The Terminator of a Sequence: A Constructive Approach to Infinity

Gabriele Bonamini

2025/02/18

Abstract

In this paper, we introduce and develop the concept of *"the terminator of a sequence"*, defined as the term that would be reached by extending the construction of a given infinite succession indefinitely. Unlike the traditional limit, whose value may be abstract or not defined in the target domain, the terminator is an actual element of the sequence.

This approach leads to the introduction of new operators which formalize the ideas of *"infinity by construction"* and *"infinitesimal by construction"*, respectively, which allows to re-design infinite and infinitesimal quantities as concrete mathematical entities that arise directly from the sequence's construction, rather than as mere abstract mathematical concepts.

A key example is the infinite sequence of natural numbers, whose terminator is denoted as $\boldsymbol{N}$ and termed *"the fundamental terminator"*. We explore the properties of $\boldsymbol{N}$, showing that it is unique, it belongs to the sequence, it is a natural number, and, remarkably, it is prime!

The paper then establishes the *Fundamental Axiom of Terminators (FAT)*: for any function $f : \mathbb{N} \to \mathbb{R}$, the terminator of the sequence $f(n)$ satisfies the following identities:

$$\Theta[f(n)] = f(\Theta(n)) = f(\boldsymbol{N}) \tag{1}$$

thus exploiting the homomorphism between the process of sequence construction and the functional application.

Several notable applications of this theory are presented: a given terminator allows to show that it does exist a one-to-one correspondence between the sets $\mathbb{R}$ and $\mathbb{Q}$ (!). Other examples show some terminators as examples of distinct Leibniz infinitesimals. The paper further investigates factorial terminators, ultimately demonstrating that $\boldsymbol{N}$ is a prime number.

Overall, the paper provides a novel framework that reinterprets classical concepts of limits, infinity, and infinitesimals through the "lens" of sequence construction. It also paves the way for a new understanding of composite numbers constructed as $[f(\boldsymbol{N}) + k]$, with $k \in \mathbb{R}$. This new perspective not only deepens the theoretical foundations of analysis and number theory, but also suggests potential applications in exploring the structure and properties of well known sequences used in the various fields of mathematics and physics.

1 Introduction

This paper introduces the concept of the *terminator* of a sequence, a novel construct distinct from the classical notion of a limit. Unlike limits, which describe asymptotic behavior, a terminator is the result obtained when a sequence is indefinitely extended while preserving its intrinsic structural properties. This approach provides a new perspective on infinity, termed *infinity by construction*, which allows for a richer characterization of infinite sequences.

Limits do not always capture structural aspects of sequences, particularly when dealing with divergence. The terminator, as an alternative, focuses on the constructive process of a sequence rather than its limiting behavior.

For example, consider the sequences: $a_n = 2n, \quad b_n = 2n + 1$.

Both diverge to infinity, yet their structures remain distinct: one is composed exclusively of even numbers, the other of odd numbers. The terminator concept ensures that these structural distinctions persist at infinity.

This novel approach has potential applications in:

- **Number Theory**: Understanding properties of integer sequences beyond their asymptotic behavior
- **Dynamical Systems**: Characterizing the fate of iterative processes without appealing to limits
- **Set Theory and Foundations of Mathematics**: Providing a constructive approach to infinity that differs from classical cardinality and ordinals.

2 Definition of the Terminator

Consider any sequence of the form $A : \mathbb{N} \to \mathbb{R}$. Define its **construction rule** as the function $A(n) = a_n$ through which the sequence is orderly built.

Define the **terminator** $\Theta(a_n)$ of the sequence A as the term that would be obtained by continuing the orderly construction of A indefinitely.

2.1 Existence of the Terminator

A rigorous proof of the terminator's existence could be based on a constructive argument:

- We define a sequence as an infinite process governed by a well-defined rule
- The terminator is the formal entity obtained by extending this process indefinitely while maintaining its structural properties
- If such a formal entity did not exist, the sequence itself would not be definable as an infinite process, which contradicts the assumption that the sequence is infinite.

Thus, the *existence* of the terminator $\Theta(a_n)$ is obvious, since the set of indices n is not bounded above and, consequently, it is possible to construct terms a_n indefinitely. Since each term of the sequence is defined in the target domain, so is the terminator too.

2.2 The Value of $\Theta(a_n)$ and the Limit of A

The determination of the *value* of $\Theta(a_n)$, which we denote by $\|\Theta(a_n)\|$, may lead to some ambiguity, since $\|\Theta(a_n)\|$ tends to the same value as the limit of the sequence (if it exists). Thus, the following cases occur:

- **Convergent Sequence**: if the sequence converges to a finite limit L, then $\|\Theta(a_n)\|$ will tend to L, without necessarily coinciding with it. (See sections 2.4 and 4.4 for the special case $L = 0$, the *infinitesimal terminator*).

