} & \frac{1}{3} & \frac{1}{3} - r - \frac{t}{h} \\ \frac{1}{3} - p - \frac{t}{f} & \frac{1}{3} + q + \frac{t}{g} & \frac{1}{3} \end{array}\right), \mathcal{E} \simeq \left(\begin{array}{c} hr - gq + \frac{1}{3}(f+g+h) \\ fp - hr + \frac{1}{3}(f+g+h) \\ gq - fp + \frac{1}{3}(f+g+h) \end{array}\right)$$

while the areal center is the limit of:

$$\left(\begin{array}{c} (2f-g-h)K + (f+g+h) \\ (2g-f-h)K + (f+g+h) \\ (2h-f-g)K + (f+g+h) \end{array}\right)$$

and thus is G when $\mathcal{S} \notin \mathcal{L}_b$, but is $\mathcal{S}$ when $\mathcal{S} \in \mathcal{L}_b$.

The hexagonal conic goes through a, b, c and $a' = b + c - G$, etc. When $t = 0$, its points at infinity are:

$$2\,q + 2\,r : -W - 2\,r : W - 2\,q$$

where $W^2 = -4\,(pq + qr + rp)$ is the limiting value of the relative area. Therefore:

Proposition 27.13.4. *Given three lines that are concurrent at G and an inscribed triangle abc it exists two families of inscribed triangles that share the same area. Their areal center is one of the points at infinity of the hexagonal conic defined by a, b, c, G. Point $\mathcal{S}$ is real if $W^2 \geq 0$. Then, the family is obtained by constructing strips of equal width (in the $\mathcal{S}$ direction).*

27.14 Concurrent hexagonal graphs

In this section, we assume that the three hexagonal graphs intersect each other at the same point $K \simeq p : q : r$.

27.14.1 Assuming that $\mathcal{S}$ is known

Lemma 27.14.1. *In any case, the line through A parallel to G_a cuts BC at $P' \simeq 0 : g : -h$,* etc. *These points P', Q', R' belong to line $[1/f,\ 1/g\,1/h]$ (the tripolar of $\mathcal{S}$). Moreover, the points $P = B + C - P'$,* etc *belong to line $[f, g, h]$ (isotomic of the tripolar).*

Proof. Quite obvious from 27.8. □

Proposition 27.14.2. *When the graphs are concurrent and $\mathcal{S} \simeq f : g : h$ is known, then K belongs to line $[f, g, h]$ (isotomic conjugate of the tripolar of $\mathcal{S}$), while $\mathcal{E} \simeq K \underset{b}{*} (2\mathcal{S} - G)$, the barycentric product of K and the anticomplement of $\mathcal{S}$. As a result, $\mathcal{E}$ belongs to the line:*

$$\Delta_S \doteq \left[\frac{f}{g + h - f}, \frac{g}{h + f - g}, \frac{h}{f + g - h}\right]$$

which is the image of $\operatorname{tripolar}(\mathcal{S})$ *by* $\operatorname{recip} \circ \operatorname{homot}(\mathcal{S}, 1/3)$.

Proof. Using 27.8, the concurrence gives an equation, and the normalization gives another. This results into a parametrization of $\mathcal{E}$ and K by a coordinate of $\mathcal{E}$ (say w). And the results follow. □

27.14.2 Assuming that K is known

Proposition 27.14.3. *When the graphs are concurrent at a known point $K \simeq p : q : r$, then $\mathcal{S}$ belongs to line $[p, q, r]$ while $\mathcal{E}$ belongs to line*

$$\Delta_K \doteq [qr\,(q + r)\,, pr\,(p + r)\,, p\,(p + q)\,q]$$

which is the image of $\operatorname{tripolar}(K)$ *by* $\operatorname{homot}(K, 2/3)$. *Moreover, a parametrization of $[\mathcal{S}, \mathcal{E}]$ by a point at infinity is:*

$$[\mathcal{S}, \mathcal{E}] \simeq \left[(p + q + r)\begin{pmatrix} q\,r \\ r\,p \\ p\,q \end{pmatrix} \underset{b}{*} \begin{pmatrix} 1 \\ \mu \\ -1 - \mu \end{pmatrix}, \begin{pmatrix} p \\ q \\ r \end{pmatrix} \underset{b}{*} \begin{pmatrix} -pq\,(\mu + 1) + \mu\,pr - qr \\ -pq\,(\mu + 1) - \mu\,pr + qr \\ +pq\,(\mu + 1) + \mu\,pr + pq + qr \end{pmatrix}\right]$$

Proof. From previous proposition, $f\,p + g\,q + h\,r = 0$. Thus the parametrization $1 : \mu : -1 - \mu$ of $\mathcal{L}_b$ induces $\mathcal{S} \simeq qr : \mu rp : -(1 + \mu)\,pq$. Added to $G_a \cdot K = 0$, etc and the normalization rule, this leads to the remaining results. □

27.14.3 Assuming that K is the center of gravity

Fact 27.14.4. *When K =X(2), then $\mathcal{S} \in \mathcal{L}_b$ while $\mathcal{E} = (-2/3)\,\mathcal{S}$. Moreover, the areas of all the inscribed triangles abc, and of all the $\alpha\beta\gamma$, are equal to $(2/9)\,S$.*

Fact 27.14.5. *Then $M_\infty = \mathcal{S}^*$ belongs to the circumcircle, while $O_a = (2A + M_\infty)/3$,* etc. *This provides the circle of similarity. And everything flows from this result.*

27.15 When the graphs are given

In the previous sections, the graphs were the result of the pre-existing mappings $a \longleftrightarrow b \longleftrightarrow c \longleftrightarrow a$. Let us now examine what happens when these graphs are chosen from the beginning.

27.15.1 Catalan graphs

Proposition 27.15.1. *When three lines Δ_j are parallel to the sidelines, they are the Catalan graphs of a LFIT if and only if the bisectors of strips (Δ_a, BC), etc are concurrent. And then $\mathcal{S}$ can be chosen at will (outside of $\mathcal{L}_b$).*

Proof. The necessity comes from the required symmetry wrt $\omega = (\mathcal{S} + \mathcal{E})/2$. Since ω is at finite distance, $\mathcal{S}, \mathcal{E}$ cannot be chosen at infinity. □

Proposition 27.15.2. *Define the collineation μ^* by $\mathcal{L}_b \mapsto \mathcal{L}_b$, $BC \mapsto \Delta_A = [p_0 q_0, q_0]$, $CA \mapsto \Delta_B = [r_1, q_1, r_1]$, $AB \mapsto \Delta_C = [p_2, p_2, r_2]$. Then its matrix (acting over the lines !) is*

$$\boxed{\mu^*} \simeq \begin{pmatrix} \dfrac{p_0}{q_0 - p_0} & \dfrac{q_0}{q_0 - p_0} & \dfrac{q_0}{q_0 - p_0} \\ \dfrac{r_1}{r_1 - q_1} & \dfrac{q_1}{r_1 - q_1} & \dfrac{r_1}{r_1 - q_1} \\ \dfrac{p_2}{p_2 - r_2} & \dfrac{p_2}{p_2 - r_2} & \dfrac{r_2}{p_2 - r_2} \end{pmatrix}$$

We have $\chi(X) = (X+1)^2 (X - \lambda)$ Then $\lambda = 1$, i.e. μ is a central symmetry, when

$$2\, p_0\, q_1\, r_2 + q_0\, r_1\, p_2 - p_0\, q_1\, p_2 - p_0\, r_1\, r_2 - q_0\, q_1\, r_2 = 0$$

In any case, $\omega \doteq ker(\mu - 1) \simeq \dfrac{q_0}{p_0 - q_0} : \dfrac{r_1}{q_1 - r_1} : \dfrac{p_2}{r_2 - p_2}$. When ω is the center, we have:

$$\mathcal{E} = \mu(\mathcal{S}) \simeq \frac{f p_0 + g q_0 + h q_0}{p_0 - q_0} : \frac{f r_1 + g q_1 + h r_1}{q_1 - r_1} : \frac{f p_2 + g p_2 + h r_2}{r_2 - p_2}$$

Proof. Computations are straightforward. □

27.15.2 Hexagonal graphs

Lemma 27.15.3. *When $\mathcal{S}, \mathcal{E}$ are know, the hexagonal graphs have the following equations:*

$$\begin{pmatrix} G_a \\ G_b \\ G_c \end{pmatrix} \simeq \begin{bmatrix} g + h - u & g - u & h - u \\ f - v & h + f - v & h - v \\ f - w & g - w & f + g - w \end{bmatrix} \tag{27.17}$$

Proof. From the very definition, we have $\alpha_t \doteq B + C - a_t$, etc. And then, we use the asymmetric parametrization. □

Lemma 27.15.4. *When $G_c \simeq [p_3, q_3, r_3]$ is given, then both relations hold:*

$$g = f\, \frac{r_3 - p_3}{r_3 - q_3} \;;\; w = f\, \frac{r_3 - p_3 - q_3}{r_3 - q_3}$$

Proof. Graph G_c describes $a_t \mapsto b_t$. Start from $a_t \simeq 0 : t/f, 1 - t/f$. Compute $\gamma_t \simeq G_c \wedge \left(a_t \wedge \overrightarrow{AC}\right)$. Then identify $C + \gamma_t$ with the $a_t + b_t$from the usual parametrization. □

Proposition 27.15.5. *Let be given three lines $\Delta_j \simeq [p_j, q_j, r_j]$ in general position and define*

$$\begin{pmatrix} f \\ g \\ h \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} p_1 - r_1 \\ p_1 - r_1 \\ p_1 - q_1 \\ p_1 - q_1 - r_1 \\ p_1 - q_1 \\ p_1 - r_1 \end{pmatrix} \underset{b}{*} \begin{pmatrix} q_2 - p_2 \\ q_2 - p_2 \\ q_2 - p_2 \\ q_2 - p_2 \\ q_2 - p_2 - r_2 \\ q_2 - p_2 \end{pmatrix} \underset{b}{*} \begin{pmatrix} r_3 - q_3 \\ r_3 - p_3 \\ r_3 - p_3 \\ r_3 - p_3 \\ r_3 - p_3 \\ r_3 - q_3 - p_3 \end{pmatrix}$$

Then the three lines are the hexagonal graphs (in that order) of a LFIT if and only if

$$\frac{(p_1 - r_1)(q_2 - p_2)(r_3 - q_3)}{(p_1 - q_1)(q_2 - r_2)(r_3 - p_3)} = +1 \quad \text{and} \quad f + g + h = u + v + w$$

In such a case, f, g, h, u, v, w are the synchronized barycentrics of the centers $\mathcal{S}, \mathcal{E}$ of this LFIT.

Proof. Caveat: this formula is not symmetric but is nevertheless the right one. The first condition is required by Lemma 27.15.4 if we want that $x = f$ in the chain $f \mapsto_3 g \mapsto_1 h \mapsto_2 x$. The second is the required synchronization. The converse is left as an exercise. □

Exercise 27.15.6. Use sagemath. Use the nine coefficients of matrix :

$$\begin{pmatrix} g+h-u & g-u & h-u \\ f-v & h+f-v & h-v \\ f-w & g-w & f+g-w \end{pmatrix} - \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} \cdot \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix}$$

together with $f + g + h - u - v - w$ and $f - 1$. Use these 11 polynomials to generate an ideal $\mathcal{I}$ over $\mathbb{Q}$. Eliminate $f, g, h, u, v, w, x, y, z$. This gives an ideal $\mathcal{J}$. Build the ideal $\mathcal{K}$ generated by the two conditions of the proposition. Divide $\mathcal{J}$ by $\mathcal{K}$, and obtain 1 (condition was necessary). Divide $\mathcal{K}$ by $\mathcal{J}$. The result is generated by three polynomials. And now, be more precise about the "in general position" used in the above proposition.

Proposition 27.15.7. *Let us now suppose that $\mathcal{S}$ is given. This determines the direction of the graphs, and therefore requires the existence of three numbers ρ, σ, τ such that :*

$$(G) \simeq \begin{bmatrix} -\rho & h-\rho & g-\rho \\ h-\sigma & -\sigma & f-\sigma \\ g-\tau & f-\tau & -\tau \end{bmatrix} \tag{27.18}$$

The condition for these three lines to be the hexagonal graphs of a LFIT is $f + g + h = \rho + \sigma + \tau$, i.e. the synchronization rule (27.6). And then, the equicenter $\mathcal{E}$ and the pillar point Ω are given by

$$\mathcal{E} = \begin{pmatrix} h+g-\rho \\ h+f-\sigma \\ g+f-\tau \end{pmatrix} \; ; \; \Omega = \begin{pmatrix} \rho \\ \sigma \\ \tau \end{pmatrix}$$

so that $\mathcal{S} + \mathcal{E} + \Omega = (f + g + h)(A + B + C)$ as required at (27.3).

Proof. The directions of the three hexagonal graphs are those of the anti-cevian triangle of $\mathcal{S}$. Let $\mathcal{T}_{\mathcal{S}}^*$ be the corresponding trigone, i.e. the three sidelines of this triangle. We have:

$$\mathcal{T}_{\mathcal{S}}^* \doteq \begin{pmatrix} -f & f & f \\ g & -g & g \\ h & h & -h \end{pmatrix}^* \simeq \begin{pmatrix} 0 & h & g \\ h & 0 & f \\ g & f & 0 \end{pmatrix}$$

The remaining computations are straightforward. □

Remark 27.15.8. This can be written as $\mathcal{S} \in \mathfrak{D}$ where

$$\mathfrak{D} \simeq \left[\frac{f-\rho}{f}, \frac{g-\sigma}{g}, \frac{h-\tau}{h} \right] \tag{27.19}$$

but the geometric interpretation of this line is not so clear.

27.15.3 The marvelous formula

Lemma 27.15.9. *When the homologue sidelines of two trigones are parallel, then the two triangles are perspective.*

Proof. The sidelines are perspective from $\mathcal{L}_b$. Another proof. The triangle $\boxed{G^*}$, obtained as the dual of trigone 27.18, is ever perspective with its model, the anticevian triangle of $\mathcal{S}$, and the perspector is

$$Q \doteq \begin{pmatrix} -\rho f + g\sigma + h\tau \\ +\rho f - g\sigma + h\tau \\ +\rho f + g\sigma - h\tau \end{pmatrix} \underset{b}{*} \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$

□

Proposition 27.15.10. *Let G be a trigone whose sidelines are respectively parallel to the sidelines of the $\mathcal{S}$-anticevian triangle and consider the collineation μ defined by $\mathcal{L}_b \mapsto \mathcal{L}_b$, $\mathcal{T}_{\mathcal{S}}^* \mapsto (G)$. This is ever an homothety. And then G describes the graphs of LFIT if and only if $stein_{\mathcal{S}} \cdot \boxed{\mu^*} \simeq dual_{\mathcal{S}}$ where*

$$dual_{\mathcal{S}} \simeq [f, g, h] \;\; ; \;\; stein_{\mathcal{S}} \doteq \text{anticomplem}\,(dual_{\mathcal{S}}) \simeq [g+h,\; h+f,\; f+g]$$

Proof. Describing μ by its action over the lines, we obtain the matrix :

$$\boxed{\mu^*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{2fgh}\, Q \cdot \mathcal{L}_b$$

where Q is the perspector defined just above. Therefore, μ is an homothety centered at Q, whose line-ratio is: $1/k \doteq 1 - (\mathcal{L}_b \cdot Q)\,/(2fgh)$... while $\boxed{\mu} \doteq$ Adjoint $\boxed{\mu^*}$, when acting on columns, describes the point-homothety homot (Q, k) (see (7.29), which uses k as point-ratio).

Line $dual_{\mathcal{S}} \simeq [f, g, h]$ is the tripolar of the isotomic of this point. Its image by the anticomplem transform is $stein_{\mathcal{S}} \doteq [g+h,\; h+f,\; f+g]$ (using ratio -2 on points, but ratio $-1/2$ on lines). We call this line $stein_{\mathcal{S}}$ since this is the conipolar of $\mathcal{S}$ w.r.t. the Steiner out-ellipse. And then one can see that $stein_{\mathcal{S}} \cdot \boxed{\mu^*} = dual_{\mathcal{S}}$ if and only if $f + g + h = \rho + \sigma + \tau$ is satisfied. □

Corollary 27.15.11. *Consider $\mu_G \doteq \mu \circ$anticomplem. This is another homothety (with point-ratio $-2k$), and we have:*

$$\boxed{\mu_G^*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{2fgh}\, Q_G \cdot \mathcal{L}_b \qquad \text{where } Q_G \doteq \begin{bmatrix} 2\,fgh + (g-h)\,(g\sigma - h\tau) - (h+g)\,f\rho \\ 2\,fgh + (h-f)\,(h\tau - f\rho) - (h+f)\,g\sigma \\ 2\,fgh + (f-g)\,(f\rho - g\sigma) - (f+g)\,h\tau \end{bmatrix}$$

And now the requirement is the global invariance of $dual_{\mathcal{S}}$ by μ_G, i.e. $Q_G \in dual_{\mathcal{S}}$. We are back with the same question: a geometric interpretation would be great.

27.16 Observers (about perspectivities)

Definition 27.16.1. We say that a (fixed) triangle $T_a T_b T_c$ **observes** the LFIT when the fixed triangle is in perspective with all of the triangles of the LFIT.

Proposition 27.16.2. *An observer $\mathcal{O}$ is necessarily parallelogic with the ABC triangle. This implies the existence of a point M_o (the ray source) that sees ABC with rays parallel to the sidelines of $\mathcal{O}$.*

Proof. The observer must be in perspective with $\mathcal{T}_\infty$, i.e. with $(\delta_{BC}, \delta_{CA}, \delta_{AB})$. This asserts the existence of the other center of parallelogy and, due to the symmetry of the relation, this implies the existence of M_o. □

Our intent is to specify some notations and prove the following theorem:

Theorem 27.16.3. *There are three kinds of observers, each family being parametrized by a generic point M in the triangle plane.*

1. *The metric observers, are using a generic point as $\mathcal{S}_H^*$ in order to obtain the P_a^H, P_b^H, P_c^H of Proposition 27.6.7. And then the locus of the perspectors is the conic through the seven M, P_j^H, O_j^H (here $\mathcal{S}_H^*$ is the other parallelogy center, not the ray source).*

2. *The Poulbot's observers, are using M as ray source in order to obtain a trigone that is tangent to each of the parabolas $\mathfrak{P}_j$ described at Proposition 27.11.1. And then, the locus of the perspectors is a straight line.*

3. *The three families of singular observers, using M as ray source in order to obtain a triangle with one side tangent to one of the parabolas, and the opposite vertex on the related sideline of ABC.*

Conversely, assume that T_a is generic, i.e. T_a is not on BC while AT_a is not tangent to $\mathfrak{P}_b$ or $\mathfrak{P}_c$. Then there are five observers that share T_a as first vertex: one metric and four Poulbot. The B-vertices of the Poulbot triangles (say B_j,j=1..4) are the common points of the tangents to $\mathfrak{P}_B$ from P_a^H and to $\mathfrak{P}_A$ from P_b^H , etc. for the vertices C_k. And then the B_j and the C_k are paired by the fact that B_jC_j is tangent to $\mathfrak{P}_A$.

In what follows, we are using the asymmetric parametrization (27.5), discarding the case $\mathcal{S} = \mathcal{E} \in \mathcal{L}_b$ (constant area) and assuming $f+g+h-u-v-w=0$. Proceeding that way, the case $\mathcal{S} \neq \mathcal{E}$; $\mathcal{S}, \mathcal{E} \in \mathcal{L}_b$ can be treated like the other cases. Let us recall that $\Omega = g+h-u : h+f-v : f+g-w$.

27.16.1 Metric observer

By itself, the theory of the LFIT is a projective one, and doesn't necessitate to use a metric or another. Nevertheless, we have encountered interesting properties when using a metric (circle of similarity). In order to reuse these properties, we are now investigating what happens when the metric is taken as a parameter.

27.16.1.1 Forcing the orthocenter

Definition 27.16.4. When an euclidian structure is given on the ABC plane, then triangle ABC receives an orthocenter $H \simeq 1/S_a : 1/S_b : 1/S_c$. Reverting the process, i.e. saying that a given point $H \simeq \rho : \sigma : \tau$ **is** the orthocenter of ABC, determines a metric structure on the ABC plane. This process is called "forcing the orthocenter". The resulting objects will be super-scripted with a "H", like in X^H (except from the isogonal conjugacy, noted $\mathcal{S}_H^*$, in order to avoid a tower of superscripts $\mathcal{S}^{*H}$).

Proposition 27.16.5. *Define O^H as the complement of H, and Γ^H (the forced circumcircle) as the circumconic whose perspector is $K^H \doteq H \underset{b}{*} O^H$. Then*

$$O^H \simeq \left(\begin{array}{c} \sigma+\tau \\ \tau+\rho \\ \rho+\sigma \end{array} \right), K^H \simeq \left(\begin{array}{c} \rho\,(\sigma+\tau) \\ \sigma\,(\tau+\rho) \\ \tau\,(\rho+\sigma) \end{array} \right) ; \Gamma^H \simeq \left[\begin{array}{ccc} 0 & \tau\,(\rho+\sigma) & \sigma\,(\tau+\rho) \\ \tau\,(\rho+\sigma) & 0 & \rho\,(\sigma+\tau) \\ \sigma\,(\tau+\rho) & \rho\,(\sigma+\tau) & 0 \end{array} \right]$$

Moreover Γ^H is the center of the conic, which goes through $2H_A - H$, etc where $H_A \doteq AH \cap BC$, etc.

Proof. Direct application of Theorem 12.7.4. □

Definition 27.16.6. The *forced isogonal transform* is defined as the ABC-isoconjugacy that exchanges H and O_H. In other words, $(x:y:z)_H^* \simeq \rho\,(\sigma+\tau)\,yz : \sigma\,(\tau+\rho)\,zx : \tau\,(\rho+\sigma)\,xy$.

27.16.1.2 Forcing the isogonal center

Definition 27.16.7. Forcing the isogonal center of a LFIT is making an arbitrary choice $M \simeq x : y : z$ for the point $\mathcal{S}_H^*$. This amounts to force the metric to $a^2 : b^2 : c^2 \simeq xf : yg : zh$. The values a, b, c are not intended to be real, only different from 0.

Proposition 27.16.8. *The "fixed points" P_a, etc(see Proposition 27.6.7) are obtained from $\mathcal{S}^*$ by collineations that doesn't depend from a choice of metric. Indeed, we have $P_a = \boxed{\phi_a} \cdot \mathcal{S}^*$ where:*

$$\boxed{\phi_a} \simeq \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1-\dfrac{w}{f} & \dfrac{v}{f} \\ 0 & \dfrac{w}{f} & 1-\dfrac{v}{f} \end{array}\right]$$

This is an affinity whose charpoly is $(X-1)^2(X-k)$ where $k = \dfrac{u-g-h}{f}$. Proper point associated to k is the direction δ_{BC}, while the proper line is $[0,-w,v]$ i.e. the line AE.

Proof. Obvious from (27.13) □

Proposition 27.16.9. *For any point M, the triangle $\phi_a M, \phi_b M, \phi_c M$ is an observer of the LFIT. The locus (on t) of the perspector K_t^H is a conic, that goes through $M = \mathcal{S}_H^*$, the three P_j^H and the three O_j^H.*

Proof. This is only reformulating some already proven properties (see Proposition 27.6.5) □

Proposition 27.16.10. *The $\mathcal{O} \mapsto ABC$ para-center of a metric observer is $M \simeq x : y : z$ itself. The $ABC \mapsto \mathcal{O}$ para-center (i.e. the ray center) is*

$$\varphi(M) \simeq \left[\begin{array}{c} f(uz-wx)(uy-vx) \\ g(vx-uy)(vz-wy) \\ h(wy-vz)(wx-uz) \end{array}\right]$$

We have $\varphi(\mathcal{E}) = 0:0:0$. Two M that share the same $\varphi(M)$ are aligned with $\mathcal{E}$, while all the $\varphi(M)$ belong to the circumconic γ with perspector $\mathcal{S} \underset{b}{*} \mathcal{E}$. Moreover $\varphi(M)$ belongs to Γ^H, the forced circumcircle. Therefore $\varphi(M)$ is the forced gudulic center of conic γ.

Proof. Existence is Proposition 27.16.2. Computing the value is easy. The result clearly depends on $M \wedge \mathcal{E}$, hence the alignment. □

27.16.2 Cevenol graphs

Definition 27.16.11. Let α_t' be the point where the parallel to BH through b_t cuts the parallel to CH through c_t. As time t flows, the point α_t' draws a straight line cev_a^H. We call it the A-Cevenol graph (related to the forced orthocenter $H \simeq \rho : \sigma : \tau$).

Proof. Straight line cev_a^H is obtained as:

$$\left[u-g-h, \frac{(u-g)\rho + u\tau}{\tau+\rho}, \frac{(u-h)\rho + u\sigma}{\rho+\sigma}\right]$$ □

Proposition 27.16.12. *The line cev_a^H is nothing but the line that joins O_a^H and Q_a^H, the centers of the direct and reverse H-similarities that generalizes the correspondence $CA \longmapsto AB : b \mapsto c$.*

Proof. Obvious from (27.12) and (27.16). □

Proposition 27.16.13. *The threeH-Cevenol graphs concur at a point G_c that belongs to the H-conic of similarity.*

$$\left[\begin{array}{c} (\sigma+\tau)\left((gv+hw-fu-gh)\rho^2 + (gv-fu-gh)\rho\sigma + (hw-fu-gh)\rho\tau - f\sigma\tau u\right) \\ (\tau+\rho)\left((hw+fu-gv-hf)\sigma^2 + (hw-gv-hf)\sigma\tau + (fu-gv-hf)\rho\sigma - g\rho\tau v\right) \\ (\rho+\sigma)\left((fu+gv-hw-fg)\tau^2 + (fu-hw-fg)\tau\rho + (gv-hw-fg)\tau\sigma - hw\rho\sigma\right) \end{array}\right]$$

Proof. Properties of G_c are easily computed. □

27.16.3 Poulbot observers

Definition 27.16.14. The Poulbot observer related to a ray source $M \simeq p : q : r$ is the only triangle P_j such that $P_b P_c$ is together parallel to MA and tangent to $\mathfrak{P}_A$ and circ. for B and C.

Proof. There is one and only one tangent to a parabola that contains a given direction. □

Proposition 27.16.15. *The "Poulbot observer"* observes *the LFIT. The equations of this observing* trigone *are:*

$$\mathcal{O}^* \simeq \left[\begin{array}{ccc} qr\,(f-v-w) & r\,(u-g)\,q - gr^2 & (u-h)\,rq - hq^2 \\ r\,(v-f)\,p - r^2 f & rp\,(g-w-u) & (v-h)\,rp - hp^2 \\ q\,(w-f)\,p - q^2 f & (w-g)\,qp - gp^2 & pq\,(h-u-v) \end{array}\right]$$

while the locus of the perspectors is the straight line:

$$loc_K \simeq {}^t\left[\begin{array}{c} ghp^2 - fuqr + (h-v)\,gpr + (g-w)\,hpq \\ hfq^2 - gvpr + (f-w)\,hpq + (h-u)\,fqr \\ fgr^2 - hwpq + (g-u)\,fqr + (f-v)\,gpr \end{array}\right] \qquad (27.20)$$

Proof. Write that $M \wedge A + \lambda \mathcal{L}_b$is tangent to $\mathfrak{P}_A$. Since $\mathcal{L}_b$ itself is a tangent, we obtain a first degree equation, leading to $\mathcal{O}^*$. Then we take the adjoint and the perspectivity, for all t, is easy to check. □

Proposition 27.16.16. *Assume that triangle $P_a P_b P_c$ is a Poulbot observer and use $P_a = P_a^H$ to force the metric, determining P_b^H and P_c^H. Then lines $P_b P_b^H$ and $P_c P_c^H$ are tangent to $\mathfrak{P}_A$ (or degenerate !)*

Proof. Note $p_1 : q_1 : r_1$ the coordinates of P_a and $\delta_b \doteq t_b : -1 - t_b : 1$ and $\delta_c \doteq 1 : t_c : -1 - t_c$ the directions of $P_a P_b$ and $P_a P_c$. Write the conditions h_b and h_c for lines $P_a \delta_b$ and $P_a \delta_C$ to be tangent to the required parabolas. On the other hand, the parallelogy center is $t_b : t_b t_c : 1$, and the direction of the third tangent is $-1 - t_b t_c : t_b t_c : 1$. Draw this tangent, say Δ and obtain the expressions in p_1, q_1, r_1, t_b, t_c of P_b, P_c. Obtaining those of P_b^H, P_c^H is obvious. It only remains to write the contact condition (length ≈120000) and take the Euclidean remainder modulo h_1 and then modulo h_2. This gives 0. □

27.16.4 Singular observers

Definition 27.16.17. When the A-vertex of an observer $\mathcal{O}$ belongs to the sideline BC, we say that $\mathcal{O}$ is an A-singular observer, etc (observers with two vertices on ABC sidelines are to be discarded).

Proposition 27.16.18. *Let line Δ_a be tangent to $\mathfrak{P}_A$. Note P_b the intersection of Δ_a and ϑ_B the second tangent to $\mathfrak{P}_A$ from B,* etc *for P_c. Then $P_a P_b P_c$ is an observer for any $P_a \in BC$. Conversely, any degenerate observer is obtained that way.*

Proof. Substitute $p_1 = 0$ in the perspectivity equation and assume that all the four coefficients of t are 0. Solving the system leads to triangles with two vertices on the sidelines of ABC and to:

$$\mathcal{O} \simeq \left[\begin{array}{ccc} 0 & u & u \\ q_1 & gh\,s & h-u \\ r_1 & g-u & 1/s \end{array}\right]$$

where s is a parameter. One can check $P_b \in \vartheta_B$, $P_c \in \vartheta_C$ and $P_b P_c$ tangent to $\mathfrak{P}_a$. □

27.16.5 Proof of the theorem

The ray center is $M \simeq p : q : r$ and the LFIT is described using the asymmetric parametrization (27.5). The parallelogy condition is described by the existence of α, β, γ such that:

$$P_c - P_b = \alpha(M - A), P_a - P_c = \beta(M - B), P_b - P_a = \gamma(M - C)$$

Adding member to member, one concludes that $(\alpha, \beta, \gamma) = k\,(p, q, r)$,where $k \neq 0$. Thus the observer can be written as:

$$\mathcal{O} \simeq \left[\begin{array}{ccc} 1 - r_1 - q_1 & 1 - r_1 - q_1 + rp\,k & 1 - r_1 - q_1 - qp\,k \\ q_1 & q_1 + qr\,k & q_1 + q\,(p + r)\,k \\ r_1 & -r\,(p+q)\,k + r_1 & r_1 - qr\,k \end{array}\right]$$

The perspectivity, for all t, with $\mathcal{T}_t$ leads to three equations, whose sizes are:

	length	fgh	vw	k	pqr	$q_1 r_1$
t^0	1155	3	2	1	4	2
t^1	1141	2	1	1	4	2
t^2	149	1	0	1	4	1

The elimination of q_1, r_1 leads to a 30837-sized compatibility condition, that splits into a number of factors:

length	fgh	vw	pqr	k
5	3	0	2	0
45	1	1	2	1
45	1	1	2	1
61	1	1	2	1
1101	3	1	4	1
137	2	1	2	0

- factors (f, g, h, p, q) are only mirroring implicit hypotheses.
- the three small factors, being of first degree in k, are giving three values for k and lead to the following three *singular observers*:

$$X_a X_b X_c,\; Y_a Y_b Y_c,\; Z_a Z_b Z_c \simeq$$

$$\left[\begin{array}{ccc} 0 & -ur & uq \\ \dfrac{uqr}{p} + hq & hq & hq - uq \\ -\dfrac{uqr}{p} - gr & ur - gr & -gr \end{array}\right] \left[\begin{array}{ccc} -hp & -hp - \dfrac{prv}{q} & vp - hp \\ vr & 0 & -pv \\ fr - vr & fr + \dfrac{prv}{q} & fr \end{array}\right] \left[\begin{array}{ccc} -gp & (w-g)\,p & -gp - \dfrac{wqp}{r} \\ (f-w)\,q & fq & fq + \dfrac{wqp}{r} \\ wq & -pw & 0 \end{array}\right]$$

- the biggest factor gives another value for k, and lead to the triangle $U_a U_b U_c$ dual of the trigone $X_b X_c$, $Y_c Y_a$, $Z_a Z_b$. This triangle is nothing but the Poulbot observer directed by M.
- the last factor describes the condition $fu\,qr + gv\,rp + hw\,pq = 0$, i.e. $M \in \gamma$ where γ is the circumconic whose perspector is $\mathcal{S} \underset{b}{*} \mathcal{E}$.

Proposition 27.16.19. *Assume that $M \in \gamma$ and consider the line Δ_M through $\mathcal{E}$ whose tripole is $M \underset{b}{\div} \mathcal{S}$. Then Δ_M is the locus of the perspectors of $\mathcal{T}_t$ with the Poulbot observer $U_a U_b U_c$. Moreover, this line is the locus of all the S_v^* whose associated metric observer $\mathcal{O}_V$ admits M as ray center. And we have*

$$S_v^* \simeq (u : v : w) + \mu\,(p\,(gr - hq) : q\,(hp - fr) : r\,(fq - gp))$$

$$\mathcal{O}_v \simeq \left(\begin{array}{ccc} u & u & u \\ v & v & v \\ w & w & w \end{array}\right) + \mu \left(\begin{array}{ccc} p\,(gr - hq) & -p\,(fr + hq + hr) & p\,(fq + gq + gr) \\ q\,(gr + hr + hp) & q\,(hp - fr) & -q\,(fp + fr + gp) \\ -r\,(gq + hq + gp) & r\,(fp + fq + hp) & r\,(fq - gp) \end{array}\right)$$

Proof. Parametrize with $p : q : r \simeq fu : gv/s : -hw/(1+s)$, and substitute into (27.20) and obtain

$$\Delta_M \simeq [vw,\; suw,\; -uv\,(1+s)]$$

From Proposition 27.16.10, we know that the all metrics observers whose S_H^* belongs to Δ_M share the same ray center M_Δ. The new fact is that $M_\Delta = M$ (direct computation). □

27.16.6 Reciprocal

1. *Four equations.* We start from $P_a \simeq p_1 : q_1 : r_1$, $P_b \simeq p_2 : q_2 : r_2$, $P_c \simeq p_3 : q_3 : r_3$ (not on the sidelines) and we write that $\mathcal{O} \doteq P_a P_b P_c$ observes the LFIT family. The corresponding determinant is a 3 degree polynomial in t. This gives four equations with the following degrees:

	length	p_1, q_1, r_1	p_2, q_2, r_2	p_3, q_3, r_3	f, g, h	v, w
t^0	823	1	1	1	0	2
t^1	2075	1	1	1	2	2
t^2	1404	1	1	1	1	1
t^3	286	1	1	1	0	0

2. The t^3 equation is nothing but the condition of parallelogy:

$$\text{trace}\,(\mathcal{O}) = \frac{p_1}{p_1+q_1+r_1} + \frac{q_2}{p_2+q_2+r_2} + \frac{r_3}{p_3+q_3+r_3} = 1$$

3. Three cubics. Then we solve two of these first degree equations in $p_3 q_3 r_3$, and substitute in the other two. This leads to the following three cubics:

name	*solving*	*length*	f, g, h	p, q, r	p_1, q_1, r_1	p_2, q_2, r_2
K_1	t^2, t^3	6398	2	1	3	3
K_2	t^2, t^3	7865	2	2	3	3
K_3	t^0, t^3	12382	3	3	3	3

4. Excluding. Line AC is asymptote to K_1 and K_2, but not to K_3: one occurrence of δ_{AC} must be excluded. Moreover, point $p_1\left(g q_1 + (g-f)\, r_1\right) : g\left(q_1+r_1\right)^2 : f p_1 q_1$ is on K_1 and K_2 but not on K_3.

5. It remains 7 common points to the first two cubics. Eliminating q_2, the equation in p_2, r_2 splits into four factors whose degrees are: 1,1,1,4 . This allows to identify:

   (a) Point P_1 itself is a possibility for P_2.

   (b) Point at infinity δ_B of AC belongs to the three cubics.

   (c) Point P_b^H. Indeed, once $P_1 \simeq p_1 : q_1 : r_1$ is known, it exists exactly one metric such that $P_1 = P_a^H$. This leads to:

$$\begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix} \simeq \begin{pmatrix} f\,(v+w-f)\,p_1 \\ g\,(v-f)\,q_1 + g v\, r_1 \\ h w q_1 + h\,(w-f)\,r_1 \end{pmatrix}$$

   Substituting into P_b, we obtain P_b^H. This point belongs to all three cubics.

   (d) It remains a group of four points that belongs to the three cubics. Since we already know these points, they are the Poulbot observers associated with the metric observer.

27.17 Orthojoin

orthopole of the tripolar of the isogonal conjugate. It is not clear if this concept is really useful.

Chapter 28

Quadrilaterals

28.1 Immortal glory of our ancestors

Many things were summarized in Ripert (1901). A generation later, a founding overview was given by Clawson (1919), with other notations.

Here, the transversal $\mathcal{L}_0$ is described as the tripolar of a point $P \underset{b}{\simeq} p : q : r$. In other words, $\mathcal{L}_0 = l_4 \simeq [qr, rp, pq]$.

Here	Ripert	Clawson	name
$\mathcal{L}_A, \mathcal{L}_0$	a, b, c, d	l_1, l_2, l_3, l_4	lines
ABC	ABC	$A_{23}A_{13}A_{12}$	vertices
$A'B'C'$	$A'B'C'$	A_{14}, A_{24}, A_{34}	vertices
O_j, Γ_j	O_j	C_j, $\mathcal{C}_j$	circumcirc
AA', BB', CC'	$AA' \ldots$	$n_j = A_{j4}A_{kn}$	diagonal lines
was D_A		$D_{12} = n_1 \cap n_2$	diagonal vertices
	G_n	U_n	X(2) of $l_i l_j l_k$
N_a, N_b, N_c	m_j	$B_1B_2B_3$	mid diagonal points
		G_j	proj of F on l_j
		F_j	$2O_j - F \in \Gamma_j$

refs: (#) Clawson's v. here		name here		Rip	Clawson's name	
1	Proposition 28.4.2	M_q	Miquel center	F	F	focal point
2	Exercice 28.4.8					
3	Exercice 28.4.9					
4	Proposition 28.4.2	Γ_M	Miquel circle	O	$\mathcal{C}$	circumcentric circle
5a	Corollary 28.4.3	S_n	slowness			$\mathcal{S}_n = \Gamma_M \cap \Gamma_n \setminus M_q$
5b	Corollary 28.4.4	S'_n				$\mathcal{S}'_n = 2O_j - \mathcal{S}_j$
6	Exercice 28.4.10	?				
7	ohlala					
8	Proposition 28.2.7	δ	Newton axis	δ	m	mid-diagonal line
9	gravity center					
10	Exercice 28.4.11					
11	Proposition 28.4.5		pedal line		p	pedal line
12	diametrical circles					$FG_j \perp l_j$
13	ohlala					
14	?					
15	Proposition 28.2.9	h	Steiner axis	h	o	orthocentric line
16	Proposition 28.3.2	ν_a, ν_b, ν_c	Newton circles			
17	Proposition 28.4.6		Miquel parabola			
18	Construction 28.8.8	$\mathcal{E}_j$	orthopole		V_j	
19	Proposition 28.4.7	$\mathcal{D}$	diagonal circle			$\mathcal{D}$
20			$Z'_1 Z'_2 Z'_3$			???

28.2 Lines only

Definition 28.2.1. A transversal is a moving line $\mathcal{L}_0$ that cuts the sidelines of a fixed triangle ABC in three other points. When the four lines $\mathcal{L}_0$, $\mathcal{L}_A = BC$, $\mathcal{L}_B = CA$, $\mathcal{L}_C = CA$ are assumed to play the same role, this situation is called a **quadrilateral**.

Notation 28.2.2. Descriptions are easier when using the symmetry of the situation. Therefore, we will often use the Clawson (1919) notations, where the 4 lines are $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ and the 6 vertices are $A_{jk} \doteq \mathcal{L}_j \cap \mathcal{L}_k$. When using indices $ijkn$, property $\{i, j, k, n\} = \{0, 1, 2, 3\}$ is ever assumed. When an 3-fold object depends on how indices are paired, as in $(ij)\,(k0)$, the result will be indexed by k (the one paired with 0).

Notation 28.2.3. Without any other notice, computations (barycentrics, etc.) are relative to triangle ABC, using $\mathcal{L}_0 \simeq$ tripolar $(P) \underset{b}{\simeq} [qr, rp, pq]$, and therefore breaking the symmetry. It will be convenient to introduce the quantities

$$\begin{aligned} k^3 &\doteq (p-q)\,(q-r)\,(r-p) \\ \Phi_p &\doteq a^2 qr - (S_b\, q + S_c\, r)\, p, \text{ etc} \end{aligned}$$

with the following properties:

$$\sum \Phi_p = 0 \; ; \; \sum qr\Phi_p = 2S\mathcal{L}_0 \cdot \boxed{\mathcal{M}_b} \cdot {}^t\mathcal{L}_0$$

Definition 28.2.4. A quadrilateral induces four **embedded triangles** and we denote $\mathcal{T}_j$ the triangle which is associated with the trigone "all the four lines except from $\mathcal{L}_j$".

