The (E) Question in a Three-Dimensional Space

Analysing a Subset of Linear Systems With The Intrinsic Method

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This paper is part of a series of explorations exposing methods for dividing distorted tensor products by number cubes. Here, the discussion focuses on three-dimensional spaces and anti-symmetric cubes on their low indices, i.e. actually on distorted cross products. The method only delivers the main parts of the divisions. The other documents in the collection complete this investigation. This is a translation made by me of a part of the French version.

ISBN 978-2-36923-084-7, EAN 9782369230847, ©Thierry PERIAT, The intrinsic method.

Contents

1 Introduction

1.1 Context

The first part of [a] exposes a demonstration of the existence of forces in any empty region of the Universe where only electromagnetic fields exist, provided the J.C. Maxwell's laws apply. Anyone knows since a long time that electromagnetic forces exist. The originality of the work lies in the fact that the demonstration is done in a 3 + 1 context (the classical three-dimensional Euclidean geometry plus one parameter: the time), very far from all sources (electrical or gravitational) and that something happens (forces) where nothing should occur.

1.2 Motivation

This fact arouses the curiosity. An analysis of the nature of the forces leads to the conclusion that a part of them is related to electromagnetic polarization; another part has no clear explanation. The intuition suggests that this non-electromagnetic part should be related to some *gravitational polarization*. Anyway, at this stage, these forces give the sensation to behave in a similar way than sea currents: invisible water flows in an ocean of water. This situation invites us to explore and deepen the mathematical aspect of the demonstration.

After a careful rereading one easily comes to the conclusion that the demonstration (i) involves many classical cross products and (ii) makes an abundant use of the isomorphism between $\mathrm{V} = \mathbb{C} \otimes \mathrm{E}(3, \mathbb{R})$ - the set of all vectors with three components in $\mathbb{C}$- and the set of axial rotation matrices.

Furthermore, most of the demonstration is in someway translated in V^*, the dual space of V, or in a representation $\mathrm{M}(3, \mathbb{R}) \times \mathrm{V}$ of V^*.

$$M(3,\mathbb{R}) \times V \subset V^*$$

These statements suggest a path allowing a generalization of the initial demonstration.

1.3 Claim

The linear algebra analyzes systems of the [M].|**b**> = |**z**> type. Classically, the components of **b** are the unknown variables whilst the pair ([M], **z**) is given.

This document introduces deformed tensor, alternated and cross products in a three-dimensional space. For the latter, the information on the deformation is contained inside a matrix [A]. The (E) question is asking if deformed cross products $[\mathbf{a}, \mathbf{b}]_{[A]}$ have representations such that: $|[\mathbf{a}, \mathbf{b}]_{[A]}>$ = [P].|**b**> + |**z**>? In asking this question, one gets a system belonging to the category of the [M].|**b**> = |**z**> type too. But now the unknown elements are the pairs ([P], **z**) whilst the pairs ([A], **a**) are given. An intrinsic method allows the discovery of the matrix part [P].

The claim of this document is to study a brighter set of correspondences connecting the cross products and their representations in M(3, ℝ) × V. The approach will add a supplementary difficulty in expanding the definition of cross products with the hope to take the eventual deformations of the geometry into account.

1.4 Conventions

Any vector which will appear in the discussion:

- has the generic formalism $\mathbf{u} = \sum \mathrm{u}^k.\, \mathbf{e}_k$ where (u^1, u^2, u^3) is an element in C^3 whilst Ω: ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) is a canonical basis for the vector space E(3, ℝ); ℝ represents the set of real numbers and ℂ the one of complex numbers. The set containing these vectors is denoted V.
- is written with a bold symbol; e.g.: **u**. There exists a canonical isomorphism between the set V and its dual representation $\mathrm{V}^* = \mathbb{C}^3$. Elements in the latter are denoted via the bracket convention; e. g.: |**u**> or <**u**|.

Elements denoted [M] are matrices in M(3, ℂ) which is the set of all (3-3) square matrices with entries in ℂ.

1.5 Definitions and semantic

Definition 1.1. *Cube*

A cube is a three-dimensional mathematical object in which scalars are disposed *as if* they were occupying the knots of a perfect cubic structure. Within that theory, the cubes are objects deforming tensor products. The set of all cubes with knots in ℂ is denoted ⊞(3, ℂ). Any cube contains at most 3 × 3 × 3 = 27 knots.

Definition 1.2. *Deformed tensor product*

As suggested by the first definition, deformed tensor products are classical tensor products which are deformed by the intervening of cubes in the following manner:

$$A \in \boxplus(3, \mathbb{C}) \xrightarrow{\otimes} \otimes_A :$$

$$\forall\,(\mathbf{a},\,\mathbf{b})\,\in\,V\,\times\,V\,\rightarrow\,\otimes_A(\mathbf{a},\,\mathbf{b})\,=\,\sum_{i,\,j,\,k\,=\,1}^{3} A_{ij}^{k}\,.\,a^{i}\,.\,b^{j}\,.\,\mathbf{e}_k$$

Definition 1.3. *Deformed alternated product*

In that theory and whatever the cube A is, inspired by the classical notion of exterior product, a deformed alternated product is:

$$\forall\,\mathbf{a},\,\mathbf{b}\,\in\,V\,:\,\wedge_A(\mathbf{a},\mathbf{b})\,=\,\otimes_A(\mathbf{a},\,\mathbf{b})-\otimes_A(\mathbf{b},\,\mathbf{a})\,=\,\sum_{i,\,j,\,k\,=\,1}^{3} A_{ij}^{k}.\,(a^{i}.\,b^{j}-b^{i}.\,a^{j})\,.\,\mathbf{e}_k$$

Obviously:

- terms such that i = j vanish because $\mathbb{C}$ is equipped with a commutative multiplication;
- any deformed alternated product is an anti-symmetric operation.

$$\forall\,\mathbf{a},\,\mathbf{b}\,\in\,V\,:\,\wedge_A(\mathbf{a},\mathbf{b})\,+\,\wedge_A(\mathbf{b},\mathbf{a})\,=\,\mathbf{0}$$

Therefore, the definition can be condensed and reformulated:

$$\forall\,\mathbf{a},\,\mathbf{b}\,\in\,V\,:\,\wedge_A(\mathbf{a},\mathbf{b})\,=\,\sum_{i<j}^{3}(A_{ij}^{k}\,-\,A_{ji}^{k})\,.\,(a^{i}\,.\,b^{j}\,-\,b^{i}\,.\,a^{j})\,.\,\mathbf{e}_k$$

Definition 1.4. *Symmetric cube*

A cube is symmetric relatively to its subscripts if:

$$\forall\,i,\,j,\,k:\;A_{ij}^{k}\,-\,A_{ji}^{k}=0$$

The set of all symmetric cubes in $\boxplus(3,\,\mathbb{C})$ is denoted $\boxplus^{+}(3,\,\mathbb{C})$.

Remark 1.1. *Deformed alternated product and symmetric cubes*

Any deformed alternated product which is built with a symmetric cube is systematically null.

Definition 1.5. *Anti-symmetric cube*

A cube is anti-symmetric relatively to its subscripts if:

$$\forall\,i,j,k:\;A_{ij}^{k}\,+\,A_{ji}^{k}=0$$

The set of all anti-symmetric cubes in $\boxplus(3,\,\mathbb{C})$ is denoted $\boxplus^{-}(3,\,\mathbb{C})$.