- **Divergent Sequence**: If the sequence diverges, $\|\Theta(a_n)\|$ will tend to infinity while remaining an explicit term of the sequence, conceptually distinct from the limit. To avoid ambiguity, we introduce the following notation:

$$\text{if } \lim_{n\to\infty} a_n = \pm\infty \text{ then } \|\Theta(a_n)\| = \pm \bowtie \tag{2}$$

  where the symbol $\bowtie$ is read: *infinity by construction*.

- **Oscillating Sequence**: if the sequence oscillates or varies aperiodically, the limit does not exist; however, there may be several values, each of which can be viewed as a terminator in a certain context, by specifying additional constraints imposed on its constructive process.

Furthermore, the exact value of the terminator is of much less interest than its other properties, which we shall define as characteristics specific to the construction or, more simply, *specific characteristics* of the terminator.

2.3 The Terminator as a Constructed Entity

Since the terminator is a term of the sequence, it *inherits* all the properties of a generic term of the sequence from which it comes. This idea opens the door to treating infinity, and infinitesimals, by introducing operators that apply these concepts in a way different from the traditional analytic approach. Let us examine in detail the implications of this reasoning.

As noted, $\Theta(a_n)$ is a term of the sequence and, by definition, is a value obtained through a process of construction. If the sequence is defined for every integer index, then every term is a well-defined value in the co-domain of the sequence. Therefore, the terminator is not an abstract concept, but a concrete number originating from the sequence itself. This implies that:

- The terminator cannot be something "external" to the sequence; it is a value directly extracted from the terms that compose it.

- If the terminator possesses special characteristics—such as being an infinite or infinitesimal number—this status depends on the construction of the terms themselves, that is, on the evolution rules of the sequence.

This concept of the "inheritance" of a term's properties is very powerful because it allows us to introduce the notions of "infinity by construction" and "infinitesimal by construction" not as abstractions, but as concrete mathematical entities that satisfy some of the properties of finite numbers.

2.4 "By Construction" Operators: $\bowtie$ and $\Diamond$

For a divergent sequence, the expression

$$\|\Theta(a_n)\| = \pm \bowtie \tag{3}$$

is read as: "The value of the terminator of the sequence $A(n)$ is (plus or minus) infinity by construction," i.e., an infinite quantity obtained through a well-defined, orderly construction. Similarly, for an infinitesimal sequence, the expression

$$\|\Theta(a_n)\| = \Diamond \tag{4}$$

is read as: "The value of the terminator of the sequence $A(n)$ is infinitesimal by construction." In this case, any + or – sign placed in front of the empty diamond indicates asymptotic convergence to zero from the right or left, respectively. This represents the idea of an infinitesimal quantity that has a defined and constructed origin rather than being purely abstract.

3 The Fundamental Terminator $\boldsymbol{N}$

Define the **fundamental sequence** as

$$a_n = n \quad , \quad n \in \mathbb{N} \tag{5}$$

which corresponds trivially to the ordered sequence of natural numbers.

The terminator of this sequence is denoted as follows:

$$\Theta(n) \equiv \boldsymbol{N} \quad \text{the fundamental terminator} \tag{6}$$

According to the previous discussion, we can also write:

$$\|\Theta(n)\| = \|\boldsymbol{N}\| = + \bowtie \tag{7}$$

3.1 Properties of the Fundamental Terminator

The specific characteristics of the fundamental terminator $\boldsymbol{N}$ are as follows:

1. $\exists!\ \boldsymbol{N}$: the terminator of $a_n = n$ exists and is unique.
2. $\boldsymbol{N} \in A$: the terminator is a term of the sequence $a_n = n$.
3. $\boldsymbol{N} \in \mathbb{N}$: the terminator is a natural number.
4. $\|\boldsymbol{N}\| = 1 \cdot (+\bowtie)$, which necessarily implies that:
5. $\|\boldsymbol{N}/\boldsymbol{N}\| = 1\cdot \bowtie /1\cdot \bowtie = 1$ since this is the ratio of two quantities identical by construction.
6. as shown in §5, $\boldsymbol{N}$ must be a prime number.