Proposition 28.2.5. *The diagonals AA', BB', CC' define a trigone. It's dual D_a, D_b, D_c is called the diagonal triangle. Seen from $\mathcal{T}_0$, this triangle is the anticevian of P. Then $(A, A', D_b, D_c) = -1$ so that each diagonal is harmonically divided by the other two.*

Proof. One has $A' \simeq 0 : q : -r$ and $D_b \simeq -p : q : r$. And the result follows. □

Proposition 28.2.6. *Let* G_d, etc *be the barycenter of triangle* $\mathcal{T}_d$, etc. *Then*

$$G_d\,[1:1:1]\;\;;\;G_a\left[-3p^2+2\,p\,(q+r)-qr:q\,(p-r):r\,(p-q)\right]$$

The barycenter G *of these four points is also the barycenter of the three* m_j *or the barycenter of the six* A_{jk}. *And we have:*

$$G\simeq\left[(q-r)\left(3p^2-2\,p\,(q+r)+qr\right)::\right]$$

Proof. Obvious from the coordinates. ☐

Proposition 28.2.7. ***Newton axis*** *(lowbrow version). Consider the traces of the transversal* $\mathcal{L}_0$ *on the other three lines, i.e.* $A'=\mathcal{L}_0\cap\mathcal{L}_A$, etc. *Lines* AA', etc *are the so-called diagonals, while points* $N_a\doteq(A+A')/2$, etc *are the so-called mid-diagonal points. These three* N_j *are aligned on what is called the Newton axis of the quadrilateral. Its barycentrics are:*

$$\delta\simeq\operatorname{cevadiv}(\mathcal{L}_0,\mathcal{L}_b)\simeq[rq-p\,(q+r)\,,\;pr-q\,(r+p)\,,\;pq-r\,(p+q)]$$

Proof. One has $m_a\simeq q-r:+q:-r$, etc and the result follows. ☐

Proposition 28.2.8. ***Reciprocal lines.*** *Let* $\mathcal{R}_j$ *be the reciprocal of line* $\mathcal{L}_j$ *wrt triangle* $\mathcal{T}_j$. *They are parallel to each other. Moreover, their equibarycenter is the Newton axis.*

Proof. Remember that $\mathcal{R}_0$ is the line through $B+C-A'$, etc. The barycentrics of these lines are:

$$\begin{array}{ccccccc}\mathcal{R}_0&\simeq&p&,&q&,&r\\ \mathcal{R}_A&\simeq&-p&,&q-2p&,&r-2p\\ \mathcal{R}_B&\simeq&p-2q&,&-q&,&r-2q\\ \mathcal{R}_C&\simeq&p-2r&,&q-2r&,&-r\\ \delta&\simeq&p-q-r&,&q-r-p&,&r-p-q\end{array}$$

The first one comes from the very definition. The other three are obtained by the usual conjugacy methods, and adjusted so that all the $\mathcal{R}_j\wedge\mathcal{L}_b$ are equal (and not simply proportional), so that computing $\mathcal{R}_0+\mathcal{R}_A+\mathcal{R}_B+\mathcal{R}_C$ makes sense. ☐

Proposition 28.2.9. ***Steiner axis*** *(lowbrow version). The orthocenters of the four embedded triangles belong to a same line, called the Steiner axis of the quadrilateral. Its barycentrics are:*

$$h\doteq\text{Steiner\_axis}\simeq[(q-r)\,S_a,\;(r-p)\,S_b,\;(p-q)\,S_c]$$

Proof. Direct computation using

$$H_0\simeq\left[\begin{array}{c}S_b\,S_c\\ S_a\,S_c\\ S_a\,S_b\end{array}\right]\;;\;H_a\simeq\left[\begin{array}{c}S_bb^2\,rp+S_cc^2\,pq-4\,S^2p^2-S_b\,S_c\,rq\\ S_a\,\Phi_q\\ S_a\,\Phi_r\end{array}\right]$$

A more stratospherical proof will be given at Section 28.4, using the fact that the polar circle of a triangle is centered at the orthocenter (see Section 13.7). ☐

28.3 Newton stuff

Remark 28.3.1. The centers of the Newton pencil $\mathfrak{N}$ belong to the Newton line δ. The centers of the Steiner pencil $\mathfrak{N}^\perp$ belong to the Steiner line h. That is the reason why, as a cycle, h belongs to $\mathfrak{N}$ while, as a cycle δ belongs to $\mathfrak{N}^\perp$. As a rule of thumb, ever consider such a line as a line of centers and never as a radical axis (i.e. as an ordinary member of their pencil). Another method: stop being dyslexic.

Proposition 28.3.2. *The conics which are tangent to the four lines $\mathcal{L}_j$ form a linear pencil. Their orthoptic cycles (see Section 12.25) belong to a linear pencil of cycles, called the **Newton pencil** of the quadrilateral. In the $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^4\right)$ barycentric space of cycles, the matrix of this pencil is*

$$\boxed{\mathfrak{N}_b} \simeq \begin{bmatrix} 0 & (p-q)\,S_c & (p-r)\,S_b & -pS_b\,S_c \\ (q-p)\,S_c & 0 & (q-r)\,S_a & -qS_a\,S_c \\ (r-p)\,S_b & (r-q)\,S_a & 0 & -rS_a\,S_b \\ pS_b\,S_c & qS_a\,S_c & rS_a\,S_b & 0 \end{bmatrix}$$

Proof. Consider one of the other tangents to such a conic, say tripolar $(u:v:w)$. Then we have:

$$\boxed{\mathcal{C}^*} \simeq \begin{bmatrix} 0 & pv-qu & ru-pw \\ pv-qu & 0 & qw-rv \\ ru-pw & qw-rv & 0 \end{bmatrix}$$

From (12.17) we have

$$\mathfrak{O} \simeq \left(\begin{array}{c} f\,S_a \\ g\,S_b \\ h\,S_c \\ f+g+h \end{array}\right) \simeq \left(\begin{array}{c} (qw-rv)\,S_a \\ (ru-pw)\,S_b \\ (pv-qu)\,S_c \\ (qw-rv)+(ru-pw)+(pv-qu) \end{array}\right)$$

Using tripolar $(u':v':w')$ leads to $\mathfrak{O}'$. And we can see that $\boxed{\mathfrak{N}_b} \doteq \left(\mathfrak{O} \underset{6}{\wedge} \mathfrak{O}'\right)$ doesn't depend on the auxiliary tangents. □

Definition 28.3.3. Once again, the line of centers of the Newton pencil is the already defined **Newton axis** δ (see Proposition 28.4.6 for the parabolic case). On the other hand, the radical axis h of this pencil ($h \in \mathfrak{N}$!) is the already defined **Steiner axis** of the quadrilateral.

Corollary 28.3.4. *The diametral circles ν_a, ν_b, ν_c, i.e. the circles having the* $[AA']$, etc *as diameters belong to the Newton pencil.*

Proof. The set C_a^* of all the lines through A or through A' is one of the involved tangential conics. And the diametral circle is obviously it's orthoptic circle.

Nevertheless, a direct proof is possible, using

$$\nu_a, \nu_b, \nu_c \simeq \begin{bmatrix} 0 \\ +q\,S_b \\ -r\,S_c \\ q-r \end{bmatrix} \begin{bmatrix} -p\,S_a \\ 0 \\ +r\,S_c \\ r-p \end{bmatrix} \begin{bmatrix} +p\,S_a \\ -q\,S_b \\ 0 \\ p-q \end{bmatrix} \tag{28.1}$$

□

Exercise 28.3.5. Determine the 4-tangent conic such that $\Omega_{\mathcal{C}} = G$.

Exercise 28.3.6. Determine the locus of the ABC perspector of the conics tangent to the four lines

28.4 Steiner stuff

Proposition 28.4.1. *The four polar circles γ_j relative to the four embedded triangles belong to a same pencil, called the **Steiner pencil**. . This pencil is orthogonal to the Newton pencil, and contains the Newton line as radical axis. The barycentrics of these cycles are:*

$$\gamma_0 = \begin{bmatrix} S_a \\ S_b \\ S_c \\ 1 \end{bmatrix} \;;\; \gamma_A = \gamma_0 - \frac{p\,S_a}{(p-q)\,(p-r)}\,\delta, \text{ etc };\; \delta = \begin{bmatrix} p-q-r \\ q-r-p \\ r-p-q \\ 0 \end{bmatrix}$$

In the $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^4\right)$ barycentric space of cycles, the matrix of this pencil is

$$\boxed{\mathfrak{N}_b^{\perp}} \simeq \begin{bmatrix} 0 & r-p-q & p-q+r & a^2(q-r)+(c^2-b^2)p \\ * & 0 & p-q-r & b^2(r-p)+(a^2-c^2)q \\ * & * & 0 & c^2(p-q)+(b^2-a^2)r \\ * & * & * & 0 \end{bmatrix}$$

Proof. From Section 12.25, each orthoptic circle $\mathfrak{O} \in \mathfrak{N}$ is orthogonal to the polar circle of any of the embedded triangles. For γ_0, see Section 13.7. For γ_A, make a change of algebraic basis. Another method: compute η_A, the NPC of triangle $\mathcal{T}_A$ and use $\gamma_A = 2\eta_A - \Gamma_A$. Finally, the simplest method to obtain $\boxed{\mathfrak{N}_b^{\perp}}$ is the general formula $\boxed{\underset{b}{\mathcal{Q}}} \cdot \operatorname{dual}\left(\boxed{\mathfrak{N}_b}\right) \cdot \boxed{\underset{b}{\mathcal{Q}}}$, but a direct computation is also possible. □

Proposition 28.4.2. *The four circumcenters of the four embedded triangles $\mathcal{T}_j$ are on a same circle that is called their **Miquel circle**. The four circumcircles have a common point, called their **Miquel point**. Wrt any of the four triangles, this point is the isogonal conjugate of δ_∞. Moreover, the Miquel point belongs to the Miquel circle. The ABC-barycentrics of these objects are:*

$$\Gamma_M \simeq \begin{bmatrix} b^2c^2(q-r)\Phi_p \\ c^2a^2(r-p)\Phi_q \\ a^2b^2(p-q)\Phi_r \\ -8\,S^2k^3 \end{bmatrix} \; ; \; M_q \simeq \begin{bmatrix} \dfrac{a^2}{(q-r)} \\ \dfrac{b^2}{(r-p)} \\ \dfrac{c^2}{(p-q)} \end{bmatrix}$$

Proof. Using wedge, Veronese and wedge3, one obtains the ABC-barycentrics of the four circumcircles and the four centers:

$$\Gamma_0 \simeq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \; ; \; \Gamma_A \simeq \begin{bmatrix} 0 \\ \dfrac{pc^2}{p-q} \\ \dfrac{pb^2}{p-r} \\ 1 \end{bmatrix} \; ; \; \Gamma_B \simeq \begin{bmatrix} \dfrac{qc^2}{q-p} \\ 0 \\ \dfrac{qa^2}{q-r} \\ 1 \end{bmatrix} \; ; \; \Gamma_C \simeq \begin{bmatrix} \dfrac{rb^2}{r-p} \\ \dfrac{ra^2}{r-q} \\ 0 \\ 1 \end{bmatrix}$$

$$O_0 \simeq \begin{bmatrix} a^2S_a \\ b^2S_b \\ c^2S_c \end{bmatrix} \; ; \; O_A \simeq \begin{bmatrix} 8p(q-p)S^2 - a^2\Phi_r - c^2\Phi_p \\ b^2\Phi_r \\ c^2\Phi_q \end{bmatrix}, \text{ etc}$$

And, some wedge, Veronese and wedge3 later, the result is obtained. Other proofs are possible (Ehrmann, 2004). □

Corollary 28.4.3. *Point $\mathcal{S}_n$, the other intersection of Γ_n and Γ_M, belongs to the three lines O_iA_{jk} (remember the convention $\{n,i,j,k\} = \{0,1,2,3\}$).*

$$\mathcal{S}_0 \simeq \frac{a^2}{\Phi_p} : \frac{b^2}{\Phi_q} : \frac{c^2}{\Phi_r}$$

Proof. Compute $\mathcal{S}_0$ as the intersection of O_1A_{23} and O_2A_{13}. By symmetry, $\mathcal{S}_0 \in O_3A_{12}$. Its isogonal conjugate wrt ABC is at infinity. Conclude by checking that $Ver(\mathcal{S}_0) \cdot \Gamma_M = 0$. □

Corollary 28.4.4. *Define $\mathcal{S}'_n = 2O_n - \mathcal{S}_n$ (on Γ_n). Then $\mathcal{S}'_nA_{jk}$ is tangent to Γ_i at A_{jk}. As an example, $\mathcal{S}'_aA$ is tangent to Γ_0, $\mathcal{S}'_aB'$ is tangent to $\Gamma_c = (A'B'C)$ at B' and $\mathcal{S}'_aC'$ is tangent to $\Gamma_b = (A'BC')$ at C'.*

$$\mathcal{S}'_0 \simeq \frac{a^2}{p(q-r)} : \frac{b^2}{q(r-p)} : \frac{c^2}{r(p-q)} \; ; \; \mathcal{S}'_a \simeq \frac{-a^2(p^2+qr) + 2\,S_b\,pq + 2\,S_c\,pr}{q-r} : pb^2 : -pc^2$$

Proposition 28.4.5. *The Miquel point has the same Simson line with respect to all four embedded triangles $\mathcal{T}_j$. This line is also called the* ***Wallace line****. or the* ***pedal line*** *of the quadrilateral. By Proposition 10.2.1, the image of this line by the homothety $\mathfrak{h}(M_q, 2)$ contains the orthocenters H_j of the triangles $\mathcal{T}_j$ and is, therefore, the Steiner axis h of the quadrilateral.*

Proof. A line is defined by two points, and three projections are on each Simson line. □

Proposition 28.4.6. *The* ***parabola*** *tangent to the four $\mathcal{L}_j$ has the Miquel point M_q as focus and the Steiner axis has directrix. The pedal line is a tangent.*

Proof. Well known property ! □

Proposition 28.4.7. ***Diagonal triangle****. The diagonal triangle $\mathcal{T}_{dia}$ is the dual of the diagonal trigone AA', BB', CC'. This triangle is the anticevian wrt $\mathcal{T}_0$ of $P =$ tripolar$(\mathcal{L}_0)$ and is also the diagonal triangle of the quadrangle of the four contact points with the inscribed parabola.*

Its circumcircle $\mathcal{D}$ belongs to the Steiner pencil and one has

$$\mathcal{D} = \gamma_0 - \frac{S_a p^2 + S_b q^2 + S_c r^2}{(q + r - p)(r + p - q)(p + q - r)}\, \delta$$

Proof. Straightforward computation. □

Exercise 28.4.8. Clawson (1919, prop. 2) Consider the 3 circles $\vartheta_A \doteq (AA'M_q)$. Call $\vartheta_{K;j}$ the point where ϑ_K cuts again line $\mathcal{L}_j$. There are 12 of such points. Each of them characterizes a tangent to one circle Γ_j at one of its eponymous points. More precisely, when $M \in \Gamma_j$, line $M\vartheta_{M,j}$ is tangent to circle Γ_j at M. Otherwise, line $M'\vartheta_{M',j}$ is tangent to circle Γ_j at M'.

Exercise 28.4.9. Clawson (1919, prop. 3) Each of the 6 circles through A_{ij}, $(A_{ik} + A_{in})/2$, $(A_{jk} + A_{jn})/2$ goes through M_q.

Exercise 28.4.10. Each of the 6 points $Q_{ij} = med[A_{ik}, A_{jk}] \cap med[A_{in}, A_{jn}]$ belongs to Γ_M. Examples: $Q_{bc} = med[A_{ba}, A_{ca}] \cap med[A_{b0}, A_{c0}] = med[B, C] \cap med[B', C']$, $Q_{0a} = med[BC'] \cap med[B'C]$.

Exercise 28.4.11. The line joining $(B + A')/2$ and $(B' + A)/2$ cuts the line joining $(B + A)/2$ and $(B' + A')/2$ on the Newton axis.

28.5 Lubin cookbook for quadrilaterals

Remark 28.5.1. This section collects all the quadrilateral formulas of the Chapter. Some of them will only be proven later. Here, they are organized according to the status given to the fourth line, that is, transversal treatment versus symmetric treatment.

Remark 28.5.2. How to describe the transversal $\mathcal{L}_0 \underset{b}{\simeq} [qr, rp, pq]$ when using the Lubin-1 formalism?

1. Choice 1. Using the turns κ, ν where $\mathcal{L}_0$ cuts Γ. These turns are visible... or not, so that $|\kappa| = 1 = |\nu|$ is not assumed.
2. Choice 2. Using $\mathcal{L}_0 \underset{z}{\simeq} [1, m, n]$, i.e. $\kappa\nu = n$, $\kappa + \nu = -m$. With the rules: conjugate$(n) = 1/n$; conjugate$(m) = m/n$
   This was the package `quadlat` . Not so convincing
3. Using $\mathcal{L}_0 \underset{z}{\simeq} [\eta, \mathrm{t}, \mathrm{y}]$. At least, this doesn't break the informal rules about degrees ! This is package `quadlat2`

28.5.1 Separate treatment of the transversal

Fact 28.5.3. *Barycentrics v/s Lubin and conversely*

1. $\mathcal{L}_0 \underset{b}{\simeq} \left[p = \dfrac{1}{\eta\alpha + \mathrm{t} + \dfrac{\mathrm{y}}{\alpha}}, q = \dfrac{1}{\eta\beta + \mathrm{t} + \dfrac{\mathrm{y}}{\beta}}, r = \dfrac{1}{\eta\gamma + \mathrm{t} + \dfrac{\mathrm{y}}{\gamma}} \right]$

2. $\begin{cases} \eta = & qr\,\alpha\,(\beta-\gamma)+rp\,\beta\,(\gamma-\alpha)+pq\,\gamma\,(\alpha-\beta) \\ \mathrm{t} = & qr\,\alpha\left(\gamma^2-\beta^2\right)+rp\,\beta\left(\alpha^2-\gamma^2\right)+pq\,\gamma\left(\beta^2-\alpha^2\right) \\ \mathrm{y} = & \alpha\beta\gamma\,(qr\,(\beta-\gamma)+rp\,(\gamma-\alpha)+pq\,(\alpha-\beta)) \end{cases}$

Fact 28.5.4. *Basic objects*

1. $A \simeq \alpha : 1 : 1/\alpha$, etc, *together with* $A' \simeq \mathcal{L}_0 \wedge \mathcal{L}_a$, etc

2. *Newton* $\delta \underset{1}{\simeq} {}^t\begin{pmatrix} 2\,n^2-2\,n\,s_2-2m\,s_3 \\ -n^2s_1+n\,s_1\,s_2-\left(m^2-n\right)s_3-s_2s_3 \\ -2\,m\,n\,s_3-2\,n\,s_1\,s_3+2\,s_3^2 \end{pmatrix}$

3. *Steiner* $h \underset{1}{\simeq} {}^t\begin{pmatrix} n^2-n\,s_2-m\,s_3 \\ -n^2\,s_1-m\,ns_2+m\,s_1\,s_3+s_2\,s_3 \\ m\,ns_3+n\,s_1\,s_3-s_3^2 \end{pmatrix}$

4. *Miquel* $M_{\mathrm{q}} \underset{1}{\simeq} \begin{pmatrix} \dfrac{s_2\mathrm{y}\eta+s_3\mathrm{t}\eta-\mathrm{y}^2}{-s_3\,\eta^2+(s_1\eta+\mathrm{t})\,\mathrm{y}} \\ 1 \\ \dfrac{-s_3\,\eta^2+(s_1\eta+\mathrm{t})\,\mathrm{y}}{s_2\mathrm{y}\eta+s_3\mathrm{t}\eta-\mathrm{y}^2} \end{pmatrix}$; $\Gamma_{\mathrm{q}} \underset{1}{\simeq} \begin{pmatrix} \left(s_1\mathrm{y}\eta-s_3\,\eta^2+\mathrm{ty}\right)\mathrm{y} \\ 0 \\ -s_3\eta\left(s_2\mathrm{y}\eta+s_3\mathrm{t}\eta-\mathrm{y}^2\right) \\ \mathrm{y}^3+s_1s_3\mathrm{y}\,\eta^2-s_3^2\eta^3-s_2\,\mathrm{y}^2\eta \end{pmatrix}$

5. *Centers of circumcircles:* $O_0 \simeq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; $O_a \underset{1}{\simeq} \begin{pmatrix} \alpha\,\mathrm{y}^2-\alpha s_2\mathrm{y}\eta-\alpha s_3\mathrm{t}\eta \\ \mathrm{y}^2+\alpha^2\mathrm{y}\eta-\alpha s_1\mathrm{y}\eta+s_3\alpha\,\eta^2 \\ -\mathrm{yt}-s_1\mathrm{y}\eta+s_3\,\eta^2 \end{pmatrix}$

6. *Orthocenters:* $H_0 \underset{1}{\simeq} \begin{pmatrix} s_1\,s_3 \\ s_3 \\ s_2 \end{pmatrix}$; $H_a \underset{1}{\simeq} \begin{pmatrix} \alpha s_1\,\mathrm{y}^2+(\alpha s_2-s_3)\,\mathrm{yt}+\alpha^3 s_3\,\eta^2 \\ \alpha\,(\mathrm{y}-\alpha\gamma\eta)\,(\mathrm{y}-\alpha\beta\eta) \\ \mathrm{y}^2+(\alpha s_2-s_3)\,\mathrm{t}\eta+\alpha^2 s_2\,\eta^2 \end{pmatrix}$

7. *Mean orthocenter* $H_m = \frac{1}{4}\sum H_j \underset{1}{\simeq}$

$$\begin{bmatrix} 2s_1\,\mathrm{y}^3-s_2\,(s_1\eta-\mathrm{t})\,\mathrm{y}^2+s_3\eta\left(s_1^2\eta-s_1\mathrm{t}-s_2\eta\right)\mathrm{y}-s_1\,s_3^2\eta^3 \\ 2\;(\mathrm{y}-\eta\beta\gamma)\,(\mathrm{y}-\alpha\gamma\eta)\,(\mathrm{y}-\alpha\beta\eta) \\ s_2/s_3\,\mathrm{y}^3-2s_2\,s_3\eta^3+s_1\,(s_2\mathrm{y}-s_3\mathrm{t})\,\eta^2+\left(s_1\,\mathrm{y}^2-s_2^2/s_3\,\mathrm{y}^2+s_2\mathrm{ty}\right)\eta \end{bmatrix}$$

Fact 28.5.5. *In the* $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^4\right)$ *space of cycles, the Newton pencil is described by the anti-symmetric matrix:* $\boxed{\mathfrak{N}} = \boxed{E_{jk}}$ *while the Steiner pencil is described by* $\boxed{\mathfrak{N}^\perp} \doteq \boxed{\underset{z}{\mathcal{Q}}}\cdot\mathrm{dual}\left(\boxed{\mathfrak{N}}\right)\cdot\boxed{\underset{z}{\mathcal{Q}}} = \boxed{S_{jk}}$.

$$\left[\begin{array}{c|c|c} S_{12} & -E_{12} & -2s_3\left(-s_3\,\eta^2+(s_1\eta+\mathrm{t})\,\mathrm{y}\right) \\ S_{13} & +E_{24} & \left(s_2\,\eta^2+\mathrm{t}^2-\mathrm{y}\eta\right)s_3-s_1\mathrm{y}\,(s_2\eta-\mathrm{y}) \\ S_{14} & +E_{14} & s_1s_3\,\mathrm{t}^2+\mathrm{y}\,(s_1s_2+s_3)\,\mathrm{t}+s_1^2\mathrm{y}^2-\eta^2 s_3^2 \\ S_{23} & +E_{23} & (-2s_2\mathrm{y}-2s_3\mathrm{t})\,\eta+2\mathrm{y}^2 \\ S_{24} & +E_{13} & (-2s_1\,s_3\eta+2s_2\mathrm{y})\,\mathrm{t}-2s_2s_3\,\eta^2+2s_1\,\mathrm{y}^2 \\ S_{34} & -E_{34} & -s_2^2\eta^2-\mathrm{t}\,(s_1\eta+\mathrm{t})\,s_2-s_3\mathrm{t}\eta+\mathrm{y}^2 \end{array}\right] \tag{28.2}$$

Proposition 28.5.6. *When using the Lubin-1 representation, the orthopole transform is:*

$$\Delta=[1,\,m,\,n]\mapsto\mathcal{E}_\Delta \underset{1}{\simeq} \frac{1}{2}\left(\begin{pmatrix} \dfrac{s_3}{n} \\ 1 \\ \dfrac{n}{s_3} \end{pmatrix}+\begin{pmatrix} -m \\ 1 \\ -\dfrac{m}{n} \end{pmatrix}+\begin{pmatrix} s_1 \\ 1 \\ \dfrac{s_2}{s_3} \end{pmatrix}-\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)=\frac{1}{2}\left(\mathcal{S}'+\sigma_\Delta\,(O)\right)+\overrightarrow{ON}$$

Here $\mathcal{S}' \doteq \dfrac{s_3}{n} : 1 : \dfrac{n}{s_3}$ *is the isogonal conjugate of the direction of* Δ *(and the antipode of* $\mathcal{S}$ *in the circumcenter) while* σ_Δ *is the symmetry wrt line* Δ*. Finally, N is X(5), the Euler center.*

Proof. Direct computation is easy. One can prefer a change of basis applied to (28.5). □

28.5.2 From the past: another way of doing

Definition 28.5.7. The four c_j were defined as:

$$\begin{cases} c_3 & \doteq \alpha^2\left(\gamma-\beta\right)qr+\beta^2\left(\alpha-\gamma\right)rp+\gamma^2\left(\beta-\alpha\right)pq \\ c_2 & \doteq \alpha\,\left(\gamma-\beta\right)qr+\beta\,\left(\alpha-\gamma\right)rp+\gamma\,\left(\beta-\alpha\right)pq \\ c_1 & \doteq \left(\gamma-\beta\right)qr+\left(\alpha-\gamma\right)rp+\left(\beta-\alpha\right)pq \\ c_0 & \doteq \frac{1}{\alpha}\,\left(\gamma-\beta\right)qr+\frac{1}{\beta}\,\left(\alpha-\gamma\right)rp+\frac{1}{\gamma}\,\left(\beta-\alpha\right)pq \end{cases}$$

where $qr:rp:pq$ are the barycentrics of tripolar $(\mathcal{L}_0)$. See Theorem 3.5.5 for more details.

Fact 28.5.8. *Since the c_k were used '"homogeneously", one can also use*

$$\begin{cases} c_3 & \simeq \sigma_1\sigma_3\,\eta+\sigma_3\mathrm{t} \\ c_2 & \simeq \sigma_3\eta \\ c_1 & \simeq \mathrm{y} \\ c_0 & \simeq \left(\sigma_2/\sigma_3\right)\,\mathrm{y}+\mathrm{t} \end{cases}$$

Their conjugates are obtained as : $\overline{c_k}=c_{3-k}/\sigma_3$ while

$$c_2\,\frac{\sigma_1}{\sigma_3}+c_0 \quad = \quad c_3\,\frac{1}{\sigma_3}+c_1\,\frac{\sigma_2}{\sigma_3}\in i\mathbb{R}$$

Lemma 28.5.9. *In the Lubin frame, the Newton line δ, the Miquel point M_{q} and the Clawson-Schmidt homography Ψ of the quadrilateral are :*

$$\text{Newton}\underset{z}{\simeq}\left[2\,c_2,\ -s_2\,c_1-c_3,\ 2\,s_3\,c_1\right]$$

$$M_{\mathrm{q}}\underset{z}{\simeq}\left(\begin{array}{c} c_2/c_1 \\ 1 \\ c_1/c_2 \end{array}\right)\ ;\ \Psi\left(\begin{array}{c} \mathbf{Z} \\ \mathbf{T} \\ \overline{\mathcal{Z}} \end{array}\right)\underset{z}{\simeq}\left[\begin{array}{c} \dfrac{\mathbf{Z}c_2-\mathbf{T}c_3}{\mathbf{Z}c_1-\mathbf{T}c_2} \\ 1 \\ \dfrac{c_1\overline{\mathcal{Z}}-c_0\mathbf{T}}{c_2\overline{\mathcal{Z}}-c_1\mathbf{T}} \end{array}\right]$$

28.5.3 Symmetric treatment of the four lines

Definition 28.5.10. As stated in Musslman (1937), one can revert some paradigms and start from the Miquel parabola. In the Morley space $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$, this leads to:

parabola focus (Miquel) $M_{\mathrm{q}}\simeq 0:1:0$; directrix (Steiner) $h\simeq[1,-2,1]$

$$\boxed{\mathcal{C}}\simeq\left[\begin{array}{ccc} 1 & -2 & -1 \\ -2 & 4 & -2 \\ -1 & -2 & 1 \end{array}\right]\ ;\ \boxed{\mathcal{C}^*}\simeq\left[\begin{array}{ccc} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{array}\right]$$

generic point_on_$\mathcal{C}$ $M\simeq-\dfrac{\left(y-\boldsymbol{i}\right)^2}{2}:1:-\dfrac{\left(y+\boldsymbol{i}\right)^2}{2}\simeq\dfrac{2}{\left(\tau-1\right)^2}:1:\dfrac{2\tau^2}{\left(\tau-1\right)^2}$

generic_line_on_$\mathcal{C}^*$ $\mathcal{L}\left(\tau\right)\simeq\left[\tau\left(\tau-1\right),2\tau,-\tau+1\right]$
where y is real –the M ordinate– and τ is a turn –the conjugate of the $\mathcal{L}\left(\tau\right)$ clinant.

Definition 28.5.11. Symetrical objects are defined using

3 items $s_1=\sum_3\tau_j$, $s_2\doteq\sum_3\tau_j\tau_k$, $s_3\doteq\tau_1\tau_2\tau_3$, $\pi_3\doteq\left(1-\tau_1\right)\left(1-\tau_2\right)\left(1-\tau_3\right)$
Used to describe the objects where $\mathcal{L}_4$ is **the** transversal, and the other three are only ordinary lines

4 items $q_1\doteq\sum_4\tau_j$, $q_2\doteq\sum_6\tau_j\tau_k$, $q_3\doteq\sum_4\tau_j\tau_k\tau_m$, $q_4\doteq\tau_1\tau_2\tau_3\tau_4$
$\pi_4\doteq\left(1-\tau_1\right)\left(1-\tau_2\right)\left(1-\tau_3\right)\left(1-\tau_4\right)$

conjugates $s_1 \mapsto s_2/s_3, s_2 \mapsto = \dfrac{s_1}{s_3}, s_3 = \dfrac{1}{s_3}, vdm = -\dfrac{vdm}{s_3^2}, s_4 = \dfrac{s_4}{s_3^2}$
$\pi_3 \mapsto -\pi_3 \div \tau_1\tau_2\tau_3 \; ; \; \pi_4 \mapsto \pi_4 \div \tau_1\tau_2\tau_3\tau_4$

birap When dealing with the four pedal LFIT of quadrilateral, many objects are appearing in packs of four, most of the time being inscribed in a cycle, so their cross-ratio is real.

Fact 28.5.12. *Basic objects (same order as 27.5.4)*

1. *vertices* $A_{jk} \simeq 2 : (\tau_j - 1)(\tau_k - 1) : 2\tau_j\tau_k$ *($j \neq k$ is ever assumed)*
2. *Newton* $\delta \simeq \left[1, \dfrac{2q_4 - q_3 + q_1 - 2}{\pi_4}, -1\right]$
3. *Steiner* $h \simeq [1, -2, 1]$*, directrix of the parabola (by definition !)*
4. *Miquel* $M_q \simeq 0 : 1 : 0$*, focus of the parabola (by definition !)*
   Miq circle $\Gamma_q \simeq -2q_4 : 0 : -2 : \pi_4$ *; circle center* $O_q \simeq 2 : \pi_4 : 2q_4$
5. *Circumcircles and centers* Γ_j, O_j
   $\Gamma_4 \simeq 2s_3 : 0 : -2 : \pi_3 \; ; \; O_4 \simeq 2 : \pi_3 : -2s_3 \; ; \; \mathrm{birap}\,[O_j] = \mathrm{birap}\,[\tau_j]$
6. *Orthocenters* H_j
   $H_4 \simeq 2(1 - s_1) : \pi_3 : 2(s_2 - s_3) \; ; \; \mathrm{birap}\,[H_j] = \mathrm{birap}\,[\tau_j^2]$
7. *Mean orthocenter* $H_m = \frac{1}{4}\sum H_j \underset{1}{\simeq} \begin{bmatrix} q_2 - 2q_1 + 2 \\ \pi_4 \\ 2q_4 - 2q_3 + q_2 \end{bmatrix}$

Fact 28.5.13. *Newton and Steiner pencils ; other properties (See Musslman, 1937)*

1. *Newton* $\boxed{\mathfrak{N}} \simeq \begin{bmatrix} 0 & \pi_4 & 2\pi_4 & 4 - 2q_1 \\ -\pi_4 & 0 & \pi_4 \int & 2 - q_1 + q_3 - 2q_4 \\ -2\pi_4 & -\pi_4 & 0 & 2q_3 - 4q_4 \\ 2q_1 - 4 & q_1 - q_3 + 2q_4 - 2 & 4q_4 - 2q_3 & 0 \end{bmatrix}$
   Steiner $\boxed{\mathfrak{N}^\perp} \simeq \begin{bmatrix} 0 & \pi_4 & q_1 - q_3 + 2q_4 - 2 & 2q_3 - 4q_4 \\ -\pi_4 & 0 & -\pi_4 & 2\pi_4 \\ 2 - q_1 + q_3 - 2q_4 & \pi_4 & 0 & 2q_1 - 4 \\ -2q_3 + 4q_4 & -2\pi_4 & 4 - 2q_1 & 0 \end{bmatrix}$
2. *Polar circles: member of SteinerPen, center* H_j
   $$\mathrm{cirpol}_4 \simeq 2(s_3 - s_2) : -2(s_2 s_1 - 2s_3 s_1 - 2s_2 + 3s_3)/\pi_3 : 2(s_1 - 1) : \pi_3$$
3. *NPC* $\mathrm{npc}_4 \simeq \pi_3(2s_3 - s_2) : (2s_3 - s_2)s_1 + 2s_2 - 3s_3 : \pi_3(s_1 - 2) ; \pi_3^2$.
4. h_j $h_4 \simeq 2(1 - s_1) : \pi_4 : 2\tau_4(s_3 - s_2)$ *is the orthocenter of* $O_1O_2O_3$. $\mathrm{birap}\,[h_j] = \mathrm{birap}\,[\tau_j]$.
5. *pt52* $2 - q_1 : \pi_4 : 2q_4 - q_3$. *"Clawson center", midpoint of* $[O_j, h_j]$*, and of* $[O_q, \mathrm{Hervey}]$
6. *Hervey* $2(1 - q_1) : \pi_4 : 2(q_4 - q_3)$. *Perpendicular bisectors of the* $[O_jH_j]$ *are crossing here. This is also the center of the circle through the* h_j $(\rho^2 = 4q_4/\pi_4^2)$
7. *yptR* $2 + q_2 - 2q_1 : \pi_4 : 2q_4 + q_2 - 2q_3$. *"Morley center". Perpendiculars from* $(O_j + H_j)/2$ *to* $\mathcal{L}_j$ *are crossing there. Isobarycenter of the* H_j*. On the Steiner line.*
8. *Slowness centers* $\mathcal{S}_4 \simeq 2(1 + \tau_4) : \pi_3 : -2(1 + \tau_4)s_3 \div \tau_4$.
   Each of them on Γ_j *while all of them on* Γ_q ; $\mathrm{birap}\,[\mathcal{S}_j] = \mathrm{birap}\,[\tau_j^2]$.
9. *Orthopoles (on the Steiner line)* $\mathcal{E}_4 \simeq \tau_4^3 + (s_1 - 2)\tau_4^2 + (s_2 + 2 - 2s_1)\tau_4 - s_3 : \tau_4\pi_4 : -\tau_4^3 + (s_1 - 2s_2 + 2s_3)\tau_4^2 + (s_2 - 2s_3)\tau_4 + s_3$
   And we have $\mathrm{birap}\,[\mathcal{E}_j] = \mathrm{birap}\,[\tau_j^2] \times (\tau_1\tau_3 + \tau_2\tau_4)^2 \div (\tau_1\tau_4 + \tau_2\tau_3)^2$.

10. vanRees $\pi_4\left(\mathbf{Z}^2\overline{\mathcal{Z}}-\mathbf{Z}\,\overline{\mathcal{Z}}^2\right)+4\left(\overline{\mathcal{Z}}-q_4\,\mathbf{Z}\right)\mathbf{T}^2+(2q_1-2q_3+4q_4-4)\,\mathbf{Z}\overline{\mathcal{Z}}\,\mathbf{T}$

Exercise 28.5.14. Prove the following assertions – source= Musslman (1937)

1. Line $\mathcal{L}_j$ is the locus of $2\div(\tau_1-1)(\widehat{\tau}-1)$ where $\widehat{\tau}$ is a turn.
2. Circle Γ_4 is the locus of $2(1-\widehat{\tau})\div\pi_3=2(1-\widehat{\tau})(1-\tau_4)\div\pi_4$
3. Circle $\Gamma_{\rm q}$ is the locus of $2(1-\widehat{\tau})\div\pi_4$

28.6 Simson lines (using barycentrics)

Proposition 28.6.1. ***Simson line.*** *The pedal vertices of a point U are collinear if and only if the point is on the circumcircle or on the line at infinity. When it exists, this line is called the Simson line of U. When $U\in\mathcal{L}_b$, $Simson\,(U)=\mathcal{L}_b$. When U belongs to the circumcircle of ABC, the barycentrics of it's Simson line are:*

$$U_t\doteq a^2:\frac{b^2}{t}:\frac{-c^2}{1+t}\mapsto Simson\,(U)\simeq\left[\frac{1}{a^2t+S_c},\frac{-t}{S_c\,t+b^2},\frac{1+t}{S_b\,t-S_a}\right] \tag{28.3}$$

Proof. Collinearity condition is the same as for the Steiner line. Everything else is either the result of straightforward computations, or obtained from the Steiner line equation (10.2), right multiplied by $\mathfrak{h}\,(U,2)$ of (7.29). □

Proposition 28.6.2. *When U is on the circumcircle, the two intersections of $Simson\,(U)$ and the nine points circle γ are (i) the midpoint M of $[U,X_4]$ (ii) the intersection L of $Simson\,(U)$ and $Simson\,(U')$ where U' is the Γ-antipode of U.*

Proof. By definition, $Simson\,(U)$ is obtained from $Steiner\,(U)$ by the point-transformation $\mathfrak{h}\,(U,1/2)$. Since $X_4\in Steiner\,(U)$, point M is on $Simson\,(U)$. Using now $h\,(X_4,1/2)$, $U\in\Gamma$ becomes $M\in\gamma$ and (i) is proved. Therefore midpoint M' of $U'X_4$ is the γ-antipode of M, while $LM\perp LM'$. □

Proposition 28.6.3. *When $\Delta\simeq[p,q,r]$ is a Simson line, then we have:*

$$p\left(q^2+r^2\right)S_a+q\left(p^2+r^2\right)S_b+r\left(p^2+q^2\right)S_c-2\,S_\omega\,pqr=0 \tag{28.4}$$

In other words, tripole (Δ) *belongs to the Simson cubic K010.*

Proof. See Section 22.5.3. □

28.7 The Steiner deltoid (using Lubin-1)

Proposition 28.7.1. *When τ is a turn, its Simson line wrt triangle ABC is*

$$\Delta\,(\tau)\simeq[2\,\tau^2,\ s_3+\tau\,s_2-\tau^2s_1-\tau^3,-2\,s_3\,\tau]$$

The envelope of all these lines is the so-called Steiner's deltoid while the contact point of $\Delta\,(\tau)$ with its envelope is $\delta\,(\tau)=\frac{1}{2}\,s_1+\tau+\dfrac{s_3}{2}\,\dfrac{1}{\tau^2}$. This deltoid is a bicircular quartic, whose implicit equation (using coordinates $\mathbf{Z}=\frac{1}{2}\,s_1+z$, etc*) is:*

$$16\,\mathbf{Z}^2\overline{\mathcal{Z}}^2-32\left(\frac{1}{s_3}\,\mathbf{Z}^3+s_3\,\overline{\mathcal{Z}}^3\right)\mathbf{T}+72\,\mathbf{Z}\overline{\mathcal{Z}}\,\mathbf{T}^2-27\,\mathbf{T}^4=0$$

Proof. The contact point comes from $\delta\,(\tau)=\Delta\,(\tau)\wedge\frac{\partial}{\partial\tau}\Delta\,(\tau)$. □

Proposition 28.7.2. *This deltoid is the roulette created by a point on the circumference of a circle $\rho=1/2$ as it rolls without slipping along the inside of the fixed circle $\rho=3/2$ centered at X(5), the Euler's point. Cups are at $\rho=3/2$, innermost points are at $\rho=1/2$.*

Proof. Let Θ be the fixed circle and E_x its center, while θ is the moving circle and X its center. The locus of contact is point $Y = (3X - E_x)/2$. Let Y_Θ and $Y_\theta$the material points of both circles that are at place Y at the moment of contact. As ever, speed of Y_Θ is zero. But, by composition of movements, speed of Y_θ vanishes at the moment of contact, proving the "without slipping" assertion. □

Proposition 28.7.3. *Any line $\Delta(\tau)$ tangent to the deltoid contains the following points:*

$$\delta_\tau \doteq \frac{1}{2}s_1 + \tau + \frac{1}{2}\frac{s_3}{\tau^2} \; ; \; m_\tau, n_\tau \doteq \frac{1}{2}s_1 \pm \sqrt{\frac{s_3}{\tau}} + \frac{1}{2}\tau \; ; \; h_\tau \doteq \frac{1}{2}s_1 + \frac{1}{2}\tau \; ; \; k_\tau \doteq \frac{1}{2}s_1 - \frac{1}{2}\frac{s_3}{\tau^2}$$

Points δ, δ, m, n are the four points in common with the curve (δ is the contact point, m, n are the so called extremities of the tangent). Point $h_\tau, k_\tau$are the two points in common with the inscribed circle. Moreover h_τ is both the middle of $[m_\tau, n_\tau]$ and of $[\delta_\tau, k_\tau]$.