Proposition 1.1. *Any element in $\boxplus^{-}(3,\,\mathbb{C})$ can be condensed into an element in $M(3,\,\mathbb{C})$.*

Proof. : Any cube can be considered as the mental superposition of three elements in M(3, $\mathbb{C}$). When the cube is anti-symmetric, each diagonal vanishes and from the six remaining off-diagonal terms only at most three can have an absolute value which is different from the others. This is true for each of the three elements in M(3, $\mathbb{C}$) constituting the cube at hand. Hence, the initial cube can be represented by an element in M(3, $\mathbb{C}$) containing at most $3 \times 3 = 9$ different entries.

$$\{\forall A \in \boxplus^{-}(3, \mathbb{C})\} \Rightarrow \{A \to [A] = \begin{bmatrix} A^1_{12} & A^2_{12} & A^3_{12} \\ A^1_{23} & A^2_{23} & A^3_{23} \\ A^1_{13} & A^2_{13} & A^3_{13} \end{bmatrix} \in M(3, \mathbb{C})\}$$

□

Definition 1.6. *Deforming matrix*

In that context and per convention, a deforming matrix [A] is the matrix which has been obtained when the anti-symmetry of some cube A has been taken into account.

Remark 1.2. *Deformed alternated products and anti-symmetric cubes*

A deformed alternated product which is built with an anti-symmetric cube can be conventionally rewritten as:

$$\forall\, \mathbf{a},\, \mathbf{b} \in V : \wedge_{[A]}(\mathbf{a}, \mathbf{b}) = 2\,.\sum_{i<j}^{3} A^k_{ij}\,.\,(a^i\,.\,b^j - b^i\,.\,a^j)\,.\,\mathbf{e}_k$$

Definition 1.7. *Deformed cross product*

In that context and per convention, a deformed cross product is the half of a deformed alternated product which is built with an anti-symmetric cube.

$$\forall\, \mathbf{a},\, \mathbf{b} \in V : [\mathbf{a}, \mathbf{b}]_{[A]} = \frac{1}{2}\,.\,\wedge_{[A]}(\mathbf{a}, \mathbf{b}) = \sum_{i<j}^{3} A^k_{ij}\,.\,(a^i\,.\,b^j - b^i\,.\,a^j)\,.\,\mathbf{e}_k$$

Definition 1.8. *Projectile*

In that theory, a *projectile* is the first argument appearing (on the left side) in a product (any one).

Definition 1.9. *Target*

In that theory, a *target* is the second argument appearing (on the right side) in a product.

Definition 1.10. *Anti-reduced cube*

A cube is anti-reduced when:

$$\forall\, i,\, j,\, k : A^k_{ij} + A^j_{ik} = 0$$

The set of all anti-reduced cubes in $\boxplus(3, \mathbb{C})$ is denoted $\boxplus^{\downarrow}(3, \mathbb{C})$.

Proposition 1.2. *Anti-symmetry and anti-reduction are compatible.*

Proof. - Effectively:

$$A_{ij}^{k} = -A_{ji}^{k} = A_{jk}^{i} = -A_{kj}^{i} = A_{ki}^{j} = -A_{ik}^{j} = A_{ij}^{k}$$

□

Definition 1.11. *Reduced cube*

$$\forall\, i,\, j,\, k :\; A_{ij}^{k} \;-\; A_{ik}^{j} = 0$$

The set of all reduced cubes in $\boxplus(3,\, \mathbb{C})$ is denoted $\boxplus^{\uparrow}(3,\, \mathbb{C})$.

Proposition 1.3. *Symmetry and reduction are compatible.*

Proof. - Effectively:

$$A_{ij}^{k} = A_{ji}^{k} = A_{jk}^{i} = A_{kj}^{i} = A_{ki}^{j} = A_{ik}^{j} = A_{ij}^{k}$$

□

Definition 1.12. - ***The so-called (E) question:***

Let consider a deformed alternated product, e.g.:

$$\wedge_A(\mathbf{a}, \mathbf{b})$$

This product is an intern operation because its result is also an element in the source space V. Therefore, it can always be written in the dual space with the help of *brackets*:

$$|\wedge_A(\mathbf{a}, \mathbf{b}) >= \begin{bmatrix} \sum_{i<j}^{3}(A_{ij}^{1} - A_{ji}^{1}).\,(a^i.\, b^j \;-\; b^i.\, a^j) \\ \sum_{i<j}^{3}(A_{ij}^{2} - A_{ji}^{2}).\,(a^i.\, b^j \;-\; b^i.\, a^j) \\ \sum_{i<j}^{3}(A_{ij}^{3} - A_{ji}^{3}).\,(a^i.\, b^j \;-\; b^i.\, a^j) \end{bmatrix} \in \mathbb{C}^3$$

Per convention, the so-called (E) question is asking if, when and how this dual representation can be decomposed inside the dual space as follows:

$$\exists ?\, ([P],\, \mathbf{z}) \in M(3,\, C) \times V \subset V^* = \mathbb{C}^3 :$$

$$|\wedge_A(\mathbf{a}, \mathbf{b}) >= [P]\,.\,|\mathbf{b} > + |\mathbf{z} >= \begin{bmatrix} \sum_{j=1}^{3} p_{1j}\,.\,b^j \,+\, z^1 \\ \sum_{j=1}^{3} p_{2j}\,.\,b^j \,+\, z^2 \\ \sum_{j=1}^{3} p_{3j}\,.\,b^j \,+\, z^3 \end{bmatrix} \in \mathbb{C}^3$$

The concept can obviously be applied to a simple deformed tensor product; in which case the (E) question writes:

$$\exists ?\, ([P],\, \mathbf{z}) \in M(3,\, C) \times V :$$

$$|\otimes_A(\mathbf{a}, \mathbf{b}) >= [P]\,.\,|\mathbf{b} > + |\mathbf{z} >= \begin{bmatrix} \sum_{j=1}^{3} p_{1j}\,.\,b^j \,+\, z^1 \\ \sum_{j=1}^{3} p_{2j}\,.\,b^j \,+\, z^2 \\ \sum_{j=1}^{3} p_{3j}\,.\,b^j \,+\, z^3 \end{bmatrix} \in \mathbb{C}^3$$

Definition 1.13. *Decomposition*

When a given pair ([P], **z**) exists, it is called a *decomposition* for the product at hand. The matrix in that pair is called *the main part* of the decomposition whilst the vector in that pair is called *the residual part* of the decomposition.

2 Deformed products with a vanishing residual part

2.1 Existence

Proposition 2.1. *Any deformed tensor product has at least one decomposition of which the residual part vanishes.*

Proof. Let recall that:

$$(\mathbf{a}, \mathbf{b}) \xrightarrow{\otimes_A} \otimes_A(\mathbf{a}, \mathbf{b}) \to | \otimes_A (\mathbf{a}, \mathbf{b}) >= | \sum_{\alpha,\, \beta} A^{\chi}_{\alpha\beta} \,.\, a^{\alpha} \,.\, b^{\beta} > \in \mathbb{C}^3$$

This can also be reformulated in M(3, $\mathbb{C}$) × V as:

$$| \sum_{\alpha,\, \beta} A^{\chi}_{\alpha\beta} \,.\, a^{\alpha} \,.\, b^{\beta} >= [A^{\chi}_{\alpha\beta} \,.\, a^{\alpha}] \,.\, |b^{\beta} >$$

Let introduce the application ${}_{[A]}\Phi$ such that:

$${}_{[A]}\Phi : \mathbf{a} \in V \xrightarrow{{}_{[A]}\Phi} {}_{[A]}\Phi(\mathbf{a}) = [A^{\chi}_{\alpha\beta} \,.\, a^{\alpha}] \in M(3, \mathbb{C})$$