4 Fundamental Theorem of Terminators

Given the sequence $A = f(n)$ where f is any function associating to each natural number a real number, the following identities always hold:

$$\Theta[f(n)] = f[\Theta(n)] = f(\boldsymbol{N}) \tag{8}$$

$$\|\Theta[f(n)]\| = \|f[\Theta(n)]\| = \|f(\boldsymbol{N})\| = f\|\boldsymbol{N}\| \tag{9}$$

4.1 Elementary Properties

The following properties can be verified $\forall j, k \in \mathbb{R}$, where the most significant specific characteristics can be easily identified:

$$\Theta(k \cdot n) = k \cdot \boldsymbol{N} \tag{10}$$

$$\Theta(j \cdot n) + \Theta(k \cdot n) = (j + k) \cdot \boldsymbol{N} \tag{11}$$

$$\Theta(j \cdot n) \cdot \Theta(k \cdot n) = (j \cdot k) \cdot \boldsymbol{N}^2 \tag{12}$$

$$\Theta(j \cdot n)/\Theta(k \cdot n) = j/k \tag{13}$$

$$\Theta(n^k) = \boldsymbol{N}^k \tag{14}$$

$$\Theta(k^n) = k^{\boldsymbol{N}} \tag{15}$$

$$\Theta(n!) = \boldsymbol{N}! \tag{16}$$

$$\Theta(n^n) = \boldsymbol{N}^{\boldsymbol{N}} \tag{17}$$

$$\Theta(f(g(n))) = f(g(\boldsymbol{N})) \tag{18}$$

5 Some Notable Terminators

5.1 One-to-One Correspondence Between Real and Rational Numbers

Consider the sequence $A(n) = 10^n$. By the fundamental theorem, its terminator is:

$$\Theta(10^n) = 10^{\boldsymbol{N}} \tag{19}$$

whose value can also be expressed with the alternative notation:

$$\|10^{\boldsymbol{N}}\| = 1\overline{0} \tag{20}$$

This indicates a 1 followed by infinitely many zeros. The specific characteristics are:

- $\exists\ 10^{\boldsymbol{N}}$
- $10^{\boldsymbol{N}} \in \mathbb{N}$ is infinite by construction, is an integer, and is necessarily a multiple of 10.

Thus, this terminator satisfies the following property:

$$\forall r \in \mathbb{R}, \exists s = (r \times 10^{\boldsymbol{N}}) \in \mathbb{N} \tag{21}$$

For example, one can write:

$$\sqrt{2} \times 10^{\boldsymbol{N}} \in \mathbb{N} \tag{22}$$

and consequently

$$\sqrt{2} = \frac{\sqrt{2} \times 10^{\boldsymbol{N}}}{10^{\boldsymbol{N}}} \in \mathbb{Q} \tag{23}$$

Ultimately, this allows for a one-to-one association between the sets $\mathbb{Q}$ and $\mathbb{R}$.

5.2 The Largest Existing Number

Consider the sequence

$$\chi(n) = \frac{10^n}{3^{(10)^{-n}}} \tag{24}$$

By the fundamental theorem, its terminator is:

$$\Theta(\chi(n) = \frac{10^{\boldsymbol{N}}}{3^{(10)^{-\boldsymbol{N}}}} \tag{25}$$

which represents the largest existing real number, expressible with the following alternative notation:

$$\Theta(\chi(n)) = \frac{10^{\boldsymbol{N}}}{3^{(10)^{-\boldsymbol{N}}}} = \overline{9} \tag{26}$$

that is, the natural number formed by infinitely many nines.

5.3 The Fundamental Infinitesimal Terminator

Define the **fundamental infinitesimal terminator** as follows:

$$\Theta\left(\frac{1}{n}\right) = \frac{1}{\Theta(n)} = \frac{1}{\boldsymbol{N}} \tag{27}$$

whose value, infinitesimal by construction, is:

$$\left\|\frac{1}{\boldsymbol{N}}\right\| = \Diamond \tag{28}$$

This terminator satisfies the definition of a Leibniz infinitesimal, since it is smaller than any positive real number and is still greater than zero. By inverting some of the terminators listed in section 3.1, we obtain several other distinct *infinitesimal by construction* terminators :

$$\frac{1}{k \cdot \boldsymbol{N}} \tag{29}$$

$$\frac{1}{(j+k)\cdot \boldsymbol{N}} \tag{30}$$

$$\frac{1}{(j\cdot k)\cdot \boldsymbol{N}^2} \tag{31}$$

$$\frac{1}{\boldsymbol{N}^k} \tag{32}$$

$$\frac{1}{k^{\boldsymbol{N}}} \tag{33}$$

$$\frac{1}{\boldsymbol{N}!} \tag{34}$$

$$\frac{1}{\boldsymbol{N}^{\boldsymbol{N}}} \tag{35}$$

Note also that

$$\boldsymbol{N}\cdot \Diamond = 1 \tag{36}$$

5.4 The Factorial Terminator

Define the *"factorial sequence* Π*"* as:

$$\Pi(n) = n! \tag{37}$$

By the fundamental theorem, its terminator is:

$$\Theta(\Pi(n)) = \Pi(\Theta(n)) = \boldsymbol{N}! \tag{38}$$

whose value, infinite by construction, represents the only integer divisible by *all* natural numbers. It is certainly even, and can be defined as "the anti-prime number par excellence."