Proof. Obvious computations. □

Proposition 28.7.4. *Use index $j = 9$ to save the original triangle as $A_9B_9C_9$. Consider four turns τ_j ($j = 0, a, b, c$) and the corresponding Simson lines Δ_j. Note $A \cdots C'$ their six intersections and use*

$$q_1 = \sum_4 \tau_a, \; q_2 = \sum_6 \tau_a\tau_b, \; q_3 = \sum_4 \tau_a\tau_b\tau_c, \; q_4 = \tau_a\tau_b\tau_c\tau_0$$

Let O_j, H_j, E_j be the circumcenters, orthocenters and NPC of triangles $\mathcal{T}_j$. One has:

$$O_j = \frac{1}{2}(s_1 + q_1) - \frac{1}{2}\tau_j \; ; \; H_j = \frac{1}{2}s_1 + s_3\,(q_1 - \tau_j)\,\tau_j \div (2q_4)$$

while the perpendicular bisector of $[O_jH_j]$ and the perpendicular from $E_j = (O_j + H_j)/2$ to Δ_j are:

$$\begin{aligned} med_j &\simeq \left[2\,\tau_j^2 - 2\,\tau_j q_1 + 2\,q_2,\; -\tau_j^2 s_1 + (q_1\,s_1 - s_2)\,\tau_j + s_2\,q_1 - s_1\,q_2,\; 2\,\tau_j s_3 - 2\,s_3\,q_1\right] \\ perp_j &\simeq \left[4\,q_4\,\tau_j,\; -(s_3\,q_2 + 2\,s_1\,q_4)\,\tau_j - (q_2 + 2\,s_2)\,q_4,\; 4\,s_3\,q_4\right] \end{aligned}$$

Proof. Use the cookbook, or write down the six intersections $A = \frac{1}{2}\left(s_1 + \tau_b + \tau_c + \dfrac{s_3}{\tau_b\tau_c}\right)$, etc and compute directly. □

Theorem 28.7.5. Morley (1903) *The four lines med_j concur at $E_9 = \frac{1}{2}s_1$, i.e. at the center of the deltoid, while the four lines $perp_j$ concur at $H_m = \frac{1}{2}s_1 + s_3\,q_2/(4q_4) = \frac{1}{4}\sum H_j$, on the Steiner axis.*

Proof. As ever, a theorem is a key result, not necessarily something difficult to prove. Indeed, computations are easy ! □

Construction 28.7.6. *Construct the deltoid tangent to four given lines. Obtain the O_j and H_j and then E_x, H_m from the Morley theorem. Let ω be the center of the Miquel circle and ρ its radius. Consider the points $h_j \doteq E_x + \omega - O_j$. For each j, h_j belongs to $\mathcal{L}_j$, while the four h_j are on the circle (E_x, ρ). Define the turns $\tau_j = (\omega - O_j)/\rho = (h_j - E_x)/\rho$ and compute the q_k. Then*

$$\alpha \mapsto \delta_\alpha \doteq E_x + 2\,\rho\,\alpha + \psi\,\frac{\rho}{\alpha^2} \qquad \text{where } \psi = \frac{1}{\rho}\,\overrightarrow{E_xH_m}\,\frac{2\,q_4}{q_2}$$

is the parametric equation of the deltoid tangent to the four lines. The contact points are obtained at $\alpha = \tau_j$

Proof. One obtains the following values

$$W = \sqrt{\frac{(m\,n + n\,s_1 - s_3)\left(ms_3 + n\,s_2 - n^2\right)}{n^2 s_3}}$$

$$\rho = \frac{i\,\sqrt{n}\,n\,s_3\,W}{n^3 - n^2 s_2 + n\,s_1\,s_3 - s_3^2} \; ; \; \psi = i\,\frac{m\,s_3 + n\,s_2 - n^2}{\sqrt{n}\,W}$$

□

Remark 28.7.7. Points h_j are also the orthocenters of the four triangles $O_i O_k O_n$, with affixes:

$$h_0 = \frac{(ns_1 - s_3)\left(ms_3 - n^2 + ns_2\right)}{(\alpha\,\gamma - n)\,(\alpha\,\beta - n)\,(\beta\,\gamma - n)} \;;\; h_a = \frac{n\,(s_1 - \alpha)\left(ms_3 - n^2 + ns_2\right)}{(\alpha\,\gamma - n)\,(\alpha\,\beta - n)\,(\beta\,\gamma - n)}$$

Following Proposition 28.7.3, we can also obtain k_j as the second intersection of $\mathcal{L}_j$ with the circle, and then the δ_j as $2h_j - k_j$.

28.8 Orthopole and pedal LFIT

Remark 28.8.1. In this section, line $\mathcal{L}_0 = \Delta$ is perceived as a special line, while the sidelines $\mathcal{L}_a = BC$, etc are perceived as ordinary lines. In a later section, we will discuss how the four LFIT behave with respect to each other.

28.8.1 Trigone and transversal

Definition 28.8.2. The **orthopole** of a line $\mathcal{L}_0 = \Delta \neq \mathcal{L}_b$, is the Neuberg center of the LFIT provided by the pedal triangles of its points. Some orthopoles are given in Table 1.1.

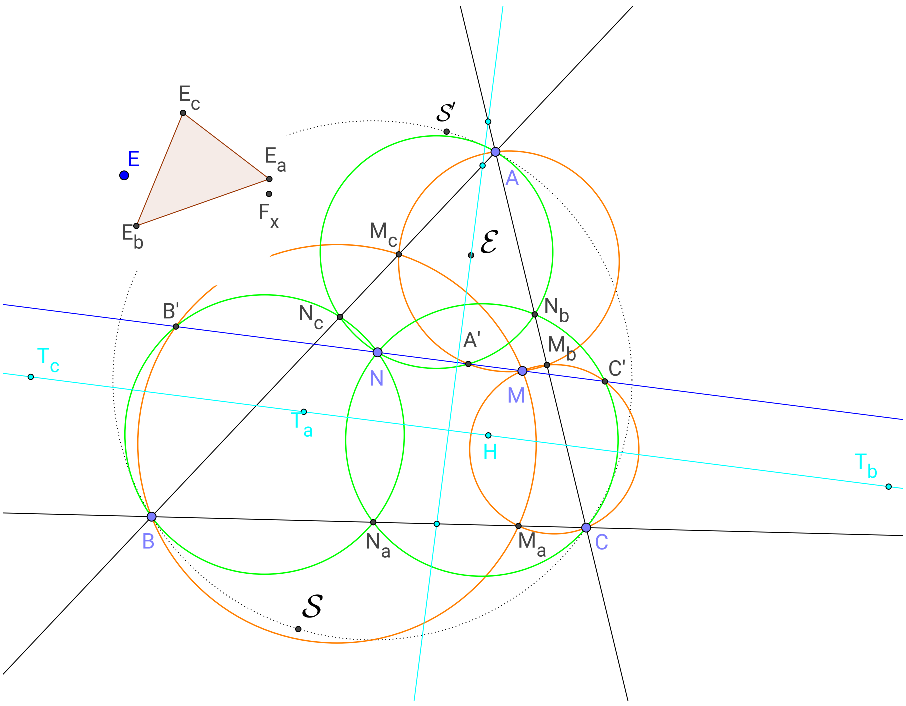


Figure 28.1: Point $\mathcal{E}$ is the orthopole of line $A'B'C'$

Proposition 28.8.3. *Given a pedal* LFIT*, the slowness center $\mathcal{S}$ lies on the circumcircle, while the orthopole $\mathcal{E}$ lies on the Simson line of $2O - \mathcal{S}$. More precisely, synchronized coordinates of the three centers associated to this* LFIT *are given by:*

$$\begin{aligned}
\mathcal{S}(\Delta) &= \frac{-1}{2S} \times \text{isogon}\left(\boxed{\mathcal{M}_b} \cdot {}^t\Delta\right) \\
\mathcal{E}(\Delta) &= {}^t\!\left(\Delta \cdot \boxed{\mathcal{M}_b}\right) \underset{b}{*} {}^t\!\left(\Delta \cdot \boxed{\mathcal{N}}\right) \\
\Omega(\Delta) &= {}^t\!\left(\Delta \cdot \boxed{\mathcal{M}_b}\right) \underset{b}{*} {}^t(\Delta) \qquad (28.5) \\
f + g + h &= \left(\Delta \cdot \boxed{\mathcal{M}_b} . {}^t\Delta\right) \quad \text{the usual norm}
\end{aligned}$$

where isogon *is understood as* isogon $(f:g:h) = a^2gh : b^2hf : c^2fg$ *while matrix* $\boxed{\mathcal{M}_b}$ *is defined by* (7.20) *and matrix* $\boxed{\mathcal{N}}$ *is defined by :.*

$$\boxed{\mathcal{N}} = 1 - \frac{2}{\mathcal{L}_b \cdot O} O \cdot \mathcal{L}_b = \frac{1}{4S^2} \begin{pmatrix} S_bS_c & -a^2S_a & -a^2S_a \\ -b^2S_b & S_cS_a & -b^2S_b \\ -c^2S_c & -c^2S_c & S_aS_b \end{pmatrix} \tag{28.6}$$

The corresponding complex coordinates will be given at Subsection 28.10.1.

Proof. We have the normalized coordinates

$$M_a^t \underset{b}{\simeq} 0 : \frac{+q}{q-r} + \left(a^2\,qr - S_bpq - S_crp\right)\frac{\theta}{a^2} : \frac{-r}{q-r} - \left(a^2\,qr - S_bpq - S_crp\right)\frac{\theta}{a^2}$$

A straightforward computation leads to the Neuberg coefficients where $f:g:h$ can be identified as the isogonal conjugate of orthodir (Δ). The $\mathcal{E}$ formula follows. □

Remark 28.8.4. The normalization of $\boxed{\mathcal{N}}$ is comes from $\boxed{\mathcal{N}} \cdot \boxed{\mathcal{N}} = \boxed{1_4}$, while the normalization of $\boxed{\mathcal{M}_b}$ comes from (7.22). Therefore, they cannot be changed. Since a precise definition of matrices $\boxed{\mathcal{M}_b}$, $\boxed{\mathcal{N}}$ is required to provide synchronized values for $\mathcal{S}$ and $\mathcal{E}$, the common factor $32S^3$ which appears when computing the quantities of formula (28.5) cannot be avoided. Afterwards, one can only proceed to a global simplification, in order to provide better looking formula.

Remark 28.8.5. Quantity $f+g+h = u+v+w = \rho+\sigma+\tau$ vanishes for isotropic lines, i.e. lines trough an umbilic. Aren't they special lines ?

Exercise 28.8.6. Let α, β, γ be the orthogonal projections of A, B, C onto Δ. Then $\mathcal{S}$ and $\mathcal{E}$ are, respectively, the pole and the elop of the homography $A \mapsto \alpha$, etc.

Exercise 28.8.7. Consider the Euler line, using for example P_1=X(2) and P_2=X(4). Obtain $\mathcal{S} = 110$, $\mathcal{E} = \Omega = 125$, $\omega = 5972$. Intersections with circumcircle: X(1113), X(1114). Check the three cevians:

pedal	3	4	20
cevian	2	4	69

Construction 28.8.8. *Construct the orthopole $\mathcal{E}$ of Δ wrt triangle ABC. Project A, B, C onto Δ and obtain Q_a, Q_b, Q_c. Draw Δ_a by Q_a orthogonally to BC, etc. Then the three lines $\Delta_a, \Delta_b, \Delta_c$ concur at $\mathcal{E}$. This property was at the origin of the name "orthopole".*

Proof. Straightforward computation. □

28.8.2 Using the flat pedal triangles

It has been already stated that a LFIT is best characterized by its degenerate triangles. And we know that a pedal triangle is flat if and only if its center is on the circumcircle. Thus we introduce points P_3, P_4 that are the intersections (visible or not) of line $\Delta = P_0P_1$ and the circumcircle Γ.

Proposition 28.8.9. *Let $\mathcal{T}_3 = (a_3b_3c_3)$, $\mathcal{T}_4 = (a_4b_4c_4)$ be the pedal (flat) triangles of P_3, P_4. Then $\mathcal{E}$ is the intersection of the Simson lines $\mathcal{T}_3$ and $\mathcal{T}_4$, while $\omega = (\mathcal{S}+\mathcal{E})/2$ is the intersection of the Newton lines $\mathcal{N}_3$ and $\mathcal{N}_4$ where $\mathcal{N}_3$ goes through the aligned points $a_3 + A$; $b_3 + B$; $c_3 + C$,* etc.

Proof. General property of the degenerate triangles of an LFIT, see Section 27.10. □

Proposition 28.8.10. *As already said, point $\mathcal{S}$ lies on the circumcircle. Its Simson line is parallel to Δ. Point $S' = 2O - \mathcal{S}$ is the isogonal conjugate of the direction of Δ. The Simson line of S' is perpendicular to Δ and, moreover, this line goes through $\mathcal{E}$, the orthopole of Δ.*

Proof. Everything is obvious, except the last point. And every computation is straightforward. One can also follow the Steiner movie, given at Corollary 10.2.4. □

Proposition 28.8.11. *The equicenter (orthopole) $\mathcal{E}$ lies on the reciprocal to the image of Δ in the circumcenter O.*

Proof. Matrix $\boxed{\mathcal{N}}$ takes that image. And then, point $pu : qv : rw$ belongs to line $[1/u, 1/v, 1/w]$ when $p+q+r=1$. And, here, $\boxed{\mathcal{M}_b} \cdot {}^t\Delta \in \mathcal{L}_b$. □

Exercise 28.8.12. Vertex-Miquel circles. In the pedal-LFIT relative to Δ, the circles (A, b_t, c_t) are the circles having $[A, M_t]$ as diameter. Therefore the Q_A, etc points of Proposition 27.6.1 are the projection of the vertices on Δ. The Vertex-Miquel point is simply M_t... whose locus is nothing than line $(Q_A, Q_B, Q_C) = \Delta$, whose direction is the orthopoint of $\mathcal{S}^*$. This doesn't contradict the former assumption $\mathcal{S}^* \in \mathrm{cycle}\,(Q_A, Q_B, Q_C)$ since this cycle is $\mathcal{L}_b \cup \Delta$! And $\mathcal{S}^*$ remains the perspector of triangles ABC and $Q_AQ_BQ_C$. See Figure 28.1.

Exercise 28.8.13. The fixed points (as characterized by concurrent corresponding lines) are $P_a = dir\,(AH) \in \mathcal{L}_z$, etc, and P_a, $\mathcal{E}$, O_a, O'_a are aligned as ever.

Proposition 28.8.14. *The area of the P-pedal triangle is* $\dfrac{S}{4}\left(1 - \dfrac{|OP|^2}{R^2}\right)$.

Proof. This is obviously true for $P = O$. Otherwise, draw the line $\Delta \doteq OP$: It cuts Γ at P_3, P_4. Then we parametrize the line Δ by $2P = (1+t)\,P_3 + (1-t)\,P_4$, so that $|t| = |OP|\,/R$. The family of all the pedal triangles $\mathcal{T}_t$ of the $P \in \Delta$ forms a $LFIT$ since formula (9.2) is of first degree in t. As a result, $area\,(\mathcal{T}_t)$ is a t-polynomial of degree 2, while our formula is already true for three points (pedal triangles $\mathcal{T}_{\pm 1}$are flat). □

28.9 Sister Marie Cordia Karl

Proposition 28.9.1. *Let $\mathcal{L} = [u, v, w]$ and $\Delta = [p, q, r]$ be two lines. The condition for the orthopole of $\mathcal{L}$ belongs to Δ can be written as*

$$\Delta \cdot \boxed{\mathfrak{H}} \cdot \mathfrak{V}\,(\mathcal{L}) = 0 \qquad \text{where } \boxed{\mathfrak{H}} \simeq$$

$$\left[\begin{array}{cccccc} 2\,a^2\,S_bS_c & 2\,b^2\,S_bS_c & 2\,c^2\,S_bS_c & b^2S_b^2 + c^2S_c^2 & -S_c\,\left(a^2c^2 + S_b^2\right) & -S_b\,\left(a^2b^2 + S_c^2\right) \\ 2\,a^2\,S_cS_a & 2\,b^2\,S_cS_a & 2\,c^2\,S_cS_a & -S_c\,\left(b^2c^2 + S_a^2\right) & a^2S_a^2 + c^2S_c^2 & -S_a\,\left(a^2b^2 + S_c^2\right) \\ 2\,a^2\,S_aS_b & 2\,b^2\,S_aS_b & 2\,c^2\,S_a\,S_b & -S_b\,\left(b^2c^2 + S_a^2\right) & -S_a\,\left(a^2c^2 + S_b^2\right) & a^2S_a^2 + b^2S_b^2 \end{array}\right]$$

$$\text{and} \qquad \mathfrak{V}\,([u, v, w]) \simeq {}^t\left[u^2, v^2, w^2, 2vw, 2wu, 2uv\right]$$

Proof. Obvious from (28.5).

□

Definition 28.9.2. Map $\mathcal{L} \mapsto \mathfrak{V}\,(\mathcal{L})$ is the full-Veronese map (whose signature is: row(3) $\mapsto$column(6), we are building conics here, not circles). For a given Δ, equation $\Delta \cdot \boxed{\mathfrak{H}} \cdot \mathfrak{V}\,(\mathcal{L}) = 0$ is the equation of a tangential conic. We call it the associated parabola of Δ and note it: $\mathfrak{H}\,(\Delta)$, while we call $\boxed{\mathfrak{H}}$ the Sister Mary Cordia Karl's matrix, due to Karl, E. (Sister Mary Cordia), 1932.

Proposition 28.9.3. *Conic $\mathfrak{H}\,(\Delta)$is a parabola. Its point at infinity is $\mathfrak{H}_\infty \doteq \boxed{\mathcal{M}_b} \cdot {}^t\Delta$, while its focus $\mathfrak{H}_\omega$ is the isogonal conjugate of $\mathfrak{H}_\infty$. Moreover, its directrix (polar of the focus) is homothetic to Δ from $X(4)$ (ratio=2).*

Proof. Compute $\mathfrak{H}\,(\Delta) \cdot {}^t\mathcal{L}_b$ and recognize the orthodir $\boxed{\mathcal{M}_b} \cdot {}^t\Delta$, proving the contact. Then write that the isotropic lines of the focus belongs to the conic, and obtain the given result. For the last point, you find it by computing the directrix, obtaining a linear formula, extracting its matrix and recognizing $homot\,(H, 2)$... but you prove it simpler by $\mathfrak{H}\,(\Delta) \cdot {}^t(\Delta \cdot h\,(H, 1/2)) \simeq \mathfrak{H}_\omega$, using (7.29) (and 1/2, we are acting on lines, not on points). □

Example 28.9.4. Taking the sidelines and $\mathcal{L}_b$ as examples, one obtains:

$$\begin{array}{rcl} \mathfrak{H}\,(BC)\,(u, v, w) & = & \left(-a^2u + S_c\,v + S_b\,w\right)\left(S_bS_c\,u - b^2S_b\,v - c^2S_c\,w\right) \\ \mathfrak{H}\,(CA)\,(u, v, w) & = & \left(-b^2v + S_a\,w + S_c\,u\right)\left(S_cS_a\,v - c^2S_c\,w - a^2S_a\,u\right) \\ \mathfrak{H}\,(AC)\,(u, v, w) & = & \left(-c^2w + S_b\,u + S_a\,v\right)\left(S_aS_b\,w - a^2S_a\,u - b^2S_b\,v\right) \\ \mathfrak{H}\,(\mathcal{L}_b)\,(u, v, w) & \simeq & \left(u\,(S_b - 2\,iS) + v\,(S_a + 2\,iS) - c^2w\right)\left(u\,(S_b + 2\,iS) + v\,(S_a - 2\,iS) - c^2w\right) \end{array}$$

the last one being the conic of isotropic lines.

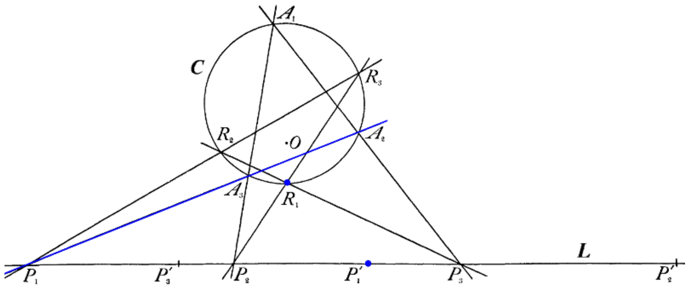


Figure 28.2: Revisiting the Sister Mary Cordia Karl's correspondence

Proposition 28.9.5. *The $\mathfrak{H}(\Delta)$ conic is degenerate if and only if Δ is the Simson line of some point M (on Γ, the circumcircle). In such a case, the pivots are M and its isogonal conjugate (the orthodir of Δ, on $\mathcal{L}_b$).*

Proof. Compute the determinant and obtain (28.4). Since this equation is invariant by isotomic conjugacy, this tells us that *tripolar* (Δ) is on K010, so that Δ is a Simson line. Conversely, computing the conic from parametrization (28.3) of *Simson* (M_t), leads to M_t and $M_t^* \in \mathcal{L}_b$. □

Remark 28.9.6. When points are opposite on Γ, Simson lines are orthogonal and also reciprocate (characteristic property).

Proposition 28.9.7. *Three Simson lines (visible or not) are going through any generic point. By the orthopole of a known line $\mathcal{L}$, we have the Simson line orthogonal to $\mathcal{L}$, given by:*

$$\Delta_1 \simeq \left[\frac{q-r}{S_b\, r + S_c\, q - a^2 p}, \frac{r-p}{S_a\, r + S_c\, p - q b^2}, \frac{p-q}{S_a\, q + S_b\, p - c^2 r}\right] \;;\; t = -\frac{p-r}{q-r}$$

and the Simson lines of the points of $\Gamma \cap \mathcal{L}$.

Proof. Degree of equation is 3. For an orthopole, the equation splits. The first degree factor leads to Δ_1. The other part is exactly the condition for $M_t \in \mathcal{L}$. □

Construction 28.9.8. *Construct the three Simson lines through a given point P. Let $Q = A+B+C-2P$. Draw the hyperbola $\mathcal{H}_A$ having $[A,Q]$ as diameter and $I_0 I_a, I_b I_c$ as asymptotic directions. The three hyperbolas $\mathcal{H}_A$, $\mathcal{H}_B$, $\mathcal{H}_C$ have four points in common, Q itself and three others R_j, which are on the circumcircle. Then the Simson lines of the $R'_j \doteq 2O - R_j$ are going through P. Moreover the three R_j are visible when P is inside of the Steiner deltoid, while only one is visible when P is outside of the deltoid.*

Proof. Let $P = z : t : \zeta$ be a generic point in the plane, and M a point on the unit circle (turn τ). Then P belongs to the Simson line of $2O - M$ when

$$\Theta \doteq \left(2\,\tau^2\right) z + \left(\tau^3 - \tau^2 s_1 - \tau s_2 + s_3\right) t + \left(2\,\tau s_3\right) \zeta = 0$$

On the other hand, asymptotes δ_1, δ_2 of $\mathcal{H}_A$ are the parallels to the bisectors of (AB, AC) drawn through $(A+Q)/2$. Let $Q_a = 2A + B + C - 2P$ and $\delta_1 = Q_a \wedge (+\omega : 0 : 1)$, $\delta_2 = Q_a \wedge (-\omega : 0 : 1)$. Then

$$\mathcal{H}_A \simeq \left({}^t\delta_1 \cdot \delta_2 + {}^t\delta_2 \cdot \delta_1\right) - K\left({}^t\mathcal{L}_z \cdot \mathcal{L}_z\right)$$

where $\omega^2 = \alpha^2 \beta\gamma$ while K is determined by $A \in \mathcal{H}_A$. This results into:

$$\mathcal{H}_A \simeq \left[\begin{array}{ccc} -2\,t & -2\,z + (s_1+\alpha)\,t & 0 \\ -2\,z + (s_1+\alpha)\,t & 4\,\alpha\, z + 2\left(\beta\gamma - \alpha^2\right) t - 4\, s_3\, \zeta & -(s_3 + \alpha s_2)\, t + 2\,\alpha s_3\, \zeta \\ 0 & -(s_3+\alpha s_2)\, t + 2\,\alpha s_3\, \zeta & 2\,\alpha\, s_3\, t \end{array}\right]$$

And we can check that $M \in \mathcal{H}_A$ is either $\tau = \alpha$ or $\Theta = 0$. As a result, the three hyperbolas belong to the same pencil. This can be checked by $\sum (\beta - \gamma)\, \mathcal{H}_A = 0$. The deltoid appears when one computes the discriminant of Θ. □

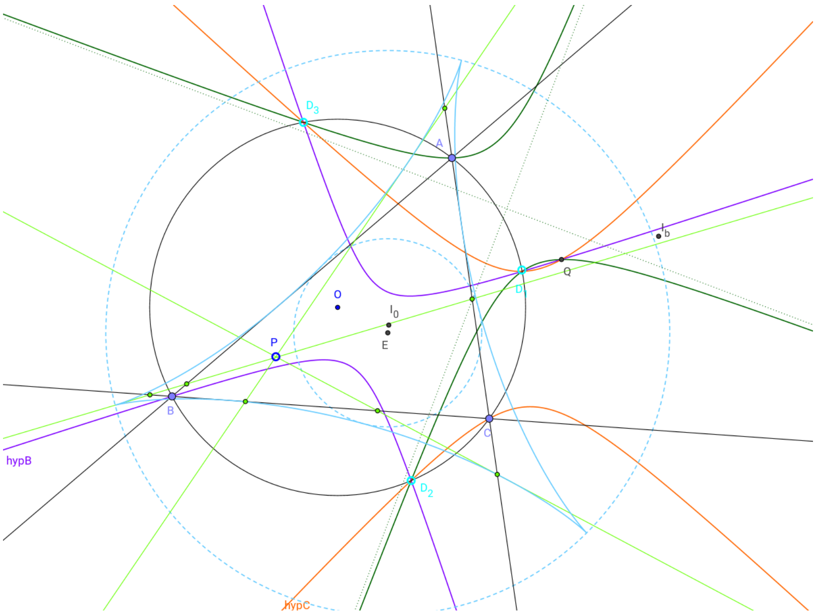


Figure 28.3: Construct the 3 Simson lines through a given point

28.10 The four pedal LFIT of a quadrilateral

Proposition 28.10.1. *Consider the four* LFIT *created by the pedal triangles of points on line $\mathcal{L}_j$ wrt the trigone $\mathcal{T}_j$ of the other three lines. The four slowness centers $\mathcal{S}_j$ are on the common Miquel circle, while the four Neuberg centers $\mathcal{E}_j$ are on the Steiner axis.*

Proof. We already know that $\mathcal{S}_0 \in$ Miquel and $\mathcal{E}_0 \in$ Steiner. The obvious symmetry of the configuration proves the rest. One can also use the fact that $\mathcal{S}_0$ is the perspector of ABC and $O_aO_bO_c$. □

Corollary 28.10.2. *Point $\mathcal{S}_j$ is therefore the second intersection of the Miquel circle (O_0, O_a, O_b, O_c) and the circumscribed circle (O_j) , while $\mathcal{E}_j$ is the intersection of the Simson line of $2O_j - \mathcal{S}_j$ with the line (H_0, H_a, H_b, H_c).*

Fact 28.10.3. *The coordinates of the four slowness/Neuberg centers are*

$$\mathcal{S}_0 \underset{1}{\simeq} \left(\begin{array}{c} \dfrac{-s_3}{n} \\ 1 \\ \dfrac{-n}{s_3} \end{array}\right) \; ; \; \mathcal{S}_a \underset{1}{\simeq} \left(\begin{array}{c} \left(m\left(s_3+n\,\alpha\right)+n\left(\alpha^2+s_2\right)\right)\alpha \\ -\left(\alpha\,\gamma-n\right)\left(\alpha\,\beta-n\right) \\ \left(m\left(s_3+n\,\alpha\right)+n\left(\alpha^2+s_2\right)\right)\frac{1}{\alpha} \end{array}\right)$$

$$\mathcal{E}_0 \underset{1}{\simeq} \left(\begin{array}{c} s_3\left(s_1 \mathrm{y}\eta + s_3\,\eta^2 - \mathrm{yt}\right) \\ 2s_3\mathrm{y}\eta \\ \mathrm{y}^2 + s_2\mathrm{y}\eta - s_3\mathrm{t}\eta \end{array}\right)$$

$$\mathcal{E}_a \underset{1}{\simeq} \left(\begin{array}{c} 2s_1s_3\,\mathrm{y}^2 + s_3\left(\alpha^2+s_2\right)\mathrm{yt} + s_3s_1\left(3\alpha^2 - 2\alpha s_1 + s_2\right)\mathrm{y}\eta - s_3^2\left(\alpha^2 - 2\alpha s_1 + s_2\right)\eta^2 \\ 2s_3\,\mathrm{y}^2 + 2s_3\alpha\left(-s_1+\alpha\right)\mathrm{y}\eta + 2s_3^2\alpha\,\eta^2 \\ \left(s_2-\alpha^2\right)\mathrm{y}^2 + s_2\left(3\alpha^2 - 2\alpha s_1 + s_2\right)\mathrm{y}\eta + s_3\left(\alpha^2+s_2\right)\mathrm{t}\eta + 2\alpha s_2 s_3\,\eta^2 \end{array}\right)$$

Proposition 28.10.4. *When point M moves along the line $\mathcal{L}_0$, the orthopole $\mathcal{E}_0$ has a constant power ρ^2 with respect to the pedal circle of M wrt $\mathcal{T}_0 = ABC$. The value of ρ^2 is ((Gulasekharam, 1941))*

$$\rho^2=\frac{\left(\mathcal{L}_0\cdot\begin{bmatrix}a^2S_a\\b^2S_b\\c^2S_c\end{bmatrix}\right)\left(\mathcal{L}_0\cdot\begin{bmatrix}-a^2\\S_c\\S_b\end{bmatrix}\right)\left(\mathcal{L}_0\cdot\begin{bmatrix}S_c\\-b^2\\S_a\end{bmatrix}\right)\left(\mathcal{L}_0\cdot\begin{bmatrix}S_b\\S_a\\-c^2\end{bmatrix}\right)}{16\,S^4\left(\mathcal{L}_0\cdot\boxed{\mathcal{M}_b}\cdot{}^t\mathcal{L}_0\right)^2} \tag{28.7}$$

Moreover ρ^2 is (-2) times the algebraic product of dist $(O_0,\mathcal{L}_0)$ *and* dist $(\mathcal{E}_0,\mathcal{L}_0)$.
Rem: first column is X(3), the other 3 are the directions of the altitudes.

Proof. Start from triangle $\boxed{\mathcal{T}(t)}$, then compute its circumcircle and apply the definition of ρ^2. It happens that all the t cancel. Moreover, we have the formulas:

$$\begin{aligned}\operatorname{dist}(\mathcal{L}_0,O_0)&=\sqrt{2S}\,\frac{\left(S_a\,a^2qr+b^2S_b\,rp+c^2S_c\,pq\right)}{8\,S^2\sqrt{\mathcal{L}_0\cdot\boxed{\mathcal{M}_b}\cdot{}^t\mathcal{L}_0}}\\ \operatorname{dist}(\mathcal{L}_0,\mathcal{E}_0)&=\sqrt{2S}\,\frac{\prod\left(a^2\,qr-S_c\,rp-S_b\,pq\right)}{8S^3\left(\sqrt{\mathcal{L}_0\cdot\boxed{\mathcal{M}_b}\cdot{}^t\mathcal{L}_0}\right)^3}\end{aligned}$$

And we can check the homogeneous degrees: in a: $(4+2+2+2)-8=2$ (since ρ is a length); in p: $8-8$ (as required since $\mathcal{L}_0$ is projectively defined). □

Remark 28.10.5. This result can be restated using some trigonometry, leading to (Goormaghtigh, 1926, p. 81)

$$\operatorname{dist}(\mathcal{L}_0,\mathcal{E}_0)=2R_0\cos(\mathcal{L}_0,\mathcal{L}_a)\cos(\mathcal{L}_0,\mathcal{L}_b)\cos(\mathcal{L}_0,\mathcal{L}_c)$$

Corollary 28.10.6. *Concerning the other* LFIT*, when point M moves along line $\mathcal{L}_A$, the orthopole $\mathcal{E}_A$ has a constant power ρ_A^2 with respect to the pedal circle of M wrt trigone $\mathcal{T}_A$.*

$$\rho_A^2=\frac{a^2\left(pq+rp-qr\right)S_a+S_b\,b^2pq+S_c\,c^2pr-8S^2p^2}{4a\,S\,(p-r)\,(p-q)}\times\frac{S_bS_c\left(-a^2qr+S_c\,pr+S_b\,pq\right)}{2a^3\,S\,(p-r)\,(p-q)}$$

Proof. Simply using the product of distances. One can also recompute everything, and/or using an algebraic change of parameters, followed by a collineation. □

Proposition 28.10.7 (Lemoyne's theorem)**.** *When point M moves along $\mathcal{L}_j$, its pedal circumcircle wrt triangle $\mathcal{T}_j$ remains orthogonal to a fixed circle centered at $\mathcal{E}_j$ and whose radius is $\sqrt{\rho_j^2}$. Let us call it the j-**orthopedal circle**, noted $\mathcal{G}_j$. This circle is virtual when $O_j,\mathcal{E}_j$ are on the same side of $\mathcal{L}_j$.*

Proof. Immediate from the invariance of ρ_j^2. See also Thebault, 1946. □

Proposition 28.10.8. *The four orthopedal circles belong to the Steiner pencil (generated by the polar circles).*

Proof. When T_0, the generic point on $\mathcal{L}_0$, comes at C', we have $C'T_a\perp T_aC$ and $C'T_b\perp T_bC$. Thus T_a,T_b are on the circle with diameter $[T_cC]$ and the circle $(T_aT_bT_c)$ is the circle with diameter $[C'C]$. This occurs also with the other two circles defining the Newton pencil. One can also use (28.2). □

28.10.1 Lubin coordinates

Fact 28.10.9. *Specific to the pedal LFIT $(ABC,\mathcal{L}_0)$*

1. Synchronized barycentrics for $(\mathcal{S}_0,\mathcal{E}_0)$

$$\begin{pmatrix}f\\g\\h\\u\\v\\w\end{pmatrix}\underset{b}{\simeq}\begin{matrix}2\begin{pmatrix}\gamma-\beta\\\alpha-\gamma\\\beta-\alpha\end{pmatrix}\underset{b}{*}\begin{pmatrix}\alpha\beta+n\\\beta\gamma+n\\\gamma\alpha+n\end{pmatrix}\underset{b}{*}\begin{pmatrix}\gamma\alpha+n\\\alpha\beta+n\\\beta\gamma+n\end{pmatrix}\\\begin{pmatrix}\gamma-\beta\\\alpha-\gamma\\\beta-\alpha\end{pmatrix}\underset{b}{*}\begin{pmatrix}\beta\gamma+n\\\gamma\alpha+n\\\alpha\beta+n\end{pmatrix}\underset{b}{*}\begin{pmatrix}\alpha\,m-n-\alpha^2\\\beta\,m-n-\beta^2\\\gamma\,m-n-\gamma^2\end{pmatrix}\end{matrix}$$

2. Similarities $\sigma_a \underset{1}{\simeq} \begin{pmatrix} \dfrac{\alpha\beta+\kappa\nu}{\alpha\gamma+\kappa\nu} & \dfrac{\left(\alpha^2+\alpha\kappa+\alpha\nu-\kappa\nu\right)(\gamma-\beta)}{2(\alpha\gamma+\kappa\nu)} & 0 \\ 0 & 1 & 0 \\ 0 & \dfrac{\left(\alpha^2-\alpha\kappa-\alpha\nu-\kappa\nu\right)(\gamma-\beta)}{2\alpha\beta(\alpha\gamma+\kappa\nu)} & \dfrac{\gamma(\alpha\beta+\kappa\nu)}{\beta(\alpha\gamma+\kappa\nu)} \end{pmatrix}$

3. Centers are the projections of ABC on $\mathcal{L}_0$. $Q_a \underset{1}{\simeq} \begin{pmatrix} \left(\alpha^2+\alpha(\kappa+\nu)-\kappa\nu\right)\kappa\nu \\ 2\alpha\kappa\nu \\ -\alpha^2+\alpha(\kappa+\nu)+\kappa\nu \end{pmatrix}$

4. Power of $\mathcal{E}_0$ wrt each of the pedal circles related to some $M_t \in \mathcal{L}_0$:

$$\begin{aligned} \rho_0^2 &= \frac{-m\left(n^3+n^2 s_2+n\, s_1 s_3+s_3^2\right)}{4n^2 s_3} \\ &= \frac{-m(\beta\gamma+n)(\gamma\alpha+n)(\alpha\beta+n)}{4n^2 s_3} R^2 \end{aligned}$$

28.10.2 The so-called paralogic triangles

Definition 28.10.10. Line Δ_{jk} is defined as the line through $\mathcal{L}_j \cap \mathcal{L}_k$ and perpendicular to $\mathcal{L}_k$. Point Q_{jk} is defined as $\Delta_{jm} \cap \Delta_{jn}$. For a given j, there are 3 Q_{jk} defining a triangle $\mathfrak{W}_j$, called paralogic in Johnson, 1929, p. 258. Obviously, the 3 Δ_{jk} are the sidelines of the $\mathfrak{W}_j^*$ trigon.

Proposition 28.10.11. *We have the following properties:*

1. Triangles $\mathcal{T}_j$ and $\mathfrak{W}_j$ are parallelogic. Orthocenter H_j sees $\mathcal{T}_j$ along the sidelines of $\mathfrak{W}_j$, while the orthocenter H_j' of $\mathfrak{W}_j$ sees $\mathfrak{W}_j$ along the sidelines of $\mathcal{T}_j$. Moreover $\left(H_j+H_j'\right)/2$ belongs to $\mathcal{L}_j$ (Sondat's theorem, see Goormaghtigh, 1946a).

2. Triangles $\mathcal{T}_j$ and $\mathfrak{W}_j$ are directly similar, with center M_q and angle $+90°$.

3. Triangles $\mathcal{T}_j$ and $\mathfrak{W}_j$ are in perspective from $\mathcal{S}_j$ (the slowness center).

4. Circle γ_j circumscribed to $\mathfrak{W}_j$, is centered at $D_j = 2\Omega - O_j$ (the circum-antipode of O_j), goes through M_q and $\mathcal{S}_j$ and is orthogonal to Γ_j.

5. Point Q_{jk} belongs to Γ_k.

Proof. Due to symmetry, only the case $j = 0$ is to be proven. Some properties are self evident. Existence of a similitude comes from the orthogonality of the respective sidelines of $\mathcal{T}_j$ and $\mathfrak{W}_j$. Parallelogy is also evident. The other properties are straightforward computations. □

28.11 Van Rees cubic

Definition 28.11.1. As defined in (Van Rees, 1829), the **Van Rees cubic** v$\mathrm{R}\mathcal{K}$ associated to points $M_1N_2M_3N_4$ is the locus of the points P such that

$$\begin{aligned} (PM_1, PM_3)+(PN_2, PN_4) &= 0 \\ (PM_1, PN_4)+(PN_2, PM_3) &= 0 \end{aligned} \tag{28.8}$$

Remark 28.11.2. Spoiler: more than often, such a cubic will be seen from a triangle ABC and described as

$$\left[nkcub, \#F \simeq \begin{bmatrix} a \\ b \\ c \end{bmatrix}, U \simeq \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \widehat{X} \simeq \begin{bmatrix} q-r \\ r-p \\ p-q \end{bmatrix} \right]$$

where U is the ABC tripole of a line $A'B'C'$, while vertices are renamed into $M_1 \doteq A, N_2 \doteq A', M_3 \doteq B, N_4 \doteq B'$.

Remark 28.11.3. This cubic also appears as the locus of the focuses of a LLLL pencil of conics. See Section 12.27.

28.11.1 Ordered quadrangle (complex coordinates)

Definition 28.11.4. An ordered quadrangle is an (ordered) list of four points M_j. Then use $M_5 = M_1M_4 \cap M_2M_3$, $M_6 = M_1M_3 \cap M_2M_4$, and note Ψ the correspondence $1 \leftrightarrow 2$, $3 \leftrightarrow 4$, $5 \leftrightarrow 6$. By definition, the diagonals of the associated quadrilateral are M_1M_2, M_3M_4 and M_5M_6 (while the four remaining lines form a quadrilateral).

Thus $M_1M_3M_5$, is a triangle –and a circle– (odd number of odd indices, like $146, 236, 245$), while $M_2M_4M_6$ is a line (odd number of even ones, like $235, 145, 136$).

Proposition 28.11.5. *The Miquel point and the direction of the Newton line of the induced quadrilateral are :*

$$M_{\mathrm{q}} \simeq \left(\begin{array}{c} \dfrac{t_1\,t_2\,z_3\,z_4 - t_3\,t_4\,z_1\,z_2}{t_1\,t_2\,t_3\,z_4 + t_1\,t_2\,t_4\,z_3 - t_1\,t_3\,t_4\,z_2 - t_2\,t_3\,t_4\,z_1} \\ 1 \\ \dfrac{t_1\,t_2\,\zeta_3\,\zeta_4 - t_3\,t_4\,\zeta_1\,\zeta_2}{t_1\,t_2\,t_3\,\zeta_4 + t_1\,t_2\,t_4\,\zeta_3 - t_1\,t_3\,t_4\,\zeta_2 - t_2\,t_3\,t_4\,\zeta_1} \end{array}\right) \simeq \left(\begin{array}{c} \dfrac{z_3\,z_4 - z_1\,z_2}{z_3 + z_4 - z_1 - z_2} \\ 1 \\ \dfrac{\zeta_3\,\zeta_4 - \zeta_1\,\zeta_2}{\zeta_3 + \zeta_4 - \zeta_1 - \zeta_2} \end{array}\right) \tag{28.9}$$

$$\Delta_\infty \simeq \left(\begin{array}{c} t_1\,t_2\,t_3\,z_4 + t_1\,t_2\,t_4\,z_3 - t_1\,t_3\,t_4\,z_2 - t_2\,t_3\,t_4\,z_1 \\ 0 \\ t_1\,t_2\,t_3\,\zeta_4 + t_1\,t_2\,t_4\,\zeta_3 - t_1\,t_3\,t_4\,\zeta_2 - t_2\,t_3\,t_4\,\zeta_1 \end{array}\right) \simeq \left(\begin{array}{c} z_3 + z_4 - z_1 - z_2 \\ 0 \\ \zeta_3 + \zeta_4 - \zeta_1 - \zeta_2 \end{array}\right)$$

Proof. Δ_∞ is obvious, while computing circles $M_1M_3M_5$,$M_1M_4M_6$, etc leads directly to M_{q}. □

Proposition 28.11.6. *The* ***isoptic*** *definition* (28.8) *given by* Van Rees (1829) *leads to the not so huge equation (length = 574)*

$$\begin{gathered} \mathrm{vR}\mathcal{K}_1 \doteq \mathbf{Z}\overline{\mathcal{Z}} \times \left((\zeta_1 + \zeta_2 - \zeta_3 - \zeta_4)\,\mathbf{Z} - (z_1 + z_2 - z_3 - z_4)\,\overline{\mathcal{Z}}\right) \\ + \left((\zeta_3\,\zeta_4 - \zeta_1\,\zeta_2)\,\mathbf{Z}^2 + ((z_1 + z_2)\,(\zeta_3 + \zeta_4) - (z_3 + z_4)\,(\zeta_1 + \zeta_2))\,\overline{\mathcal{Z}}\,\mathbf{Z} + (z_1\,z_2 - z_3\,z_4)\,\overline{\mathcal{Z}}^2\right)\mathbf{T} \\ + ((z_3 + z_4)\,\zeta_1\,\zeta_2 - (z_1 + z_2)\,\zeta_3\,\zeta_4)\,\mathbf{Z}\mathbf{T}^2 + (z_3\,z_4\,(\zeta_1 + \zeta_2) - z_1\,z_2\,(\zeta_3 + \zeta_4))\,\overline{\mathcal{Z}}\mathbf{T}^2 \\ + (z_1\,z_2\,\zeta_3\,\zeta_4 - \zeta_1\,\zeta_2\,z_3\,z_4)\,\mathbf{T}^3 = 0 \end{gathered} \tag{28.10}$$

One can check that this vanRees cubic goes through the six points M_j. Points at infinity are both umbilics and Δ_∞. Moreover, point M_{q} is a singular focus (and belongs to the curve). Additionally, the cubic goes through the six U_j, where $U_j = M_jM_{\mathrm{q}} \cap \Psi(M_j)\,\Delta_\infty$ (remember: $\Psi(M_j)$ is the formal notation of M_j').

