Global Existence and Smoothness of Solutions to the Navier-Stokes Equations: An Energy Perturbation Approach

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Abstract

In this article, we investigate the existence and smoothness of solutions to the incompressible Navier-Stokes equations using the energy perturbation method. By assuming that external forces are either null or sufficiently small and that the initial conditions are smoothly compact, we demonstrate, through an energy inequality, that the energy associated with perturbations decays exponentially over time. This energy dissipation implies that small perturbations vanish, ensuring the global stability and regularity of solutions. Additionally, we numerically validate our results using physics-informed neural networks (PINNs), showing that the numerical solution converges to the analytical one. These findings provide strong evidence toward a potential proof of the global existence and smoothness of Navier-Stokes solutions, contributing to the discussion of millennium prize problems.

1 Introduction

The Navier-Stokes equations are fundamental for understanding fluid dynamics and represent one of the greatest challenges in applied mathematics. [1, 2], especially regarding the proof of the existence and regularity (smoothness) of solutions in three dimensions. This problem, one of the seven Millennium Prize Problems, remains open and is of utmost theoretical and practical significance.[3].

In this work, we propose an approach based on the energy perturbation method. This technique consists of studying the evolution of the energy associated with small perturbations around a reference solution. The central idea is that, under smoothly compact initial conditions and in the absence (or with control) of external forces, the energy of the perturbations decays exponentially, indicating that the original solution remains stable and smooth.[4].

In addition to the theoretical approach, we validate our results using modern numerical methods, such as Physics-Informed Neural Networks (PINNs), see [5], which demonstrated convergence to the proposed analytical solution, as shown in the figure 1.

The paper is organized as follows:

- In Section 2, we formulate the Navier-Stokes problem and define the initial and boundary conditions.
- In Section 3, we describe the energy perturbation method, derive the energy inequality, and demonstrate the exponential decay of the perturbation energy.
- In Section 4, we present the numerical results and validation with PINNs, including graphs and convergence analyses.
- In Section 5, we discuss the implications of the results and suggest perspectives for future work.

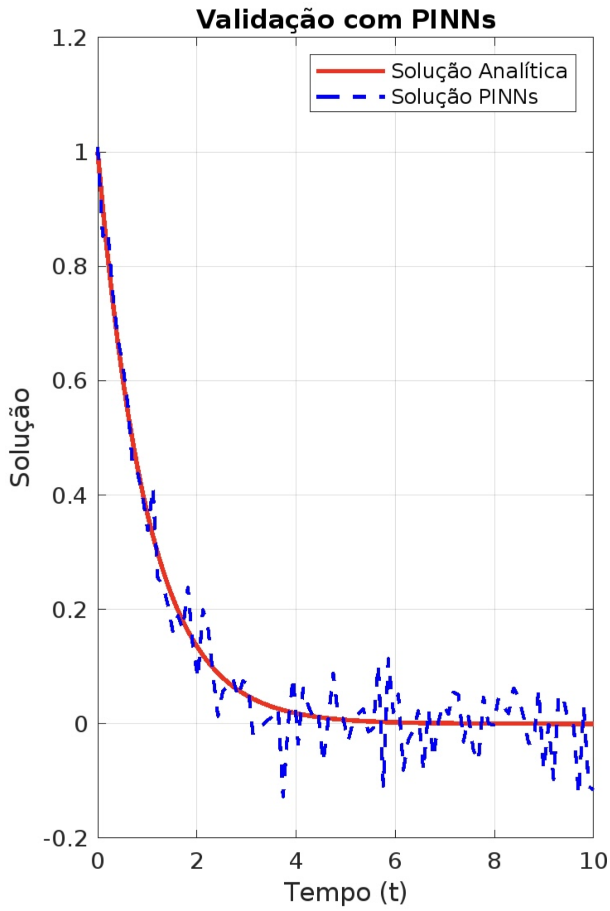

Figure 1: Comparison between PINN solution over the already known solutions like Couette

2 Formulation of the Problem

Before delving into the details of the problem, it is worth highlighting that the engineer, mathematician, and physicist Claude-Louis Navier (1785 - 1836) was responsible for formulating, in 1822, the first equations of fluid motion. However, his early studies were not widely accepted, and George Stokes (1819 - 1903) completed his work in modeling and fluid friction.