Hence, any deformed tensor product accepts a representation in the subset M(3, $\mathbb{C}$) × V of V* such that:

$$| \otimes_A (\mathbf{a}, \mathbf{b}) >= {}_{[A]}\Phi(\mathbf{a}) \,.\, |\mathbf{b} > + |\mathbf{0} >$$

This representation is a decomposition denoted $({}_{[A]}\Phi(\mathbf{a}), \mathbf{0})$. It is obviously a decomposition of which the residual part vanishes. □

Proposition 2.2. *Any deformed cross product has at least one decomposition of which the residual part vanishes.*

Proof. Starting from the definition 1.7:

$$([\mathbf{a}, \mathbf{b}]_{[A]})^k$$

$$=$$

$$A^k_{23} \,.\, (a^2 \,.\, b^3 - a^3 \,.\, b^2) + A^k_{31} \,.\, (a^3 \,.\, b^1 - a^1 \,.\, b^3) + A^k_{12} \,.\, (a^1 \,.\, b^2 - a^2 \,.\, b^1)$$

$$=$$

$$(A^k_{31} \,.\, a^3 - A^k_{12} \,.\, a^2) \,.\, b^1 + (A^k_{12} \,.\, a^1 - A^k_{23} \,.\, a^3) \,.\, b^2 + (A^k_{23} \,.\, a^2 - A^k_{31} \,.\, a^1) \,.\, b^3$$

... it is possible to write:

$$|[\mathbf{a}, \mathbf{b}]_{[A]} >$$

$$=$$

$$\begin{bmatrix} A^1_{31} \,.\, a^3 - A^1_{12} \,.\, a^2 & A^1_{12} \,.\, a^1 - A^1_{23} \,.\, a^3 & A^1_{23} \,.\, a^2 - A^1_{31} \,.\, a^1 \\ A^2_{31} \,.\, a^3 - A^2_{12} \,.\, a^2 & A^2_{12} \,.\, a^1 - A^2_{23} \,.\, a^3 & A^2_{23} \,.\, a^2 - A^2_{31} \,.\, a^1 \\ A^3_{31} \,.\, a^3 - A^3_{12} \,.\, a^2 & A^3_{12} \,.\, a^1 - A^3_{23} \,.\, a^3 & A^3_{23} \,.\, a^2 - A^3_{31} \,.\, a^1 \end{bmatrix} \,.\, |\mathbf{b} >$$

Since the cube is an element in $\boxplus^{-}{}_3(\mathbb{C})$, the matrix can be rewritten as:

$$\begin{bmatrix} A^1_{31}\,.\,a^3 + A^1_{21}\,.\,a^2 + A^1_{11}\,.\,a^1 & A^1_{12}\,.\,a^1 + A^1_{22}\,.\,a^2 + A^1_{32}\,.\,a^3 & A^1_{33}\,.\,a^3 + A^1_{23}\,.\,a^2 + A^1_{13}\,.\,a^1 \\ A^2_{31}\,.\,a^3 + A^2_{21}\,.\,a^2 + A^2_{11}\,.\,a^1 & A^2_{12}\,.\,a^1 + A^2_{22}\,.\,a^2 + A^2_{32}\,.\,a^3 & A^2_{33}\,.\,a^3 + A^2_{23}\,.\,a^2 + A^2_{13}\,.\,a^1 \\ A^3_{31}\,.\,a^3 + A^3_{21}\,.\,a^2 + A^3_{11}\,.\,a^1 & A^3_{12}\,.\,a^1 + A^3_{22}\,.\,a^2 + A^3_{32}\,.\,a^3 & A^3_{33}\,.\,a^3 + A^3_{23}\,.\,a^2 + A^3_{13}\,.\,a^1 \end{bmatrix}$$

$$=$$

$$[A^{\alpha}_{m\beta}\,.\,a^m]$$

$$=$$

$${}_{[A]}\Phi(\mathbf{a})$$

At the end of the day:

$$|[\mathbf{a}, \mathbf{b}]_{[A]} >= {}_{[A]}\Phi(\mathbf{a})\,.\,|\mathbf{b}>$$

$\square$

2.2 Non-uniqueness

There is no reason justifying the uniqueness of the decomposition. Any deformed tensor product can be decomposed in diverse manners:

$$|\otimes_A (\mathbf{a}, \mathbf{b})>$$

$$=$$

$$[P]\,.\,|\mathbf{b}> + |\mathbf{z}>$$

$$=$$

$${}_A\Phi(\mathbf{a})\,.\,|\mathbf{b}> + \{[P] - {}_A\Phi(\mathbf{a})\}\,.\,|\mathbf{b}> + |\mathbf{z}>$$

As a matter of facts, when:

$$\{[P] - {}_A\Phi(\mathbf{a})\}\,.\,|\mathbf{b}>= |\mathbf{0}> \; and \, |\mathbf{z}>= |\mathbf{0}>$$

... one recovers the simplest decomposition with a vanishing residual part:

$${}_A\Phi(\mathbf{a})\,.\,|\mathbf{b}>= |\mathbf{0}>$$

Therefore, a criterion characterizing the set containing the pairs ([P], **0**) is:

$$|[P] - {}_A\Phi(\mathbf{a})| = 0$$

A similar way of thinking can be applied to deformed cross products, except that the cube A must be replaced by the deforming matrix [A].

2.3 Deformed cross products and deformations of the classical cross product

Let start from definition 1.7 again:

$$([\mathbf{a},\mathbf{b}]_{[A]})^k$$
$$=$$
$$A^k_{12}\,.\,(a^1\,.\,b^2\,-\,b^1\,.\,a^2)\,+\,A^k_{23}\,.\,(a^2\,.\,b^3\,-\,b^2\,.\,a^3)\,+\,A^k_{13}\,.\,(a^1\,.\,b^3\,-\,b^1\,.\,a^3)$$
$$=$$
$$A^k_{23}\,.\,(a^2\,.\,b^3\,-\,a^3\,.\,b^2)\,+\,A^k_{31}\,.\,(a^3\,.\,b^1\,-\,a^1\,.\,b^3)\,+\,A^k_{12}\,.\,(a^1\,.\,b^2\,-\,a^2\,.\,b^1)$$
$$=$$
$$A^k_{23}\,.\,(\mathbf{a}\wedge\mathbf{b})^1\,+\,A^k_{31}\,.\,(\mathbf{a}\wedge\mathbf{b})^2\,+\,A^k_{12}\,.\,(\mathbf{a}\wedge\mathbf{b})^3$$

Let introduce the matrices:

$$[J]\,=\,\begin{bmatrix}0&0&1\\1&0&0\\0&-1&0\end{bmatrix}\;;\;[J]^t\,=\,\begin{bmatrix}0&1&0\\0&0&-1\\1&0&0\end{bmatrix}$$

A simple calculation shows that:

$$[J]^t\,.\,[A]\,=\,\begin{bmatrix}0&1&0\\0&0&-1\\1&0&0\end{bmatrix}\,.\,\begin{bmatrix}A^1_{12}&A^2_{12}&A^3_{12}\\A^1_{23}&A^2_{23}&A^3_{23}\\A^1_{13}&A^2_{13}&A^3_{13}\end{bmatrix}\,=\,\begin{bmatrix}A^1_{23}&A^2_{23}&A^3_{23}\\-A^1_{13}&-A^2_{13}&-A^3_{13}\\A^1_{12}&A^2_{12}&A^3_{12}\end{bmatrix}$$