Proof. Direct substitutions are easy... even when a direct examination is equally easy ! □

Proposition 28.11.7. *Let M_j be four generic points in the plane, and define*

$$\begin{aligned} s_1 &= \frac{(z_1 + z_2)\,\zeta_3\,\zeta_4 - (z_3 + z_4)\,\zeta_1\,\zeta_2}{\zeta_3\,\zeta_4 - \zeta_1\,\zeta_2} \\ s_2 &= \frac{z_1\,z_2\,(\zeta_3 + \zeta_4) - z_3\,z_4\,(\zeta_1 + \zeta_2)}{\zeta_3 + \zeta_4 - \zeta_1 - \zeta_2} \\ s_3 &= \frac{z_1\,z_2 - z_3\,z_4}{(\zeta_3 + \zeta_4) - (\zeta_1 + \zeta_2)} \;;\; s_3' = \frac{(z_3 + z_4) - (z_1 + z_2)}{\zeta_1\,\zeta_2 - \zeta_3\,\zeta_4} \end{aligned} \tag{28.11}$$

Then $s_3 = s_3'$ if and only if M_{q} belongs to the unit circle Γ. And then it exist a triangle α, β, γ inscribed in Γ such that the three pairs (M_{2j+1}, M_{2j+2}) are isogonal wrt $\alpha\beta\gamma$. Moreover, the Van Rees cubic $\mathrm{vR}\mathcal{K}\,(M_1 \cdots M_4)$ goes also through the points α, β, γ.

Proof. A simple elimination gives the condition and the values of the s_j. This doesn't imply that all these three points are visible points. Concerning the last assertion, convert $n\mathcal{K}\,(\alpha)\,n\mathcal{K}\,(\beta)\,n\mathcal{K}\,(\gamma)$ using the symmetric functions s_j, substitute (28.11) and check the appearance of the cube of the condition $M_{\mathrm{q}} \in \Gamma$.

□

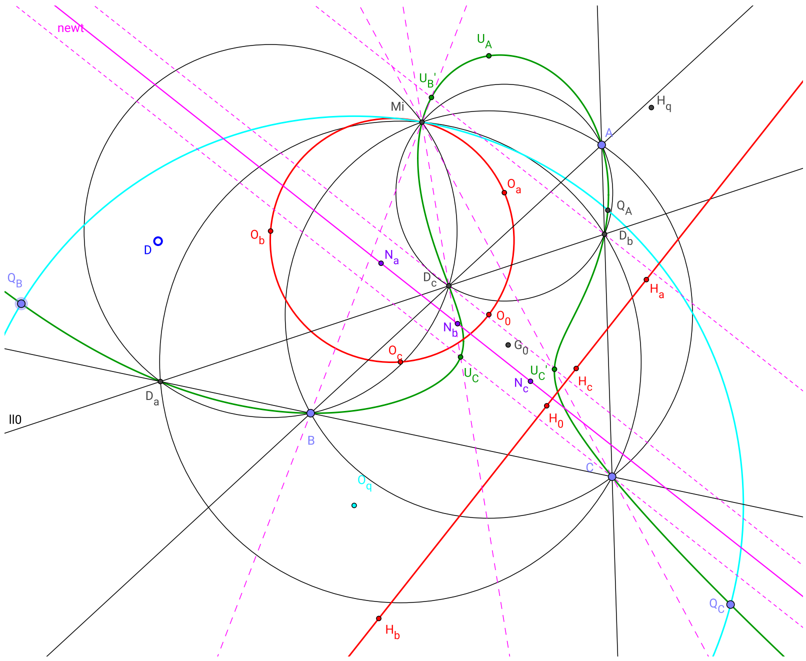


Figure 28.4: Starting from the four M_j.

28.11.2 Cartesian centered equation

Proposition 28.11.8. *In a cartesian frame $X : Y : \mathbf{T}$ where $M_{\mathrm{q}} \simeq 0:0:1$ and $\Delta_\infty \simeq 1:0:0$, the equation of the cubic can be written as:*

$$\mathrm{vR}\mathcal{K}_c\left(X,Y,\mathbf{T}\right) \doteq \left(X^2+Y^2\right)\left(Y-A\mathbf{T}\right)+\mathbf{T}^2XB+\mathbf{T}^2YC=0$$

where $A,B,C\in\mathbb{R}$ and a parametrization is:

$$x=\frac{B+\sqrt{B^2+4y\left(A-y\right)C-4y^2\left(A-y\right)^2}}{2\left(A-y\right)}$$

A first proof. Substitute $\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}$ by $\mathbf{Z}+\dfrac{\mathbf{T}\left(z1\,z2-z4\,z3\right)}{-z3-z4+z1+z2} : \mathbf{T} : \overline{\mathcal{Z}}+\dfrac{\mathbf{T}\left(zz1\,zz2-zz3\,zz4\right)}{-zz3-zz4+zz1+zz2}$ and obtain

$$\overline{\mathcal{Z}}\,\mathbf{Z}^2c_{210}+\overline{\mathcal{Z}}^2\mathbf{Z}c_{120}+\mathbf{T}\overline{\mathcal{Z}}\mathbf{Z}c_{111}+\mathbf{T}^2\overline{\mathcal{Z}}c_{012}+\mathbf{T}^2\mathbf{Z}c_{102}$$

Substitute $\mathbf{Z}=\kappa\left(X+\boldsymbol{i}Y\right)$, etc and identify. This gives

$$\kappa^2=-\frac{c_{120}}{c_{210}},A=\frac{-\boldsymbol{i}c_{111}\kappa}{2c_{120}},B=\frac{\boldsymbol{i}\left(c_{210}c_{012}-c_{102}c_{120}\right)}{2c_{120}c_{210}},C=\frac{c_{102}c_{120}+c_{210}c_{012}}{2c_{120}c_{210}}$$

And then, one can check that A,B,C are real. □

Another proof. Substitute the values of M_q,Δ_∞ into 28.9 and solve for z_2,ζ_2,z_4,ζ_4. Substitute the result, together with $\mathbf{Z}=X+iY$, $\overline{\mathcal{Z}}=X-iY$ into 28.10, and obtain the required $\mathrm{vR}\mathcal{K}_c\left(X,Y,\mathbf{T}\right)$ formula with real values for coefficients A,B,C. □

Theorem 28.11.9 (Van Rees, 1829)**.** *When one of these points is on the cubic, so are the other three:*

$$M\doteq\begin{pmatrix}X\\Y\\\mathbf{T}\end{pmatrix};\ \mathfrak{d}\left(M\right)\doteq\begin{pmatrix}\dfrac{B\mathbf{T}^2}{A\mathbf{T}-Y}-X\\Y\\\mathbf{T}\end{pmatrix};\ \mathfrak{f}\left(M\right)\simeq\begin{pmatrix}\dfrac{X\left(A\mathbf{T}-Y\right)}{Y}\\A\mathbf{T}-Y\\\mathbf{T}\end{pmatrix};\ \left(\mathfrak{d}\mathfrak{f}\right)\left(M\right)=\left(\mathfrak{f}\mathfrak{d}\right)\left(M\right)$$

Moreover, we have the equivalence:

$$(\mathfrak{d}\mathfrak{f})\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix} \in n\mathcal{K} \Longleftrightarrow \begin{cases} y_1 + y_2 & = A \\ x_1 y_2 + x_2 y_1 & = B \\ y_1 y_2 - x_1 x_2 & = C \end{cases} \tag{28.12}$$

Proof. Direct substitution. These formulas were derived by Van Rees (1829) from the fact that $y_1 + y_2 = A$ leads to the same radical W. One can check that eliminating x_2, y_2 in 28.12 leads precisely to $M_1 \in n\mathcal{K}$. □

Definition 28.11.10. Transformation $\mathfrak{d}\mathfrak{f}$ is called **intrinsic conjugacy**. Points $M, \mathfrak{d}(M)$ are aligned with Δ_∞ while points $M, \mathfrak{f}(M)$ are aligned with M_q. In other words: start from M, towards Δ_∞ and cut the curve at $\mathfrak{d}(M)$. Then start towards M_q and cut the curve at $\mathfrak{d}\mathfrak{f}(M)$. But the other road is fine either.

Theorem 28.11.11. *Intrinsic conjugates are isogonal conjugates wrt any triangle inscribed in the vanRees cubic.*

Proof. This is the right place to state this theorem, while the proof will come later, at Remark 28.11.15. □

Proposition 28.11.12. *When P, Q are intrinsic conjugates, then*

1. Points P, Q have the same tangential
2. $|M_q P| \times |M_q Q| = \sqrt{B^2 + C^2}$
3. The bisectors of $(M_q P, M_q Q)$ have constant slopes wrt the x-axis, namely $\left(C \pm \sqrt{B^2 + C^2}\right)/B$.

Proof. Direct computation is easy from (28.12). □

Proposition 28.11.13. *When assuming $P \in \mathrm{vR}\mathcal{K}$, the defining property (28.8) , i.e. the equivalent assertions:*

$$\begin{aligned} (PM_1, PM_3) + (PN_2, PN_4) &= 0 \\ (PM_1, PN_4) + (PN_2, PM_3) &= 0 \end{aligned}$$

hold for any pairs of conjugates $M_1 \longleftrightarrow N_2$ and $M_3 \longleftrightarrow N_4$.

Proof. Write down $\tan(PM_1, PM_3) + \tan(PN_2, PN_4)$ using $P \simeq X : Y : \mathbf{T}$, $M_j \simeq x_j : y_j$ and simplify using the Van Rees method (28.12) i.e. $y_1 + y_2 = A$, etc. This gives the required equation times $x_1 + x_2 - x_3 - x_4$. □

Theorem 28.11.14. *When three points are on a vanRees cubic, their conjugates are colinear if and only if the given points are cocyclic with the Miquel point.*

Proof. First part. Use $N_j \simeq x_j : y_j : 1$ $(j = 2, 4, 6)$ and cut the curve by the line $uX + vY + w$. Then $X = -(vY + w)/u$ while

$$-\left(u^2 + v^2\right) Y^3 + \left(A\,u^2 + A\,v^2 - 2vw\right) Y^2 + \left(2Avw + Buv - C\,u^2 - w^2\right) Y + A\,w^2 + Buw = 0$$

Substitute X in (28.12) and obtain

$$\Psi(N_j) \simeq v(A - y_j) - w + (Aw + Bu)/y_j \;:, u(A - y_j) : u$$

Write down the cocyclicity condition, substitute the y_j symmetric functions obtained from the Y equation, and obtain 0 as asserted.

Second part. Use $M_j \simeq x_j : y_j : 1$ $(j = 1, 3, 5)$ and cut the curve by the circle $uX + vY + \left(X^2 + Y^2\right) w$. Then $X = Y(Av + Cw - vY) \div (u(Y - A) - Bw)$ while

$$-\left(u^2 + v^2\right) Y^3 + \left(2\left(u^2 + v^2\right) A + 2w\,(Bu + Cv)\right) Y^2 + \text{osotros} = 0$$

Substitute X in (28.12) and obtain

$$\Psi\left(M_j\right) \simeq \left[\begin{array}{c} v y_j^3+(-2 A v-C w) y_j^2+\left(A^2 v+A C w-B u\right) y_j+A B u+B^2 w \\ y_j\left(u A+B w-u y_j\right)\left(-y_j+A\right) \\ y_j\left(u A+B w-u y_j\right) \end{array}\right]$$

Write down the colinearity condition, substitute the y_j symmetric functions obtained from the Y equation, and obtain 0 as asserted.
Remark: don't be surprised by the appearance of $\operatorname{vdM}\left(y_j\right)$! □

Remark 28.11.15. This finalizes the proof of the theorem about isogonal conjugacy.

28.11.3 Barycentric version

Proposition 28.11.16. *Consider the quadrilateral $\mathcal{L}_j$ where $\mathcal{L}_0=A'B'C' \simeq [qr, rp, pq]$. Let δ, F be resulting Newton line and focus, i.e.*

$$\delta \simeq [-p+q+r, \text{ etc}] \;\; ; \;\; F^* \simeq \delta_\infty \simeq (q-r) ::$$

Use $M_1=A, M_2=A'$, $M_3=B, M_4=B'$, $M_5=C', M_6=C$ (mind the order !) and apply the construction of the former section. Define

$$U_A \doteq (A \wedge F) \wedge\left(A' \wedge \delta_\infty\right), \text{ etc} \;;\; U_A' \doteq\left(A' \wedge F\right) \wedge\left(A \wedge \delta_\infty\right), \text{ etc.}$$

Then the 12 points A, A', etc, U_A, U_A', etc are on the $\mathrm{vR}\mathcal{K}$ *whose barycentric equation wrt ABC is:*

$$\mathrm{vR}\mathcal{K}_b=\left[nkcub, \#F \simeq\left[\begin{array}{c} a \\ b \\ c \end{array}\right], U \simeq\left[\begin{array}{c} p \\ q \\ r \end{array}\right], \widehat{X} \simeq\left[\begin{array}{c} q-r \\ r-p \\ p-q \end{array}\right]\right]$$

$$\simeq \sum p x\left(b^2 z^2+c^2 y^2\right)+2\left(\sum S_a\, p\right) x y z$$

Moreover, points $F, \delta_\infty=F^*$ and both umbilics are also on this curve.

Proof. Direct computation. □

Exercise 28.11.17. Prove that A, etc, A', etc, $\Omega_x \Omega_y, F$ are con-cubic with any tenth point in the plane (see Proposition 22.1.4) and are not sufficient to define the $\mathrm{vR}\mathcal{K}$. Prove also that, when A is restricted to the cubic, then $A^*=A'$.

Exercise 28.11.18. Consider two pairs $M_1=P, M_2=P^*, M_3=Q, M_4=Q^*$ of isogonal conjugates wrt triangle ABC. Define $M_5=M_1M_4 \cap M_2M_3$, $M_6=M_1M_3 \cap M_2M_4$. Then

1. Miquel: circles $\mathcal{C}_{135}, \mathcal{C}_{146}, \mathcal{C}_{236}, \mathcal{C}_{245}$ concur at some F;

2. Newton: midpoints $m_{12}=\left(M_1+M_2\right)/2$, etc are aligned on some δ ;

3. F, δ_∞ are isogonal conjugates, and so are M_5, M_6 ;

4. Let $K_aK_bK_c$ be the re-intersections of the sidelines with the cubic $\mathrm{vR}\mathcal{K}$ through A, B, C, F, M_j. The K_j belong to the ABC tripolar of U, the root of the cubic, while their reciprocals $L_a=B+C-K_a$, etc are aligned on the line $\mathfrak{h}(G,-2)(\delta)$.

Exercise 28.11.19. Let ABC be the reference triangle, R_a the foot of the A-altitude and $A'=2O-A$. Compute the locus of points P such that $(PB, PR_a)=(PA', PC)$ (1) wrt ABC (and obtain a $p\mathcal{K}$ cubic) ; (2) wrt $A'BC$ (and obtain a $n\mathcal{K}$ cubic) ; (3) compare the pivot and the root.

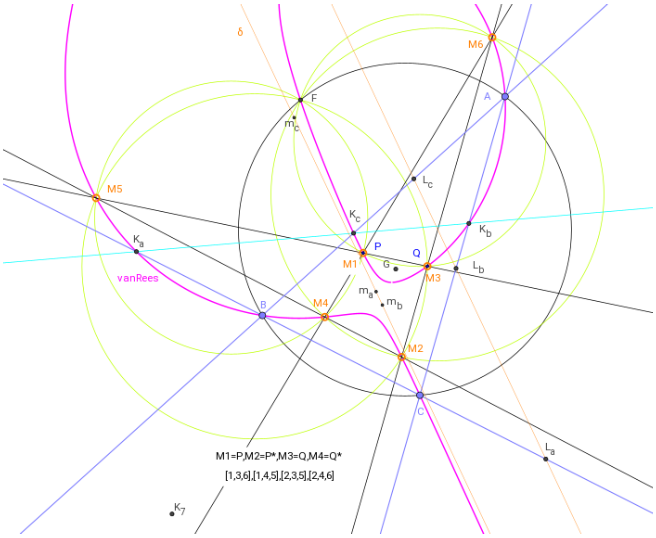


Figure 28.5: van Rees cubic defined by two isogonal pairsSection 12.27

28.12 Exercises

Exercise 28.12.1. Use the Lubin-1 representation, and write the transversal as the t-locus of

$$ik\tau + t\tau : 1 : \frac{-ik}{\tau} + \frac{t}{\tau}$$

where k is real and τ is a turn. Recompute everything, especially $dist\,(\Delta, \mathcal{E})$... that only depends on τ (see Goormaghtigh, 1939).

Exercise 28.12.2. Point O_0 is the perspector of $\mathcal{S}_A, \mathcal{S}_B, \mathcal{S}_C$ with ABC.

Exercise 28.12.3. Let P_0, P_a, P_b, P_c be the projections of M_q on the sidelines. There is a similarity $P_a \mapsto A'$, $P_b \mapsto B'$, $P_c \mapsto C'$. Elaborate further.

Exercise 28.12.4. Let be Q_0, Q_a, Q_b, Q_c the reflections of M_q about the sidelines.
The similarity σ defined by $Q_a \mapsto A'$, $Q_b \mapsto B'$ is centered at M_q, maps the Steiner line onto $\mathcal{L}_0$, sends Q_c to C', the circle ABC onto the Miquel circle and $A \mapsto \mathcal{S}_A$, etc.

Exercise 28.12.5. When points U_1 and U_2 are antipodes on the circumcircle, the orthopole of $Simson\,(U_1)$ is the intersection of $Simson\,(U_2)$ with $Steiner\,(U_1)$.

Exercise 28.12.6. When a line goes through the circumcenter, its orthopole belongs to the nine points circle.

Exercise 28.12.7. When a line goes through a fixed point P, its orthopole belongs to the conic centered at $(P+H)/2$ and passing through the projections of P on the sidelines.

Proposition 28.12.8. *When X moves on a line through the centroid X_2, then orthopole of $trilipo\,(X)$ moves on a line through the orthocenter X_4. For example :*

X on line	*orthopole (trilipo (X)) on line*
L(2,1)	*L(4,9)*
L(2,3)	*L(4,6)*
L(2,6)	*L(4,3)=L(2,3)*
L(2,7)	*L(1,4)*

Proof. Suppose X is not X_2 and does not lie on a sideline of triangle ABC. Then, using barycentrics, we have :

$$orthopole(trilipo(X)) \wedge X_4 =$$
$$S_a\,(y-z) : S_b\,(z-x) : S_c\,(x-z) = (S_a : S_b : S_c) \underset{b}{*} (X \wedge X_2)$$

□

28.13 Diagonal triangle

Notation 28.13.1. In this section, A, B, C is the diagonal triangle of the quadrilateral, with its a, b, c, S_a, S_b, S_c. The four lines are

$$\mathcal{L}_0, \mathcal{L}_a, \mathcal{L}_b, \mathcal{L}_c \simeq \left[\begin{array}{ccc} \rho & \sigma & \tau \end{array}\right], \left[\begin{array}{ccc} -\rho & \sigma & \tau \end{array}\right], \left[\begin{array}{ccc} \rho & -\sigma & \tau \end{array}\right], \left[\begin{array}{ccc} \rho & \sigma & -\tau \end{array}\right]$$

while the six vertices are noted by:

$$A'', B'', C'', A', B', C' \simeq \left[\begin{array}{c} 0 \\ \tau \\ \sigma \end{array}\right], \left[\begin{array}{c} \tau \\ 0 \\ \rho \end{array}\right], \left[\begin{array}{c} \sigma \\ \rho \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ -\tau \\ \sigma \end{array}\right], \left[\begin{array}{c} -\tau \\ 0 \\ \rho \end{array}\right], \left[\begin{array}{c} -\sigma \\ \rho \\ 0 \end{array}\right]$$

so that relation $A', B', C' \in \mathcal{L}_0$ remains valid.

Fact 28.13.2. *Basic objects*

1. *Newton* $\underset{d}{\simeq} \left[\rho^2, \sigma^2, \tau^2\right]$, *while* $(A'' + A')/2 \simeq 0 : -\tau^2 : \sigma^2$
2. *Newton pencil* $B_x = b^2\left(\rho^2 - \sigma^2\right) + c^2\left(\tau^2 - \rho^2\right)$; $E_x = a^2\left(a^2\rho^2 - \sigma^2 b^2 - \tau^2 c^2\right)$
3. *Steiner pencil* $B_x = \rho^2$; $E_x = 0$
4. *Steiner line* $\left[b^2\left(\rho^2 - \sigma^2\right) + c^2\left(\tau^2 - \rho^2\right), \text{etc}\right]$
5. *Miquel* $M_q \underset{d}{\simeq} \left[\begin{array}{c} \left(\sigma^2 - \tau^2\right)\left(b^2\left(\rho^2 - \sigma^2\right) + c^2\left(\tau^2 - \rho^2\right)\right) \\ \left(\tau^2 - \rho^2\right)\left(c^2\left(\sigma^2 - \tau^2\right) + a^2\left(\rho^2 - \sigma^2\right)\right) \\ \left(\rho^2 - \sigma^2\right)\left(a^2\left(\tau^2 - \rho^2\right) + b^2\left(\sigma^2 - \tau^2\right)\right) \end{array}\right]$; $\text{length}(\Gamma_M) \approx 4000$
6. *Centers of circumcircles:* length ≈ 1500
7. *Orthocenters:*
   $H_0 \underset{d}{\simeq} \left((\sigma - \tau)\left(b^2 - c^2\right) + (2\rho + \sigma + \tau)a^2\right)\left(\left(\rho^2 + \sigma\tau\right)a^2 - (\sigma + \tau)\left(\sigma b^2 + \tau c^2\right)\right)$, etc
8. *vanRees cubic:* $\sum_3 x\left(-\rho^2 x^2 + \sigma^2 y^2 + \tau^2 z^2\right) S_a + xyz \sum_3 \rho^2 a^2 = 0$

28.14 Inscribed (ordered) quadrangle

Definition 28.14.1. An inscribed ordered quadrangle is an (ordered) list of four points A, B, C, D inscribed in a same circle. By definition, the diagonals of the associated quadrilateral are AB, CD (and EF) where $E = AD \cap BC$ and $F = AC \cap BD$. Triangles are $\mathcal{T}_0 = ACE$, $\mathcal{T}_a = ADF$, $\mathcal{T}_c = BCF$, $\mathcal{T}_e = BDE$.

Proposition 28.14.2. *Properties of the general quadrangle obviously apply. As a result:*

1. $M_1 \doteq (A + B)/2$, $M_3 = (C + D)/2$, $M_5 = (E + F)/2$ are aligned along the Newton axis

$$\delta \simeq \left[\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\gamma} - \frac{1}{\delta}, \frac{(\alpha + \beta)(\gamma + \delta)(\alpha\beta - \gamma\delta)}{2\alpha\beta\gamma\delta}, -\alpha - \beta + \gamma + \delta\right]$$

2. Triangles $\mathcal{T}_j$ are defining 4 circles $\mathcal{C}_j$ which concur at the Miquel point:

$$M_{\mathrm{q}} \simeq \frac{\alpha\beta - \gamma\delta}{\alpha + \beta - \gamma - \delta} : 1 : \frac{\alpha\beta - \gamma\delta}{\alpha\beta\delta + \alpha\beta\gamma - \delta\alpha\gamma - \gamma\delta\beta}$$

3. Orthocenters $H_0 =_z \dfrac{\alpha^2\delta + \alpha\beta\delta - \gamma\delta\beta - \gamma^2\beta}{\alpha\delta - \beta\gamma}$ are aligned on the Steiner axis:

$$h \simeq \left[\frac{1}{\gamma} + \frac{1}{\delta} - \frac{1}{\alpha} - \frac{1}{\beta}, \quad \frac{(\alpha\gamma - \beta\delta)(\alpha\delta - \beta\gamma)}{\gamma\alpha\beta\delta}, \quad -\alpha - \beta + \gamma + \delta\right]$$

4. The four centers of the $\mathcal{C}_j$, the Miquel point $M_{\rm q}$ and the quadrilateral center O are on the same Miquel circle

$$O_0 \underset{z}{\simeq} \frac{\alpha\,\gamma\,(\delta-\beta)}{\alpha\,\delta-\beta\,\gamma}\;;\;\mathcal{C}_{\rm Mq}\simeq\left(\begin{array}{c}\alpha\,\beta\,\delta+\beta\,\gamma\,\alpha-\delta\,\alpha\,\gamma-\gamma\,\delta\,\beta\\ 0\\ -\alpha\,\gamma\,\delta\,\beta\,(\alpha+\beta-\gamma-\delta)\\ (\alpha\,\delta-\beta\,\gamma)\,(\alpha\,\gamma-\delta\,\beta)\end{array}\right)$$

5. The vR$\mathcal{K}$ cubic, defined by $(PA, PC) + (PB, PD) = 0$ goes through the 6 vertices, δ^∞, $M_{\rm q}$, Ω_x, Ω_y and can be geogebra-drawn using some of the six $V_a = AM_{\rm q} \cap B\delta^\infty$. Caveat: V_e, V_f are nothing but vertices F, E.

6. When restricted to vR$\mathcal{K}$, the Clawson-Schmidt involution Ψ is nothing but the isogonal conjugacy wrt any of the four imbedded triangles

Proposition 28.14.3. *When E, F are at finite distance, $M_q \in EF$ characterizes the inscribed $ABCD$ quadrilaterals.*

Proof. Use general formulas where $A, A', B, B', C, C' = E, F, A, B, C, D$. □

Proposition 28.14.4. *The following properties are specific to the inscribed quadrangles:*

1. $M_{\rm q}$ belongs to diagonal EF

2. Let $Q_j = \delta \cap \mathcal{L}_j$. Then assuming $jk = 0e, ac$ then

$$\left|\frac{(\delta-\gamma)\,(\alpha-\beta)\,(\alpha+\beta-\gamma-\delta)}{2\,(\alpha\,\delta-\beta\,\gamma)\,(\alpha\,\gamma-\delta\,\beta)}\right|^2 = \overline{M_eM_a}\cdot\overline{M_eM_c} = \overline{M_eQ_j}\cdot\overline{M_eQ_k}$$

28.15 Bicentric quadrilaterals

Definition 28.15.1. A quadrilateral (Josefsson, 2023) is said **bicentric** when it has an inscribed circle $(\mathcal{I})$ and a circumscribed circle (O) as well.

Notation 28.15.2. In what follows, $(\mathcal{I})$ is the unit circle (thus $\mathcal{I} \simeq 0:1:0$). Sidelines $\mathcal{L}_1 \cdots \mathcal{L}_4$ are the tangents at the contact points $E_1 \cdots E_4$, parametrized by $\alpha, \beta, \gamma, \delta$. Everything else is indiced using the Clawson scheme. The vertices are $A_{jk} = \mathcal{L}_j \wedge \mathcal{L}_k$. For example: $A_{12} = \dfrac{2\alpha\beta}{\alpha+\beta} : 1 : \dfrac{2}{\alpha+\beta}$.

Fact 28.15.3. *Since cocyclicity of $A_{12}, A_{23}, A_{34}, A_{41}$ is assumed, we have $\alpha\gamma + \beta\delta = 0$. And then*

$$O\simeq\left[\begin{array}{c}2\beta\gamma\alpha\left(\alpha\beta-\alpha\gamma+\beta^2+\beta\gamma\right)\\ (\alpha-\beta)\,(\alpha+\beta)\,(\beta+\gamma)\,(\beta-\gamma)\\ 2\beta\left(\alpha\beta+\alpha\gamma-\beta^2+\beta\gamma\right)\end{array}\right]\;;\;\Delta\doteq O\mathcal{I}\simeq{}^t\left(\begin{array}{c}\alpha\beta+\alpha\gamma-\beta^2+\beta\gamma\\ 0\\ -\alpha\gamma\left(\alpha\beta-\alpha\gamma+\beta^2+\beta\gamma\right)\end{array}\right)$$

The rotation operator is $\mathrm{rotq}: \alpha \mapsto \beta \mapsto \gamma \mapsto -\alpha\gamma/\beta$.

Fact 28.15.4. ***Diagonals***. *The contact diagonals E_1E_3, E_2E_4 are orthogonal at*

$$P\simeq\left[\begin{array}{c}\alpha\gamma\left(\alpha\beta-\alpha\gamma+\beta^2+\beta\gamma\right)\\ 2\alpha\beta\gamma\\ \alpha\beta+\alpha\gamma-\beta^2+\beta\gamma\end{array}\right]\;;\;P\in\Delta$$

The main diagonals $A_{12}A_{34}, A_{23}A_{41}$ are also going through P, while the "third diagonal" is common to both quadrilaterals and goes through A_{13} and A_{24} together with $E_5 = E_1E_4 \cap E_2E_3$ and $E_6 = E_1E_2 \cap E_3E_4$.

Fact 28.15.5. ***Miquel circles***. *Trigone $\mathcal{T}_1^*$ is "all lines except $\mathcal{L}_1$". We have:* $\mathrm{kir}_1 \doteq (A_{23}, A_{34}, A_{42}) \simeq$

$$2\beta\left(\alpha\beta+\alpha\gamma-\beta^2\right) : -4\alpha\,\beta^2\gamma : 2\beta\gamma\alpha\left(\beta^2+\beta\gamma-\alpha\gamma\right) : (\alpha-\beta)\,(\beta+\gamma)\left(\alpha\gamma-\beta^2\right)$$

Therefore their common point is $M_{\rm q} \simeq \dfrac{2\alpha\beta\gamma}{\alpha\beta+\alpha\gamma-\beta^2+\beta\gamma} : 1 : \dfrac{2\beta}{\alpha\beta-\alpha\gamma+\beta^2+\beta\gamma} \in \Delta$

Fact 28.15.6. ***AEF circles***. *These circles are defined by:* $\mathrm{cir}_{12} = (E_1, A_{12}, E_2)$. *Their centers are* $O_{12} \simeq \alpha\beta : \alpha + \beta : 1$.

As a result, we have $T \doteq O_{12}O_{34} \wedge O_{23}O_{41} \simeq \left[\begin{array}{c} \alpha\gamma\left(\alpha\beta - \alpha\gamma + \beta^2 + \beta\gamma\right) \\ 4\alpha\beta\gamma \\ \alpha\beta + \alpha\gamma - \beta^2 + \beta\gamma \end{array}\right] \in \Delta$

Fact 28.15.7. ***PEF Lemoines***. L_{12} *is the Lemoine point of triangle* PE_1E_2. *We have:*

$$L_{12} = |PE_1|^2 E_2 + |PE_2|^2 E_1 + |E_1E_2|^2 P \simeq \left[\begin{array}{c} -\alpha\beta\left(\alpha\,\beta^2 - 4\alpha\beta\gamma + 3\alpha\,\gamma^2 - \beta^3 - 4\beta^2\gamma - 3\gamma^2\beta\right) \\ 8\alpha\gamma\,\beta^2 \\ 3\alpha\,\beta^2 + 4\alpha\beta\gamma + \alpha\,\gamma^2 - 3\beta^3 + 4\beta^2\gamma - \gamma^2\beta \end{array}\right]$$

These four points are on a same circle centered at $\Omega_L \doteq \left[\begin{array}{c} 3\alpha\gamma\left(\alpha\beta + \gamma\beta - \alpha\gamma + \beta^2\right) \\ 8\alpha\beta\gamma \\ 3\left(\alpha\beta + \gamma\beta + \alpha\gamma - \beta^2\right) \end{array}\right] \in \Delta$

Fact 28.15.8. ***PAB circles***. *These circles are defined by* $\mathrm{cir}_2 = (P, A_{12}, A_{23})$. *Their centers are:*

$$O_2 \simeq \left[\begin{array}{c} \beta^6 + 2\beta^3\left(\alpha + \gamma\right)\left(\beta^2 + \alpha\gamma\right) + \left(\alpha^2 + 4\alpha\gamma + \gamma^2\right)\beta^4 + 5\alpha^2\gamma^2\beta^2 \\ 2\beta^5 + 2\beta^2\left(\alpha + \gamma\right)\left(\beta^2 + \alpha\gamma\right) + 4\alpha\,\beta^3\gamma + 2\alpha^2\beta\,\gamma^2 \\ 5\beta^4 + 2\beta\left(\alpha + \gamma\right)\left(\beta^2 + \alpha\gamma\right) + \left(\alpha^2 + 4\alpha\gamma + \gamma^2\right)\beta^2 + \alpha^2\gamma^2 \end{array}\right]$$

As a result, we have $S \doteq O_1O_3 \wedge O_2O_4 \simeq \left[\begin{array}{c} \alpha\gamma\left(\alpha\beta + \alpha\gamma - \beta^2 + \beta\gamma\right)\left(\alpha\beta - \alpha\gamma + \beta^2 + \beta\gamma\right)^2 \\ 4\alpha\gamma\beta\left(\beta - \gamma\right)\left(\beta + \gamma\right)\left(\alpha + \beta\right)\left(\alpha - \beta\right) \\ \left(\alpha\beta + \alpha\gamma - \beta^2 + \beta\gamma\right)^2\left(\alpha\beta - \alpha\gamma + \beta^2 + \beta\gamma\right) \end{array}\right] \in \Delta$

Fact 28.15.9. ***PAB incenters***. *Let* I_2 *be the incenter of triangle* $A_{12}PA_{23}$. *Thus* I_2 *is one of the points* Z *verifying:*

$$\tan\left(PA_{12}, PZ\right) + \tan\left(PA_{23}, PZ\right) = 0$$

This amongs to the product of two lines, one of them going through E_2. *Writing* Z *as* $kP + E_2$ *and solving*

$$\tan\left(A_{12}P, A_{12}Z\right) + \tan\left(A_{12}A_{23}, A_{12}Z\right) = 0$$

leads to $k^2 = \left(\beta^4 - \left(\alpha^2 - 4\alpha\gamma + \gamma^2\right)\beta^2 + \alpha^2\gamma^2\right) \div 4\alpha\,\beta^2\gamma$. *This quantity is invariant by* rotq. *Therefore, the* I_j *are* P*-homothetic with the* E_j *and the center of the* (I_j) *circle is on* Δ. *The other value of* k *leads to the excenters* $J_k \in PE_j$.

Fact 28.15.10. ***Shadows***. *The shadow* A'_{jk} *of vertex* $A_{jk} \in \Gamma$ *is the other intersection between* (O) *and line* $\mathcal{I}A_{jk}$. *Then* $A'_{12}A'_{34}$ *and* $A'_{23}A'_{41}$ *are orthogonal diameters of* (O). *Moreover,*

$$W_1 \doteq \left(A_{12} \wedge A'_{23}\right) \wedge \left(A'_{34} \wedge A_{41}\right) \in \Delta, \text{ etc}$$

Fact 28.15.11. ***Points*** G_j. *They are the intersections of each line* $\mathfrak{L}_j \doteq A_{jk}A'_{lm}$ *with the next one. So that:*

$$G_2 \simeq 4\alpha\beta\gamma : \left(\alpha + \beta\right)\left(\beta + \gamma\right) : 4\beta, \text{ etc}$$

The four lines E_jG_j *are concurrent at*

$$Q \doteq \left[\begin{array}{c} 4\beta\gamma\alpha\left(\alpha\beta - \alpha\gamma + \beta^2 + \beta\gamma\right) \\ \alpha^2\beta^2 - \alpha^2\gamma^2 + 8\alpha\,\beta^2\gamma - \beta^4 + \beta^2\gamma^2 \\ 4\beta\left(\alpha\beta + \alpha\gamma - \beta^2 + \beta\gamma\right) \end{array}\right] \in \Delta$$

while the G_j *are concyclic around* $V \doteq 2O - \mathcal{I} \in \Delta$.

Fact 28.15.12. ***Pencil***. *Cycles* $(P; 0)$, $(\mathcal{I})$, (O) *and* $(M_{\mathrm{q}}; 0)$ *belong to the same pencil. None of the other circles.*

Fact 28.15.13. ***vanRees cubic*** *goes through the 6 vertices, and the six* $V_{12} = A_{12}M_{\mathrm{q}} \cap A_{34}\delta^\infty$. *Caveat:* $V_{13} = A_{24}$, $V_{24} = A_{13}$. *Moreover the cubic goes through* δ^∞, M_{q}, Ω_x, Ω_y. *And* $\mathcal{I}$ *is is the node of the cubic.*

28.16 Rigby points

Proposition 28.16.1. *Given three distinct points U_j on the circumcircle, the following conditions are equivalent:*

1. *The three Simson lines are concurrent in a point K.*
2. *Simson line of U_j is orthogonal with line $U_{j+1}U_{j-1}$*
3. *Two sidelines of $U_1U_2U_3$ have the same orthopole K (and therefore the third too).*

In such a case, the three points U_j are said to form a "Rigby triangle" , and K is their Rigby result. When using Lubin-1 representation, the condition is $\tau_1\tau_2\tau_3 = \alpha\beta\gamma$, and $z(K) = \frac{1}{2}(\alpha+\beta+\gamma+\tau_1+\tau_2+\tau_3)$.

Proof. All three properties lead to the same condition, and the same value of $z(K)$. □

Exercise 28.16.2. Prove that K is the midpoint of $[\alpha+\beta+\gamma\ ;\ \tau_1+\tau_2+\tau_3]$ and conclude (Honsberger, 1995, p. 136).

Remark 28.16.3. In ETC, the third of a Rigby triangle is called the **Simson-Rigby** point of the first two, and noted $U_3 = SR(U_1, U_2)$, while their common result is called the **Rigby-Simson** point and noted $K = RS(U_1, U_2)$.

Corollary 28.16.4. *When using barycentrics, and $U_1 \simeq p : q : r$, $U_2 \simeq u : v : w \in \Gamma$, we have:*

$$\begin{aligned} SR(P,U) &= isog\left((P \wedge U) \wedge \mathcal{L}_b\right) \\ &= \frac{a^2}{(q+r)u-(v+w)p} : \frac{b^2}{(p+r)v-(u+w)q} : \frac{c^2}{(p+q)w-(u+v)r} \end{aligned} \tag{28.13}$$

One can also use the Peter Moses (2004/10) expression :

$$U_3 \simeq \frac{qw-rv}{rwb^2-qvc^2} : \frac{ru-pw}{c^2up-a^2wr} : \frac{pv-uq}{a^2qv-b^2pu}$$

Example 28.16.5. Centers X(2677) to X(2770) are examples of SR and RS points. In the following table, the first three of a quadruple is a (sorted) triangle $U_1U_2U_3$ and its Rigby point.

74	98	691	?	99	1380	1380	?	102	104	2222	?
74	99	842	?	99	2378	2379	?	103	104	1308	?
74	110	477	3258	100	101	1308	?	104	840	1292	?
74	1113	1114	125	100	104	953	3259	104	1381	1382	11
74	1294	1304	?	100	105	840	?	107	110	1304	?
98	110	842	2682	100	109	2222	?	110	110	476	1553
98	843	1296	?	100	110	1290	?	110	112	935	1554
98	1379	1380	115	100	1381	1381	?	110	827	1287	?
99	110	691	?	100	1382	1382	?	110	930	1291	?
99	111	843	?	101	109	929	1521	110	1113	1113	?
99	1379	1379	?	102	103	929	?	110	1114	1114	?

Proposition 28.16.6. ***Third point**. For each point U on the circumcircle, it exists exactly one other point U_2 such that $SR(U, U_3) = U$. This point is the isogonal conjugate of the orthopoint of line U,X(3), belongs to line UU^* and is given by :*

$$\begin{aligned} third(U) &= \frac{a^2}{u^2(bw+cv)(bw-cv)} : \frac{b^2}{v^2(cu+aw)(cu-aw)} : \frac{c^2}{w^2(av+bu)(av-bu)} \\ &= \frac{a^2}{(b^2-c^2)u+(v-w)a^2} : \frac{b^2}{(c^2-a^2)v+(w-u)b^2} : \frac{c^2}{(a^2-b^2)w+(u-v)c^2} \end{aligned}$$

Proof. The limit of line U_1U_2 is orthogonal to line U,X(3). Everything else follows. Remark: relation $SR\left(U, \mathit{third}\left(U\right)\right) = U$ is not granted for a random point U in the triangle plane. But this relation obviously holds when restricting U to the circumcircle. □

Proposition 28.16.7. *Simson-**Moses point**. If points U_1, U_2 are on the circumcircle then, using isoconjugacy wrt pole $P = X_6$ (isogonal conjugacy), the intersection of lines $U_1 \left(U_2\right)^*_P$ and $U_2 \left(U_1\right)^*_P$ is point $SR\left(U_1, U_2\right)$. When another isoconjugacy is used, the intersection remains on the circumcircle. Its barycentrics are :*

$$S^*_P\left(U_1, U_2\right) = \frac{-w_1v_2 + v_1w_2}{qw_1w_2 - rv_1v_2} : \frac{-w_1u_2 + u_1w_2}{pw_1w_2 - ru_1u_2} : \frac{-v_1u_2 + u_1v_2}{pv_1v_2 - qu_1u_2}$$

Proof. The barycentrics are straightforward, while parametrization (7.17) leads to the other properties. □

Definition 28.16.8. In ETC, the Simson-Moses point is computed using $P = X_2$ (isotomic conjugacy) in $S^*_P\left(U_1, U_2\right)$, and noted $SM\left(U_1, U_2\right)$. Centers X(2855) to X(2868) are examples of Simson-Moses points.

Chapter 29

Morley: LyX macros, to be moved atop

ptv: space; negative space
negative thick space
quad;quad

Chapter 30

Curves connecting the Morley centers

30.1 Introduction

30.1.1 The Morley theorem

In it's simplest version, the Morley theorem can be described as in (Coxeter, 1961, p. 24): given a triangle ABC, we use a protractor to divide each angle in three equal parts. Then the intersections of the adjacent angle trisectors form an equilateral triangle (Figure 30.1a).