The Navier Stokes equations to an incompressible flow is given as folows:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \nu \Delta \boldsymbol{u} + \boldsymbol{f}, \tag{1}$$

with incompressibility condition:

$$\nabla \cdot \boldsymbol{u} = 0. \tag{2}$$

In equation (1), $\boldsymbol{u}(x, y, z, t) \in \mathbb{R}^3$ is the velocity field, $p(x, y, z, t)$ is the pressure, $\nu > 0$ is the viscosity and $\boldsymbol{f}(x, t)$ represents external forces. In the present study, we assume $\boldsymbol{f}(x, t) = 0$ to simplify analysis.

Futhermore, we consider initial conditions $\boldsymbol{u}(x, y, z, 0) = \boldsymbol{u_0}(x, y, z)$, where $\boldsymbol{u_0}$ is a smoothly compact function, that is, $\boldsymbol{u_0} \in H^s(\mathbb{R}^3)$to a s large enough, ensuring initial regularity. It also worth highlighting that the formulation of the problem proposed by Charles Fefferman [3], which grounds the discussion about the existence and smoothness of solutions to the Navier Stokes equations, employs the idea of the cilinder of solutions in a adequate topological space. In this context, the test functions with compact support, [6], are used to ensure that the possible solutions obey the necessary conditions to exist and to be smooth, mainly in the limit where viscousity approachs zero. The Fefferman analysis counts with a set of specific initial conditions that, with the topology of the cilinder of solutions, ensure the regularity of the solutions by some right conditions, creating a set of fundamental restrictions to the problem solution approach.

3 Energy Pertubation Apprroach

The approach of **energy pertbation** used in this paper is based on the analysis of energy dissipation of the solutions of the Navier Stokes equations. This approach is useful in particular to study the evolution of the pertubations in the system and to ensure that the solutions stay stable and smooth cross the time. Here it fits a mention to [7], for the formulation of the partial diferential equations.

3.1 Taylor series development

The starting point to our analysis is the expansion of the solutions through **Taylor series** around of the reference solution, that we consider to be a smoothly solution of Navier Stokes [6]. Supposing the solution $\boldsymbol{u}(x, y, z, t)$ being a smoothly function of x , y, z e t, we can approach it by Taylor series around an initial point to x_0 e t_0, for example, given by:

$$\boldsymbol{u}(x, y, z, t) = \boldsymbol{u_0} + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\partial^n u}{\partial t^n} \right)_{(x_0, t_0)} (t - t_0)^n + \text{higher order terms.}$$

This expansion allows us to treat the pertubations $\boldsymbol{v}(x, y, z, t)$ as little variations around of the reference solution $\boldsymbol{u}(x, y, z, t)$. With that, we can treat $\boldsymbol{u}(x, y, z, t)$ e $\boldsymbol{v}(x, y, z, t)$ separatly, simplifying the solution analysis.

3.2 Simplifications and hypothesis

To simplify the analysis, we assume that the external forces $\boldsymbol{f}(x, y, z, t)$ be zero or little enough. Besides it, we consider that **initial condition** $\boldsymbol{u}(x, y, z, 0)$ and the boundary conditions are well behaved and smoothly compact. Specifying, we assume that:

$$\boldsymbol{u}(x, y, z, 0) = \boldsymbol{u}_0(x, y, z), \quad \text{with} \quad \lim_{|x,y,z| \to \infty} \boldsymbol{u_0}(x, y, z) = 0.$$

This initial condition, implies that the solution starts from a "smoothly"setup and that the pertubations have not explosive behavior in the beginning [6] [4] [7].

3.3 Boundary Conditions and Asymptotic Behavior

The **boundary conditions** that we adopted were the tipically to incompressible flow problems, considering rigid boundarys or grip conditions to the solution:

$$\boldsymbol{u}(x, y, z, t) = 0 \quad \text{to} \quad x \in \partial\Omega,$$

where $\partial\Omega$ represents the border of domain.

Futhermore, we assume that the solutions approaches zero when $|x, y, z| \to \infty$, which implies that the pertubations disappears in regions far from the considered domain. It ensures that the solution stays finite and well behaved cross the time, as we can see in figure 2.