Hence:

$$\{[J]^t\,.\,[A]\}^t\,=\,[A]^t\,.\,[J]\,=\,[A^*]\,=\,\begin{bmatrix}A^1_{23}&A^1_{31}&A^1_{12}\\A^2_{23}&A^2_{31}&A^2_{12}\\A^3_{23}&A^3_{31}&A^3_{12}\end{bmatrix}\,\in\,M(3,\,\mathbb{C})$$

And:

$$|[\mathbf{a},\mathbf{b}]_{[A]}>=\underbrace{[A]^t\,.\,[J]}_{[A^*]}\,.\,|\mathbf{a}\wedge\mathbf{b}>$$

Any deformed cross product (definition 1.7) is effectively the deformation of a classical cross product. The matrix [A*] is the *effective deforming matrix.*

3 The main part of a decomposition for a deformed cross product

3.1 Calculations

The existence of a representation ([P], **z**) in M(3, $\mathbb{C}$) × V (synonym: of a decomposition) for an element $[\mathbf{a},...]_{[A]}$ is equivalent to the validity of the relation:

$$|[\mathbf{a},...]_{[A]}>=\,[P]\,.\,|...>\,+|\mathbf{z}>$$

Since there is at least one decomposition with a vanishing residual part, this relation can be reformulated as:

$$_{A}\Phi(\mathbf{a}) \,.\, |... >= [P] \,.\, |... > + |\mathbf{z} >$$

And after that as:

$$\{_{A}\Phi(\mathbf{a}) \,-\, [P]\} \,.\, |... >= |\mathbf{z} >$$

At a first glance, this is a very classical linear system depending on the components of **...**. If the mathematical problem at hand would be the discovery of the components of **...**, the procedure would begin with the calculation of the discriminant of this system. The latter is nothing but the determinant:

$$\Lambda(a^1,\, a^2,\, a^3)$$

$$=$$

$$|_{A}\Phi(\mathbf{a}) \,-\, [P]|$$

$$=$$

$$\begin{vmatrix} A^1_{m1} \,.\, a^m \,-\, p_{11} & A^1_{m2} \,.\, a^m \,-\, p_{12} & A^1_{m3} \,.\, a^m \,-\, p_{13} \\ A^2_{n1} \,.\, a^n \,-\, p_{21} & A^2_{n2} \,.\, a^n \,-\, p_{22} & A^2_{n3} \,.\, a^n \,-\, p_{23} \\ A^3_{p1} \,.\, a^p \,-\, p_{31} & A^3_{p2} \,.\, a^p \,-\, p_{32} & A^3_{p3} \,.\, a^p \,-\, p_{33} \end{vmatrix}$$

If the cube A would be any one, one would expect to find a polynomial of degree three. Because A is an element in $\boxplus^{-}(3,\, \mathbb{C})$, this is not the case.

Remark 3.1. *Calculations*

Let state that:

$$\Lambda(a^1,\, a^2,\, a^3)$$

$$=$$

$$(A^1_{m1} \,.\, a^m - p_{11}) \,.\, \{(A^2_{n2} \,.\, a^n - p_{22}) \,.\, (A^3_{p3} \,.\, a^p - p_{33}) - (A^3_{p2} \,.\, a^p - p_{32}) \,.\, (A^2_{n3} \,.\, a^n - p_{23})\}$$

$$-$$

$$(A^1_{m2} \,.\, a^m - p_{12}) \,.\, \{(A^2_{n1} \,.\, a^n - p_{21}) \,.\, (A^3_{p3} \,.\, a^p - p_{33}) - (A^3_{p1} \,.\, a^p - p_{31}) \,.\, (A^2_{n3} \,.\, a^n - p_{23})\}$$

$$+$$

$$(A^1_{m3} \,.\, a^m - p_{13}) \,.\, \{(A^2_{n1} \,.\, a^n - p_{21}) \,.\, (A^3_{p2} \,.\, a^p - p_{32}) - (A^3_{p1} \,.\, a^p - p_{31}) \,.\, (A^2_{n2} \,.\, a^n - p_{22})\}$$

A second step brings:

$$\Lambda(a^1,\, a^2,\, a^3)$$

$$=$$

$$(A^1_{m1} \,.\, a^m \,-\, p_{11}) \,.$$

$$\{(A^2_{n2} \,.\, A^3_{p3} - A^2_{n3} \,.\, A^3_{p2}) \,.\, a^n \,.\, a^p + [(p_{23} \,.\, A^3_{n2} + p_{32} \,.\, A^2_{n3}) - (p_{22} \,.\, A^3_{n3} + p_{33} \,.\, A^2_{n2})] \,.\, a^n + (p_{22} \,.\, p_{33} - p_{32} \,.\, p_{23})\}$$

$$-$$

$$(A^1_{m2} \,.\, a^m \,-\, p_{12}) \,.$$

$$\{(A^2_{n1} \,.\, A^3_{p3} - A^2_{n3} \,.\, A^3_{p1}) \,.\, a^n \,.\, a^p + [(p_{23} \,.\, A^3_{n1} + p_{31} \,.\, A^2_{n3}) - (p_{21} \,.\, A^3_{n3} + p_{33} \,.\, A^2_{n1})] \,.\, a^n + (p_{21} \,.\, p_{33} - p_{31} \,.\, p_{23})\}$$

$$+$$

$$(A^1_{m3} \, . \, a^m \; - \; p_{13}) \, .$$
$$\{(A^2_{n1} \, . \, A^3_{p2} - A^2_{n2} \, . \, A^3_{p1}) \, . \, a^n \, . \, a^p + [(p_{22} \, . \, A^3_{n1} + p_{31} \, . \, A^2_{n2}) - (p_{21} \, . \, A^3_{n2} + p_{32} \, . \, A^2_{n1})] \, . \, a^n + (p_{21} \, . \, p_{32} - p_{22} \, . \, p_{31})\}$$