5.4.1 $(\boldsymbol{N}!+1)$ is a Prime Number

Consider the sequence

$$\Pi_{+1}(n) = n! + 1 \tag{39}$$

Listing some terms for n greater than 2:

$\Pi_{+1}(3) = 3! + 1 = 7$, a prime number

$\Pi_{+1}(4) = 4! + 1 = 25$, factorable as 5×5

$\Pi_{+1}(5) = 5! + 1 = 121$, factorable as 11×11

$\Pi_{+1}(6) = 6! + 1 = 721$, factorable as 7×103

$\Pi_{+1}(7) = 7! + 1 = 5041$, factorable as 71×71

$\Pi_{+1}(8) = 8! + 1 = 40321$, a prime number

$\Pi_{+1}(9) = 9! + 1 = 362881$, factorable as 601×601

$\Pi_{+1}(10) = 10! + 1 = 3628601$, factorable as 11×329891

...

In general, if $a_n > 2$ is not prime, then it is easy to show that the smallest integer factor $m \neq 1$ of a_n must always satisfy:

$$n < m < a_n \tag{40}$$

By the definition of the terminator and the fundamental theorem of terminators, we have:

$$\Theta(\Pi_{+1}(n)) = \Theta(n! + 1) = \Theta(n!) + \Theta(1) = \Theta(n)! + 1 = \boldsymbol{N}! + 1 \tag{41}$$

but since, by the same definition of terminator, there cannot exist any $m > \boldsymbol{N}$, it follows that the terminator $[\boldsymbol{N}! + 1]$ **must be *prime***.

Remark: The introduction of terminators implies the definition of new composite numbers of the no further reducible form $[f(\boldsymbol{N}) + k]$, with $k \in \mathbb{R}$, in which an "infinite part $Ip\{\}$" and a "finite part $Fp\{\}$" are distinguished. The notation associated with the previous example is as follows:

$$\zeta = \Theta(\Pi_{+1}(n)) = [\boldsymbol{N}! + 1] \tag{42}$$

$$Ip\{\zeta\} = \boldsymbol{N}! \tag{43}$$

$$Fp\{\zeta\} = 1 \tag{44}$$

5.5 Theorem: *N* is a Prime Number

Consider the sequence

$$a_n = \frac{(n-1)! + 1}{n} \tag{45}$$

By Wilson's Theorem it is shown that

$$n \text{ is prime} \ \Leftrightarrow \ a_n \in \mathbb{N} \tag{46}$$

but the terminator of a_n is

$$\frac{(\boldsymbol{N} - 1)! + 1}{\boldsymbol{N}} = \frac{\boldsymbol{N}! - 1! + 1}{\boldsymbol{N}} = \frac{\boldsymbol{N}!}{\boldsymbol{N}} = \frac{\boldsymbol{N}(\boldsymbol{N} - 1)!}{\boldsymbol{N}} = (\boldsymbol{N} - 1)! = \boldsymbol{N}! - 1 \ \in \mathbb{N} \tag{47}$$

hence, **the Fundamental Terminator *N* is *prime***.

5.6 The Terminator of Fibonacci Numbers

Consider the well known sequence defined by recursion: $F(n) = F(n-1) + F(n-2)$, with $F(0) = 0$ and $F(1) = 1$, whose n-th term can be calculated through the Binet's formula:

$$F(n) = \frac{(\frac{(1+\sqrt{5})}{2})^n - (\frac{(1-\sqrt{5})}{2})^n}{\sqrt{5}} \tag{48}$$

Thus, the terminator of Fibonacci numbers is:

$$F(\boldsymbol{N}) = \boldsymbol{F} = \frac{(\frac{(1+\sqrt{5})}{2})^{\boldsymbol{N}} - (\frac{(1-\sqrt{5})}{2})^{\boldsymbol{N}}}{\sqrt{5}} \tag{49}$$

The first term of the numerator has $+\bowtie$ value, while the second one has $-\Diamond$ value. Thus, we can write:

$$\boldsymbol{F} \cong \frac{1}{\sqrt{5}}\phi^{\boldsymbol{N}} \tag{50}$$

where $\phi = 1,618033989...$ is the golden ratio, and

$$\|\boldsymbol{F}\| = +\bowtie +\Diamond \cong +\bowtie \tag{51}$$

6 Conclusion

The notion of the terminator offers a new way to conceptualize infinity, distinct from limits or conventional divergence. By focusing on the constructive process of sequences, this framework preserves essential properties that are lost in standard asymptotic analysis. Further research could explore formalizing terminators in different mathematical contexts and extending this approach to higher-dimensional or functional spaces.