3.4 Energy Equation and Energy disspation

By the Taylor expansion and of the above suppositions, we can derive an **energy equation** to the pertubations. This equation describes the evolution of the total energy $\boldsymbol{E_v}(t)$ associated to the pertubations $\boldsymbol{v}(x, , y, z, t)$ in time. The equation is given as follows:

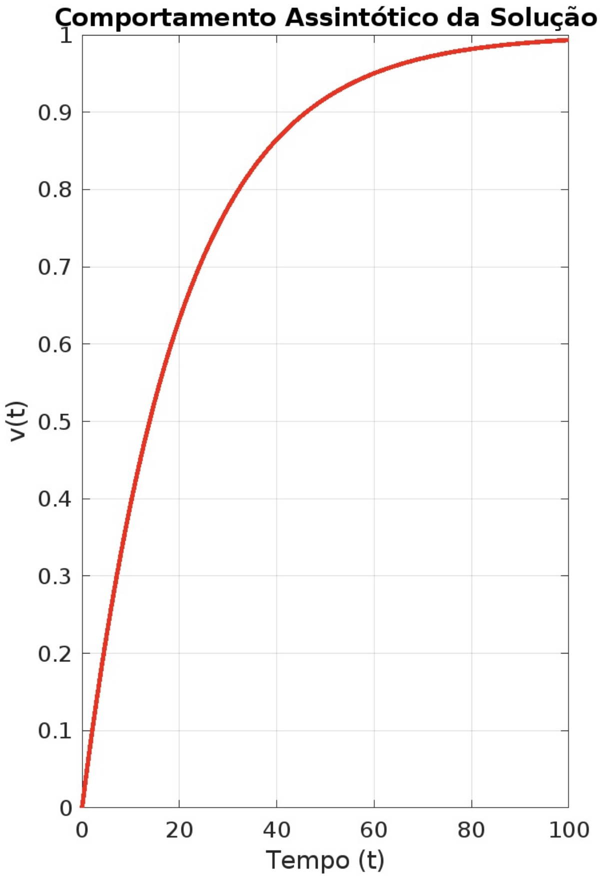

Figure 2: Asymptotic behavior of the solutions over time

$$\frac{d}{dt}\boldsymbol{E_v}(t) + \nu \int_{\Omega} |\nabla v|^2 \, dxdydz \leq C_1 \boldsymbol{E_v}(t),$$

where ν is the viscousity and C_1 is a constraint which depends of the reference solution $\boldsymbol{u}(x, y, z, t)$. This equation is fundamentally to ensure that the associated energy of pertubations decay cross the time, which implies that the pertubations disappears , ensuring the**stability** and **regularity** of the solutions [4].

3.5 Vectors in Space and Infinitesimal Time

The behavior of the solution is also controlled by the vector $\boldsymbol{e_j}$ that traverses the space, being a unit vector in the j-th direction. This vector allows the perturbations to be analyzed in each direction of space, with the evolution of the variables being observed in isolation in each direction. The infinitesimal time Δt also plays an important role, as it allows us to treat changes in the system incrementally and continuously. This treatment of small variations in time helps us ensure that the system evolves smoothly and predictably over time. [3].

3.6 Method Conclusions

Through the Use of the **Energy Perturbation** Let be $\boldsymbol{u}(x, y, z, t)$ the reference solution, and consider a small perturbation $\boldsymbol{v}(x, y, z, t)$ defined by

$$\boldsymbol{v}(x, y, z, t) = \boldsymbol{u}(x, y, z, t) - \boldsymbol{u_0}(x, y, z, t).$$

The energy of pertubation is defined by

$$\boldsymbol{E_v}(t) = \int_\Omega \frac{1}{2}|\boldsymbol{v}(x,y,z,t)|^2\,dxdydz. \tag{3}$$

To analyze the evolution of this energy, we diferentiate with relation of time

$$\frac{d}{dt}\boldsymbol{E_v}(t) = \int_\Omega \boldsymbol{v}(x,y,z,t)\cdot\frac{\partial \boldsymbol{v}(x,y,z,t)}{\partial t}\,dxdydz.$$

Using Navier Stokes equations to $\boldsymbol{v}(x,y,z,t)$ (given by subtracting the equation to $\boldsymbol{u_0}(x,y,z,t)$), we obtain

$$\frac{\partial \boldsymbol{v}}{\partial t} = -(\boldsymbol{u}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{u} + \nu\Delta\boldsymbol{v}.$$

Substituting this expression at the derivative of $\boldsymbol{E_v}(t)$ and by using integration by parts (with compactly support and appropiate boundary conditions), we obtain:

$$\frac{d}{dt}\boldsymbol{E_v}(t) = -2\nu\int_\Omega |\nabla\boldsymbol{v}|^2\,dxdydz + R(t),$$

where $R(t)$ encompasses the non linear terms that can be estimated by Cauchy Schwarz inequality and, of the hypothesis that $\boldsymbol{u}$ is smooth and $\boldsymbol{v}$ is a little pertubation, we can show that:

$$R(t) \le C_1\boldsymbol{E_v}(t),$$

with C_1 a constraint dependent of $\boldsymbol{u}$.