But a trisector is only defined up to a 120° rotation, and a line like Ab (Figure 30.1b) is yet another trisector... and triangle abc is equilateral again. Such a triangle is perspective with the reference triangle (i.e. lines Aa, Bb, Cc are concurring to the so-called perspector of first kind P Figure 30.1b), and is also in perspective with the excentral triangle, defining the so-called perspector of second kind Q.

In this context, it is interesting to obtain an exhaustive list of all the objects that shares a given set of properties. Concerning the Morley triangles, one can find many statements in the literature, that involves one, three (Kimberling, 1998-2024), five (Wikipedia: Gene Ward Smith, 2004), eighteen (Connes, 1998) and even extending them to 27 –Taylor and Marr (1913) or, differently, Viricel and Bouteloup (1993).

This variety of opinions was created by Frank Morley himself. The property he discovered circa 1899 was, in his opinion, a "simple" byproduct of more deeper geometrical results (Tecosky-Feldman, 1996). So that Morley waited a long period before publishing a separate, formal statement of his theorem : 1924 in a Japanese journal, 1929 in the US. Nevertheless, the topic became quickly a broad research topic. In 1978, Oakley and Baker published a list of 150 references on the subject. As a result of this large activity, quite each author on the topic has coined his own version of the theorem !

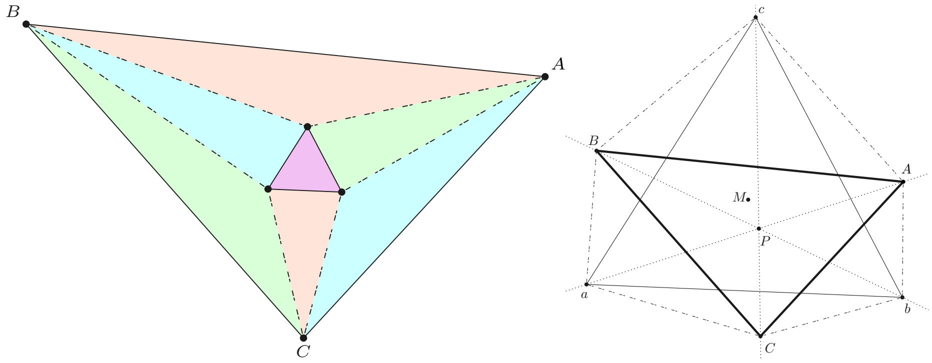
In order not to depart from this well-established tradition, let us state "our own version" of the Morley theorem.

Theorem 0 (*Former results, Morley*)**.** *Given a reference triangle ABC, it exists exactly 18 triangles $\mathcal{T}$ that verify*

1. each vertex of $\mathcal{T}$ is the intersection of two trisectors of triangle ABC;
2. triangle $\mathcal{T}$ is equilateral;
3. triangle $\mathcal{T}$ is perspective with ABC and also with the in-excentral triangles.

30.1.2 Aim of this chapter

Our aim is to investigate the Morley configuration in the context of the complex projective geometry, with a special focus towards curves that contains various classes of points. We will collect and connect some already known results, but also present and prove the following new results :



(a) $\mathcal{T}_{000}$, the flagship triangle (b) $\mathcal{T}_{111}$, another triangle

Figure 30.1: How many are the Morley triangles ?

Theorem 1 (New result)**.** *It exists a circular quintic that contains the 18 Morley centers and the 3 Morley directions of axes (see Figure 30.7). This quintic $\mathcal{L}_M$ contains also the points inventoried as X(549) and X(3534) in the Kimberling (1998-2024). When the vertices belong to the unit circle, the equation of $\mathcal{L}_M$ only depends from the symmetric functions of z_A,* etc.

Theorem 2 (New result)**.** *It exists a circular quintic that contains the 27 secondary perspectors of the Morley configuration. This quintic $\mathcal{L}_Q$ contains also the vertices ABC, the inexcenters, the centroid of $I_aI_bI_c$ and the Bevan points. When $z_A = \alpha^2$,* etc *belong to the unit circle, the equation of $\mathcal{L}_Q$ only depends from the symmetric functions of α, β, γ.*

Theorem 3 (New result)**.** *Suppose that the isocubic pivoting around point U contains an equilateral triangle centered at U. Then: (1) the triangle is homothetic to the Morley triangles; (2) the locus of U is an 8th degree algebraic curve $\mathcal{L}_U$ whose equation depends only from the symmetric functions of z_A,* etc. *The curve contains 39 identified points, the Morley centers among them.*

30.1.3 Organization of this chapter

The starting point of this chapter is Douillet (2010), that was written to explain why Morley centers must be "skew" in some manner, since they are 18 and not 9. In order to allow a more systematic use of computing tools, it has been required to develop a precise and formal frame.

In Section 30.2, we describe the theoretical tools we are using: the Lubin parameterizations and the complex projective geometry. They will be presented in a way that facilitates their utilization inside a formal computing tool like SAGE. It should be noticed that writing $z_A = \alpha^2$, etc is of standard practice when dealing with the in-excenters. Writing $z_A = \alpha^3$, etc was the choice of Lubin (1955) in his founding paper about the Morley configuration.

Using $z_A = \alpha^6$, etc. in order to cover both the 18 Morley centers and the 4 in-excenters appears to be new.

In Section 30.3, we describe the basic objects of the Morley configuration: trisectors, intersections and centers. The Taylor and Marr indices are presented. They are a triple $\mathbf{k} = k_a k_b k_c$ of integers modulo 3. It will be seen that $\sum \mathbf{k} \doteq (k_a + k_b + k_c \mod 3)$ determines three families. $\sum \mathbf{k} \equiv 0$ characterize the nine direct Morley triangles and $\sum \mathbf{k} \equiv 2$ the nine retrograde ones. The ***strange*** case $\sum \mathbf{k} \equiv 1$ will be studied either (these triangles are no more equilateral).

In Section 30.4 we will present the perspectors of these triangles with the reference triangle (first kind, $P_\mathbf{k}$) and the in-excentral triangles (second kind, $Q_\mathbf{k}$). Results about the algebraic curve that contains the $P_\mathbf{k}$ are recalled.

In Section 30.5 we prove our Theorem 1 about the circular curve that contains the 18 Morley centers, and our Theorem 2 about the 27 perspectors Q.

The next Section 30.6 examines two problematics. Firstly, we recall some properties of the pivotal isocubics, and then investigate the intersections of two Morley cubics (having a Morley center as pivot). Secondly, we examine the equilateral triangles inscribed in such cubics and prove our Theorem 3.

At the end of the chapter, we examine what remains of all these properties when the base triangle degenerates into an equilateral one.

The chapter ends by a summary section. The bibliography is integrated into the bibliography of the whole book.

30.2 Some methods

Some of the definitions and properties given here have been stated in detail in previous chapters, but they are stated again here to facilitate an independent access to this chapter.

30.2.1 The complex projective triangle plane

An efficient way to compute the various geometrical objects we are dealing with is to describe the points M by a projective column

$$M \simeq \left(\begin{array}{c} \mathbf{Z} \\ \mathbf{T} \\ \overline{\mathcal{Z}} \end{array} \right) \in \mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$$

equivalently written inline using the "colon notation" : $M \simeq \mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}$. In the simplest situations, we have either :

- $\mathbf{T} = 1$ and $\mathbf{Z} = x+iy$, $\overline{\mathcal{Z}} = x-iy$ (x, y real). In this case, M is an ordinary point, $z \doteq \mathbf{Z}/\mathbf{T}$ is the ordinary complex number describing M when the plane is seen from above and $\zeta \doteq \overline{\mathcal{Z}}/\mathbf{T}$ is the ordinary complex number describing M when the plane is seen from below.
- $\mathbf{T} = 0$ and $\mathbf{Z} = x+iy$, $\overline{\mathcal{Z}} = x-iy$ (x, y real). In this case, M is at infinity and represents the direction of a line, while the complex number $\omega^2 = \mathbf{Z}/\overline{\mathcal{Z}}$ belongs to the unit circle. The fact that ω is only known by its square is a remainder of the fact angles between straight lines are characterized by their tangent and are measured up to $\pi\mathbb{Z}$ and not up to $2\pi\mathbb{Z}$.

30.2.2 Unavoidable constants and base field

Using complex numbers in plane geometry requires a number $\boldsymbol{i}$ to describe how a quarter turn appears when seen from above the plane. Dividing the angles in three introduces a number ϕ such that $1 + \phi + \phi^2 = 0$ and therefore introducing $\omega_{12} = \phi/\boldsymbol{i}$ is unavoidable. Perceiving ω_{12} as the number usually noted $\exp(2i\pi/12)$ is a facility, but is not required, as soon as we use $\boldsymbol{i} = \omega_{12}^3$, $\phi = \omega_{12}^4$ and $\omega_{12}^4 - \omega_{12}^2 + 1 = 0$.

Beside these unavoidable constants, some parameters α, β, γ are required to describe the base triangle ABC. When explaining the rest of the Morley configuration to a formal computing tool like SAGE, we define $\mathbb{K} \doteq \mathbb{Q}(\omega_{12})(\alpha, \beta, \gamma)$ and use the following definitions:

Definition 30.2.1. A **point** P is a column $z : t : \zeta$ that belongs to $\mathbb{K}^3$ and is dealt projectively (when a common factor is detected, the expression is simplified), while a **n-curve** $\mathcal{C}$ is an homogenous polynomial of degree n, i.e. $\mathcal{C} \in \mathbb{K}_n\left[\mathbf{Z}, \mathbf{T}, \overline{\mathcal{Z}}\right]$. And, as usual $P \in \mathcal{C}$ means: substitute $\mathbf{Z} \to z$, etc in $\mathcal{C}$, then simplify and obtain 0.

Definition 30.2.2. A **line** is defined as a first degree polynomial. The line that goes through $z_1 : t_1 : \zeta_1$ and $z_2 : t_2 : \zeta_2$ is given by

$$\det \begin{vmatrix} z_1 & z_2 & \mathbf{Z} \\ t_1 & t_2 & \mathbf{T} \\ \zeta_1 & \zeta_2 & \overline{\mathcal{Z}} \end{vmatrix} \tag{30.1}$$

and the row of its coefficients can be computed using the $\wedge$ operator (the so-called wedge of the columns).

Definition 30.2.3. In the same vein, cocyclicity is described by :

$$\det_{j=1}^{4}\left(\left[z_j\, \overline{z_j},\, z_j,\, \overline{z_j},\, 1\right]\right) = 0 \tag{30.2}$$

30.2.3 Morley method to avoid conjugacy

But the full power of complex geometry is obtained by forgetting any notion of conjugacy. Indeed, this transformation is not a rational operation, while everything goes better when remaining within a fractional field.

The Morley solution to this dilemna is to restrain all of the parameters to the real line, where $\overline{z} = z$, or to the unit circle, where $\overline{z} = 1/z$. And now, we have to consider $\overline{\mathcal{Z}}$ as another independent variable, called "big zeta" and having the same status as the variable $\mathbf{Z}$, called "big Z".

This leads to a larger class of points, the former being only "the ***visible*** points". Among the useful "non-visible" objects are the umbilics of the plane, defined as :

$$\Omega_x \simeq 1 : 0 : 0,\ \Omega_y \simeq 0 : 0 : 1$$

They are also called the circular points at infinity since... they are at infinity ($\mathbf{T} = 0$) and are used to characterize the circles as the conics that goes through both Ω_x and Ω_y.

30.2.4 The Lubin parameterization

The method proposed by Lubin in his founding paper (1955) uses rational fractions $\mathbb{C}(\alpha, \beta, \gamma)$ to describe geometric objects. To obtain the representation of a barycentric point $P \underset{bar}{\simeq} u : v : w$, we need to transcribe its barycentrics u, v, w that are functions of sidelengths a, b, c and angles A, B, C into *rational* functions of the algebraic basis α, β, γ. In the simplest case where these barycentrics only depends on a^2, b^2, c^2, the choice $z_A = \alpha$, etc is clearly sufficient. For example, this happens with the gravity center, or the orthocenter.

For more complicated situations, a more powerful choice must be done, leading to the Lubin parameterizations :

$$z_A = \alpha^n, z_B = \beta^n, z_C = \gamma^n$$

When dealing with the in-excenters, n must be even. The choice $n = 2$ is of common practice (and is equivalent with the Poncelet parameterization, using the incircle as unit circle).

When dividing angles in three, n must be a multiple of 3. Indeed, this was the choice of Lubin in his founding paper (1955). In order to study together the both sort of objects, the choice $n = 6$ is unavoidable. Since the literature uses mostly $n = 2$ and $n = 3$, some confusions can occur when comparing the provided formulas. Surely, a careful reader could recognize which is which by taking into account the total degree of homogeneity. But a better practice is to use symbols $\overset{L1}{\simeq}$, $\overset{L2}{\simeq}$, $\overset{L3}{\simeq}$ and $\overset{L6}{\simeq}$ to emphasize which Lubin representation used in a given projective formula, or $\overset{L1}{=}$, etc for exact formulas (e.g. affixes).

In some occasions, it is more efficient to use A, B, C as basis, instead of Ω_x, O, Ω_y, i.e. to use the so called barycentric coordinates. This will be specified using the $\simeq_{bar}$ notation.

30.2.5 Lubin parameterization using sixth degree formulas

As said in Subsection 30.1.3, the founding paper written Lubin in 1955 was taking into account the Morley centers, but not the inexcenters. In the present context, arcs like $\widehat{BC}$ on the unit circle must be divided into six equal parts instead of three, and the Lubin theorem must be modified accordingly.

Proposition 30.2.4. *(Lubin, 1955) Given $z_0 = \alpha^6$, $z_6 = \beta^6$, $z_{12} = \gamma^6$ on the unit circle, the numbers α, β, γ are determined up to a power of $\psi \doteq \exp(2\boldsymbol{i}\pi/6) = \omega_{12}^2$, and so are the affixes z_j, $0 \leq j \leq 17$ (see Figure 30.2). There is no choice for α, β, γ that gives symmetric expressions for the 18 affixes. On the contrary, α, β, γ can be chosen in order to provide the values written on Figure 30.2 (where $\phi = \psi^2 = \omega_{12}^4$).*

Proof. Consider indices as given modulo $18 = 6 \times 3$. A ratio like

$$\rho_j \doteq (z_j\, z_{j+6}\, z_{j+12}) \div (z_0\, z_6\, z_{12})$$

varies continuously when vertices A, B, C are displaced along the unit circle without crossing each other. But, obviously, ρ_j is a power of ψ and therefore a constant that can be computed in the equilateral case. This gives $\rho_j = \psi^j$.

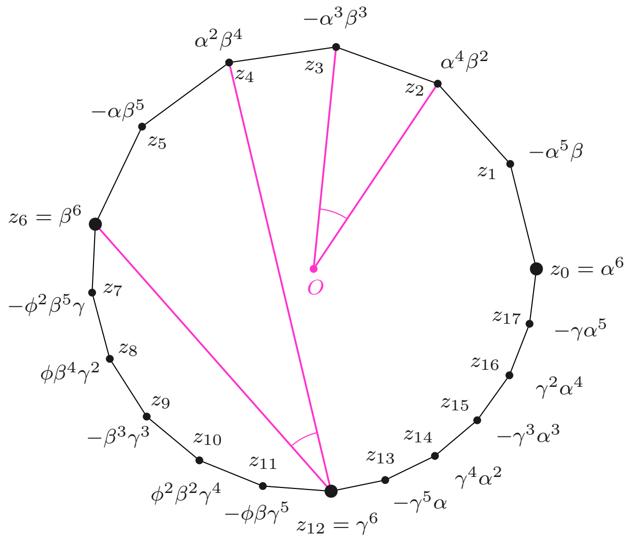


Figure 30.2: The Lubin choices of orientation

(a) Coefficients involved for z_2 and z_{16} are powers of $\phi = \psi^2$. Using transformations $\beta \to \psi\beta$ and $\gamma \to \psi\gamma$ allows to enforce coefficients $+1, +1$ (cosmetic choice).

(b) Coefficients involved for z_3, z_9, z_{15} are ± 1. Transformation $\beta \to -\beta$ and $\alpha \to \psi\alpha$ allows respectively to enforce $z_9 = -\beta^3\gamma^3$ and $z_3 = -\alpha^3\beta^3$ (cosmetic choices). Then $\rho_3 = -1$ ensures the value of z_{15}.

(c(And now all coefficients are fixed: z_1, z_4, z_5 are deduced from z_3/z_2 while z_{17}, z_{15}, z_{14} are deduced from z_{16}/z_{15} and all the others from the ρ_j. □

As a summary, when $A = z_0 = \alpha^6$, $B = z_6 = \beta^6$, $C = z_{12} = \gamma^6$ are given, we can chose α, β, γ in order that mid-arc points z_3, z_9, z_{15} have symmetric expressions $-\alpha^3\beta^3$, $-\beta^3\gamma^3$, $-\gamma^3\alpha^3$. This will lead to a symmetric expression for the coordinates of the incenter. On the contrary, no symmetrical expression can be found for the Morley centers.

30.2.6 Symmetric expressions

When possible, we will use the symmetric functions of α, β, γ to obtain shorter and better looking expressions. The usual notations, s_1, s_2, s_3 will have a meaning that depends of the degree of the used representation. In other words, we note:

$$s_1 \overset{Lk}{=} \alpha + \beta + \gamma \ ; \ s_2 \overset{Lk}{=} \alpha\beta + \beta\gamma + \gamma\alpha \ ; \ s_3 \overset{Lk}{=} \alpha\beta\gamma$$

On the contrary, the σ_j will be reserved for the symmetric expressions of the first degree, i.e. relative to the vertices themselves :

$$\sigma_1 \doteq z_A + z_B + z_C \overset{L6}{=} \alpha^6 + \beta^6 + \gamma^6 \ ; \ \sigma_2 \overset{L6}{=} \alpha^6\beta^6 + \beta^6\gamma^6 + \gamma^6\alpha^6 \ ; \ \sigma_3 \overset{L6}{=} s_3^6$$

30.3 The basic objects

30.3.1 The trisectors

When dealing with numerous objects, the choice of the naming convention is crucial. The following are taken from Taylor and Marr (1913).

Definition 30.3.1. Let $u : A \to B \to C \to A$ be the upwards permutation, and $d = u^{-1}$ the downwards permutation. Trisectors are named $D_k^{\mathbf{x}}$ where $D \in \{A, B, C\}$, $x \in \{u, d\}$ and $k \in \{0, 1, 2\} \in \mathbb{Z}/3\mathbb{Z}$. Value $k = 0$ is used for trisectors obtained by joining points of Figure 30.2, while trisector $D_0^{\mathbf{u}}$ and $D_0^{\mathbf{d}}$ are, respectively "near $u(D)$" and "near $d(D)$". Trisectors $D_k^{\mathbf{u}}$ are k-indexed clockwise, while trisectors $D_k^{\mathbf{d}}$ are k-indexed with the other orientation. When necessary, index k relative to vertex A will be named k_a, etc.

For example, trisector $B_0^{\mathbf{u}}$ is the line E_6E_{14}, while $B_0^{\mathbf{d}}$ is the line E_6E_{16} (see Figure 30.2). Let us compute their equations. We have:

$$E_6 \overset{L6}{\simeq} \begin{pmatrix} \beta^6 \\ 1 \\ \beta^{-6} \end{pmatrix} \simeq \begin{pmatrix} \beta^{12} \\ \beta^6 \\ 1 \end{pmatrix} ; \quad E_{14} \overset{L6}{\simeq} \begin{pmatrix} \gamma^4\alpha^2 \\ 1 \\ \gamma^{-4}\alpha^{-2} \end{pmatrix} \simeq \begin{pmatrix} \gamma^8\alpha^4 \\ \gamma^4\alpha^2 \\ 1 \end{pmatrix}$$

We have chosen to write the variables $\mathbf{Z}, \mathbf{T}, \overline{\mathcal{Z}}$ in that order to facilitate the control the homogeneous degrees appearing in vector and matrices: a difference of six units should remain constant between two adjacent places. And we have :

$$B_q^{\mathbf{u}} \simeq E_6 \wedge E_{14} \overset{L6}{\simeq} \left[\ \phi^q\beta^6 - \alpha^2\gamma^4 \ ; \ -\phi^q\beta^{12} + \alpha^4\gamma^8\phi^{2\,q} \ ; \ -\alpha^4\beta^6\gamma^8\phi^{2\,q} + \alpha^2\beta^{12}\gamma^4\ \right]$$

The $B_q^{\mathbf{d}}$ trisectors are obtained in the same way. After simplifications, we have :

$$\begin{aligned} B_q^{\mathbf{u}} &\overset{L6}{\simeq} \left[\ 1 \ ; \ -\alpha^2\gamma^4\phi^{2q} - \beta^6 \ ; \ \alpha^2\beta^6\gamma^4\phi^{2q}\ \right] \\ B_q^{\mathbf{d}} &\overset{L6}{\simeq} \left[\ 1 \ ; \ -\alpha^4\gamma^2\phi^{\ q} - \beta^6 \ ; \ \alpha^4\beta^6\gamma^2\phi^{\ q}\ \right] \end{aligned} \tag{30.3}$$

When rotating α, β, γ and indices p, q, r we obtain the other trisectors, with the following restriction. Indices k described in Definition 30.3.1 are independent of the cosmetic choices of Figure 30.2. But these choices have created powers of ϕ in expressions of z_8, z_{10}, and the following correspondence must be used:

$$k_a = p + 1,\ k_b = q,\ k_c = r$$

30.3.2 The 27 Morley vertices

By intersecting two trisectors issued from two different vertices of ABC, we obtain $(18 \times 12)/2 = 108$ points. Among them, 27 play a special role and provide remarkable patterns when drawn[1] altogether (Figure 30.3). This depends on the following lemma:

Proposition 30.3.2. *Define the Morley vertices by* $A_{jk} \doteq B_j^{\mathbf{u}} \cap C_k^{\mathbf{d}}$, etc*: i.e. given a vertex, take an **up** trisector from the following vertex and a **down** trisector from the preceding vertex. Then affixes of all these points are polynomials in* $\alpha, \beta, \gamma, \phi$*:*

$$A_{qr} \overset{L6}{=} \alpha^2\beta^4\phi^r + \alpha^2\beta^2\gamma^2\phi^{q+2r} + \alpha^2\gamma^4\phi^{2q} - \beta^4\gamma^2\phi^{q+r} - \beta^2\gamma^4\phi^{2\,q+2\,r}$$

Proof. The computation is straightforward from $A_{qr} \doteq B_q^{\mathbf{u}} \wedge C_r^{\mathbf{d}}$ and (30.3). □

30.3.3 The Lubin proof of the Morley theorem

On Figure 30.3, we can see triangles like $A_{22}A_{10}A_{31}$ that are equilateral but are obtained from only two beams of trisectors. They are called the lighthouse triangles, and are not what we are interested with. Taylor and Marr (1913) have also proposed the following:

Definition 30.3.3. Given the triple of indexes $\mathbf{k} = k_ak_bk_c$, the corresponding Taylor-Marr triangle $\mathcal{T}_{\mathbf{k}}$ is defined as $(A_{k_bk_c}, B_{k_ck_a}, C_{k_ak_b}) = (A_{q;r}, B_{r;p+1}, C_{p+1;q})$.

The Morley theorem can now be stated as:

Theorem 30.3.4. *(Morley) When* $\sum \mathbf{k} \doteq k_a + k_b + k_c \equiv 0 \mod 3$*, triangle* $\mathcal{T}_{\mathbf{k}}$ *is equilateral direct. When* $\sum \mathbf{k} \equiv 2$*, triangle* $\mathcal{T}_{\mathbf{k}}$ *is equilateral retrograde. When* $\sum \mathbf{k} \equiv 1$*, the triangle is not equilateral.*

[1] All our figures are drawn using :

$$\alpha = -1,\ \beta = (15 + 8\,i)/17,\ \gamma = (63 - 16\,i)/65,\ \phi = \left(-1 + i\sqrt{3}\right)/2$$

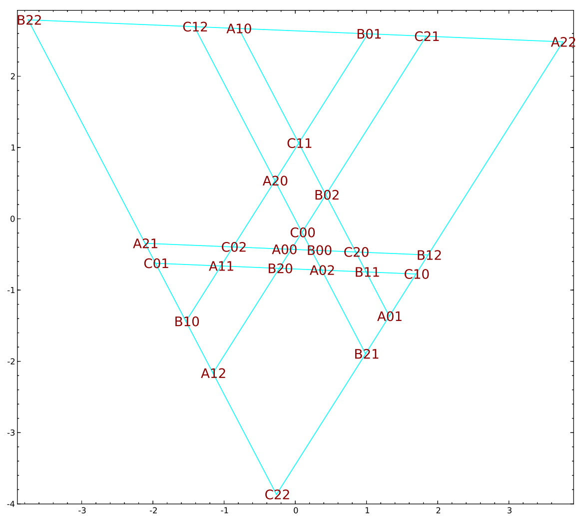


Figure 30.3: Morley equilateral triangles

Proof. (Lubin) Collecting $A_{qr}+\phi B_{r;p+1}+\phi^2 C_{p+1;q}$ one obtains:

$$\left(\phi^{2\,S+2}+\phi^{S+1}+1\right)\alpha^2\beta^2\gamma^2\phi^{r-p-2} - \left(\phi^S-1\right)\left(\frac{\alpha^4\beta^2}{\phi^{r+2}} - \frac{\alpha^4\gamma^2}{\phi^{p+r+2}} - \frac{\alpha^2\beta^4}{\phi^{p+q+1}} + \frac{\alpha^2\gamma^4}{\phi^q} + \frac{\beta^4\gamma^2}{\phi^{p+1}} - \frac{\beta^2\gamma^4}{\phi^{q+r}}\right)$$

where $S=\sum\mathbf{k}=p+q+r+1$. The other orientation is treated using $A_{qr}+\phi^2 B_{r;p+1}+\phi C_{p+1;q}$ □

Remark 30.3.5. When working on triangles $M_{\mathbf{k}}$ where $\mathbf{k}=k_a k_b k_c$ one can adopts two different strategies. Here, we have chosen to use formulas parametrized by the k_j. One other strategy would have been to treat only the indices $\mathbf{k}=000$ (Morley), $\mathbf{k}=100$ (strange) and then propagate the results using the action of some group acting transitively on the corresponding orbit :

$$even=\begin{array}{cc} & \rightarrow\quad r \\ \downarrow & 000\ 120\ 102 \\ m & 222\ 012\ 021 \\ & 111\ 201\ 210 \end{array}\quad odd=\begin{array}{cc} & \rightarrow\quad r \\ \downarrow & 200\ 110\ 101 \\ m & 122\ 002\ 020 \\ & 011\ 221\ 212 \end{array}\quad strange=\begin{array}{cc} & \rightarrow\quad f \\ \downarrow & 100\ 010\ 001 \\ m & 022\ 202\ 220 \\ & 211\ 121\ 112 \end{array}$$

This other strategy is described in Douillet (2014b).*** chapter ***

Proposition 30.3.6. *The Morley center $M_{\mathbf{k}}$, defined as the center of the corresponding Morley triangle $\mathcal{T}_{\mathbf{k}}$, is described by :*

$$M_{\mathbf{k}} \overset{L6}{=} \frac{1}{3}\sum_3\left(\left(\phi^r-\phi^{-p-q}\right)\alpha^2\beta^4-\left(\phi^{p+q}-\phi^{-r}\right)\alpha^4\beta^2\right)\ \ assuming \sum\mathbf{k}\not\equiv 1 \qquad (30.4)$$

The centers of the strange triangles collapse by three and are described by the strange formula :

$$center\left(\mathcal{T}_{\mathbf{k}}\right) \overset{L6}{=} \frac{1}{3}\left(\sum_3\phi^{p-q}\right)\alpha^2\beta^2\gamma^2=\phi^{k_b-k_c}\alpha^2\beta^2\gamma^2\ \ assuming \sum\mathbf{k}\equiv 1$$

Proof. In the general case, writing $\left(A_{qr}+\phi B_{r;p+1}+\phi^2 C_{p+1;q}\right)/3$ gives the sum of these two terms. And it ever happens that one of them vanishes, depending of the value of $\sum\mathbf{k}$. When $p+q+r\equiv 0$, we have $p-q\equiv q-r$, etc. Therefore, a less strange naming for these points is :

$$G_a \overset{L6}{=} \alpha^2\beta^2\gamma^2\ ;\ G_b \overset{L6}{=} \phi G_a\ ;\ G_c \overset{L6}{=} \phi^2 G_a \qquad (30.5)$$

□

30.3.4 The barycentric formula

When considering the 18 genuine Morley centers, Taylor and Marr have obtained the following barycentric formula :

$$M_{\mathbf{k}} \underset{bar}{\simeq} \sin A \left(\cos \frac{A+2\,k_a\pi}{3} + 2\,\cos \frac{B+2\,k_b\pi}{3} \cos \frac{C+2\,k_c\pi}{3} \right) : \text{ etc} : \text{ etc}$$

When trying to express the involved trigonometric quantities in function of α, β, γ, the inscribed angle theorem leads to :

$$\exp \boldsymbol{i}\, \frac{A+2k_a}{3} \overset{L6}{\equiv} -\frac{\gamma}{\beta}\, \phi^p, \text{ etc} \tag{30.6}$$

masking the lack of symmetry by using $p \equiv k_a - 1$, $q = k_b$, $r = k_c$. After some substitutions, we obtain the "symmetrical looking" formula :

$$M_{\mathbf{k}} \underset{bar}{\overset{L6}{\simeq}} \left(\frac{\beta^3}{\gamma^3} - \frac{\gamma^3}{\beta^3} \right) \left(\left(\frac{\alpha\phi^q}{\gamma} + \frac{\gamma}{\alpha\phi^q} \right) \left(\frac{\beta\phi^r}{\alpha} + \frac{\alpha}{\beta\phi^r} \right) - \frac{\gamma\phi^p}{\beta} - \frac{\beta}{\gamma\phi^p} \right) : \text{ etc} : \text{ etc} \tag{30.7}$$

This formula shows what is left in the background when points like M_{000}, M_{111}, M_{222} are described as "the first, second and third Morley centers", and inventoried as X(356), X(3277) and X(3276) in the Kimberling (1998-2024): an expression like $\cos(A/3)$ is not rational in the coordinates of the vertices (nor in the sidelengths of triangle ABC). Therefore a given choice of an algebraical branch cannot be preserved when points are dragged-and-dropped by a dynamical drawing tool, or when a rational parameterization is used.

One can check the equivalence with (30.4) except from the strange case. In the later case, we obtain three points at infinity verifying :

$$\omega^2 (M_{\mathbf{k}}) = \alpha^4\beta^4\gamma^4\phi^{p-q} \tag{30.8}$$

Definition 30.3.7. In order to restrict the notation $M_{\mathbf{k}}$ to the sole and only 18 Morley centers, we define δ_a, δ_b, δ_c (where δ stays for direction) as the points at infinity such that :

$$\omega^2 (\delta_a) \overset{L6}{=} +\alpha^4\beta^4\gamma^4 \; ; \; \omega^2 (\delta_b) \overset{L6}{=} \phi^2\omega (\delta_a) \; ; \; \omega^2 (\delta_c) \overset{L6}{=} \phi^4\omega (\delta_a)$$

Remark 30.3.8. One can see that $\operatorname{isogon} G_a \overset{L6}{\simeq} -\alpha^4\beta^4\gamma^4 : 0 : 1$ is the direction of the sideline $B_{00}C_{00}$. Therefore, $\operatorname{isogon}(-G_a) = \delta_a$ and the δ_x are the directions of the axes common to the Morley triangles.

Contrary to the incenter X(1), there is no cosmetic arrangements that can gives a symmetrical formula for z_{000}. Let us insist once again on this negative result. When dealing with the inexcenters, we can use the Lemoine transforms that replace α by $-\alpha$, etc. If we start from a symmetrical formula, this will generate four distinct points (4, not 8 due to projective properties) and four is the required number of the inexcenters. Concerning the Morley centers, we can replace α by $\alpha\phi$ or $\alpha\phi^2$, etc. Starting from a symmetrical formula, this will generate nine points (9, not 27, due to projective properties) and this is what happens with the strange objects. In order to generate 18 points, the initial formula must be skew.

30.4 Curves connecting the Taylor-Marr perspectors

In this section, we will see that all the 27 Taylor-Marr triangles behave the same way when perspectivity is involved. In other worlds, Morley and ***strange*** triangles have not to be separated according to $\sum \mathbf{k}$ when only alignments are taken into account. This comes from the linearity of the alignments: the powers of ϕ are linearly independent numbers (over the rationals) and the $1 + \phi + \phi^2 = 0$ property is not involved here.

On the contrary, we will see in Subsection 30.5.2 that the family of the Morley centers behaves another way and cannot be enlarged by encompassing the centers G_x of the strange triangles. The enlargement will come from the directions δ_x.

30.4.1 The primary perspectors

Any Taylor-Marr triangle $\mathcal{T}_\mathbf{k}$ is perspective with the reference triangle ABC. The corresponding perspector is noted $P_\mathbf{k}$. Three of them are inventoried in the Kimberling database, where P_{000} =X(357), P_{111} =X(1134) and P_{222} =X(1136). Their barycentrics are given by :

$$P_\mathbf{k} \underset{bar}{\overset{L6}{\simeq}} \frac{\left(\gamma^6 - \beta^6\right)}{\beta^2\gamma^4\phi^p + \beta^4\gamma^2\phi^{-p}} : \text{etc} : \text{etc} \tag{30.9}$$

Figure 30.4 shows the numerous alignments with the vertices ABC. For example, on each line issued from A, there are three $P_\mathbf{k}$: an even (circle), an odd (box) and a strange (cross) one. It can be seen that all these 27 points belong to a same quintic $\mathcal{L}_P$. This is not a new result, this circular quintic is described by its barycentric equation :

$$\mathcal{L}_P : \left(b^2r^2 - c^2q^2\right)p^3S_a + \left(c^2p^2 - a^2r^2\right)S_bq^3 + \left(a^2q^2 - b^2p^2\right)r^3S_c = 0$$

as $Q003$ in (Gibert, 2004-2024), where 95 points have been identified.

From Figure 30.4, we can see that vertices A, B, C are singular points, so that all alignments described above contains in fact five points of the quintic. The complex equation of $\mathcal{L}_P$ is given Figure 30.4, where $\Im(E) = (E - \text{conj}(E))/2$ is the "imaginary part" operator, producing an object F such that $F/\text{conj}(F) = -1$.

$$\mathcal{L}_P \overset{L1}{\simeq} \Im \left(\begin{array}{c} \frac{1}{\sigma_3}\mathbf{Z}^4\overline{\mathcal{Z}} + \frac{\sigma_2}{3\sigma_3}\mathbf{Z}^3\overline{\mathcal{Z}}^2 - \frac{\sigma_2}{3\sigma_3^2}\mathbf{Z}^4\mathbf{T} - \frac{2\sigma_1}{\sigma_3}\mathbf{Z}^3\overline{\mathcal{Z}}\mathbf{T} \\ +\frac{2\sigma_1\sigma_2 - 3\sigma_3}{3S_3^2}\mathbf{Z}^3\mathbf{T}^2 + \frac{\sigma_1^2}{\sigma_3}\mathbf{Z}^2\overline{\mathcal{Z}}\mathbf{T}^2 + \frac{7\sigma_3\sigma_1 - \sigma_1^2\sigma_2 - 2\sigma_2^2}{3\sigma_3^2}\mathbf{Z}^2\mathbf{T}^3 \\ +\frac{2\sigma_1\sigma_2^2 + \sigma_2\sigma_3 - 5\sigma_1^2\sigma_3}{3\sigma_3^2}\mathbf{Z}\mathbf{T}^4 + \frac{\sigma_1^3}{3\sigma_3}\mathbf{T}^5 \end{array} \right)$$

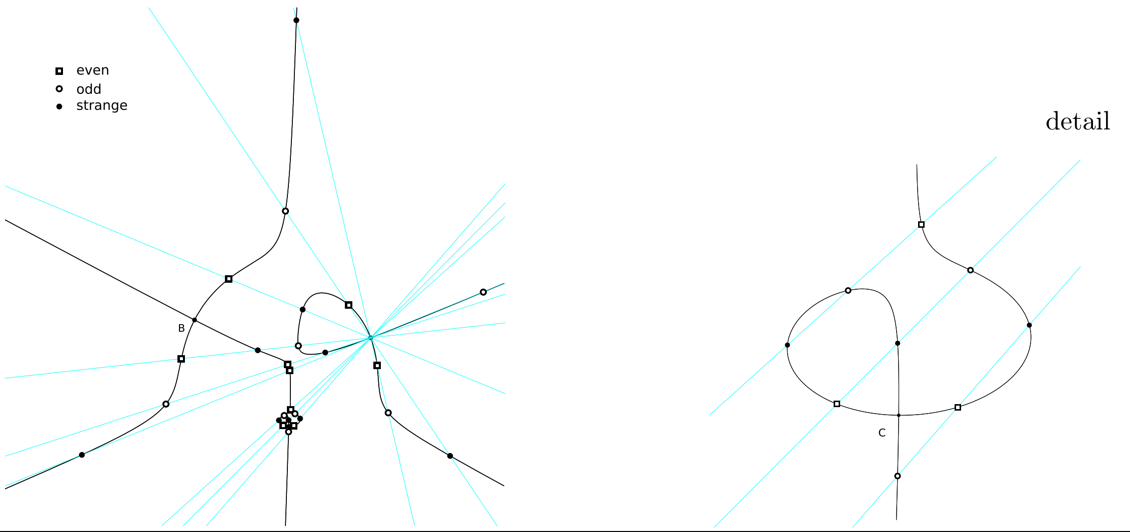


Figure 30.4: Family of X(357) : the 27 $P_\mathbf{k}$ belong to a same quintic

30.4.2 The adjunct primary perspectors

Due to obvious angle properties, the 108 Morley vertices are isogonal conjugates by pairs. For each Taylor-Marr triangle $\mathcal{T}_\mathbf{k}$, it exist an adjunct triangle $\mathcal{T}_\mathbf{k}^*$, isogonal conjugate of $\mathcal{T}_\mathbf{k}$, whose vertices are "**du**" intersections. This triangle $\mathcal{T}_\mathbf{k}^*$ is, in turn, perspective with ABC, and the perspector is nothing but the conjugate $P_\mathbf{k}^*$ of the initial perspector $P_\mathbf{k}^*$ (this is a general property of perspectivities wrt the reference triangle). For example P_{000}^* =X(358) is the perspector of $\mathcal{T}_{000}^*$ with ABC. In the same vein, we have P_{111}^* =X(1135) and P_{222}^* =X(1137).

Since $BC^2 = \left(\beta^6 - \gamma^6\right)\left(\beta^{-6} - \gamma^{-6}\right)$, we have obviously the barycentrics :

$$P_{\mathbf{k}}^{*} \overset{L6}{\underset{bar}{\simeq}} \left(\frac{\gamma^6-\beta^6}{\beta^4\gamma^4}\right)\left(\gamma^2\phi^p+\beta^2\phi^{-p}\right) : \text{etc} : \text{etc} \tag{30.10}$$

As it can be seen by computing the corresponding determinant, the points $M_{\mathbf{k}}$, $P_{\mathbf{k}}$ and $P_{\mathbf{k}}^{*}$ with the same index are aligned (the centroid of the adjunct triangle is not on that line).

The 27 $P_{\mathbf{k}}^{*}$ are aligned by triples with each vertex A, B, C. Moreover, they all belong to the isogonal conjugate of $\mathcal{L}_P$, the quartic $\mathcal{L}_P{}^{*}$ whose barycentric equation is :

$$\mathcal{L}_P{}^{*} : \left(b^2r^2-c^2q^2\right)S_a a^4 qr - \left(a^2r^2-c^2p^2\right)S_b b^4 pr + \left(a^2q^2-b^2p^2\right)S_c c^4 pq = 0$$

This quartic is described as $Q002$ in (Gibert, 2004-2024). Its complex equation is :

$$\Im\left(\begin{array}{c}\frac{\sigma_1}{2\,\sigma_3}\,\mathbf{Z}^3\overline{\mathcal{Z}} - \frac{\sigma_2}{6}\,\mathbf{Z}\overline{\mathcal{Z}}^3 - \frac{1}{\sigma_3}\,\mathbf{Z}^3\mathbf{T} - \frac{\sigma_2}{\sigma_3}\,\mathbf{Z}^2\overline{\mathcal{Z}}\mathbf{T} \\ +\frac{\sigma_1}{\sigma_3}\,\mathbf{Z}^2\mathbf{T}^2 + \frac{\sigma_2-\sigma_1^2}{3\,\sigma_3}\,\mathbf{Z}\mathbf{T}^3\end{array}\right) = 0$$

while its rectangular asymptotes are crossing at $3\left(\sigma_2^2-\sigma_1\sigma_3\right) \div 2\,\sigma_1\sigma_2$=X(3292).

Let us recall that the degree of the isogonal conjugate of a curve is twice the degree of the original curve. In the case of an isogonal cubic (like the $p\mathcal{K}_{\mathbf{k}}^{\mathcal{M}}$ that are presented later), the three sidelines appear in factor and the degree reduces to $6-3=3$ (curve $p\mathcal{K}_{\mathbf{k}}^{\mathcal{M}}$ is invariant). Concerning $\mathcal{L}_P$, the vertices count twice: when taking the isogonal conjugate of this curve, the sidelines appear twice in factor, and the degree reduces to $2\times 5-2\times 3=4$. This is the reason why the curve $\mathcal{L}_P{}^{*}$ is a quartic and not a curve of higher degree.

30.4.3 The secondary perspectors and their adjuncts

Any Morley triangle $\mathcal{T}_{\mathbf{k}}$ is also perspective with the excenter triangle $I_aI_bI_c$. The corresponding perspector is noted $Q_{\mathbf{k}}$. Point Q_{000} is inventoried in the Kimberling database as X(1507). For the general case, we have the following barycentrics :

$$Q_{\mathbf{k}} \overset{L6}{\underset{bar}{\simeq}} \left(\frac{\gamma^3}{\beta^3}-\frac{\beta^3}{\gamma^3}\right)\left(1+\frac{\phi^p\gamma}{\beta}+\frac{\beta}{\phi^p\gamma}-\frac{\phi^q\alpha}{\gamma}-\frac{\gamma}{\phi^q\alpha}-\frac{\phi^r\beta}{\alpha}-\frac{\alpha}{\phi^r\beta}\right) : \text{etc} : \text{etc} \tag{30.11}$$

Figure 30.5 shows the numerous alignments with the excenters $I_aI_bI_c$. For example, on each line issued from I_a, there are three $Q_{\mathbf{k}}$: an even (circle), an odd (box) and a strange one (cross). It can be seen that all these 27 points belong to a same circular quintic $\mathcal{L}_Q$. This is a new result, and will be expanded in Subsection 30.5.4. On this quintic, points I_a, I_b, I_c are singular points, so that all alignments of the former paragraph contains in fact five points of the quintic.