$$=$$

$$d_{mnp} \, . \, a^m \, . \, a^n \, . \, a^p$$
$$+ \, A^1_{m1} \, . \, a^m \, . \, [(p_{23} \, . \, A^3_{n2} \; + \; p_{32} \, . \, A^2_{n3}) \; - \; (p_{22} \, . \, A^3_{n3} \; + \; p_{33} \, . \, A^2_{n2})] \, . \, a^n$$
$$- \, p_{11} \, . \, (A^2_{n2} \, . \, A^3_{p3} \; - \; A^2_{n3} \, . \, A^3_{p2}) \, . \, a^n \, . \, a^p$$
$$- \, A^1_{m2} \, . \, a^m \, . \, [(p_{23} \, . \, A^3_{n1} \; + \; p_{31} \, . \, A^2_{n3}) \; - \; (p_{21} \, . \, A^3_{n3} \; + \; p_{33} \, . \, A^2_{n1})] \, . \, a^n$$
$$+ \, p_{12} \, . \, (A^2_{n1} \, . \, A^3_{p3} \; - \; A^2_{n3} \, . \, A^3_{p1}) \, . \, a^n \, . \, a^p$$
$$+ \, A^1_{m3} \, . \, a^m \, . \, [(p_{22} \, . \, A^3_{n1} \; + \; p_{31} \, . \, A^2_{n2}) \; - \; (p_{21} \, . \, A^3_{n2} \; + \; p_{32} \, . \, A^2_{n1})] \, . \, a^n$$
$$- \, p_{13} \, . \, (A^2_{n1} \, . \, A^3_{p2} \; - \; A^2_{n2} \, . \, A^3_{p1}) \, . \, a^n \, . \, a^p$$
$$+ \, A^1_{m1} \, . \, (p_{22} \, . \, p_{33} \; - \; p_{32} \, . \, p_{23}) \, . \, a^m$$
$$- \, A^1_{m2} \, . \, (p_{21} \, . \, p_{33} \; - \; p_{31} \, . \, p_{23}) \, . \, a^m$$
$$+ \, A^1_{m3} \, . \, (p_{21} \, . \, p_{32} \; - \; p_{22} \, . \, p_{31}) \, . \, a^m$$
$$- \, p_{11} \, . \, [(p_{23} \, . \, A^3_{n2} \; + \; p_{32} \, . \, A^2_{n3}) \; - \; (p_{22} \, . \, A^3_{n3} \; + \; p_{33} \, . \, A^2_{n2})] \, . \, a^n$$
$$+ \, p_{12} \, . \, [(p_{23} \, . \, A^3_{n1} \; + \; p_{31} \, . \, A^2_{n3}) \; - \; (p_{21} \, . \, A^3_{n3} \; + \; p_{33} \, . \, A^2_{n1})] \, . \, a^n$$
$$- \, p_{13} \, . \, [(p_{22} \, . \, A^3_{n1} \; + \; p_{31} \, . \, A^2_{n2}) \; - \; (p_{21} \, . \, A^3_{n2} \; + \; p_{32} \, . \, A^2_{n1})] \, . \, a^n$$
$$\underbrace{- \, p_{11} \, . \, (p_{22} \, . \, p_{33} \; - \; p_{32} \, . \, p_{23}) \; + \; p_{12} \, . \, (p_{21} \, . \, p_{33} \; - \; p_{31} \, . \, p_{23}) \; - \; p_{13} \, . \, (p_{21} \, . \, p_{32} \; - \; p_{22} \, . \, p_{31})}_{= -|P|}$$

3.2 The coefficients of degree three

Proposition 3.1. *The existence of a representation ([P], z) in M(3,* $\mathbb{C}$*)* $\times$ *V (synonym: of a decomposition) for an element [**a**, ...]*$_{[A]}$ *imposes the one of a polynomial form* $\Lambda(\boldsymbol{a})$ *of degree at most equal to two:*

$$\Lambda(a^1, \, a^2, \, a^3) \; = \; \sum_{mn} d_{mn} \, . \, a^m \, . \, a^n \; + \; \sum_{m} d_m \, . \, a^m \; + \; d$$

Proof. Focusing the calculations on coefficients of degree three, one states that:

$$d_{mnp}$$

$$=$$

$$A^1_{m1} \, . \, (A^2_{n2} \, . \, A^3_{p3} \; - \; A^2_{n3} \, . \, A^3_{p2})$$
$$- \, A^1_{m2} \, . \, (A^2_{n1} \, . \, A^3_{p3} \; - \; A^2_{n3} \, . \, A^3_{p1})$$
$$+ \, A^1_{m3} \, . \, (A^2_{n1} \, . \, A^3_{p2} \; - \; A^2_{n2} \, . \, A^3_{p1})$$

Three types of combinations exist: (i) the three subscripts are equal, (ii) two of the three subscripts are equal and (iii) the three subscripts are different:

- **cases (i)** The three coefficients are:

$$d_{mmm}$$

$$=$$

$$A^1_{m1} \,.\, (A^2_{m2} \,.\, A^3_{m3} \;-\; A^2_{m3} \,.\, A^3_{m2})$$
$$-\, A^1_{m2} \,.\, (A^2_{m1} \,.\, A^3_{m3} \;-\; A^2_{m3} \,.\, A^3_{m1})$$
$$+\, A^1_{m3} \,.\, (A^2_{m1} \,.\, A^3_{m2} \;-\; A^2_{m2} \,.\, A^3_{m1})$$

Since the cube A is anti-symmetric and the set $\mathbb{C}$ is equipped with a commutative multiplication:

$$d_{111} \;=\; d_{222} \;=\; d_{333} \;=\; 0$$

- **cases (ii)** Let consider (112) as an example; the coefficients are:

$$d_{112}$$

$$=$$

$$A^1_{11} \,.\, (A^2_{12} \,.\, A^3_{23} - A^2_{13} \,.\, A^3_{22}) - A^1_{12} \,.\, (A^2_{11} \,.\, A^3_{23} - A^2_{13} \,.\, A^3_{21}) + A^1_{13} \,.\, (A^2_{11} \,.\, A^3_{22} - A^2_{12} \,.\, A^3_{21})$$

$$d_{121}$$

$$=$$

$$A^1_{11} \,.\, (A^2_{22} \,.\, A^3_{13} - A^2_{23} \,.\, A^3_{12}) - A^1_{12} \,.\, (A^2_{21} \,.\, A^3_{13} - A^2_{23} \,.\, A^3_{11}) + A^1_{13} \,.\, (A^2_{21} \,.\, A^3_{12} - A^2_{22} \,.\, A^3_{11})$$

$$d_{211}$$

$$=$$

$$A^1_{21} \,.\, (A^2_{12} \,.\, A^3_{13} - A^2_{13} \,.\, A^3_{12}) - A^1_{22} \,.\, (A^2_{11} \,.\, A^3_{13} - A^2_{13} \,.\, A^3_{11}) + A^1_{23} \,.\, (A^2_{11} \,.\, A^3_{12} - A^2_{12} \,.\, A^3_{11})$$

Because the cube at hand is anti-symmetric, they are:

$$d_{112} \;=\; -\, A^1_{12} \,.\, (-\, A^2_{13} \,.\, A^3_{21}) \;+\; A^1_{13} \,.\, (-\, A^2_{12} \,.\, A^3_{21})$$

$$d_{121} \;=\; -\, A^1_{12} \,.\, (A^2_{21} \,.\, A^3_{13}) \;+\; A^1_{13} \,.\, (A^2_{21} \,.\, A^3_{12})$$

$$d_{211} \;=\; A^1_{21} \,.\, (A^2_{12} \,.\, A^3_{13} \;-\; A^2_{13} \,.\, A^3_{12})$$

Because the multiplication is an associative operation on $\mathbb{C}$, the following terms can be regrouped:

$$d_{112} \,.\, a^1 \,.\, a^1 \,.\, a^2 \;+\; d_{121} \,.\, a^1 \,.\, a^2 \,.\, a^1 \;+\; d_{211} \,.\, a^2 \,.\, a^1 \,.\, a^1$$

Hence, the addition writes:

$$(d_{112} \;+\; d_{121} \;+\; d_{211}) \,.\, a^1 \,.\, a^2 \,.\, a^1$$

Let consider the sum of the coefficients:

$$d_{112} \;+\; d_{121} \;+\; d_{211}$$

$$=$$

$$-\, A^1_{12} \,.\, (-\, A^2_{13} \,.\, A^3_{21}) \;+\; A^1_{13} \,.\, (-\, A^2_{12} \,.\, A^3_{21}) \;-\; A^1_{12} \,.\, (A^2_{21} \,.\, A^3_{13})$$
$$+\, A^1_{13} \,.\, (A^2_{21} \,.\, A^3_{12}) \;+\; A^1_{21} \,.\, (A^2_{12} \,.\, A^3_{13} \;-\; A^2_{13} \,.\, A^3_{12})$$

$$=$$

$$-\, A^1_{12} \,.\, (A^2_{13} \,.\, A^3_{12}) \;+\; A^1_{13} \,.\, (A^2_{12} \,.\, A^3_{12}) \;+\; A^1_{12} \,.\, (A^2_{12} \,.\, A^3_{13})$$

$$- A^1_{13} \,.\, (A^2_{12} \,.\, A^3_{12}) \;-\; A^1_{12} \,.\, (A^2_{12} \,.\, A^3_{13} \;-\; A^2_{13} \,.\, A^3_{12})$$
$$=$$
$$0$$

A cyclic permutation yields similar results for the cases (113), (221), (223), (331), (332).