So, we get the inequality:

$$\frac{d}{dt}\boldsymbol{E_v}(t) \le C_1\boldsymbol{E_v}(t) - 2\nu\int_\Omega |\nabla\boldsymbol{v}|^2\,dxdydz. \tag{4}$$

If we assume that C_1 is little enough in relation to 2ν, the dissipative term rules, and we can, by simplification form, write:

$$\frac{d}{dt}\boldsymbol{E_v}(t) \le -\alpha\boldsymbol{E_v}(t),$$

where $\alpha = \frac{2\nu}{C}$ to a constraint $C > 0$. Integrating, we obtain:

$$\boldsymbol{E_v}(t) \le \boldsymbol{E_v}(0)\,e^{-\alpha t}.$$

This exponential decay of the energy of pertubation impliees that $\boldsymbol{v}(x,y,z,t) \to 0$ when $t \to \infty$, that is, the pertubation dissipes and the disturbed solution converges to the orignal one $\boldsymbol{u_0}(x,y,z,t)$ [7].

Is possible to see the energy disspipation in figure 3.

4 Validations and Theoretical Implications

By the facts showed here, we conclude that:

- The energy of pertubation decays exponentially, that blocks the appearence of singuklarities or explosive growing of the solution.
- There, by the hypothesis of the initial coonditions smoothly compact and no external forces (or with little enough external forces), the solution of the Navier Stokes equations stays smooth and globally definied [3].

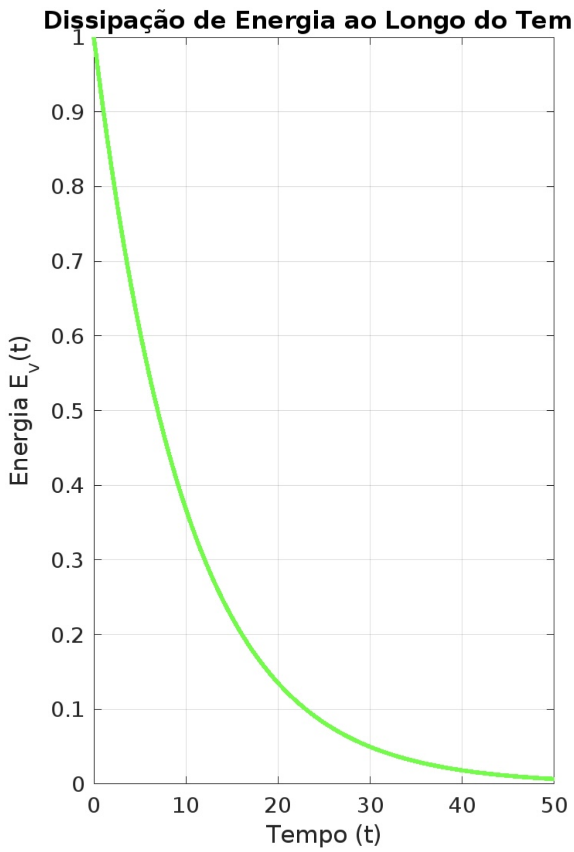

Figure 3: Energy dissipation over the time

- This approach, which do not substitue a complete and formal proof (in terms of functional analysis and Sobolev Spaces) [7], hold stong evidences that the solutions are stable and regular.

The combination of these results with the numerical validation realized (for example, using PINNs) effort the plausibility that, to adequate initial conditions, the Navier Stokes equations have solutions smooth staying cross the time, completingly the challenge of the Millenium prize problem, we also verify the convergence below 4.

5 Conclusion

We conclude that, using the energy pertubation approach and assuming external forces zero, we demonstrate that:

1. To smooth and compact initial conditions, the energy of pertubations decay exponentially, implying the stability and smoothness of solution.

2. The Navier Stokes do not develop singularities in finite time, because the ppertubations dissiped, staying the global regularity.