30.4.4 The adjunct secondary perspectors

It happens that any adjunct Morley triangle $\mathcal{T}_{\mathbf{k}}^{*}$ is also perspective with the excenter triangle $I_aI_bI_c$. The corresponding perspector is noted $R_{\mathbf{k}}$. Point R_{000} is inventoried as X(1508) in Kimberling database. For the general case, we have the following barycentrics :

$$R_{\mathbf{k}} \overset{L6}{\underset{bar}{\simeq}} \left(\frac{\gamma^3}{\beta^3}-\frac{\beta^3}{\gamma^3}\right)\left(1+\frac{\beta\gamma}{\phi^p\gamma^2+\beta^2\phi^{-p}}-\frac{\gamma\alpha}{\phi^q\alpha^2+\gamma^2\phi^{-q}}-\frac{\alpha\beta}{\phi^r\beta^2+\alpha^2\phi^{-r}}\right) : \text{etc} : \text{etc}$$

These points are aligned by triples (even, odd, fake) with the excenters. Moreover, I_0, $P_{\mathbf{k}}$, $R_{\mathbf{k}}$ are aligned for each $\mathbf{k}$. Nevertheless, there is no quintic that contains all the 27 $R_{\mathbf{k}}$, while the circular quintic that contains the 18 Morley perspectors contains no other named centers.

30.5 Two new orbital curves

30.5.1 How to discover a curve that contains a set of points

Definition 30.5.1. A N-sized set of points is said to be "co-n-curve" if it exists an algebraic n-th degree curve that contains the set, and verifies $(n+1)(n+2) < 2N$.

$\mathcal{L}_Q \overset{L2}{\simeq} \mathcal{Q}_1 + \mathcal{Q}_2$ where :

$$\mathcal{Q}_1 \overset{L2}{=} \Im\left(\begin{array}{l}\dfrac{3}{s_3^2}\,\mathbf{Z}^4\overline{\mathcal{Z}} - \dfrac{s_1}{s_3^3}\,\mathbf{Z}^4\mathbf{T} - \dfrac{12}{s_3^2}\,\mathbf{Z}^3\mathbf{T}^2 - \dfrac{4\,s_1 s_3 + 6\,\sigma_2}{s_3^2}\,\mathbf{Z}^2\overline{\mathcal{Z}}\mathbf{T}^2 \\ + \dfrac{12\,\sigma_1 s_3 + 2\,\sigma_2 s_1 + 4\,s_2 s_3}{s_3^3}\,\mathbf{Z}^2\mathbf{T}^3 - \dfrac{(3\,\sigma_1 + 4\,s_2)\,\sigma_1}{s_3^2}\,\mathbf{Z}\mathbf{T}^4 + \dfrac{\sigma_1^2 s_2}{s_3^2}\,\mathbf{T}^5\end{array}\right)$$

$$\mathcal{Q}_2 \overset{L2}{=} \left(-\frac{s_1}{s_3}\,\mathbf{Z} + s_2\overline{\mathcal{Z}}\right)\left(\mathbf{Z}\overline{\mathcal{Z}} - \frac{s_1}{s_3}\,\mathbf{Z}\mathbf{T} - s_2\,\overline{\mathcal{Z}}\mathbf{T} + \left(\frac{s_1 s_2}{s_3} - 4\right)\mathbf{T}^2\right)^2$$

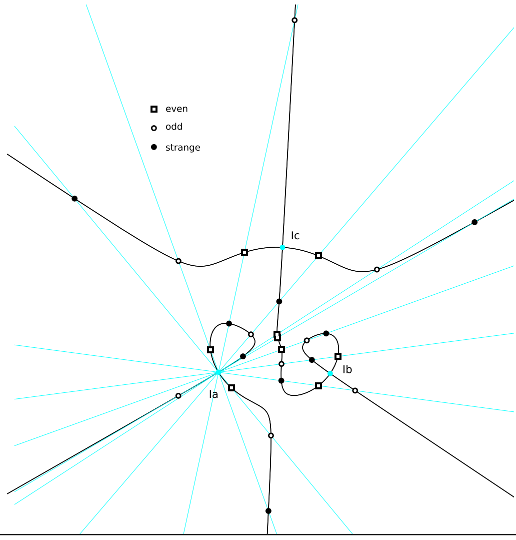


Figure 30.5: Family of X(1507) : the 27 $Q_{\mathbf{k}}$ belong to a same quintic

Figure 30.6: A 10th degree that doesn't go through the 72 R perspectors

Without the condition related to the degree, such property would be trivial. On the contrary, discovering (and then proving) that a given set is co-n-curve is rather difficult when the set is large. Numerical investigations can be carried in order to suggest the existence of such a curve (or to prove the lack of existence). Given an set K and a degree n, we choose a proper subset K' such $(n+1)\,(n+2)\,/2 < |K'|$ and we minimize

$$\chi^2 \doteq \frac{1}{|K'|}\sum_{m\in K'} f^2\left(x_m, y_m\right) \quad \text{where} \quad f\left(x, y\right) = \sum_{j+k\leq n} c_{jk}x^j y^k$$

The quantity χ^2 is a quadratic function of the c_{jk}, and the minimization has to be conducted under a condition of normalization. Condition $\sum c_{jk}^2 = 1$ would be the best theoretical one, but we better guess and check for a non vanishing coefficient, and force it to 1.

These computations are repeated with increasing numerical precision, say $\epsilon = 1/2^{digits}$. When χ^2 decreases with ϵ, and remains of the same relative order, this is a good omen. It remains to check if $f^2(m) \approx \chi^2$ when m is chosen in $K \setminus K'$.

When, for any single numerical attempt, χ^2 remains quite equal to a non-zero constant as $\epsilon \to 0$, this prove that the requested algebraic curve does not exists. For example, the 18-sized set $\{R_{\mathbf{k}}, \sum \mathbf{k} \not\equiv 1\}$ is not co-4-curve, while sets $K \cup ABC$ or $K \cup I_a I_b I_c$ are not co-5-curve. . In Figure 30.6, we give the result of an attempt to include $\mathcal{PG}_{72}(R_0)$ in a tenth degree curve. When using the already given values, we obtain a curve that looks nice, but χ^2 remains around $6E-8$ even if when we increase the precision and use $\epsilon = 1E-200$.

30.5.2 The quintic of the Morley centers

As stated in our Theorem 1, the 18 Morley centers belongs to a circular quintic $\mathcal{L}_M$. Its equation can be obtained "in brute force" by solving a formal system of 20 equations in 20 unknowns whose coefficients are polynomials in α, β, γ (with degrees up to 15).

$$\mathcal{L}_M \overset{L1}{\simeq} \Im \left(\begin{array}{l} \dfrac{1}{\sigma_3}\,\mathbf{Z}^4\overline{\mathcal{Z}} + \dfrac{\sigma_2}{2\,\sigma_3^2}\,\mathbf{Z}^4\mathbf{T} - \dfrac{\sigma_1\sigma_2 + 27\sigma_3}{9\,\sigma_3^2}\,\mathbf{Z}^3\mathbf{T}^2 + \dfrac{\sigma_2}{\sigma_3}\,\mathbf{Z}^2\overline{\mathcal{Z}}\mathbf{T}^2 \\ \qquad + \dfrac{3\,\sigma_3\sigma_1 - 5\sigma_2}{6\,\sigma_3^2}\,\mathbf{Z}^2\mathbf{T}^3 + \dfrac{8\,\sigma_3\sigma_1^2 - 9\,\sigma_2\sigma_3 + \sigma_1\sigma_2^2}{6\sigma_3^2}\,\mathbf{Z}\mathbf{T}^4 - \dfrac{2\,\sigma_1^3}{9\,\sigma_3}\,\mathbf{T}^5 \end{array} \right)$$

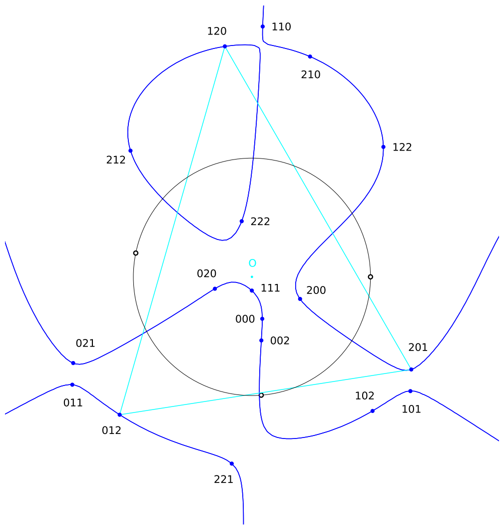


Figure 30.7: The Quintic of the Morley's centers

The computation is not so easy to conduct since intermediate expressions are really huge, but the result is quite simple when expressed in terms of the z_A, etc themselves. One obtains Figure 30.7. The five points at infinity are the umbilics together with the directions of the Morley axes. The circular asymptotes concur at $z = -\sigma_1/2$ =X(550) while the ordinary ones concur at $z = +\sigma_1/6$=X(549). The later is on the curve, the former is not. Moreover point $z = -2\sigma_1/3$=X(3534) also belongs to $\mathcal{L}_M$.

The barycentric equation of this quintic is antisymmetric (when a, b, c and x, y, z are rotated in the same manner). One obtains Figure 30.8 (top).

30.5.3 A lemma about the Bevan centers

The circle through the excenters is called the Bevan circle. Using determinant (30.2), its equation is easy to compute, leading to :

$$B_{circle} \overset{L2}{\simeq} \mathbf{Z}\overline{\mathcal{Z}} - \frac{s_1}{s_3}\,\mathbf{Z}\mathbf{T} - s_2\,\overline{\mathcal{Z}}\mathbf{T} + \left(\frac{s_1 s_2}{s_3} - 4\right)\mathbf{T}^2 = 0$$

$$\mathcal{L}_M \simeq \sum_3 \frac{b^2-c^2}{a^2}\left(\begin{array}{c} 56\,S^2 S_a\,x^5 - S_a\left(8\,S_bS_c + 62\,S_a\left(S_b+S_c\right)\right)x^3yz \\ +a^2\left(21\,S_bS_c + 102\,S_a\left(S_b+S_c\right)\right)xy^2z^2 \end{array}\right)$$

$$+\mathfrak{S}_6 \frac{1}{a^2}\left(\begin{array}{c} 34\,S_a^2S_b^2 + 6\,S_a^2S_b\,S_c - 28\,S_a^2S_c^2 + 48\,S_a\,S_b^3 + 103\,S_a\,S_b^2S_c \\ +47\,S_a\,S_b\,S_c^2 + 75\,S_b^3S_c + 75\,S_b^2S_c^2 \end{array}\right)x^3y^2$$

$$+\mathfrak{S}_6\left(25\,S_a^2S_c^2 - 4\,S_a^2S_b^2 + 21\,S_a^2S_b\,S_c + 5\,S_a\,S_b^2S_c + 61\,S_a\,S_b\,S_c^2 + 36\,S_b^2S_c^2\right)\frac{x^4y}{a^2}$$

where $\mathfrak{S}_6$ denotes a sum of six terms taking the signatures into account, and $\sum_3$ denotes an ordinary cyclic sum of three terms. Moreover, S is the area of ABC and $S_a = \left(b^2+c^2-a^2\right)/2$, etc. are the Conway symbols.

$$\mathcal{L}_Q \simeq -\left(a+b+c\right)xyz\sum_3 ab\left(a-b\right)\left(a+b-c\right)\left(2\,abz^2 - 3\,c^2xy\right)$$

$$+\mathfrak{S}_6\, b^2c^3\left(a+b-c\right)\left(a-b+c\right)x^4y$$

$$+\mathfrak{S}_6\, bc^3\left(2\,a^3 + a^2b - a^2c - 2\,ab^2 + 4\,abc - 2\,ac^2 - b^3 - b^2c + bc^2 + c^3\right)x^3y^2$$

Figure 30.8: Barycentric equations of $\mathcal{L}_M$ and $\mathcal{L}_Q$

This can be rewritten as $(\mathbf{Z}-s_2)\operatorname{conj}(\mathbf{Z}-s_2)=4$. Therefore, the center of this circle, known as X(40) in Kimberling, is the reflection of I_0 in O while the radius of the circle is twice those of the circumcircle.

The line I_0O is called the Bevan axis. It is a diameter of the Bevan circle and its equation is $s_1\mathbf{Z} - s_2s_3\overline{\mathcal{Z}} = 0$. The Bevan points are the intersection of the axis and the circle and listed as X(2448), X(2449). We obtain :

$$z\left(B_\pm\right) \overset{L2}{=} s_2 \pm 2\,\frac{s_3}{s_1}\sqrt{\frac{s_1\,s_2}{s_3}} = s_2 \pm 2\,\frac{s_3}{s_1} \div \left|\frac{s_3}{s_1}\right|$$

where the radicand is $s_1\,\overline{s_1} \geq 0$. Written using the first expression, both points cannot be distinguished in the Lubin(2) representation. When using the second one, we are describing the Bevan points as the center plus or minus the oriented radius, at the cost of leaving the algebraical domain.

30.5.4 The quintic of the secondary perspectors

A numerical study shows that all the 27 secondary perspectors $Q_{\mathbf{k}}$ belongs to a same quintic $\mathcal{L}_Q$ (see Figure 30.5). This time, it would be foolish to proceed in brute force. The Maple length of $z\left(Q_{000}\right)$ is 1583, while the Maple length of $z\left(M_{000}\right)$ was only 197. Moreover, since the incenter don't play the same role as the excenters, the equation of $\mathcal{L}_Q$ cannot be expressed as a Lubin(3) expression, and computations have to be done using Lubin(6).

But there is another way of attack. We can obtain a numerical equation of $\mathcal{L}_Q$ using $a,b,c = 6,9,13$ (the standard Kimberling triangle) and then use the Kimberling inventory to detect which simpler triangle centers belong to $\mathcal{L}_Q$. We obtain X(1)=incenter, X(164), X(165), X(2448), X(2449). X(165) is the centroid of $I_aI_bI_c$ and its affix is $-z(1)/3 = s_2/3$, while X(2448) and X(2449) are the Bevan points described in the former section. Point X(164), which is the incenter of $I_aI_bI_c$ must be discarded since its affix does not belongs to Lubin(6).

Let us resume what can be guessed, and count the resulting equations. We have $A, B, C \in \mathcal{L}_Q$ (3), $I_0 \in \mathcal{L}_Q$ (1), each excenter is singular (3×3), $\mathcal{L}_Q$ is circular (2), X(165) is the singular focus and belongs to the curve (2+1), the Bevan pair X(2444), X(2445) belongs to $\mathcal{L}_Q$ (but counts only for 1) : this amounts to 19, exactly one minus the number of independent equations required to determine a quintic.

Therefore we can conduct the computations using $z_A = \alpha^2$ (i.e. the 2nd degree Lubin parameterization), leaving one coefficient undetermined. After what, it remains to change to 6th Lubin and fix the remaining coefficient so that Q_{100} belongs to the curve. It remains to check, by an explicite computation, that Q_{000} belongs also to the curve. But one can check also that curve $\mathcal{L}_Q$ remains rational over 2nd degree Lubin, and can be rewritten as such. By algebraic symmetry, this prove the result for the 7+17 remaining perspectors, so that 47 points of the curve have been identified.

In the equation given Figure 30.5, quintic $\mathcal{Q}_1$ is circular, and admits X(165), $z \overset{L2}{=} s_2/3$, as singular focus. The real points at infinity are the directions of the Morley axes, and the corresponding asymptotes concur at $z \overset{L2}{=} -s_2/9$, not on the curve (and not in ETC). All the inexcenters are singular points for this curve. On the other hand, quintic $\mathcal{Q}_2$ is the product of the Bevan line and the Bevan circle (counted twice).

The barycentric equation of this quintic is antisymmetric (when a, b, c and x, y, z are rotated in the same manner). This leads to the new result given Figure 30.8.

30.6 Two properties concerning the Morley cubics

30.6.1 Description of a pivotal cubic

Let us consider a fixed point $U \simeq z : t : \zeta$. The set of all the points $P \simeq \mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}$ that are aligned with U and their isogonal conjugate isogon P is a cubic (called the iso-cubic related to the pivot U and noted $p\mathcal{K}_U$). Its equation is easy to obtain from the corresponding determinant, leading to :

$$p\mathcal{K}_U\left(\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}\right) \overset{L1}{\simeq} \Im\left(\zeta \mathbf{Z}^2 \overline{\mathcal{Z}} - \frac{z}{\sigma_3}\mathbf{Z}^2\mathbf{T} + \left(\frac{\sigma_1}{\sigma_3}z - 2\zeta\right)\mathbf{Z}\mathbf{T}^2 + \sigma_1 \zeta\,\mathbf{T}^3\right) + t\,\Im\left(\frac{1}{\sigma_3}\mathbf{Z}^3 - \frac{\sigma_1}{\sigma_3}\mathbf{Z}^2\mathbf{T} + \frac{\sigma_2}{\sigma_3}\mathbf{Z}\mathbf{T}^2\right) \quad (30.12)$$

When obtained along this curve, the point isogon A is together on BC and on AU, so that $BC \cap AU$, etc (the so-called vertices of the U-Cevian triangle) belong to $p\mathcal{K}_U$. From isogon $I_0 = I_0$, points U, I_0, isogon I_0 are aligned, and the four inexcenters belong to the cubic. When making $\mathbf{T} = 0$, one sees that the three points at infinity $z_j : 0 : \zeta_j$ (j=1,2,3) must verify: $\sigma_3^2 \zeta_1\zeta_2\zeta_3 = z_1 z_2 z_3$. Conversely, three such points can be used to characterize a $p\mathcal{K}$.

When U is the circumcenter $(0 : 1 : 0)$, only the second term of (30.12) remains. One obtains the $p\mathcal{K}_O$ cubic, inventoried as K003 in Gibert (2004-2024). This cubic has three asymptotes in the directions δ of the Morley axes since the $\omega_j^2 : 0 : 1$ where $\omega_j^2 \overset{L6}{=} \phi^j \alpha^4\beta^4\gamma^4$ are clearly the roots of the leading part. When U is at infinity, the two others are the umbilics and $p\mathcal{K}$ is circular (the McKay cubics).

General properties of such cubics are well known. For example, $p\mathcal{K}_U$ cuts the circumcenter, apart from A, B, C in three points Γ_n (n=1,2,3) such that the U is the orthocenter of triangle $(\Gamma_1, \Gamma_2, \Gamma_3)$ and lines $U\Gamma_n$ are parallel to the asymptotes Gibert (2007). Beside that, when $U \neq V$, the remaining common points of $p\mathcal{K}_U$ and $p\mathcal{K}_V$ are isogonal conjugates and belong to the line UV.

Let us now consider the Morley cubics $p\mathcal{K}_{\mathbf{k}}^{\mathcal{M}}$, obtained by taking the corresponding Morley center $M_{\mathbf{k}}$as pivot. Many points are known to be long to these cubics. For example:

Proposition 30.6.1. *Each cubic $p\mathcal{K}_{\mathbf{k}}^{\mathcal{M}}$ is circumscribed to the corresponding Morley triangle $\mathcal{T}_{\mathbf{k}}$.*

Proof. One can conduct the 18×3 computations... or check that $A_{00} \in p\mathcal{K}_{000}^{\mathcal{M}}$ and propagate this property to the other objects (more details in Douillet, 2014b). ☐

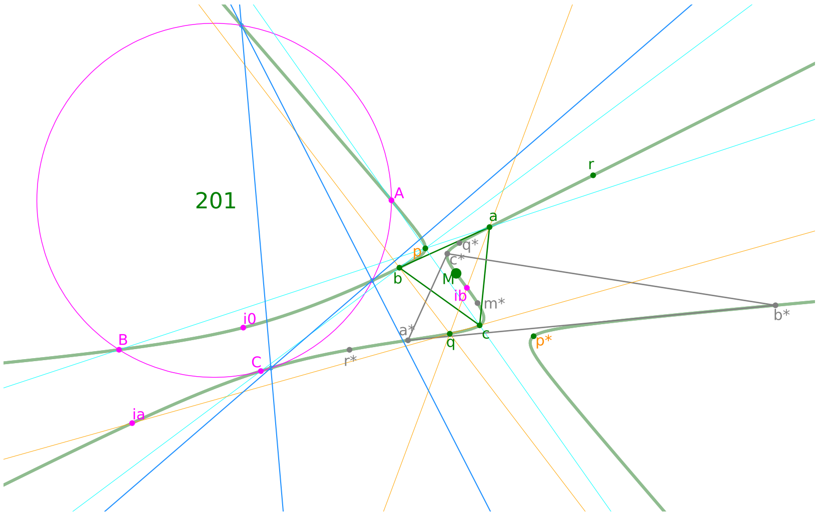


Figure 30.9: The Morley isocubic KM201 ($j = 03$), viewed as an ordinary isocubic

30.6.2 Intersections between Morley cubics

It can be seen that operator $\mathbf{s} : \alpha \to \beta;\ \beta \to \phi^2\gamma;\ \gamma \to \alpha$ (the so called "strange" operator) fixes the points M_{000} and I_0, and rotates the triples (A, B, C) and (I_a, I_b, I_c). Therefore, the cubic $p\mathcal{K}_{000}^{\mathcal{M}}$ is globally invariant by $\mathbf{s}$.

Let us now consider the intersections $W_{\mathbf{k}\kappa}$ between two Morley cubics. From the degree of both curves, nine points are required. Points $A, B, C, I_0 I_a I_b I_c$ are obvious, and it remains two other points. They are isogonal conjugates, and therefore belong to the line $M_{\mathbf{k}} M_\kappa$.

$\mathbf{k}$	#	point	discriminant
222,111	18	P_{000}^*, P_{111}^*	square
120,102	18		Δ
012,210	18		$\Delta \cdot \mathbf{s}$
201,021	18		$\Delta \cdot \mathbf{s}^2$
200,020,002	9×3	A_{00}, B_{00}, C_{00}	square
011	9		δ
101	9		$\delta \cdot \mathbf{s}$
110	9		$\delta \cdot \mathbf{s}^2$
122,212,221	9×3	$\widetilde{S}$, $\widetilde{S}\,\mathbf{s}$, $\widetilde{S}\,\mathbf{s}^2$	square

Table 30.1: Intersection of the $\mathbf{k}$ cubic with the $\kappa = 000$ cubic

The equation (E) for the two points W is obtained by intersecting $p\mathcal{K}_{\mathbf{k}}^{\mathcal{M}}$ by the line $M_{\mathbf{k}} M_\kappa$ and discarding the factor relative to $W = M_{\mathbf{k}}$. Part of the time, it happens that this equation splits totally (and the discriminant of (E) is a perfect square). Otherwise, Table 30.1 show that the discriminant takes only two essentially different values, according to the parity.

Here again, the strange operator appears as the operator that allows to jump from an orbit to some others (but not to all of them). Each of the discriminants δ and Δ can be written as the product of the four Lemoine replicas of a polynomial. For example :

$$\delta_{12} = \prod_{j=0}^{j=3} L_j \left(\alpha\beta\gamma + \left(\alpha^2\gamma - 2\,\alpha\beta^2 - \beta\gamma^2 \right) \phi + \left(\beta^2\gamma - 2\,\alpha\gamma^2 - \alpha^2\beta \right) \phi^2 \right)$$

The existence of exact solutions is correlated with the fact that triangles $\mathcal{T}_{\mathbf{k}}$ and $\varphi(\mathcal{T}_{\mathbf{k}})$ are homothetic for each $\varphi \in \mathcal{PG}_{18}$, and therefore are in perspective. Their common perspector $S_{\mathbf{k}\kappa}$

can ever been obtained without radicals, and belongs necessarily to line $M_{\mathbf{k}}M_{\kappa}$. When this point belongs to one of the cubics, so does also isogon $S_{\mathbf{k}\kappa}$ and line $M_{\mathbf{k}}M_{\kappa}$ cannot have any other intersections with $p\mathcal{K}_{\mathbf{k}}^{\mathcal{M}}$.

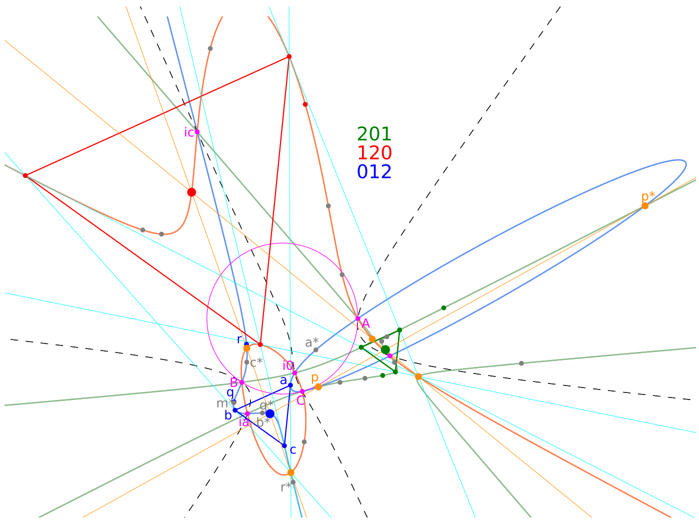


Figure 30.10: Three cubics from the same m-orbit

The first case is the most striking. We consider a m orbit. It contains a Morley center M_j, and its relatives: M_{j+} defined as $m\left(M_j\right)$ and M_{j-} defined as $m^2\left(M_j\right)$. For example, using $j=0$, the orbit is M_{000}=X(356), $M_{000+}=M_{222}$=X(3277) and $M_{000-}=M_{111}$=X(3276). The orbit M_{201}, $M_{201+}=M_{120}$ and M_{012} was used to draw Figure 30.10. The first specificity of such a orbit comes from $M_j+M_{j+}+M_{j-}=0$: the centroid of the three centers is the circumcenter. Therefore, the sum of the equations of the three Morley cubics is the McCay cubic $p\mathcal{K}_O$ (dotted line).

The common points of two cubics of the m-orbit are two Morley perspectors. More precisely, $p\mathcal{K}_{201}^{\mathcal{M}}\cap p\mathcal{K}_{120}^{\mathcal{M}}$ contains P_{201} which is the perspector of triangle $\mathcal{T}_{201}$ wrt ABC and P_{201}^{*} which is the perspector of $\mathcal{T}_{201}^{*}$ wrt ABC but also the perspector of $\mathcal{T}_{201}$ with $\mathcal{T}_{120}$.

30.6.3 Inscribed pivotal equilateral triangles

Let us now prouve our Theorem 3 (stated on page 438) and show that the equation of the locus $\mathcal{L}_U$ is as given in Figure 30.11.

Proof. Consider the line $\Delta_\tau=\{z_U+\mu\tau,\ \mu\in\mathbb{R}\}$ issued from point U and directed by the unimodular complex τ (know up to a ±1 factor). This line cuts the isocubic $p\mathcal{K}_U$ pivoting around U in three points. One of them is U itself, and the other two are the root of a second degree equation in μ: call it $E\left(\tau\right)$.

If U is the center of an inscribed equilateral triangle admitting Δ_τ as one of its axes, the equations $E\left(\tau\right),E\left(\tau\phi\right),E\left(\tau\phi^2\right)$ share a common root μ. But, from (30.12), we have :

$$E\left(\tau\right)+E\left(\tau\phi\right)+E\left(\tau\phi^2\right)=3\,\left(\tau^6-\sigma_3^2\right)\,\mu^2\,\mathbf{T}^3 \tag{30.13}$$

Obtaining a condition on τ^6 is natural: a direction is only known by its τ^2, and the condition must be symmetric on τ, $\phi\tau$, $\phi^2\,\tau$. This gives $\tau^2=\sqrt[3]{s_3^{12}}\overset{L6}{=}\phi^k s_3^4$, i.e. the Morley directions of axes, proving the first part. The equation of the locus $\mathcal{L}_U$ is obtained by eliminating μ between $E\left(\tau\phi\right),E\left(\tau\phi^2\right)$. Due to (30.13), E($\tau$) is automatically fulfilled. The last part comes Proposition 30.6.1. □

To characterize a 8th degree curve, $8\times9/2-1=35$ points are required. We can identify 39 points on $\mathcal{L}_U$, i.e. 4 more than this minimal number. We have:

18 the Morley centers $M_{\mathbf{k}}$ (where $\mathcal{T}$ is the associated Morley triangle $\mathcal{T}_{\mathbf{k}}$.

3 the vertices ABC. When the pivot is A, the cubic is the union of sideline BC and bisectors I_bI_c, I_0I_a. The common root for μ is 0 and the triangle is reduced to the pivot.

$\mathcal{L}_U = \Re(\Phi)$ where

$$\Phi_1 \overset{L1}{\simeq} \frac{1}{\sigma_3^2}\mathbf{Z}^7\overline{\mathcal{Z}} - \mathbf{Z}^4\overline{\mathcal{Z}}^4 + 3\left(-\frac{1}{\sigma_3^2}\mathbf{Z}^6 - \frac{\sigma_2}{\sigma_3^2}\mathbf{Z}^5\overline{\mathcal{Z}} + \frac{\sigma_1}{\sigma_3}\mathbf{Z}^4\overline{\mathcal{Z}}^2 + \mathbf{Z}^3\overline{\mathcal{Z}}^3\right)\mathbf{T}^2$$
$$+\left(\frac{2\,\sigma_1}{\sigma_3^2}\mathbf{Z}^5 - \frac{2\,\sigma_2}{\sigma_3}\mathbf{Z}^3\overline{\mathcal{Z}}^2\right)\mathbf{T}^3 + 3\left(\frac{2\,\sigma_2}{\sigma_3^2}\mathbf{Z}^4 + \frac{\sigma_2^2 - 2\,\sigma_1\sigma_3}{\sigma_3^2}\mathbf{Z}^3\overline{\mathcal{Z}} - \frac{\sigma_1\sigma_2}{\sigma_3}\mathbf{Z}^2\overline{\mathcal{Z}}^2\right)\mathbf{T}^4$$
$$+6\left(-\frac{\sigma_1\sigma_2}{\sigma_3^2}\mathbf{Z}^3 + \frac{\sigma_1^2}{\sigma_3}\mathbf{Z}^2\overline{\mathcal{Z}}\right)\mathbf{T}^5 + \left(\frac{\sigma_1^2\sigma_2 + \sigma_2^2}{\sigma_3^2}\mathbf{Z}^2 - \frac{\sigma_1^3\sigma_3 + 2\,\sigma_1\sigma_2\sigma_3 + \sigma_2^3}{2\,\sigma_3^2}\mathbf{Z}\overline{\mathcal{Z}}\right)\mathbf{T}^6$$

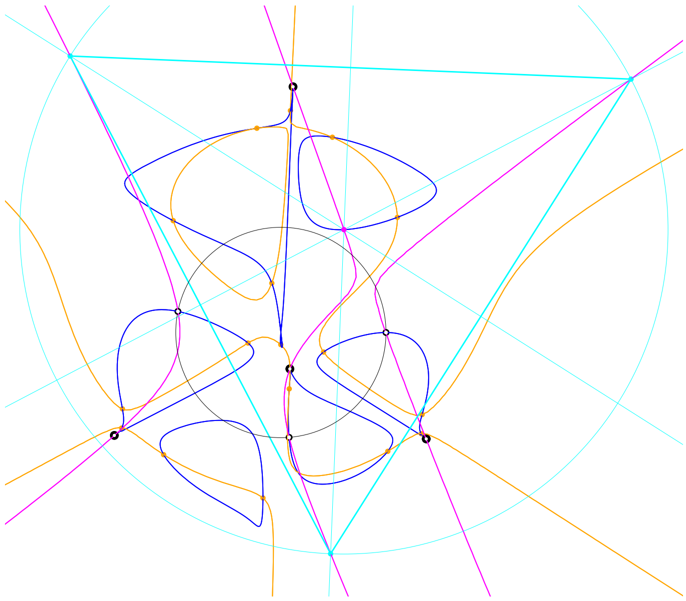


Figure 30.11: A random point on $\mathcal{L}_U$, its pivotal isocubic and the associated triangle

8 the inexcenters $I_0 I_a I_b I_c$. Each of them counts twice being a cusp (the cuspidal tangent goes through the circumcenter). When the pivot is I_j, the cubic is the union of the three bisectors that concur in I_j. Here again, $\mu = 0$ is the common root.

2 the umbilics. The circular asymptotes concur at O.

6 the Morley directions. Each counts twice (cusps again)

2 the circumcenter counts twice. One tangent is the Euler line, the other goes through $\sigma_1\sigma_3 + \sigma_2^2 : 0 : \sigma_1^2 + \sigma_2$. One of the triangle is on the circumcircle, the other is made of the Morley directions.

The curve has an equation that depends only from the full symmetric functions (the σ_j). This comes from the fact that all the 4 in-excenters are symmetrically involved and the 18 Morley centers are also symmetrically involved, so that none of them need to be distinguished from the rest of its class.

When we examine the common points between this curve and the quintic of the Morley centers, we obtain $8 \times 5 = 40$ points. The resultant in $\overline{\mathcal{Z}}$ of the corresponding polynomials splits in four parts. A 18th degree polynomial, describing the Morley centers. A factor $\mathbf{T}^7$ describes the 8 points at infinity (only one $\mathbf{T}$ for both umbilics). And there are two other irreducible polynomials, whose respective degrees are 3 and 11.

But this results only into one and five visible common points on Figure 30.11. Let us describe what happens to the 3rd degree factor $\mathbf{Z}^3\sigma_2 - 3\,\mathbf{Z}\sigma_1\sigma_3 + 2\,\sigma_1^2\sigma_3$. It gives three solutions z_0, z_1, z_2. But, in our example, the points that belong to both curves are $z_0 : 1 : \overline{z_0}$, $z_1 : 1 : \overline{z_2}$, $z_2 : 1 : \overline{z_1}$: the first point is visible, the other two aren't. After extraction of the suitable squared factors, the discriminant of the 3rd polynomial can be written as :

$$\Delta = 1 - \frac{\sigma_1\sigma_2}{\sigma_3} = 1 - |\sigma_1|^2 \in \mathbb{R}$$

Therefore, we will have different behaviors for acute and obtuse triangles (σ_1 is the affix of the orthocenter, so that $\Delta > 0$ forces the orthocenter to be inside the circumcircle).

30.7 Some concluding remarks

30.7.1 Degeneracies in the equilateral triangle

Degenerate cases are often useful to fully understand the general case. It is interesting to see what happens when triangle ABC is near to become equilateral $\alpha = -1$, $\beta = \exp(2\boldsymbol{i}\pi/18)$,$\gamma = \exp(-2\boldsymbol{i}\pi/18)$. All the points that are "triangle centers" as defined in Kimberling (1998) are collapsing to the origin, and so are doing I_0 and the Morley centers M_{000}, M_{222}, M_{111}. But the other centers continue to live a separate life, the excenters $I_a = -2A$, etc among them.

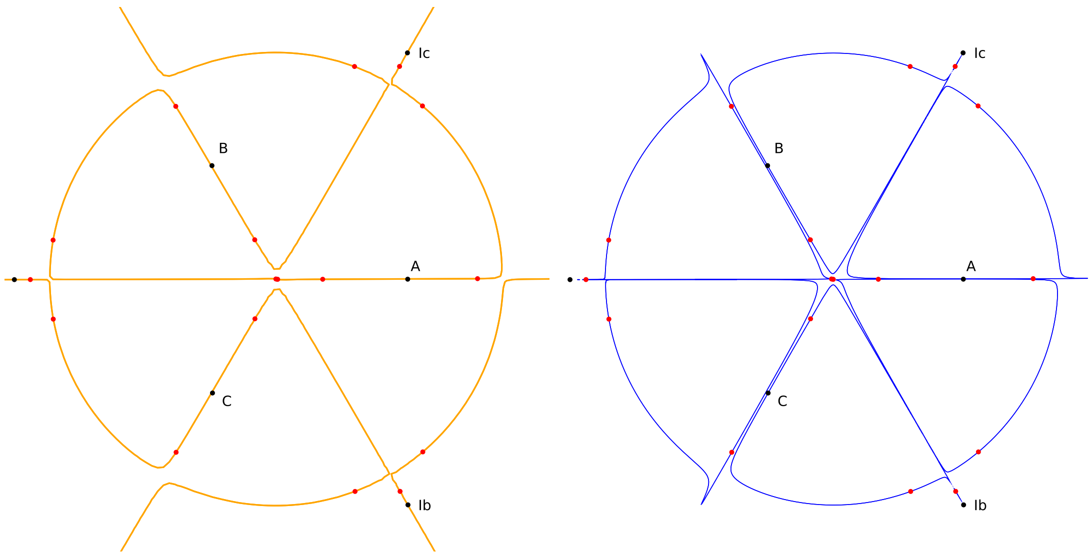


Figure 30.12: A quasi-equilateral triangle

In Figure 30.12, points are quite at the exact place they will occupy when ABC will be exactly equilateral. Points M_{120}, M_{012}, M_{201} form an equilateral triangle and M_{102}, M_{021}, M_{210} also, while the odd $M_{\mathbf{k}}$ belongs by triple to the bisectors of ABC.

Concerning the curves (left the quintic $\mathcal{L}_M$, right the octic $\mathcal{L}_U$), we have a greater sensibility near the special points while the curves are degenerating into the bisectors union a circle:

$$\begin{array}{lll} \mathcal{L}_M \mapsto \left(\mathbf{Z}^3 - \overline{Z}^3\right)\left(\mathbf{Z}\overline{Z} - 3\,\mathbf{T}^2\right) & \mathcal{L}_P \mapsto \left(\mathbf{Z}^3 - \overline{Z}^3\right)\left(\mathbf{Z}\overline{Z} - \mathbf{T}^2\right) \\ \mathcal{L}_U \mapsto \left(\mathbf{Z}^3 - \overline{Z}^3\right)^2\left(\mathbf{Z}\overline{Z} - 3\,\mathbf{T}^2\right) & \mathcal{L}_{P^*} \mapsto \left(\mathbf{Z}^3 - \overline{Z}^3\right)\mathbf{T} \\ & \mathcal{L}_Q \mapsto \left(\mathbf{Z}^3 - \overline{Z}^3\right)\left(\mathbf{Z}\overline{Z} - 4\,\mathbf{T}^2\right) \end{array}$$

30.7.2 Summarizing our results

The present chapter is about the Morley configuration, that occurs when drawing the trisectors of the angles of a triangle. In this configuration, a family of 18 triangles can be identified that are together equilateral and perspective with the original triangle.

This configuration has been studied using the formalism of the complex projective geometry, combined with the parameterization that was originally developed by Lubin (1955). Numerical investigations have been conducted to detect if an algebraic curve exists that contains various classes of points. This property was already known and proven concerning the perspectors $P_{\mathbf{k}}$ of the Morley triangles with the base triangle.

We have discover that the 18 Morley centers themselves, as well as the 3 Morley directions belong to a same circular quintic $\mathcal{L}_M$. This result is a new one. Using a formal computing tool, we have explicited the complex projective equation of this curve (i.e. relative to Ω_x, O, Ω_y: the umbilics and the circumcircle). When converting this equation into a barycentric one (i.e. relative to the vertices A, B, C), one obtains an expression whose complexity is the probable explanation of the novelty of our Theorem 1. A comparable result has been obtained concerning the secondary perspectors (Theorem 2).

Thereafter, we have classified the intersections of two Morley isocubics and described the situations where these intersections can or cannot be rationally generated from α, β, γ. We have also determined the condition for an isogonal cubic to contain an inscribed equilateral triangle having prescribed directions. We have proven that the solutions are discrete except when Morley directions are chosen. In this later case, the locus of the pivots is an algebraic 8th-degree curve containing the Morley centers (Theorem 3).

As ever, combining numerical investigations with formal computing tools has proven to be efficient.

Chapter 31

Groups acting over the Morley configuration

31.1 Introduction

31.1.1 Aim of this chapter

Our aim now is to investigate the Morley configuration in the context of the group theory, with a special focus on how properties relative to the flagship triangle (pictured in Figure 30.1a) can be propagated to the other replicas under the action of a well suited group. Moreover, broader classes of objects have been introduced by relaxing some of the required properties. For example, the strange triangles, introduced by Taylor and Marr (1913), and the lighthouse triangles introduced by Viricel and Bouteloup (1993).

We will prove following results:

Theorem 1 (*New result*)**.** *The Morley objects involving also the in-excenters are connected by the projective group identified as* $\mathcal{G}\,[\,72, 43\,]$ [1] *(thereafter named* $\mathcal{PG}_{72}$*), while its projective subgroup* $\mathcal{G}\,[\,18, 4\,]$ *(thereafter named* $\mathcal{PG}_{18}$*) is sufficient when the inexcenters are not involved. On the contrary, the* ***strange*** *objects are connected by the projective group* $\mathcal{G}\,[\,9, 2\,]$ *(thereafter named* $\mathcal{PG}_9^\bullet$*, where the bullet stays as a remainder of strangeness).*

Theorem 2 (*New result*)**.** *Under the action of a group that is 1-transitive, the orbit of a pair* $\{\kappa_1, \kappa_2\}$ *contains at least a pair* $\{0, 0\cdot\varphi\}$ *and depends only on* φ*. Under the action of* $\mathcal{PG}_{18}$*, the orbits of Morley pairs are 18-sized when* φ *is even* $(1, m, r, mr, mr^2)$ *or* 9*-sized when* φ *is odd (control:* $5 \times 18 + 9 \times 9 = 171$*).*

Under the action of $\mathcal{G}_{54}$*, most of these orbits collapse by three, and we have* O_1, O_m*: 18-sized,* O_r*: 54-sized while* O_t*,* O_{tm}*,* O_{tm^2} *are* 27*-sized (control:* $18\times 2 + 54 + 27 \times 3 = 171$*). Therefore any property that involves Morley centers by pairs, has essentially to be proven on five special cases.*

31.1.2 Morley centers as an intricate family

In order to show why the Morley centers must be considered as a family connected by the action of some group, we will consider what happens to the flagship of the family (the center of the small triangle in Figure 30.1a) when A, B, C are moving freely along a fixed circumcircle. Its barycentric coordinates are obtained by using $\mathbf{k} = 000$ (i.e. $k_a = 0$, etc) in the Taylor and Marr formula :

$$M_{\mathbf{k}} = \sin A \left(\cos\frac{A + 2\,k_a\pi}{3} + 2\,\cos\frac{B + 2\,k_b\pi}{3}\cos\frac{C + 2\,k_c\pi}{3}\right) : \text{etc} : \text{etc} \tag{31.1}$$

Written that way, the point M_{000} looks like a well defined object. It is described as X(356) in the Kimberling database of triangle centers. When written using the Lubin parameterization (α, β, γ are on the unit circle, more details will be given in Subsection 31.2.1), one obtains :

[1] The first number is the cardinal, followed by an index in the SAGE (2005-2014) database.