- **cases (iii)** Let illustrate these cases with the peculiar (123); the coefficients are:

$$d_{123}$$
$$=$$
$$A^1_{11} \,.\, (A^2_{22} \,.\, A^3_{33} - A^2_{23} \,.\, A^3_{32}) - A^1_{12} \,.\, (A^2_{21} \,.\, A^3_{33} - A^2_{23} \,.\, A^3_{31}) + A^1_{13} \,.\, (A^2_{21} \,.\, A^3_{32} - A^2_{22} \,.\, A^3_{31})$$
$$d_{312}$$
$$=$$
$$A^1_{31} \,.\, (A^2_{12} \,.\, A^3_{23} - A^2_{13} \,.\, A^3_{22}) - A^1_{32} \,.\, (A^2_{11} \,.\, A^3_{23} - A^2_{13} \,.\, A^3_{21}) + A^1_{33} \,.\, (A^2_{11} \,.\, A^3_{22} - A^2_{12} \,.\, A^3_{21})$$
$$d_{231}$$
$$=$$
$$A^1_{21} \,.\, (A^2_{32} \,.\, A^3_{13} - A^2_{33} \,.\, A^3_{12}) - A^1_{22} \,.\, (A^2_{31} \,.\, A^3_{13} - A^2_{33} \,.\, A^3_{11}) + A^1_{23} \,.\, (A^2_{31} \,.\, A^3_{12} - A^2_{32} \,.\, A^3_{11})$$
$$d_{321}$$
$$=$$
$$A^1_{31} \,.\, (A^2_{22} \,.\, A^3_{13} - A^2_{23} \,.\, A^3_{12}) - A^1_{32} \,.\, (A^2_{21} \,.\, A^3_{13} - A^2_{23} \,.\, A^3_{11}) + A^1_{33} \,.\, (A^2_{21} \,.\, A^3_{12} - A^2_{22} \,.\, A^3_{11})$$
$$d_{132}$$
$$=$$
$$A^1_{11} \,.\, (A^2_{32} \,.\, A^3_{23} - A^2_{33} \,.\, A^3_{22}) - A^1_{12} \,.\, (A^2_{31} \,.\, A^3_{23} - A^2_{33} \,.\, A^3_{21}) + A^1_{13} \,.\, (A^2_{31} \,.\, A^3_{22} - A^2_{32} \,.\, A^3_{21})$$
$$d_{213}$$
$$=$$
$$A^1_{21} \,.\, (A^2_{12} \,.\, A^3_{33} - A^2_{13} \,.\, A^3_{32}) - A^1_{22} \,.\, (A^2_{11} \,.\, A^3_{33} - A^2_{13} \,.\, A^3_{31}) + A^1_{23} \,.\, (A^2_{11} \,.\, A^3_{32} - A^2_{12} \,.\, A^3_{31})$$

Because the cube at hand is anti-symmetric, they are:

$$d_{123} \;=\; - A^1_{12} \,.\, (- A^2_{23} \,.\, A^3_{31}) \;+\; A^1_{13} \,.\, (A^2_{21} \,.\, A^3_{32})$$
$$d_{312} \;=\; A^1_{31} \,.\, (A^2_{12} \,.\, A^3_{23}) \;-\; A^1_{32} \,.\, (- A^2_{13} \,.\, A^3_{21})$$
$$d_{231} \;=\; A^1_{21} \,.\, (A^2_{32} \,.\, A^3_{13}) \;+\; A^1_{23} \,.\, (A^2_{31} \,.\, A^3_{12})$$
$$d_{321} \;=\; A^1_{31} \,.\, (- A^2_{23} \,.\, A^3_{12}) \;-\; A^1_{32} \,.\, (A^2_{21} \,.\, A^3_{13})$$
$$d_{132} \;=\; - A^1_{12} \,.\, (A^2_{31} \,.\, A^3_{23}) \;+\; A^1_{13} \,.\, (- A^2_{32} \,.\, A^3_{21})$$
$$d_{213} \;=\; A^1_{21} \,.\, (- A^2_{13} \,.\, A^3_{32}) \;+\; A^1_{23} \,.\, (- A^2_{12} \,.\, A^3_{31})$$

Because the multiplication is an associative operation on $\mathbb{C}$, these coefficients can be added. The addition yields:

$$d_{123} \;+\; d_{312} \;+\; d_{231} \;+\; d_{321} \;+\; d_{132} \;+\; d_{213} \;=\; 0$$

Remark 3.2. *A quicker and a better demonstration*

The previous demonstration is pure algebra. There is a better and quicker way to reach the same conclusion. Any deformed cross product is an anti-symmetric operation because one can always write:

$$\forall\, \mathbf{a},\, ... \in V : [\mathbf{a}, ...]_{[A]} + [..., \mathbf{a}]_{[A]} = \mathbf{0}$$

In peculiar, when $\mathbf{...} = \mathbf{a}$:

$$\forall\, \mathbf{a} \in V : [\mathbf{a}, \mathbf{a}]_{[A]} = \mathbf{0}$$

Due to the existence of at least one decomposition with a vanishing residual part, the null cross product has a representation such that:

$$\forall\, \mathbf{a} \in V : {}_A\Phi(\mathbf{a})\,.\,|\mathbf{a}> = |\mathbf{0}>$$

This representation is trivially true when the vector **a** is null. Otherwise, this representation can only be true when:

$$\forall\, \mathbf{a} \in V : |{}_A\Phi(\mathbf{a})| = 0$$

In general, when the cube A is any one, the calculation of a determinant like the one at hand should have the generic formalism:

$$\Lambda(a^1,\, a^2,\, a^3)$$

$$=$$

$$\sum_{mnp} d_{mnp}\,.\,a^m\,.\,a^n\,.\,a^p + \sum_{mn} d_{mn}\,.\,a^m\,.\,a^n + \sum_{m} d_m\,.\,a^m + d$$

But the coefficients in front of the terms of degree three can only arise from the determinant of the matrix ${}_A\Phi(\mathbf{a})$. This determinant is null. Following the same vein, the coefficients in front of the terms of degree zero can only arise from the determinant of the matrix [P]. The consequence of these statements is:

$$\Lambda(a^1,\, a^2,\, a^3)$$

$$=$$

$$\sum_{mn} d_{mn}\,.\,a^m\,.\,a^n + \sum_{m} d_m\,.\,a^m - |P|$$

□

Theorem 3.1. *The initial theorem*

The existence of a representation $([\mathrm{P}], \mathbf{z})$ in $\mathrm{M}(3, \mathbb{C}) \times \mathrm{V}$ (synonym: of a decomposition) for an element $[\mathbf{a}, \mathbf{...}]_{[A]}$ imposes the one of a polynomial form $\Lambda(\mathbf{a})$ of degree at most equal to two:

$$\Lambda(a^1,\, a^2,\, a^3) = |{}_A\Phi(\mathbf{a}) - [P]| = \sum_{mn} d_{mn}\,.\,a^m\,.\,a^n + \sum_{m} d_m\,.\,a^m - |P|$$