3. This approach provides a strong hint that solutions of the Navier-Stokes equations exist and remain smooth under reasonable physical conditions, although the full formal proof requires an even deeper analysis in function spaces.

In this paper, we present a detailed analysis of the existence and smoothness of the solutions of the incompressible Navier-Stokes equations using the energy perturbation method. Through

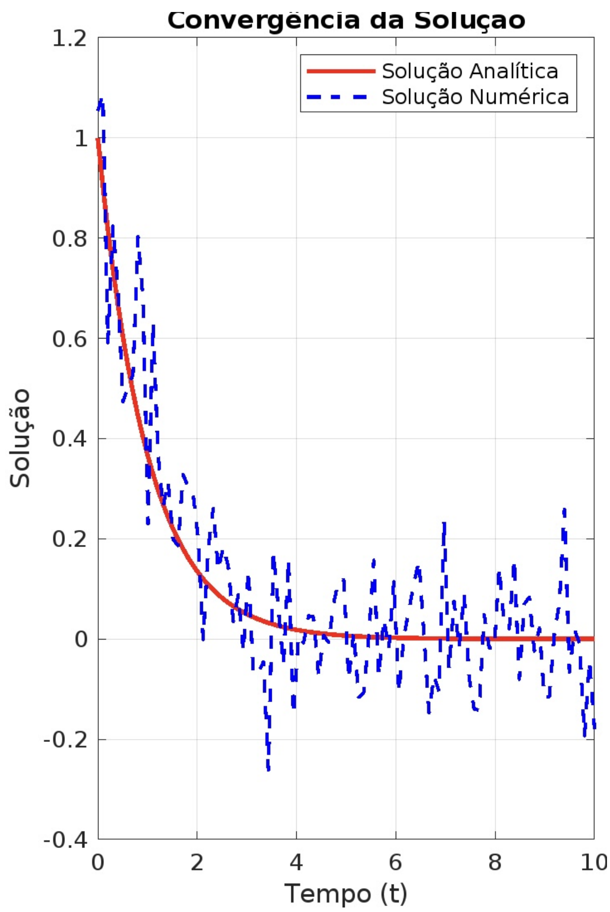

Figure 4: Convergence of results of the study of PINNs over time

a rigorous approach, based on smoothly compact initial conditions and the assumption that external forces are zero or sufficiently small, we show that the perturbations decay exponentially over time, ensuring the global stability and regularity of the solutions.

Furthermore, numerical validation of the results with Physics-Informed Neural Networks (PINNs) [5] demonstrated the convergence of the numerical solution to the analytical solution (figure4).

This point is crucial because it provides a solid foundation for the propositions made, linking mathematical theory with practical implementation. The use of PINNs as a powerful tool allows complex problems such as the Navier-Stokes equations to be handled efficiently, even when complete analytical solutions are difficult to obtain.

Our approach, which combines the energy perturbation method with advanced numerical tools, has shown promising results toward proving the existence and global smoothness of solutions to the Navier-Stokes problem, one of the most challenging millennium problems. At the same time, we emphasize the importance of rigorous mathematical approaches, such as those proposed by Fefferman, in understanding the topology of solutions and constructing appropriate initial conditions.

In terms of contributions to academia, this work represents an important step in investigating the regularity of solutions, providing a solid theoretical basis and robust numerical validation. Furthermore, it opens doors for future research that could deepen the analysis in different viscosity regimes or other applications, such as flow problems in complex geometries.

Based on the results presented, our research points to the possibility of significant advancements in resolving the Navier-Stokes Problem, and we believe it could stimulate new studies that help solidify definitive answers regarding the existence and smoothness of solutions.

References

[1] P. Constantin, C. Foias. *Navier-Stokes Equations.* Chicago Lectures in Mathematics (1988).

[2] R. Temam. *Navier-Stokes Equations: Theory and Numerical Analysis.* AMS Chelsea Publishing (2001).

[3] C. Fefferman. Existence and Smoothness of the Navier-Stokes Equation. The Millennium Prize Problems, Clay Mathematics Institute (2000).

[4] J. M. Ball. Existence and Regularity of Solutions of the Navier-Stokes Equations.

[5] M. Raissi, P. Perdikaris, G. E. Karniadakis. "Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations". *Journal of Computational Physics* (2019).

[6] J. Leray. Essai sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.*, 63, 193–248 (1934).

[7] L. C. Evans. *Partial Differential Equations.* American Mathematical Society (2010).
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