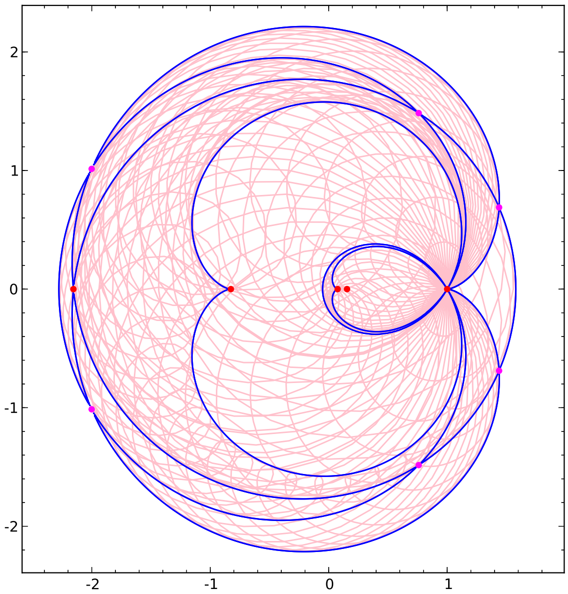


Figure 31.1: Locus of the Morley center M_{000} when A remains fixed (here, $z_A = 1$)

$$z_{000} \overset{L3}{=} \left(\frac{1-\phi^2}{3}\right)\left(\alpha^2\beta + \alpha\gamma^2 - \beta\gamma^2\right) + \left(\frac{1-\phi}{3}\right)\left(\alpha^2\gamma + \alpha\beta^2 - \beta^2\gamma\right)$$

where $\phi = \exp(2\boldsymbol{i}\pi/3)$. This formula is rather not symmetrical. If we fix α, β and move γ all along the unit circle, the point z describes a closed curve. If $\alpha = -1$ is fixed and β takes 60 values regularly disposed along the circle, we obtain the pink lines of Figure 31.1.

It can be shown that the boundary curve (in dark gray) is a ten fold circular curve of degree 20. But all the grayed area is globally connected and the same picture is obtained if one uses another value of $\mathbf{k} = k_a k_b k_c$ (see Subsection 31.4.6 for the condition $\sum \mathbf{k} \not\equiv 1!$). This shows that the Morley centers form an intricate family, that is interconnected by the transitive action of some group.

31.1.3 Organization of this chapter

This chapter is a continuation of Douillet (2010, in English) and Douillet (2014d, in French). In Section 31.2, we describe how to teach the Morley configuration to a computer. In this context, the key question is using objects that have an unique normal form, since incantations like "the computer will easily see that..." would be useless. In Section 31.3, we make some numerical explorations. Among the $108 \times 107 \times 106 \div 6$ triangles that can be formed from the 108 intersections, one can find 54 equilateral triangles. Among them, there are the 18 Morley triangles, while the remaining ones can be described as lighthouse triangles. One can also see that each intersection belongs to nine perspective triangles.

In Section 31.4, we are back to formal results and we prove our Theorem 1 concerning the groups acting over the Morley configuration. In Section 31.5 we use these results to explain why the regular Morley objects are shadowed by *strange* objects that have only reduced properties. We will show that, in fact, these objects are "not sufficiently skew".

Finally, Section 31.6 shows the efficiency of this group theoretical approach on two examples. The first is the family of Martini circles. The second is based on the fact that two equilateral triangles having the same directions and the same orientation are perspective. The classification of these equilateral perspectors is facilitated by our Theorem 2 that classifies the orbits of the pairs of centers.

The chapter ends by a summary section. The bibliography is integrated into the bibliography of the whole book.

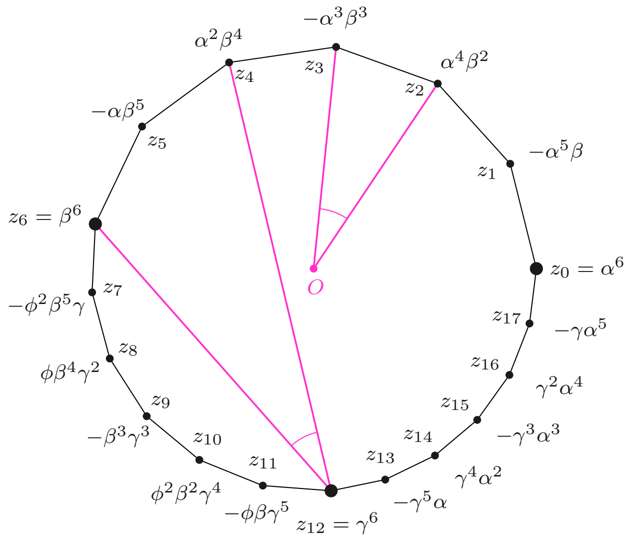


Figure 31.2: The Lubin choices of orientation

31.2 Morley configuration for computers

31.2.1 The Lubin parameterization

In this context, conjugacy is the operation that transforms what is seen from above the plane by what is seen from below. This implies $conj\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)=\overline{\mathcal{Z}}:\mathbf{T}:\mathbf{Z}$. Applied to the ordinary points, this generates two important subclasses:
• visible finite points, that can be written $z:1:\overline{z}$ where $\overline{z}$ is the $\mathbb{C}$ conjugate of z.
• visible infinite points, that can be written $\omega^2:0:1$ where $\omega\in\mathbb{C}$ is unimodular.

But, in the general case, variable $\overline{\mathcal{Z}}$ is an independent variable, with the same status as $\mathbf{Z}$ and $\mathbf{T}$. This allows to use points like the circular points at infinity (the so called-umbilics), i.e. $\Omega_x\simeq 1:0:0$, $\Omega_y\simeq 0:0:1$. These points are crucial to characterize the circles among the projective conics.

But the non-algebraic nature of the complex conjugacy impose to avoid this transformation as much as possible. This exigence can be fulfilled by identifying the circumcircle of ABC with the unit circle. This leads to ***define*** conjugacy as the action of the substitutions:

$$\mathbf{Z}\leftrightarrow\overline{\mathcal{Z}},\ \omega_{12}\to 1/\omega_{12},\ \alpha\to 1/\alpha,\ \beta\to 1/\beta,\ \gamma\to 1/\gamma$$

It only remains to explain how to algebraically divide angles into six equal parts (dividing in three is required for Morley theorem and dividing in two is required for the in-excenters). This can be done by using α^6 as the complex affix of vertex A, etc. This leads to Figure 31.2.

More details are given in Lubin (1955) for the choice $z_A=\alpha^3$, etc and in Douillet (2014a) for the choice $z_A=\alpha^6$, etc. In any cases, we must have some -1 and ϕ due to the obvious relations $z_0z_6z_{12}=-z_3z_9z_{15}=\phi^2z_2z_8z_{14}$.

31.2.2 Symmetric expressions

From the preceding choices, it results that:

Proposition 31.2.1. *The coordinates of the points generated from Figure 31.2 are homogeneous rational fractions in α,β,γ and their degrees are in progression: $dg\,(z)=6+dg\,(t)$; $dg\,(t)=6+dg\,(\zeta)$. Curves, that are homogeneous polynomials in $\mathbf{Z},\mathbf{T},\overline{\mathcal{Z}}$, are also globally homogeneous fractions when pondering variables as $\mathbf{Z}=6$, $\mathbf{T}=0$, $\overline{\mathcal{Z}}=-6$, $\alpha=1$, etc.*

In some situations, these expressions can be rewritten using the symmetric functions of α,β,γ. We define the s_j by :

$$s_1\overset{L6}{=}\alpha+\beta+\gamma,\ s_2\overset{L6}{=}\alpha\beta+\beta\gamma+\gamma\alpha,\ s_3\overset{L6}{=}\alpha\beta\gamma$$

where $\stackrel{L6}{=}$ is a reminder of the dependence[2] from the choice $z_A = \alpha^6$, etc. On the contrary, the σ_j are the symmetrical functions of the vertices themselves :

$$\sigma_1 \doteq z_A + z_B + z_C,\ \sigma_2 \doteq z_A z_B + z_B z_C + z_C z_A,\ \sigma_3 \doteq z_A z_B z_C \stackrel{L6}{=} s_3^6$$

When trying to explain this concept to a SAGE computer, and also the concept of an "auxiliary point", we have to introduce $\widehat{\mathbb{K}} \doteq \mathbb{Q}(\omega_{12})(\alpha,\, \beta,\, \gamma,\, s_1,\, s_2,\, s_3)$, then $\widehat{\mathbb{K}}\left[\mathbf{Z}, \mathbf{T}, \overline{\mathcal{Z}}, z, t, \zeta\right]$ and thereafter trans-type everything. The usual algorithm can now be used onto the *coefficients* of the polynomials.

31.2.3 Isogonal duality

We need to introduce the involution known as the "isogonal transformation" of the plane, thereafter named *isog* and shortened as $P \to P^*$. Connoisseurs are knowing that only the associated Cremona map is effectively involutive (Déserti, 2009b). A simplified presentation is as follows:

Proposition 31.2.2. *Given a triangle ABC and a finite, visible point P not on a sideline, we define line Aa by the equal inclination property $(AB, AP) = (Aa, AC)$, etc. Then lines Aa, Bb, Cc are concurring in a finite, visible point not on a sideline. This point is called P^*, the isogonal image of P. We have :*

$$isog \begin{pmatrix} \mathbf{Z} \\ \mathbf{T} \\ \overline{\mathcal{Z}} \end{pmatrix} \stackrel{L1}{\simeq} \begin{pmatrix} \sigma_3^2 \overline{\mathcal{Z}}^2 - \sigma_3 \mathbf{Z}\mathbf{T} - \sigma_3\sigma_2 \overline{\mathcal{Z}}\mathbf{T} + \sigma_3\sigma_1 \mathbf{T}^2 \\ -\sigma_3 \mathbf{Z}\overline{\mathcal{Z}} + \sigma_3 \mathbf{T}^2 \\ \mathbf{Z}^2 - \sigma_1 \mathbf{Z}\mathbf{T} - \sigma_3 \overline{\mathcal{Z}}\mathbf{T} + \sigma_2 \mathbf{T}^2 \end{pmatrix} \tag{31.2}$$

Extending this formula can be used to define P^* "quite" everywhere.

Computing $isog\left(isog\left({}^t\left[\mathbf{Z}, \mathbf{T}, \overline{\mathcal{Z}}\right]\right)\right)$ gives (algebraically) ${}^t\left[\mathbf{Z}, \mathbf{T}, \overline{\mathcal{Z}}\right]$ times the common factor : $-\prod_3 \alpha^{12} \cdot (\beta^6\gamma^6\overline{\mathcal{Z}} - \beta^6\mathbf{T} - \gamma^6\mathbf{T} + \mathbf{Z})$, which is the product of the three sidelines. Thus *isog* exchanges (in the Cremona sense) line BC with vertex A, etc while outside of the three sidelines, we have as a pointwise involution.

The four in-excenters are clearly the fixed points of this Cremona transform. The circumcircle is exchanged with the line at infinity (and, therefore, the umbilical points are swapped). This is confirmed by remarking that :$(u : v : w)^* \simeq \left(a^2/u : b^2/v : c^2/w\right)$ where a, b, c are the length of the side lines.

The appearance of this transform is unavoidable since, for example, the trisectors issued from A are equally inclined by pairs with AB and BC. Therefore the isogonal image of any intersection of trisectors is another intersection.

31.3 Numerical explorations

31.3.1 Taylor-Marr naming conventions

When dealing with numerous objects, the choice of the naming convention is crucial. The following three were chosen by Taylor and Marr in their 1913 article. The first convention applies to all the 18 trisectors, the second to 27 vertices among all the $(18 \times 12)/2 = 108$ vertices generated and the third to 27 triangles among the $(27 \times 26 \times 25)/6 = 2925$ triangles generated by this reduced set of vertices. Obviously, these objects will appear to be the most interesting ones.

Definition 31.3.1. For $D \in \{A, B, C\}$, $x \in \{u, d\}$ and $k \in \mathbb{Z}/3\mathbb{Z}$, the ***trisector*** $D_k^{\mathbf{x}}$ is defined as follows. (1) the trisector is issued from D. (2) u is the upwards permutation $A \to B \to C \to A$, while $d = u^{-1}$ is the downwards one. (3) Value $k = 0$ is used for trisectors obtained by joining points of Figure 31.2. Trisectors $D_0^{\mathbf{u}}$ and $D_0^{\mathbf{d}}$ are, respectively "near $u(D)$" and "near $d(D)$" (4) Then $D_0^{\mathbf{u}}$ is rotated by ϕ^{2k} to give $D_k^{\mathbf{u}}$ while $D_0^{\mathbf{d}}$ is rotated by ϕ^k to give $D_k^{\mathbf{d}}$, leading to opposite orientations for the two beams.

[2] The usual incenter formula $z(I_0) \stackrel{L2}{=} -s_2$ becomes $z(I_0) \stackrel{L6}{=} -\alpha^3\beta^3 - \beta^3\gamma^3 - \gamma^3\alpha^3$

Definition 31.3.2. For $D \in \{A, B, C\}$ and $j, k \in \mathbb{Z}/3\mathbb{Z}$, the Morley ***vertex*** D_{jk} is the intersection of the trisectors $u\left(D\right)_j^{\mathbf{u}}$ and $d\left(D\right)_k^{\mathbf{d}}$. By Proposition 31.2.2, the isogonal conjugate $D_{jk}{}^*$ of the Morley vertex D_{jk} is the intersection of trisectors $u\left(D\right)_j^{\mathbf{d}}$ and $d\left(D\right)_k^{\mathbf{u}}$.

Definition 31.3.3. For $\mathbf{k} = k_a k_b k_c$, where $k_a, k_b, k_c \in \mathbb{Z}/3\mathbb{Z}$, the Taylor-Marr ***triangle*** $\mathcal{T}_{\mathbf{k}}$ is the triangle whose vertices are $A_{k_b k_c}$, $B_{k_c k_b}$, $C_{k_a k_b}$.

31.3.2 Two examples

The coefficients of a given trisector are easy to compute when using (30.1). For example, using $E_6 \wedge E_{14}$ and $E_{12} \wedge E_4$ respectively, one obtains :

$$B_0^{\mathbf{u}} \overset{L6}{\simeq} \left[\ 1\,;\ -\alpha^2\gamma^4 - \beta^6\ ;\ \alpha^2\beta^6\gamma^4\ \right] \qquad C_0^{\mathbf{d}} \overset{L6}{\simeq} \left[\ 1\,;\ -\alpha^2\beta^4 - \gamma^6\ ;\ \alpha^2\beta^4\gamma^6\ \right] \tag{31.3}$$

And now, we can compute the intersections of two of these trisectors. For example, $A_{00} \doteq B_0^{\mathbf{u}} \cap C_0^{\mathbf{d}}$ is :

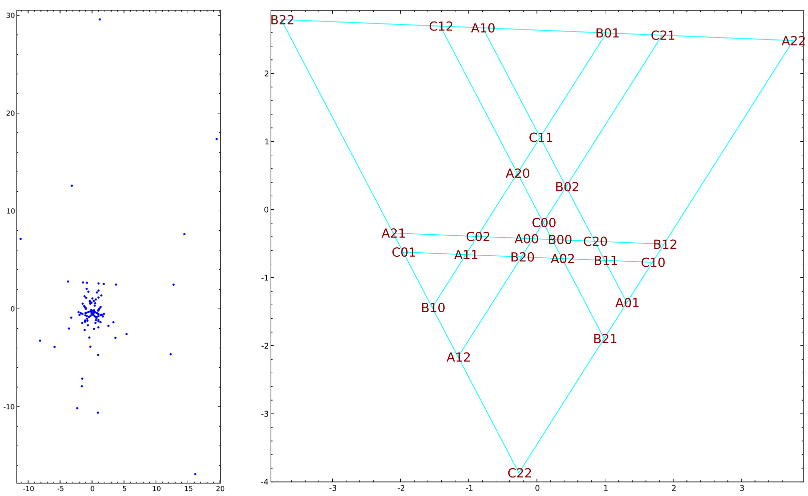
$$A_{00} \overset{L6}{\simeq} B_0^{\mathbf{u}} \wedge C_0^{\mathbf{d}} \overset{L6}{\simeq} \begin{pmatrix} \left(\alpha^2\beta^4 + \gamma^6\right)\alpha^2\beta^6\gamma^4 - \left(\alpha^2\gamma^4 + \beta^6\right)\alpha^2\beta^4\gamma^6 \\ \alpha^2\beta^6\gamma^4 - \alpha^2\beta^4\gamma^6 \\ \alpha^2\gamma^4 - \alpha^2\beta^4 + \beta^6 - \gamma^6 \end{pmatrix}$$

$$z\left(A_{00}\right) \overset{L6}{=} \alpha^2\beta^4 + \alpha^2\gamma^4 - \beta^4\gamma^2 - \beta^2\gamma^4 + \alpha^2\beta^2\gamma^2 \tag{31.4}$$

It should be noticed that not all of the 108 affixes are polynomial and that, most of the time, powers of ϕ are not canceling.

31.3.3 Hunting the equilateral triangles

When plotting altogether the 108 intersections, as in Figure 31.3a, the result does not seem remarkable. On the contrary, plotting only the 27 Morley vertices leads to Figure 31.3b.



(a) The 108 cuts (b) The 27 Morley vertices

Figure 31.3: Morley equilateral triangles

In order to detect the equilateral triangles $z_1 z_2 z_3$ that are hidden in Figure 31.3a, it would be tedious to apply the criterion $\left(z_1 + \phi z_2 + \phi^2 z_3\right)\left(z_1 + \phi^2 z_2 + \phi z_3\right) = 0$ using the 108 quantities like (31.4)! We better start by screening what happens when numerical values are assigned to α, β, γ. This requires only to compute the squared length of the $107 \times 108/2 = 5778$ segments joining the 108 vertices and to sort the list obtained. This gives small sub-lists of isometric segments, that can

be searched for segments that actually form triangles. The hunt gives two kinds of possibilities:
• 36 triangles like $(A_0^{\mathbf{u}}B_0^{\mathbf{u}},\ A_1^{\mathbf{u}}B_2^{\mathbf{u}},\ A_2^{\mathbf{u}}B_1^{\mathbf{u}})$, each of them involving only two families of trisectors. Due to Proposition 31.3.4, we call them the lighthouse triangles.
• 18 triangles among the 27 Taylor-Marr triangles $\mathcal{T}_{\mathbf{k}}$. These are the Morley triangles.

31.3.4 The lighthouse triangles

The lighthouse triangles were called "étoiles" (stars) when described for the first time by Viricel and Bouteloup (1993). Since they involve only lines issued from two fixed points, the result doesn't really involve any triangle and its general form is:

Proposition 31.3.4 (Lighthouse theorem (Guy, 2007)). *Let n be an integer, $n \geq 3$, and define a n-lighthouse as n lines $E_0, E_1, \cdots, E_{n-1}$ issued at +180°/n from each other from a given point E. Let $F_0, F_1, \cdots, F_{n-1}$ be another lighthouse. Define points $P(j,k) = E_j \cap F_k$ and consider indices modulo n. Then, starting from any $P(x,y)$, the n-gone $[P(x+k,\ y+k),\ k = 1 \cdots n]$ is regular and its circumcircle contains E and F (See Figure 31.4).*

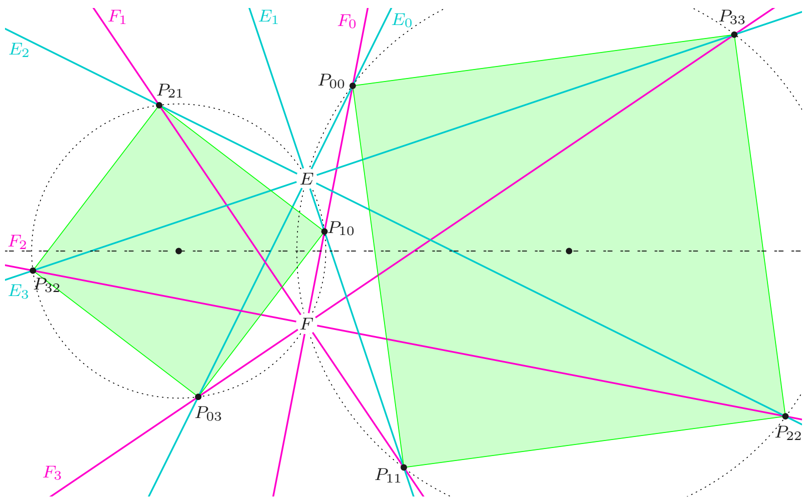


Figure 31.4: The twin lighthouses theorem

When E, F are the two vertices B, C, we have two choices of lighthouses per vertex (up and down), leading to four sets of three triangles. The sidelines of the corresponding families of triangles have the following directions :

$$\omega^2\,(\mathrm{du}) = -\phi^k\alpha^8\beta^2\gamma^2 \quad \omega^2\,(\mathrm{dd}) = -\phi^k\alpha^6\beta^4\gamma^2 \quad \omega^2\,(\mathrm{uu}) = -\phi^k\alpha^6\beta^2\gamma^4$$
$$\omega^2\,(\mathrm{ud}) \overset{L6}{=} -\phi^k\alpha^4\beta^4\gamma^4 \tag{31.5}$$

The nine lighthouse triangles that belong to the ud families can be seen on Figure 31.3b. Triangle (A_{22}, A_{10}, A_{01}) is one of them. Here indices j, k are running contrariwise, since the beams have been oriented in that manner.

31.3.5 Hunting the perspective triangles

Another possibility for detecting triangles that share some properties with the Morley triangles is to require some perspectivities. Numeric exploration proves (at least for the triangle ABC we have chosen !) that if P is one of the 108 intersections, there are 9 among all the ordered triples of intersections XYZ such that (1) P is among X, Y, Z; (2) XYZ is perspective with ABC and with $I_aI_bI_c$. For example, when $P = A_{21} = B_2^{\mathbf{u}}C_1^{\mathbf{d}}$, we have the triangles:

$$\begin{array}{lll} B_2^{\mathbf{u}}C_1^{\mathbf{d}}, B_2^{\mathbf{u}}C_1^{\mathbf{u}}, B_2^{\mathbf{d}}C_1^{\mathbf{d}}\ ; & B_2^{\mathbf{u}}C_1^{\mathbf{d}}, C_1^{\mathbf{u}}A_0^{\mathbf{u}}, A_0^{\mathbf{d}}B_2^{\mathbf{d}}\ ; & *B_2^{\mathbf{u}}C_1^{\mathbf{d}}, C_1^{\mathbf{u}}A_1^{\mathbf{d}}, A_1^{\mathbf{u}}B_2^{\mathbf{d}} \\ B_2^{\mathbf{u}}C_1^{\mathbf{u}}, B_2^{\mathbf{u}}C_1^{\mathbf{d}}, B_2^{\mathbf{d}}C_1^{\mathbf{u}}\ ; & *B_2^{\mathbf{u}}C_1^{\mathbf{d}}, C_1^{\mathbf{u}}A_0^{\mathbf{d}}, A_0^{\mathbf{u}}B_2^{\mathbf{d}}\ ; & B_2^{\mathbf{u}}C_1^{\mathbf{d}}, C_1^{\mathbf{u}}A_1^{\mathbf{u}}, A_1^{\mathbf{d}}B_2^{\mathbf{d}} \\ B_2^{\mathbf{d}}C_1^{\mathbf{d}}, B_2^{\mathbf{u}}C_1^{\mathbf{u}}, B_2^{\mathbf{u}}C_1^{\mathbf{d}}\ ; & B_2^{\mathbf{u}}C_1^{\mathbf{d}}, C_1^{\mathbf{u}}A_2^{\mathbf{u}}, A_2^{\mathbf{d}}B_2^{\mathbf{d}}\ ; & *B_2^{\mathbf{u}}C_1^{\mathbf{d}}, C_1^{\mathbf{u}}A_2^{\mathbf{d}}, A_2^{\mathbf{u}}B_2^{\mathbf{d}} \end{array}$$

Among them, 3 uses trisectors issued from only two ABC vertices. For the other six, P occupies the place of the ABC vertex it doesn't involves. When we restrain ourselves to only the 27 Morley vertices, it remains 3 possibilities par vertex, and they are the Taylor-Marr triangles allowed by the indices of P. In our example, $P = A_{21}$ and the triangles are the ones tagged by a $*$ in the above list, namely $\mathcal{T}_{021}$, $\mathcal{T}_{121}$, $\mathcal{T}_{221}$.

31.4 Groups acting over the Morley configuration

31.4.1 Motivation: the Lubin proof

Proving that the flagship triangle $\mathcal{T}_{000}$ is equilateral and oriented like ABC is easy when all the preparing work is already done. One computes also $z(B_{00})$ and $z(C_{00})$ and obtains (Lubin, 1955) :

$$\begin{array}{lllll} z(A_{00}) \overset{L6}{=} \alpha^2\beta^2\gamma^2 & -\beta^4\left(\gamma^2-\alpha^2\right) & -\gamma^4\left(\beta^2-\alpha^2\right) & & \\ z(B_{00}) \overset{L6}{=} \alpha^2\beta^2\gamma^2\phi & & +\gamma^4\left(\beta^2-\alpha^2\right)\phi^2 & -\alpha^4\left(\phi\gamma^2-\beta^2\right) & \\ z(C_{00}) \overset{L6}{=} \alpha^2\beta^2\gamma^2\phi^2 & +\beta^4\left(\gamma^2-\alpha^2\right)\phi & & & +\alpha^4\left(\phi\gamma^2-\beta^2\right)\phi^2 \end{array} \tag{31.6}$$

showing that $z(A_{00}) + \phi z(B_{00}) + \phi^2 z(C_{00}) = 0$ and proving the property. It remains to see how to propagate this property to the other triangles. Let us detail this problematic, that occurs for any object that belongs to the Morley configuration.

31.4.2 The general abstract group $\mathcal{PG}_{216}$

When dealing with the Morley configuration, one uses repetitively permutations of α, β, γ and transformations like $\alpha \mapsto \phi\alpha$. In many articles, e.g. in Taylor and Marr (1913), this aspect of the problem is hidden or limited to an heuristic role. Let us adopt the opposite behavior and bring this problematic to the foreground.

Since $z_A = \alpha^6$ only defines α up to a power of $(-\phi)$ it is natural to introduce the operator $(\alpha, \beta, \gamma) \mapsto (-\phi\alpha, \beta, \gamma)$. Mixed with the symmetric group $\mathfrak{S}_3$ acting on the parameters α, β, γ, this generates a 1296-sized group, whose center is 6-sized. Since all the objects of interest are projective, multiplying all of the α, β, γ by the same numeric quantity doesn't change anything and we better consider the associated projective group. Let us call it $\mathcal{PG}_{216}$ from its order $1296/6 = 216$.

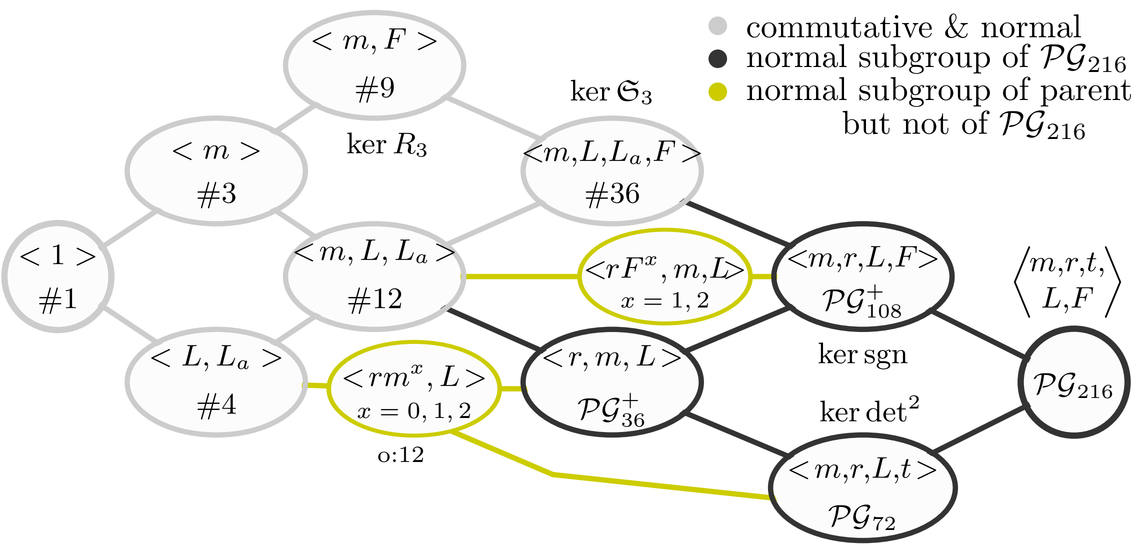


Figure 31.5: The normal subgroups of $\mathcal{PG}_{216}$

The first thing to do is obtaining the graph of the normal subgroups (see Figure 31.5) and the table of the independent characters (Table 31.1). To obtain both figures, we can represent the bigger group by matrices acting onto rows, and generate them from :

$$g_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, g_3 = \begin{pmatrix} -\phi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} g_1 = \begin{pmatrix} \beta & \gamma & \alpha \end{pmatrix}, \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} g_2 = \begin{pmatrix} \alpha & \gamma & \beta \end{pmatrix}, \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} g_3 = \begin{pmatrix} -\phi\alpha & \beta & \gamma \end{pmatrix}$$

With this convention, we have $(X.g_j).g_k = X.(g_j g_k)$: in a product, the first written (on the left) represents the first acting.

When using SAGE, a brute force method is as follows. Write down the generating matrices. Generate the group and transcribe it as a permutation group. Divide by the center and obtain $\mathcal{PG}_{216}$. Obtain the character table $\mathfrak{C}$ (spell it fraktur-C) . Use ${}^t\overline{\mathfrak{C}} \cdot \mathfrak{C}$ to discover the number of conjugacy classes and their sizes. Ask for the normal subgroups and their orders. This gives :

name	$\mathcal{PG}_{216}$	$\mathcal{PG}_{108}^+$	$\mathcal{PG}_{72}$	$\mathcal{PG}_{36}^+$	$\mathcal{N}_\Delta$	$\mathcal{N}_{12}$	$\mathcal{N}_9$	$\mathcal{N}_4$	$\mathcal{N}_3$	$\mathcal{N}_1$
#	216·92	108·22	72·43	36·11	36·14	12·5	9·2	4·2	3·1	1·1
gens	$tmrLF$	$mrLF$	$trmL$	rmL	mLL_aF	mLL_a	mF	LL_a	m	1
# quo	1·1	2·1	3·1	6·2	6·1	18·3	24·12	54·5	72·42	216·92
ident		$\mathbb{Z}/2$	$\mathbb{Z}/3$	σ	$\mathfrak{S}_3$	$--$	$\mathfrak{S}_4$	rtF	$--$	$\mathcal{PG}_{216}$

In this table, the first line gives the name used in this article. The biggest groups have specific names while the commutative ones are named $\mathcal{N}_n$. At the boundary, $\mathcal{N}_\Delta$ is commutative, and 36-sized (Δ stands for diagonal). Then comes the index order/item in the SAGE database. The next line gives the generators involved. The 4th line gives the index order/item of the quotient group $\mathcal{PG}_{216}/\mathcal{N}$. As it should be we have relations like $36 \times 6 = 216$ for the orders of subgroup and quotient. What remains now is to interpret all this information.

One generator is required for $\mathcal{N}_3$, the smallest normal subgroup, say m (order 3). And a second generator for $\mathcal{N}_9$, say F (order 3). Subgroup $\mathcal{N}_4$ is the Klein group and requires two generators, say L, L_a (order 2). Altogether, m, F, L, L_a generates the maximal commutative subgroup $\mathcal{N}_\Delta$ (size 36). As a set, $\mathcal{N}_\Delta$ encompasses all the small classes $\mathcal{C}(1), \mathcal{C}(m)$ $\#1+2$, $\mathcal{C}(F), \mathcal{C}(F^2), \mathcal{C}(y), \mathcal{C}(yFm), \mathcal{C}(yF^2m^2)$ $\#3\times 5$ and $\mathcal{C}(ym), \mathcal{C}(yF), \mathcal{C}(yF^2)$ $\#6\times 3$. From commutativity, orders are obvious.

To generate subgroup $\mathcal{PG}_{108}^+$, another generator is required, say r (order 3). As a set, $\mathcal{PG}_{108}^+$ contains $\mathcal{N}_\Delta$ together with three 24-sized classes that are: $\mathcal{C}(r)$, $\mathcal{C}(rF)$, $\mathcal{C}(rF^2)$. All the orders are 3. To generate the whole $\mathcal{PG}_{216}$, a last generator is required, say t (order 2). As a set, $\mathcal{PG}_{216}$ has to be completed by six other classes: $\mathcal{C}(t), \mathcal{C}(tF), \mathcal{C}(tF^2), \mathcal{C}(tL), \mathcal{C}(tLF), \mathcal{C}(tLF^2)$. All are 18-sized, while orders are respectively $2, 6, 6, 4, 12, 12$.

From the nature of the problem, an efficient choice of generators is as follows :

$$\begin{gathered} r = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \phi^2 & 0 \\ 0 & 0 & \phi \end{pmatrix}, F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \phi^2 \end{pmatrix} \\ L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, L_a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, L_c = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{gathered} \tag{31.7}$$

where the names r, t, m, F, L have been chosen as rotate, transpose, Morley, fake and Lemoine. In fact L is the B-Lemoine transform $\beta \to -\beta$ and $L_a = r^{-1} \cdot L \cdot r$ is the A-Lemoine transform $\alpha \to -\alpha$. The first six will be used as generators of the projective group, and they are normalized to have a 1 in the first line. The last two will be used as generators of irreducible representations: they will be used "as is" and must have the right trace (here -1).

Elements L_a, L_c are not required to generate $\mathcal{PG}_{216}$ itself, but one of them must be used to generate a subgroup that do not contain r. In fact, SAGE also considers six generators and not only five. This can be related with the following "unique factorization property": each element x of $\mathcal{PG}_{216}$ can be written *exactly in this order* as :

$$x = t^\tau \cdot r^\rho \cdot m^\mu \cdot F^\psi \cdot L^\lambda \cdot L_a^\eta \tag{31.8}$$

where the exponents range over the multiplicative order of the generator. For example, $\eta \in \mathbb{Z}/2\mathbb{Z}$ while $\psi \in \mathbb{Z}/3\mathbb{Z}$.

31.4.3 Three scalar class-invariants

Let us now consider the characters of $\mathcal{PG}_{216}$. The quotient of this group by $\mathcal{PG}_{36}^{+} = \langle r, m, L \rangle$ is cyclic. This generates six linear characters χ_j, $j = 0, 1, \cdots, 5$, that can be written:

$$\chi_j = (-1)^{j \times \tau(x)} \times (\phi)^{j \times \psi(x)}$$

where $\tau(x)$, $\psi(x)$ are the exponents appearing in (31.8). Character χ_3 can be obtained directly from the matrix of x by reducing to 1 each non zero element and taking the determinant (projective quantity). Let us call it the parity of x, notation $\operatorname{sgn} x$. Character χ_2 can be obtained directly from the matrix of x as is squared determinant (projective quantity), noted $\det^2 x$.

Finally, let us consider the odd matrices. They have exactly one diagonal non-zero element, that can be used to normalize the matrix. Proceeding that way, the determinant is now well defined and ranges over all the sixth roots of the unity. Quantity $\det^2 x$ is the already described character χ_2, but $\det^3 x$ defines another characteristic of an odd class. Let us call it the twist. It can be seen that a non twisted element is such that the two transposed variables have encountered the same parity of Lemoine sign changes, while these parities are different for twisted transforms.

31.4.4 Character table of $\mathcal{PG}_{216}$

These results allows us to organize the character table of $\mathcal{PG}_{216}$. Obtained using the methods described in Tinkham (1964) or using SAGE as a black box, the character table $\mathfrak{C}$ is a 19×19 matrix. Let us recall that $\Delta \doteq 216 \left(\overline{\mathfrak{C}} \cdot \mathfrak{C}\right)^{-1}$ is a diagonal matrix whose elements are the sizes of the corresponding classes, while $\mathfrak{C} \cdot \Delta \cdot \overline{\mathfrak{C}}$ is the scalar matrix 216 (the order of the group).

In Table 31.1, a class of conjugacy is described by a column (header: order o and cardinal #) and a character by a row (header: name). We have collected columns (except from the first) and rows (except from the last) into blocks of three.

o	1	3	2, 6, 6	6	3	2, 6, 6	4, 12, 12
#	1	2+3+3	3×3	6×3	24×3	18×3	18×3
R_{1j}	1 1 1	$\boxed{\Phi}$	$\boxed{\Phi}$	$\boxed{\Phi}$	$\boxed{\Phi}$	$+\boxed{\Phi}$	$+\boxed{\Phi}$
sR_{1j}	1 1 1	$\boxed{\Phi}$	$\boxed{\Phi}$	$\boxed{\Phi}$	$\boxed{\Phi}$	$-\boxed{\Phi}$	$-\boxed{\Phi}$
R_{2j}	2 2 2	$2\boxed{\Phi}$	$2\boxed{\Phi}$	$2\boxed{\Phi}$	$-\boxed{\Phi}$	$\boxed{0}$	$\boxed{0}$
R_{3j}	3 3 3	$3\boxed{\Phi}$	$-\boxed{\Phi}$	$-\boxed{\Phi}$	$\boxed{0}$	$\boxed{\Phi}$	$-\boxed{\Phi}$
sR_{3j}	3 3 3	$3\boxed{\Phi}$	$-\boxed{\Phi}$	$-\boxed{\Phi}$	$\boxed{0}$	$-\boxed{\Phi}$	$\boxed{\Phi}$
R_{6j}	6 6 6	−3 0 0 −3 0 0 −3 0 0	$-2\boxed{\Psi}$	$\boxed{\Psi}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$
R'_6	6	−3 0 0	6 0 0	−3 0 0	0 0 0	0 0 0	0 0 0

where $\boxed{\Phi} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \phi & \phi^2 \\ 1 & \phi^2 & \phi \end{pmatrix}$, $\boxed{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\boxed{\Psi} = \begin{pmatrix} 1 & -2 & -2 \\ 1 & -2\,\phi & -2\,\phi^2 \\ 1 & -2\,\phi^2 & -2\,\phi \end{pmatrix}$

Table 31.1: Character table of $\mathcal{PG}_{216}$

Bloc $R_{1;j}$ contains the even linear characters, i.e. describes $\mathcal{PG}_{216}/\mathcal{PG}_{72}$. These characters are duplicated by the parity s to generate the odd linear characters sR_1.

Character $R_{2;1}$ correspond to the irreducible non scalar representation of the group $\mathcal{PG}_{216}/\mathcal{N}_\Delta = \mathfrak{S}_3(A, B, C)$ and is replicated by $\det^2$ (and not by sgn since $R_{2;1}$ vanishes on the odd elements). The three characters $R_{2;j}$ can be obtained directly by considering the quotient $\mathcal{PG}_{216}/\mathcal{N}_{12}$.

Character $R_{3;1}$ is generated by the quotient $\mathcal{PG}_{216}/\mathcal{N}_9$. We have $\mathcal{PG}_{216}/\mathcal{N}_9 = \mathfrak{S}_4(I_0, I_a, I_b, I_c) \simeq \langle r, t, L\rangle$ and a representation is obtained by the morphism : $t \mapsto t$, $r \mapsto r$, $L \mapsto L_c$ together with $g \mapsto 1$, $F \mapsto 1$. This character is replicated six times by the six linear characters.

The last four representations involve 6×6 matrices. Let us define :

$$m_6 = \begin{pmatrix} \phi 1_3 & 0 \\ 0 & \phi^2 1_3 \end{pmatrix},\; F_6 = \begin{pmatrix} \phi m^2 & 0 \\ 0 & m^2 \end{pmatrix},\; L_6 = \begin{pmatrix} L_c & 0 \\ 0 & L_a \end{pmatrix},\; r_6 = \begin{pmatrix} r & 0 \\ 0 & r^2 \end{pmatrix},\; t_6 = \begin{pmatrix} 0 & 1_3 \\ 1_3 & 0 \end{pmatrix}$$

The first four matrices are obtained by replicating matrices from $R_{3;1}$ and the last one is chosen to mix these two copies into an irreducible representation. Using exactly these five matrices leads to representation $R_{6;1}$. The other two replicates are obtained by using ϕF_6 or $\phi^2 F_6$ instead of F_6. The last representation is generated by the quotient $\mathcal{PG}_{216}/\mathcal{N}_4$. It happens that $\mathcal{PG}_{216}/\mathcal{N}_4 \simeq \langle r, t, F\rangle$. Therefore R'_6 is obtained by replacing L_6 by the unit matrix. Since character R'_6 vanishes outside $\langle m, r, t\rangle$, there is no replications all.