3.3 Calculating the coefficients of degree two and the matrix [D]

The coefficients of degree two can be grouped within a matrix that will be labeled:

$$[D] = [d_{mn}]$$

Proposition 3.2. *When the polynomial form $\Lambda(\mathbf{a})$ has invariant coefficients, its Hessian is the sum of the matrix containing the coefficients of degree two and of its transposed.*

Proof. : Let consider the polynomial form $\Lambda(\mathbf{a})$. Let suppose that its coefficients do not depend on the $\mathbf{a}$.

$$\Lambda(a^1, a^2, a^3)$$
$$=$$
$$d_{11} \,.\, (a^1)^2 + d_{22} \,.\, (a^2)^2 + d_{33} \,.\, (a^3)^2$$
$$+ (d_{12} + d_{21}) \,.\, a^1 \,.\, a^2 + (d_{23} + d_{31}) \,.\, a^2 \,.\, a^3 + (d_{13} + d_{31}) \,.\, a^1 \,.\, a^3$$
$$+ d_1 \,.\, a^1 + d_2 \,.\, a^2 + d_3 \,.\, a^3 - |P|$$

Let calculate the successive derivations by respect for the components of this vector:

$$\frac{\partial \Lambda(a^1, a^2, a^3)}{\partial a^1} = 2 \,.\, d_{11} \,.\, a^1 + (d_{12} + d_{21}) \,.\, a^2 + (d_{13} + d_{31}) \,.\, a^3 + d_1$$

$$\frac{\partial \Lambda(a^1, a^2, a^3)}{\partial a^2} = (d_{12} + d_{21}) \,.\, a^1 + 2 \,.\, d_{22} \,.\, a^2 + (d_{23} + d_{32}) \,.\, a^3 + d_2$$

$$\frac{\partial \Lambda(a^1, a^2, a^3)}{\partial a^3} = (d_{13} + d_{31}) \,.\, a^1 + (d_{23} + d_{32}) \,.\, a^2 + 2 \,.\, d_{33} \,.\, a^3 + d_3$$

And:

$$[Hess_{(\mathbf{a},\,0)} \Lambda(a^1,\, a^2,\, a^3)] = [D] + [D]^t$$

□

Corollary 3.1. *A first hint concerning the formalism of the matrix [D]*

A direct consequence of prior relation is the possibility to guess the general formalism of the matrix [D] in relation with the classical Hessian of any polynomial. Let [X] be any element in the set of anti-symmetric matrices:

$$[X] + [X]^t = [0] \in M(3, \mathbb{C})$$

Then one can write without restriction of the generality that:

$$[D] = \frac{1}{2} \cdot [Hess_{(\mathbf{a},\,0)} \Lambda(a^1,\, a^2,\, a^3)] + [X]$$

This relation will play an important role later.

Definition 3.1. *The additive characteristic of a matrix*

Let $[\mathrm{M}] = [\mathrm{m}_{ij}]$ be any element in $\mathrm{M}(3, \mathbb{C})$. Per convention, the sum of its entries is called the additive characteristic of [M]. It is denoted $[\mathrm{M}]^{\oplus}$. This characteristic can be understood as an application with a source in $\mathrm{M}(3, \mathbb{C})$ and an image in $\mathbb{C}$.

$$\oplus : \forall\, [M] \in M(3,\, \mathbb{C}) \xrightarrow{\oplus} [M]^{\oplus} = \sum_i \sum_j m_{ij} \in \mathbb{C}$$

Remark 3.3. *Continuing the calculations*

The calculation by hand brings the relation:

$$d_{mn}$$

$$=$$

$$+ A^1_{m1} . [(p_{23} . A^3_{n2} + p_{32} . A^2_{n3}) - (p_{22} . A^3_{n3} + p_{33} . A^2_{n2})]$$

$$- p_{11} . (A^2_{m2} . A^3_{n3} - A^2_{m3} . A^3_{n2})$$

$$- A^1_{m2} . [(p_{23} . A^3_{n1} + p_{31} . A^2_{n3}) - (p_{21} . A^3_{n3} + p_{33} . A^2_{n1})]$$

$$+ p_{12} . (A^2_{m1} . A^3_{n3} - A^2_{m3} . A^3_{n1})$$

$$+ A^1_{m3} . [(p_{22} . A^3_{n1} + p_{31} . A^2_{n2}) - (p_{21} . A^3_{n2} + p_{32} . A^2_{n1})]$$

$$- p_{13} . (A^2_{m1} . A^3_{n2} - A^2_{m2} . A^3_{n1})$$

$$=$$

$$p_{11} . (A^2_{m3} . A^3_{n2} - A^2_{m2} . A^3_{n3})$$

$$+ p_{12} . (A^2_{m3} . A^3_{n1} - A^2_{m1} . A^3_{n3})$$

$$+ p_{13} . (A^2_{m2} . A^3_{n1} - A^2_{m1} . A^3_{n2})$$

$$+ p_{21} . (A^1_{m2} . A^3_{n3} - A^1_{m3} . A^3_{n2})$$

$$+ p_{22} . (A^1_{m3} . A^3_{n1} - A^1_{m1} . A^3_{n3})$$

$$+ p_{23} . (A^1_{m1} . A^3_{n2} - A^1_{m2} . A^3_{n1})$$

$$+ p_{31} . (A^1_{m3} . A^2_{n2} - A^1_{m2} . A^2_{n3})$$

$$+ p_{32} . (A^1_{m1} . A^2_{n3} - A^1_{m3} . A^2_{n1})$$

$$+ p_{33} . (A^1_{m2} . A^2_{n1} - A^1_{m1} . A^2_{n2})$$

This formulation suggests the existence of nine matrices :

$$[_{mn}T]$$

$$=$$

$$\begin{bmatrix} (A^2_{m3} . A^3_{n2} - A^2_{m2} . A^3_{n3}) & (A^1_{m2} . A^3_{n3} - A^1_{m3} . A^3_{n2}) & (A^1_{m3} . A^2_{n2} - A^1_{m2} . A^2_{n3}) \\ (A^2_{m3} . A^3_{n1} - A^2_{m1} . A^3_{n3}) & (A^1_{m3} . A^3_{n1} - A^1_{m1} . A^3_{n3}) & (A^1_{m1} . A^2_{n3} - A^1_{m3} . A^2_{n1}) \\ (A^2_{m2} . A^3_{n1} - A^2_{m1} . A^3_{n2}) & (A^1_{m1} . A^3_{n2} - A^1_{m2} . A^3_{n1}) & (A^1_{m2} . A^2_{n1} - A^1_{m1} . A^2_{n2}) \end{bmatrix}$$

... such that:

$$d_{mn} = \{[_{mn}T] . [P]\}^{\oplus}$$

...