31.4.5 Subgroups $\mathcal{PG}_{72}$, $\mathcal{PG}_{18}$, $\mathcal{PG}_9^\bullet$

Lets now examine what happen to objects that are fixed by some of the former transforms. The groups of interest are the Morley group $\mathcal{PG}_{18}$ obtained from $\langle t, r, m\rangle$, its extension the Lemoine group $\mathcal{PG}_{72}$ obtained from $\langle t, r, m, L\rangle$ and the strange group $\mathcal{PG}_9^\bullet$ obtained from $\langle m, \mathbf{s}\rangle$ where $\mathbf{s} \doteq Fr$ is the ***strange*** operator (relation $rF \simeq m^2 Fr$ shows that the other choice would have been equivalent). The SAGE index of these groups are $\mathcal{G}[18, 4]$, $\mathcal{G}[72, 43]$ and $\mathcal{G}[9, 2]$. We will see later that $\mathcal{PG}_{18}$ connects the Morley centers, $\mathcal{PG}_{72}$ connects the secondary Morley perspectors and $\mathcal{PG}_9^\bullet$ connects the ***strange*** objects.

o	1	3	2	6	3	3	3	2	4	*representation*			
$\#$	1	2	3	6	8	8	8	18	18				
x	1	m	L	mL	r	rm	rm^2	t	tL	m	L	r	t
id	1	1	1	1	1	1	1	1	1	1	1	1	1
sgn	1	1	1	1	1	1	1	-1	-1	1	1	1	1
R_2	2	2	2	2	-1	-1	-1	0	0	i_2	i_2	r_2	t_2
S_1	2	-1	2	-1	2	-1	-1	0	0	r_2	i_2	i_2	t_2
S_2	2	-1	2	-1	-1	2	-1	0	0	r_2	i_2	r_2	t_2
S_3	2	-1	2	-1	-1	-1	2	0	0	r_2	i_2	r_2^2	t_2
R_3	3	3	-1	-1	0	0	0	$+1$	-1	i_3	L_c	r	$+t$
sR_3	3	3	-1	-1	0	0	0	-1	$+1$	i_3	L_c	r	$-t$
R_6	6	-3	-2	1	0	0	0	0	0	m_6	L_6	r_6	t_6

where $i_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $r_2 = \begin{pmatrix} \phi & 0 \\ 0 & \phi^2 \end{pmatrix}$, $t_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and the others as above

Table 31.2: Character table of $\mathcal{PG}_{72}$

The Lemoine group is what is obtained by mapping $F \mapsto 1$. This implies $\det^2 \mapsto 1$ so that classes and characters are (roughly) collapsing by triples. This leads to the 9×9 character table given in Table 31.2

Concerning the characters, the former block R_1 lead to the identity, the former block sR_1 to the signature while each of the former blocks R_2, R_3, sR_3, R_6 leads to the single character with the same name. On the contrary, representations R'_6 splits into the three 2×2 representations S_1, S_2, S_3. One can see that representation R_6 is faithful: $\mathcal{PG}_{72} = \langle m_6, L_6, r_6, t_6\rangle$, due to $(m_6 r_6)(r_6 m_6)^{-1} = 1$, to compare with $(mr)(rm)^{-1} = \phi$ that generates a 3-sized center in $\langle m, L, r, t\rangle$ requiring a projective quotient.

Concerning the conjugacy classes, all the classes of $\mathcal{PG}_{216}$ that were generated using F have disappeared. The former even classes $\mathcal{C}(1)$, $\mathcal{C}(m)$, $\mathcal{C}(L)$, $\mathcal{C}(mL)$ and odd classes $\mathcal{C}(t)$, $\mathcal{C}(tL)$ remain unchanged. On the contrary, the former class $\mathcal{C}(r)$ doesn't remains connected and splits into $\mathcal{C}(r), \mathcal{C}(rm), \mathcal{C}(rm^2)$. We will see later that $\mathcal{PG}_{72}$ is the group connecting the secondary perspectors.

o	1	3	3	3	3	2
$\#$	1	2	2	2	2	9
x	1	m	r	$r\,m$	$r\,m^2$	t
id	1	1	1	1	1	$+1$
sgn	1	1	1	1	1	-1
R_2	2	2	-1	-1	-1	0
S_1	2	-1	2	-1	-1	0
S_2	2	-1	-1	2	-1	0
S_3	2	-1	-1	-1	2	0

Table 31.3: Character table of $\mathcal{PG}_{18}$

When the Lemoine transforms are discarded either, it remains the projective group $\mathcal{PG}_{18} = \langle r, t, m\rangle$. Its character table is the 6×6 matrix given in Table 31.3. Here again, we have a reduction of both the classes and the characters. Classes $\mathcal{C}(1), \mathcal{C}(m)$ are kept, while $\mathcal{C}(L), \mathcal{C}(mL)$ are now perceived as replicas of 1+m and disappear. Classes $\mathcal{C}(r), \mathcal{C}(rm), \mathcal{C}(rm^2)$ are kept, but reduced in size. Finally, $\mathcal{C}(t\,L)$ disappear while $\mathcal{C}(t)$ is reduced by half. This results into the disappearance of characters R_3, sR_3, R_6. It should be noticed that all 2×2 irreducible representations use only the six matrices used for the 2×2 representation of $\mathfrak{S}_3$.

When all transforms are discarded except from the m Morley and the $\mathbf{s}$ strange operators, it remains the strange group $\mathcal{PG}_9^\bullet = \langle m, Fr\rangle$, which is a 9-sized commutative product group.

31.4.6 Action of the generators on the indices

In the previous sections many transforms have been introduced. In order to see how they connect (or separate) the various Morley objects into classes, we will examine how they act on the three digits indexes. This will complete the remarks made by Taylor and Marr (1913) or by Gambier (1954).

Theorem 31.4.1. *The image of any Taylor-Marr triangle $\mathcal{T}_\mathbf{k}$ by an element of $\mathcal{PG}_{216}$ is a Taylor-Marr triangle. Indices are transformed according to:*

	k_a	k_b	k_c		k_a	k_b	k_c
L	k_a	k_b	k_c	t	$2-k_a$	$-k_c$	$-k_b$
m	k_a+2	k_b+2	k_c+2	F	k_a+2	k_b+1	k_c
r	k_c+1	k_a+2	k_b	$\mathbf{s}$	k_c	k_a	k_b

(31.9)

Proof. Computations are easier when using $p \doteq k_a - 1$, $q = k_b$, $r = k_c$. Using these indices is underlined by using adding a $'$ to the names of the objects. The action of $r : \alpha \to \beta$, $\beta \to \gamma$, $\gamma \to \alpha$ on a Taylor-Marr triangle is described on the following table.

points	*trisect*			*Lubin*		*Lubin*		*trisect*			*points*
A'_{qr}	B'	u	q	β^6	$\alpha^2\gamma^4\phi^{(2\,q)}$	γ^6	$\alpha^4\beta^2\phi^{(2\,q)}$	C'	u	q	B'_{zx}
	C'	d	r	γ^6	$\phi^r\alpha^2\beta^4$	α^6	$\phi^r\beta^2\gamma^4$	A'	d	r	
B'_{rp}	C'	u	r	γ^6	$\alpha^4\beta^2\phi^{(2\,r)}$	α^6	$\beta^4\gamma^2\phi^{(2\,r)}$	A'	u	r	C'_{xy}
	A'	d	p	α^6	$\phi^p\beta^2\gamma^4$	β^6	$\phi^p\alpha^4\gamma^2$	B'	d	p	
C'_{pq}	A'	u	p	α^6	$\beta^4\gamma^2\phi^{(2\,p)}$	β^6	$\alpha^2\gamma^4\phi^{(2\,p)}$	B'	u	p	A'_{yz}
	B'	d	q	β^6	$\phi^q\alpha^4\gamma^2$	γ^6	$\phi^q\alpha^2\beta^4$	C'	d	q	

Each vertex of $\mathcal{T}_\mathbf{k}$ (col. 1) is the intersection of two trisectors, that are named (col. 2) and then described (col. 3) as passing by two points obtained from 31.2. Then transformation is applied (col. 4) and the result is processed back (col. 5 and 6). One obtains $x = q, y = r, z = p$, leading to (31.9). □

Therefore *strange* remains ever *strange*, while *even* and *odd* $(\sum \mathbf{k} \equiv 1)$ are only permuted by *odd* (sgn $= -1$) elements of $\mathcal{PG}_{216}$.

$$even = \begin{matrix} & \rightarrow & r \\ \downarrow & 000\ 120\ 102 \\ m & 222\ 012\ 021 \\ & 111\ 201\ 210 \end{matrix},\quad odd = \begin{matrix} & \rightarrow & r \\ \downarrow & 200\ 110\ 101 \\ m & 122\ 002\ 020 \\ & 011\ 221\ 212 \end{matrix},\quad strange = \begin{matrix} & \rightarrow & Fr \\ \downarrow & 100\ 010\ 001 \\ m & 022\ 202\ 220 \\ & 211\ 121\ 112 \end{matrix} \tag{31.10}$$

When canceling all details, "r rotates the vertices, m rotates the trisectors while t swaps the orientation and therefore the ***up*** and the ***down*** trisectors".

Moreover it is obvious from (31.2) that each element of $\mathcal{PG}_{216}$ commutes with the isogonal conjugacy.

31.4.7 The Cones proof about the strange triangles

In an illuminating paper, Connes (1998) has considered the rotations g_a with center A and angle $2\left(A + k_a\pi\right)/3$, etc. Their associated turns are respectively: $\omega = \phi^p\beta/\gamma$, $\omega = \phi^q\gamma/\alpha$, $\omega = \phi^r\alpha$. Products g_cg_b, g_bg_a, g_ag_c are proper rotations and their centers are the intersections of the corresponding trisectors. The product $g_c^3g_b^3g_a^3$ is the identity since g_a^3 is the product of the reflections with respect to AB and AC, etc. Then Connes has proven that :

Theorem 31.4.2. *Let g_c, g_b, g_a be affine transforms such that g_cg_b, g_bg_a, g_ag_c, $g_cg_bg_a$ are not translations and then let $\delta = \omega\left(g_cg_bg_a\right)$. Then we have equivalence between (1) $g_c^3g_b^3g_a^3 = 1$ and (2) $\delta^3 = 1$ and $U + V\,\delta + W\,\delta^2 = 0$ where U, V, W are, respectively, the fixed points of g_cg_b, g_bg_a, g_ag_c.*

In our problem, $\omega\left(g_c, g_b, g_a\right) = \phi^{p+q+r}$, so $\mathcal{T}_\mathbf{k}$ is equilateral if and only if $k_a + k_b + k_c \not\equiv 1$ mod 3. This explains why strange triangles are nor equilateral.

31.5 Orbits

31.5.1 Orbit of a vertex

Theorem 31.4.1 has shown that Morley centers $M_\mathbf{k}$ form a single orbit under the action of $\mathcal{PG}_{18}$. Consider now the orbit of a Morley vertex. It exists a transposition that fixes this point. Therefore, its orbit under group $\mathcal{PG}_{18}$ is 9-sized. For example, the orbit $\mathcal{PG}_{18}\left(A_{00}\right)$, described in the $\langle r, m\rangle$ order, contains :

$$A_{00}, B_{01}, C_{10}\ ;\ A_{22}, B_{20}, C_{02}\ ;\ A_{11}, B_{12}, C_{21}$$

These are the vertices of the first column of the ***strange*** family of triangles, see (31.10) and Figure 31.6.

One can check that these three triangles share the same centroid $G_a \overset{L6}{=} \alpha^2\beta^2\gamma^2$, a point on the unit circle. Noting ${G_a}^*$ the isogonal conjugate of this point, one can check that $\omega^2\left({G_a}^*\right) = -\alpha^4\beta^4\gamma^4$ and that ${G_a}^*$ is the direction of the line $B_{00}C_{00}$. Therefore, the direction of the axes of the Morley triangles are given by : $\delta_a \doteq (-G_a)^*$, etc.

From (31.9), we can see that the ***strange*** operator $s \doteq Fr$ rotates the vertices of the flagship triangle $\mathcal{T}_{000}$ according to: $A_{00} \to B_{00} \to C_{00} \to A_{00}$. For the other Morley triangles, s has to be transformed by $\mathcal{PG}_{18}$-conjugacy, leading to $r^{\{1,2\}}F^{\{0,1,2\}}$. As barycenter of an orbit, G_A is invariant by $\mathcal{PG}_{18}$ and so are the three δ, while they are rotated by s.

Since the circumcenter (origin) is the barycenter of the three G_A, it is also the barycenter of the nine M_{even} and the barycenter of the nine M_{odd}.

31.5.2 Orbits of the Morley and the Taylor-Marr centers

When using barycentric coordinates, $U \simeq_{bar} u : v : w$ has to be interpreted as $U = \left(uA + vB + wC\right)/(u + v + w$. Taylor and Marr have proven that :

$$M_\mathbf{k} \underset{bar}{\simeq} \sin A\left(\cos\frac{A + 2\,k_a\pi}{3} + 2\,\cos\frac{B + 2\,k_b\pi}{3}\cos\frac{C + 2\,k_c\pi}{3}\right) : \text{etc} : \text{etc} \tag{31.11}$$

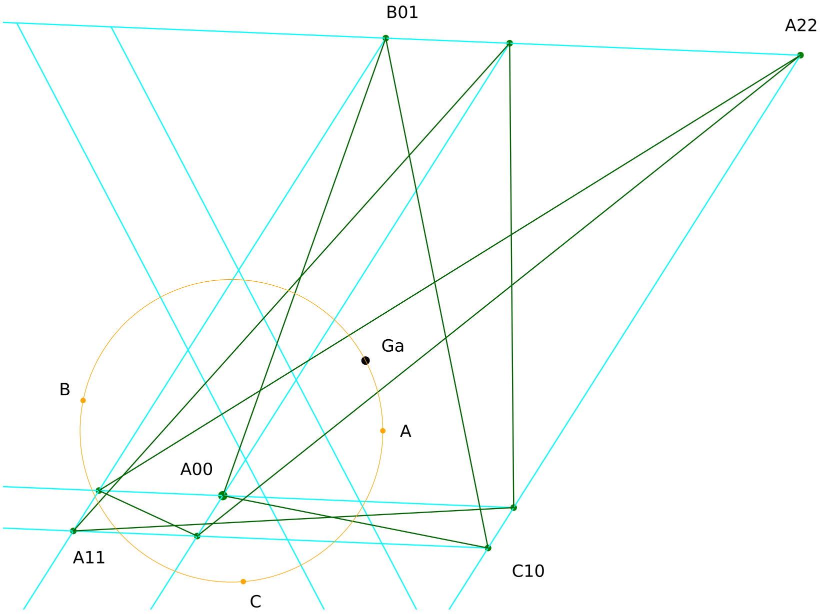


Figure 31.6: The vertex A_{00} and its 9-sized orbit

But this should not give the impression of a 27-sized family. When replacing sin and cos using $\exp i\ (A+2\,k_a)/3 \overset{L6}{=} -\phi^p\,\gamma/\beta$, etc from Figure 31.2 and the inscribed angle theorem, one re-obtains the Connes (1998) dichotomy. The Morley case is:

$$M_{\mathbf{k}} \overset{L6}{=} \frac{1}{3}\sum_{3}\left(\left(\phi^r-\phi^{-p-q}\right)\,\alpha^2\beta^4-\left(\phi^{p+q}-\phi^{-r}\right)\,\alpha^4\beta^2\right) \quad \text{when } p+q+r\not\equiv 0$$

(remember: $q=k_b$, but $,p=k_a-1$), while the M_{100} obtained from (31.11) is not the center G_A of the strange triangle $\mathcal{T}_{100}$, but the point at infinity δ_a, i.e. the direction of the perpendicular bisector of segment $[B_{00}C_{00}]$. This strange behavior is due to cancellation of ϕ-powers according to $1+\phi+\phi^2=0$. The barycentric formula looks symmetric, but an effective symmetry would only allow a 9-sized $\mathcal{PG}_{18}$-orbit for M_{000}, not to a 18-sized one.

Let us consider the polynomial $\Phi(\mathbf{Z})$ enumerating the Morley affixes. We have :

$$\Phi(\mathbf{Z}) \doteq \prod_{g\in\mathcal{PG}_{18}}\left(\mathbf{Z}-g\left(z_{000}\right)\right)$$

This polynomial is obviously invariant under the action of $\mathcal{PG}_{18}$ and therefore can be expressed using only the symmetrical functions σ_j. One obtains :

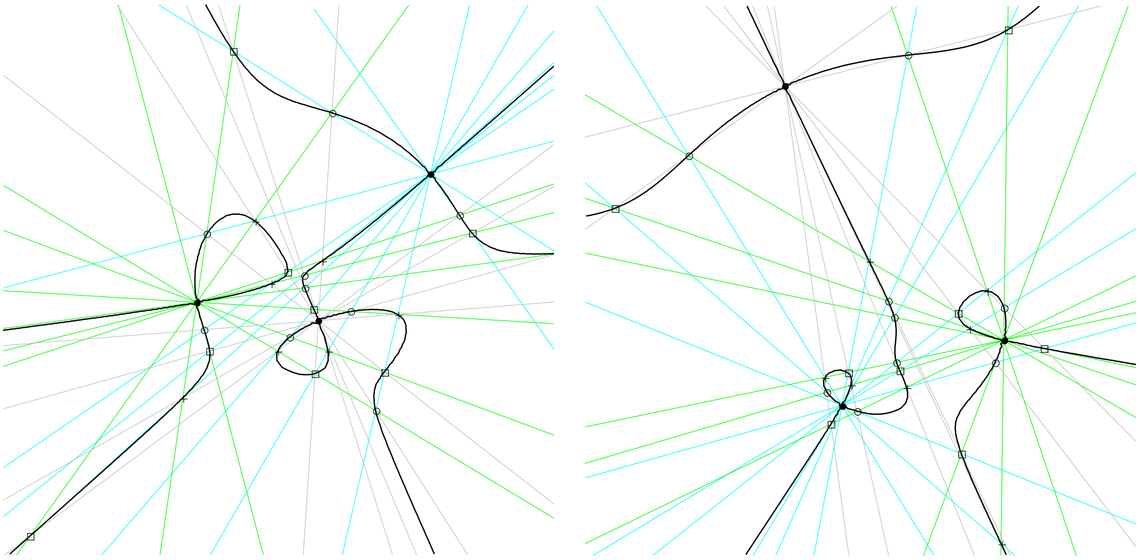
$$\Phi(\mathbf{Z}) = \mathbf{Z}^{18}-6\,\mathbf{Z}^{16}\sigma_2+12\,\mathbf{Z}^{15}\sigma_3+15\,\mathbf{Z}^{14}\sigma_2^2-60\,\mathbf{Z}^{13}\sigma_2\sigma_3+(\text{rather large expression})$$

This expression cannot be factored in $\mathbb{Q}[z_A,z_B,z_C]$ and therefore

31.5.3 Orbits of the primary perspectors

As already said, each of the 27 triangles $\mathcal{T}_{\mathbf{k}}$ are perspective with ABC: lines $AA_{\mathbf{k}}$, $BB_{\mathbf{k}}$, $CC_{\mathbf{k}}$ are concurring at a same point, the perspector $P_{\mathbf{k}}$. A "good looking formula" is Kimberling, 1998-2024 : $P_{\mathbf{k}} \simeq_{bar} \sin A\ \div\cos\left(\left(A+2\,k_a\pi\right)/3\right)$: etc : etc. But the Cones dichotomy is ever present. Using again Figure 31.2, one obtains $P_{100} \overset{L6}{=} S_3-S_1S_2+S_1^2S_3/S_2$ where $S_1=\alpha^2+\beta^2+\gamma^2$, etc, result that can be propagated to all the $P_{strange}$ via the strange group $\mathcal{PG}_9^{\bullet}$. On the contrary, P_{000} has a rather tedious affix (SAGE-length 761), and its $\mathcal{PG}_{18}$-orbit is 18-sized.

The barycentric formula explains nevertheless why all the 27 $P_{\mathbf{k}}$ share the same alignment properties. For example, on each line issued from A, there are three $P_{\mathbf{k}}$: an even one (circle), an odd one (box) and a strange one (cross). Moreover, all the 27 $P_{\mathbf{k}}$ are on a same circular quintic (the curve on Figure 31.7a). Due to the invariance under $\mathcal{PG}_{18}$, the equation of this curve can



(a) Family of X(357) is inscribed in a quintic

(b) Family of X(1507) is inscribed in a quintic

Figure 31.7: Perspectors of first and second kind

be written using only the full symmetric functions σ_j. See Gibert (2004-2024, reference Q003) or Douillet (2014a) for more details.

In Subsection 31.2.3, we have already introduced the adjunct triangles $\mathcal{T}_{\mathbf{k}}^*$ obtained by considering ***down-up*** intersections, instead of the ***up-down*** ones. These triangles are not equilateral, but are nevertheless perspective with ABC with respect to $(P_{\mathbf{k}})^*$ (general property of the isogonal duality). Points P_{000}^*, P_{111}^* and P_{222}^* are X(358), X(1135) and X(1137) in the Kimberling database. Obviously, $P_j^* \simeq_{bar} = \sin A \, \times \cos\left(\left(A + 2\,k_a \pi\right)/3\right) : \text{etc} : \text{etc}$

By isogonal conjugacy, curve Q003 transforms into an algebraic curve of degree 10. But A is a double point of Q003, so that $(BC)^2$ is a factor of $Q003^*$ and it remains an algebraic curve of degree 4, which contains all the 27 $P_{\mathbf{k}}^*$. See Gibert (2004-2024, reference Q002) or Douillet (2014a) for more details. Here again, the coefficients only depends of the σ_j.

As it can be seen, by computing the corresponding determinant or otherwise, the points $M_{\mathbf{k}}$, $P_{\mathbf{k}}$ and $P_{\mathbf{k}}^*$ that belong to the same index are aligned.

31.5.4 Orbits of the secondary perspectors

The parameterization $z_A = \alpha^6$, etc was tailored in order to access individually each of the in-excenters. The group $\mathcal{PG}_{18}$ leaves invariant the incenter I_0, and induces $\mathfrak{S}_3$ on the excenters $I_a I_b I_c$. The full $\mathfrak{S}_4$ group is obtained by introducing the Lemoine transforms, leading to the group $\mathcal{PG}_{72}$. Called "transformation continue" by Lemoine (1900), the B-Lemoine transform moves these in-excenters according to $I_0 \longleftrightarrow I_b$, $I_a \longleftrightarrow I_c$ while its action on the barycentrics is $a : b : c \mapsto -a : b : c$ and its action on α, β, γ has already been described using the L matrix given in (31.7).

It happens that triangles $\mathcal{T}_{\mathbf{k}}$ are perspective with $I_a I_b I_c$: lines $I_a A_{\mathbf{k}}$, $I_b B_{\mathbf{k}}$, $I_c C_{\mathbf{k}}$ are concurring at a same point, the secondary perspector $Q_{\mathbf{k}}$. A direct computation gives $Q_{100} \overset{L6}{=} -s_1 s_2 s_3 + s_2^3 + s_3^2 - 2\, s_2^2 s_3 / s_1$ for the strange perspector, while the affix of Q_{000} is skew enough ((length 1042) to generate a 18-size orbit.

In fact, triangles $\mathcal{T}_{\mathbf{k}}$ are not only perspective with $I_a I_b I_c$, as carried by respectively by groups $\mathcal{PG}_9^\bullet$ and $\mathcal{PG}_{18}$. They are also perspective with triangles $I_0 I_c I_b$, etc as carried by groups $\mathcal{PG}_{72}$ and $\mathcal{PG}_{36}^+$.

These 27 $Q_{\mathbf{k}}$ are involved in alignments with $I_a I_b I_c$ as shown in Figure 31.7b. For example, on each line issued from I_a, there are three $Q_{\mathbf{k}}$: an even one (circle), an odd one (box) and a strange one (cross). Moreover, all the 27 $Q_{\mathbf{k}}$ are on a same circular quintic (the curve on Figure 31.7a). This is a new result, more details in Douillet (2014a). Due to the alignments, the 3 excenters are singular points while the incenter is ordinary: the curve is not invariant under the Lemoine transforms and its equation must be written in Lubin(2), using the symmetrical functions S_j where $S_1 \overset{L6}{=} \alpha^3+\beta^3+\gamma^3$, etc instead of the $\sigma_1 \overset{L6}{=} \alpha^6+\beta^6+\gamma^6$, etc.

The equation of a 5-curve $\mathcal{Q}$ involves 21 coefficients and can be written as a 21×21 determinant. But the expression of Q_{000} is large (8 times larger than M_{000}) while the simplification $z_A=\alpha^3$, etc can no more be used. But numerical investigations allow to identify some points on $\mathcal{Q}$: the vertices A,B,C (3), I_0 (1), the excenters (being singular, they count as 3×3, the umbilics (2), $(I_a+I_b+I_c)/3$ who is the singular focus and belongs to the curve (2+1), while the Bevan pair counts 1 (X(2444) are indiscernible X(2445) belongs to $\mathcal{Q}$ (1) : this amounts to 19.

Therefore we can conduct the computations using Lubin(2), leaving only one coefficient undetermined. Thereafter, one can go back to Lubin(6) and use the simpler Q_{100} to determine the only possible equation. The proof comes from testing that (1) $Q_{000}\in\mathcal{Q}$ and (2) the equation can be send back to Lubin(2). Then $\mathcal{PG}_{72}$ extends the property to all the 27 $Q_{\mathbf{k}}$.

The adjunct triangles $\mathcal{T}^*_{\mathbf{k}}$ are themselves perspective with $I_aI_bI_c$, defining a family of perspectors $R_{\mathbf{k}}$ (they are not the the conjugates of the $Q_{\mathbf{k}}$). These points are aligned by triples (even, odd, strange) with the excenters. Moreover, I_0, $P_{\mathbf{k}}$, $R_{\mathbf{k}}$ are aligned for each $\mathbf{k}$. But, contrary to the other families, there is no quintic that contains all the 27 $R_{\mathbf{k}}$, while the circular quintic that contains the 18 "true" perspectors contains no Kimberling centers. Even the $1,0,0$ affix is not so simple, since one has :
$R_{100} \overset{L6}{=} \left(s_1^3s_2s_3-s_1^2s_2^3-s_1^2s_3^2+s_1s_2^2s_3+s_2^4-s_2s_3^2\right)\div\left(s_1^2-s_2\right)$

31.6 Some applications

31.6.1 Martiny circles

Let us choose a Morley vertex, say $V=A_{00}$. It belongs to exactly an even and an odd Morley triangle (here $\mathcal{T}_{000}$ and $\mathcal{T}_{200}$). The four other vertices of these two triangles are cocyclic together with the ABC vertex used to name V (here A, B_{00}, B_{02}, C_{00}, C_{20}). According to Gambier (1954), this unpublished result was obtained by H. Martiny (1882-1963). This can be checked directly, and the center of the circle is easy to find. Let us call it D_{A00}. Its images under $\langle m\rangle$ are D_{A00}, D_{A22}, D_{A11} and we have :

$$m^k\left(D_{A22}\right) \overset{L6}{=} -\alpha^2\beta^2\gamma_1^2+\left(\frac{\beta}{\gamma}\,\phi^k+\frac{\gamma}{\beta}\,\phi^{-k}\right)\beta\gamma\,\alpha^4$$

Over the $\langle m\rangle$ orbit, the parentheses sum to 0 and the centroid of these points is antipodal with the already obtained point $G_A \overset{L6}{=} \alpha^2\beta^2\gamma^2$. Moreover, these parentheses are real, and this also applies to the obvious :

$$A \overset{L6}{=} -\alpha^2\beta^2\gamma_1^2+\left(\frac{\alpha^2}{\beta\gamma}+\frac{\beta\gamma}{\alpha^2}\right)\beta\gamma\,\alpha^4$$

Therefore A, D_{A00}, D_{A22}, D_{A11} are aligned and the three circles are tangent in A. Triplicating this result by the operator F, we can generate the nine A-Martiny circles: they all go through A and the angles they form with each other are multiples of 60° (since the $-\phi^k\alpha^2\beta^2\gamma^2$ are at 120° from each other).

31.6.2 Equilateral perspectors

Specific properties appear when several Morley centers are involved at the same time. In order to illustrate the problematic, we will study two problems of this kind: the perspectivity of two Morley triangles, and the intersection of two Morley cubics. The set of all the pairs $\{\mathbf{k},\kappa\}$ where $\mathbf{k}\neq\kappa$ contains $(18\times17)/2=153$ elements. Let us say that a pair is *even* or *odd* according to $\operatorname{sgn} g$ where $\kappa=g\cdot\mathbf{k}$ and $g\in\mathcal{PG}_{18}$. Then, under the action of $\mathcal{PG}_{18}$, the *even* pairs form four 18-sized orbits, while the *odd* pairs form nine 9-sized ones (as it should be, we have $18\times4+9\times9=153$).

Let us see why. Since $\mathcal{PG}_{18}$ is faithful over the $M_{\mathbf{k}}$, $g(\{x,y\})=\{x,y\}$ implies either $xg=x$, $yg=y$ (and then $g=1$) or $xg=y$, $yg=x$ (and then $xg^2=x$, so that g is odd). Conversely, the order of any $g\in\mathcal{PG}_{18}$ is 2.

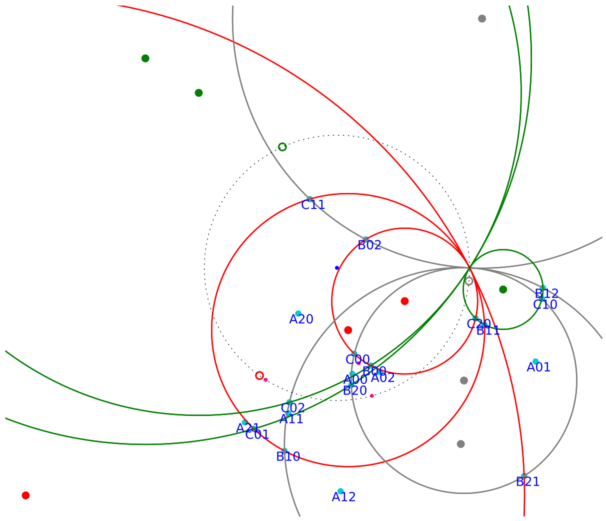


Figure 31.8: The nine A-Martiny circles and their centers

The action of $\langle \mathbf{s},\, m\rangle$over the 27 Taylor-Marr indices is summarized by the following table :

	$\mathcal{T}_{000}$	*even*					*odd*			*strange*		
		$\rightarrow$	m		$\rightarrow$	?	$\rightarrow$	m			$\rightarrow$	m
id	A_{00}	000	222	111	120	102	200	122	011	100	022	211
$\mathbf{s}$	B_{00}	000	222	111	012	210	020	212	101	010	202	121
$\mathbf{s}^2$	C_{00}	000	222	111	201	021	002	221	110	001	220	112

We will now consider the perspectivity between two Morley triangles, written as $\mathcal{T}_{000}\cdot g_{\mathbf{k}}$ and $\mathcal{T}_{000}\cdot g_{\mathbf{k}'}$: vertices are ordered according to the order transmitted by $\mathcal{PG}_{18}$ from the ordering A_{00}, B_{00}, C_{00}. Oriented that way, these triangles are homothetic and therefore in perspective. Their perspector $S_{\mathbf{k}\kappa}$ is obviously on the line $M_{\mathbf{k}}M_{\kappa}$ joining the centers. These perspectors can therefore be characterized by the real numbers $\mu_{\mathbf{k}}$ such that :

$$S_{\mathbf{k};000} = \mu\, M_{000} + (1-\mu)\, M_{\mathbf{k}}$$

After some computations, one obtains the following table :

$\mathcal{T}_{\mathbf{k}}$, the other triangle		$\mathcal{PG}_{18}$	perspector
m, m^2	222,111	18	$P^*_{000},\, P^*_{111}$
$r,\, r^2$	120,102	18	$S,\, S\,r^2$
$mr,\, m^2r^2$	012,210	18	$S\,\mathbf{s},\, S\,\mathbf{s}r^2$
$m^2r,\, mr^2$	201,021	18	$S\,\mathbf{s}^2,\, S\,\mathbf{s}^2r^2$
$tm^3,\, tm^1r^2,\, tm^1r$	200,020,002	9×3	A_{00}, B_{00}, C_{00}
$tm^1,\, tm^2r^2,\, tm^2r$	122,212,221	9×3	$\widehat{S},\, \widehat{S}\,\mathbf{s},\, \widehat{S}\,\mathbf{s}^2$
$tm^2,\, tm^3r^2,\, tm^3r$	011,101,110	9×3	$\widetilde{S},\, \widetilde{S}\,\mathbf{s},\, \widetilde{S}\,\mathbf{s}^2$

Here again, except from the first one, the orbits are grouped into triples under the action of the strange operator $\mathbf{s}$, so that we have essentially 3 new points to introduce :

$$\mu \stackrel{L6}{=} \frac{(\phi+2)\left(\alpha^2\beta^2-\gamma^4\right)}{(\beta^2-\gamma^2)\left(\gamma^2\phi^2-\alpha^2\right)} \qquad \widetilde{\mu} \stackrel{L6}{=} \frac{\alpha^2\gamma^2\phi^2+\alpha^2\beta^2\phi+2\,\alpha^4+2\,\beta^2\gamma^2}{\alpha^2\beta^2\phi^2+\alpha^2\gamma^2\phi-\alpha^4-\beta^2\gamma^2}$$

$$\widehat{\mu} \stackrel{L6}{=} \frac{\left(\alpha^4\beta^2-2\,\alpha^2\gamma^4+\beta^4\gamma^2\right)\phi^2-\left(\alpha^4\gamma^2-2\,\alpha^2\beta^4+\beta^2\gamma^4\right)\phi}{\alpha^2\gamma^4\phi^2-\alpha^2\beta^4\phi+(2\,\phi+1)\,\alpha^2\beta^2\gamma^2+\alpha^4\beta^2-\alpha^4\gamma^2+\beta^4\gamma^2-\beta^2\gamma^4}$$

31.7 Summary of this chapter

When drawing the configuration formed by the bisectors of a triangle, one obtains the four points known as the in-excenters. One can see that their affixes are the roots of the polynomial :

$$\Phi_{inex}(\mathbf{Z}) = \mathbf{Z}^4 - 2\,\mathbf{Z}^2\sigma_2 + 8\,\mathbf{Z}\sigma_3 - 4\,\sigma_1\sigma_3 + \sigma_2^2$$

where the $\sigma_j$are the symmetric functions in z_A, etc. Since this polynomial doesn't factor naturally, the four in-excenters form an indiscernible family, that must be considered as a whole. These four points are connected by the Lemoine transforms, that were called "transformations continues" in the original paper (Lemoine, 1900).

The present article is about the Morley configuration, that occurs when drawing the trisectors of the angles of a triangle. In this configuration, a family of 18 triangles can be identified that are together equilateral and in perspective with the original triangle. Their centers are called the Morley centers and form a 18-sized algebraically indiscernible family .

We have investigated the operators acting over the Morley configuration. The usual approach is to only consider these operators as simple heuristics. On the contrary, we have put these operators in the foreground, and shown how they can be naturally introduced from the parameterization given by Lubin (1955)

We have elicited the structure of the groups formed by the various subsets of these operators. The reference group is $\mathcal{PG}_{216}$, inventoried as $\mathcal{G}\,[\,216, 92\,]$ in the SAGE database. The group $\mathcal{PG}_{18}$ (inventoried as $\mathcal{G}\,[\,18, 4\,]$) is the one that connects the Morley centers in a single orbit. When enlarged with the Lemoine transforms, we obtain the group $\mathcal{PG}_{72}$ (inventoried as $\mathcal{G}\,[\,72, 43\,]$) that connects the perspectors between the Morley triangles and the in-excentral triangles.

We have also examined the various attempts made to enlarge the Morley family, using equilateral triangles that are not in perspective or perspective triangles that are not equilateral. In both cases, the new objects are not connected by the Morley group, but by a smaller group that explains why these attempts are behaving strangely. In fact, they appear as too symmetric to fulfill their requirements.

Chapter 32

Using Kimberling's database into Maple

In this chapter, we give the specifications of our implementation of the Kimberling ETC. At the present moment, this implementation is restricted to a private use. The intent was to participate to a distributed maintenance of the database, together with providing "random search keys", specific to each local copy, in order to allow a distributed method of proof.

The pillar of everything is the tandem `~/etc/bar_igcta.csv` and sk that is

1. written once for ever using `mysql/phpMyAdmin`, and cured until it becomes `Maple`-compliant
2. read and parsed by Maple at the beginning of each session.

32.1 Sparse version (2019 and after)

The old procedure `relire` has been split. The new procedure `relire` deals with the formal values, stored in the file `~ipse/public_html/etc/bar_igtca.csv`. On the other hand, procedure `reliresk` reads the maple archive stored as `cat(encyfile[1 .. -9], "relire_sk/pas_toujours/relire_sk.m")`.

32.1.1 How-to update

1. copy `~ipse/docs/Cherche/Geometry/ETC_2018` towards `.../ETC_2023`. Suppress everything, except from `download.sh` and the various links (the `*.gif` are used by the etc files, the other are probably useless).
2. update the number of pages to load, update `http://` into `https://` and run the batch `download.sh`.
3. Run the builders for the search keys. They are in `relire_sk.mw`. See Alg. 6.4, Alg. 6.7

   new_build_sk(non destructive). Uses `Geometry/ETC_2023`

   newbuild_enc_sort(non destructive).
   Uses `~ipse/public_html/etc/sk_plex.csv` Each item is `1=jj ; 2,3,4= sk[jj] ; 5=xk[jj]`

   exec — save (in `relire_sk.m`)
   `fac47, smax, siz_enc, list_collisions`
   `enc_sort, fk, sk`

4. Then the formal values of the coordinates can be loaded.

32.1.2 Duplicates

From now on (2023), it will be assumed that `reliresk` contains the authoritative values, while their counterparts in `relire` are no more maintained. Shouldn't be loaded.

gerdat — list of lines going through the point. Skip it!

hc hc[j]=sk[j][1]*_facency. Skip it !

enc_sort saved in `relire_sk.m`. Removed from `relire`.

gg,tt,cc,ac isogon, isotom, complem, anticomplem. No more used

32.1.3 la table barita

j	ff	sk			xyz		
6800	0	−1.498509	−2.971024	5.469533	−1.498509	−2.971024	5.469533
6801	0	0.3093201	0.7440262	−0.05334631	0.3093201	0.7440262	−0.05334631
6802	0	−0.6812215	0.09831283	1.582909	−0.6812215	0.09831283	1.582909
6803	0	0.4724080	0.5413951	−0.01380309	0.0	0.0	0.0
6804	0	0.2233194	0.1687478	0.6079328	0.0	0.0	0.0
6805	0	0.2628773	0.2279281	0.5091946	0.0	0.0	0.0
6806	0	0.4146746	0.4550234	0.1303019	0.0	0.0	0.0
6807	0	−0.3066239	−0.6240702	1.930694	0.0	0.0	0.0
6808	0	−2.636019	−4.108946	7.744965	0.0	0.0	0.0
6809	0	1.581039	2.199954	−2.780993	0.0	0.0	0.0
6810	na	−0.03692055	−0.2205824	1.257503	6810	0	0
6811	na	−0.1950282	−0.4571182	1.652146	6811	0	0
6812	na	−0.3790703	−0.7324531	2.111523	6812	0	0
6813	na	−148.1508	−221.8054	370.9562	6813	0	0
6814	na	−1.684099	−2.684833	5.368932	6814	0	0

32.1.4 Description

When searching for points satisfying some property, we want to use a procedure like

```
localize:= proc(expr, px := vx) global myfun; local lediv;
myfun := unapply(evalf(subs(iciK, expr)), op(convert(px, list)))@OP;
lediv:= evalf(sqrt(add(myfun(sk[j])^2, j=1..10)/10));
seq(`if`(abs(myfun(sk[j]))/lediv < 1/1000000000000, j, NULL), j = 1 .. smax);
end proc;
```

where the index ranges through the $smax = 15639$ points. Thus we need a full list of numerical barycentrics (named `sk` in our code), even where we don't have created the corresponding formal values. This is done using two tables.

fk in $\{0, 1, 2\}$. Generalizes the flag `ff`, by introducing `fk=2` when the barycentrics of X(n) have complex values when using $a = 6, b = 9, c = 13$.

sk generalizes `xyz`, i.e. contains the numerical values of the barycentrics of X(n) for $n \leq 15639$. For the sake of efficiency, this table is mostly stored in `$ipse/public_html/etc/t6913.csv` (the real values), while the complex values are stored in `$ipse/public_html/etc/sk_plex.csv` (and also corrections, when required).

32.1.5 Rationales

This paradigm shift has been allowed by the fact that, nowadays, the Kimberling (1998-2024) database provides the searchkeys for the three systems $(6, 9, 13)$, $(9, 13, 6)$ and $(13, 6, 9)$, and not only for the $(6, 9, 13)$. They are buried in the `http://faculty.evansville.edu/ck6/encyclopedia/Search_6_9_13.html`, etc. tables. Put together, these three numbers are, *quite all of the time,* the normalized trilinear coordinates of the given point, and we can use

$$[x, y, z] = [6t_A, 9t_B, 13t_C]$$

To be more specific, the table `fk` is filled according to the following criteria (where $=_\epsilon$ is defined by "less different than $1E-16$"), we have three kinds of points

fk=0 where $x + y + z =_\epsilon 8\sqrt{35}$: ordinary points at finite distance, then normalized so that $x + y + z = 1$

fk=1 where $x+y+z =_\epsilon 0$: points at infinity, then normalized so that $1/x + 1/y + 1/z = 1$.

fk=2 for the other cases (complex points... or errors) 5000, 5001, 5002, 5003, 5374, 8072, 8073, 11065, 11066, 14091

We can check that (for $n \leq jmax = 6802$) all the corresponding normalized values (see Section 6.3) are equal with the tabulated xyz except from the complex points (expected behavior).

Remark 32.1.1. This gives a better method to exclude the points that are perceived as having a "bad behavior":

1. Eight points non algebraic points, tagged by `fdat[n]="special"`, namely X(n) where n= 368, 369, 370, 1144, 3232, 5373, 5394, 5626.

2. Eight points with "too long" barycentric formulas (> 735), tagged by `fdat[n]="len=..."`, namely X(n) where X(n)= 5676, 5677, 5678, 5679, 5680, 5681, 5682, 5973.

32.1.6 Usage: the new ency procedure

1. We start from [-288, 1701, -2197] or from [3538, 5293, -8831]. The NORMALIZE algorithm ALG. 6.2 uses the value of
$$\frac{|x+y+z|}{|x|+|y|+|z|}$$
to decide if $x+y+z \not\approx 0$ (i.e. $M \notin \mathcal{L}_b$) or $x+y+z \approx 0$ (i.e. $M \in \mathcal{L}_b$). And then the column is standardized according its type. In the first case, we divide x by $-288+1701-2197 = -784$. In the second case, we multiply x by the sum of inverses $(1/3538 + 1/5293 - 1/8831) \approx 0.00036$. This gives the searchkeys 0.3673469 and 1.2677958.

2. The key is compared with the existing ones. The answer is either fail (and provide an interval for the keys) or a line from `enc_sort`. First element is a true key, second element is a sequence of integers, i.e. a single integer (best case) or a sequence of several integers

3. In the second case, a subsequent test is made using the three coordinates. More details at ALG. 6.8.

32.2 Older versions (2017 and before)

this section has been cancelled (was 31.2)

32.3 Synchronizing formal barycentrics and the search keys

this section has been cancelled (was 31.3)

32.4 Requirements (all versions)

this section has been cancelled (was 31.4)

32.5 Building the database

this section has been cancelled (was 31.5)

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