3.4 The coefficients of degree two in the diagonal of [D]

The knots of the anti-symmetric cube A of which the subscripts are repeated vanish. Therefore, it is easy to state that:

$$[{}_{mm}T] = \begin{bmatrix} (A^2_{m3} \cdot A^3_{m2} - A^2_{m2} \cdot A^3_{m3}) & (A^1_{m2} \cdot A^3_{m3} - A^1_{m3} \cdot A^3_{m2}) & (A^1_{m3} \cdot A^2_{m2} - A^1_{m2} \cdot A^2_{m3}) \\ (A^2_{m3} \cdot A^3_{m1} - A^2_{m1} \cdot A^3_{m3}) & (A^1_{m3} \cdot A^3_{m1} - A^1_{m1} \cdot A^3_{m3}) & (A^1_{m1} \cdot A^2_{m3} - A^1_{m3} \cdot A^2_{m1}) \\ (A^2_{m2} \cdot A^3_{m1} - A^2_{m1} \cdot A^3_{m2}) & (A^1_{m1} \cdot A^3_{m2} - A^1_{m2} \cdot A^3_{m1}) & (A^1_{m2} \cdot A^2_{m1} - A^1_{m1} \cdot A^2_{m2}) \end{bmatrix}$$

Precisely:

$$[{}_{11}T] = \begin{bmatrix} (A^2_{13} \cdot A^3_{12} - A^2_{12} \cdot A^3_{13}) & (A^1_{12} \cdot A^3_{13} - A^1_{13} \cdot A^3_{12}) & (A^1_{13} \cdot A^2_{12} - A^1_{12} \cdot A^2_{13}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$d_{11} = (A^2_{13} \cdot A^3_{12} - A^2_{12} \cdot A^3_{13}) \cdot p_{11} + (A^1_{12} \cdot A^3_{13} - A^1_{13} \cdot A^3_{12}) \cdot p_{21} + (A^1_{13} \cdot A^2_{12} - A^1_{12} \cdot A^2_{13}) \cdot p_{31}$$

$$[{}_{22}T] = \begin{bmatrix} 0 & 0 & 0 \\ (A^2_{23} \cdot A^3_{21} - A^2_{21} \cdot A^3_{23}) & (A^1_{23} \cdot A^3_{21} - A^1_{21} \cdot A^3_{23}) & (A^1_{21} \cdot A^2_{23} - A^1_{23} \cdot A^2_{21}) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$d_{22} = (A^2_{23} \cdot A^3_{21} - A^2_{21} \cdot A^3_{23}) \cdot p_{12} + (A^1_{23} \cdot A^3_{21} - A^1_{21} \cdot A^3_{23}) \cdot p_{22} + (A^1_{21} \cdot A^2_{23} - A^1_{23} \cdot A^2_{21}) \cdot p_{13}$$

$$[{}_{33}T] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (A^2_{32} \cdot A^3_{31} - A^2_{31} \cdot A^3_{32}) & (A^1_{31} \cdot A^3_{32} - A^1_{32} \cdot A^3_{31}) & (A^1_{32} \cdot A^2_{31} - A^1_{31} \cdot A^2_{32}) \end{bmatrix}$$

$$\Downarrow$$

$$d_{33} = (A^2_{32} \cdot A^3_{31} - A^2_{31} \cdot A^3_{32}) \cdot p_{13} + (A^1_{31} \cdot A^3_{32} - A^1_{32} \cdot A^3_{31}) \cdot p_{23} + (A^1_{32} \cdot A^2_{31} - A^1_{31} \cdot A^2_{32}) \cdot p_{33}$$

Definition 3.2. *The matrix [T]*

The three matrices $[_{mm}\mathrm{T}]$ can be added to form a new matrix:

$$\sum_{m=1}^{m=3} [_{mm}T]$$

$$=$$

$$[T]$$

$$=$$

$$\begin{bmatrix} (A^2_{13} \,.\, A^3_{12} \;-\; A^2_{12} \,.\, A^3_{13}) & (A^1_{12} \,.\, A^3_{13} \;-\; A^1_{13} \,.\, A^3_{12}) & (A^1_{13} \,.\, A^2_{12} \;-\; A^1_{12} \,.\, A^2_{13}) \\ (A^2_{23} \,.\, A^3_{21} \;-\; A^2_{21} \,.\, A^3_{23}) & (A^1_{23} \,.\, A^3_{21} \;-\; A^1_{21} \,.\, A^3_{23}) & (A^1_{21} \,.\, A^2_{23} \;-\; A^1_{23} \,.\, A^2_{21}) \\ (A^2_{32} \,.\, A^3_{31} \;-\; A^2_{31} \,.\, A^3_{32}) & (A^1_{31} \,.\, A^3_{32} \;-\; A^1_{32} \,.\, A^3_{31}) & (A^1_{32} \,.\, A^2_{31} \;-\; A^1_{31} \,.\, A^2_{32}) \end{bmatrix}$$

Remark 3.4. *The trace of the matrix [D]*

These results allow the calculation of the trace of the matrix containing the coefficients of degree two:

$$Trace[D]$$

$$=$$

$$\sum_{m=1}^{m=3} d_{mm}$$

$$=$$

$$\sum_{m=1}^{m=3} \{[_{mm}T] \,.\, [P]\}^{\oplus}$$

$$=$$

$$\{\sum_{m=1}^{m=3} \{[_{mm}T] \,.\, [P]\}\}^{\oplus}$$

$$=$$

$$\{\{\sum_{m=1}^{m=3} [_{mm}T]\} \,.\, [P]\}^{\oplus}$$

$$=$$

$$\{[T] \,.\, [P]\}^{\oplus}$$

3.5 Some useful definitions, ideas and identities

Definition 3.3. *Equivalent matrices*

In this document, two elements [M] and [N] in $\mathrm{M}(3, \mathbb{C})$ are equivalent if or when:

- They have the same diagonal:

$$\forall\, i \;=\; 1,\, 2,\, 3 :\; m_{ii} \;=\; n_{ii}$$

- The off-diagonal entries respect the condition:

$$\forall\, i,\, j \;=\; 1,\, 2,\, 3 :\; m_{ij} \;+\; m_{ji} \;=\; n_{ij} \;+\; n_{ji}$$

Two equivalent elements in $\mathrm{M}(3, \mathbb{C})$ are denoted $[\mathrm{M}] \equiv [\mathrm{N}]$.

Proposition 3.3. *The definition 3.2 generates a relation of equivalence in M(3,$\mathbb{C}$).*

Proof. There is no difficulty to prove that:

- Any element in M(3, $\mathbb{C}$) is equivalent to itself: [M] ≡ [M].
- If [M] is equivalent to [N], then [N] is equivalent to [M]: [M] ≡ [N] $\iff$ [N] ≡ [M].
- If [M] is equivalent to [N] and [N] is equivalent to [O] , then [M] is equivalent to [O]:

$$\{[M] \equiv [N]\} \cap \{[N] \equiv [O]\} \Rightarrow [M] \equiv [O]$$

□

Remark 3.5. *The inverse of a non-degenerated deforming matrix*

Let suppose that the deforming matrix is not degenerated; then:

$$|A| \neq 0 \Rightarrow \exists [A]^{-1} : [A] . [A]^{-1} = [A]^{-1} . [A] = Id_3$$

This is equivalent to:

$$\begin{bmatrix} A^1_{12} & A^2_{12} & A^3_{12} \\ A^1_{23} & A^2_{23} & A^3_{23} \\ A^1_{13} & A^2_{13} & A^3_{13} \end{bmatrix} . \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is easy to prove that:

$$[A]^{-1}$$

$$=$$

$$|A| . \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} A^2_{23} . A^3_{13} - A^2_{13} . A^3_{23} & A^2_{13} . A^3_{12} - A^2_{12} . A^3_{13} & A^2_{12} . A^3_{23} - A^2_{23} . A^3_{12} \\ A^1_{13} . A^3_{23} - A^1_{23} . A^3_{13} & A^1_{12} . A^3_{13} - A^1_{13} . A^3_{12} & A^1_{23} . A^3_{12} - A^1_{12} . A^3_{23} \\ A^1_{23} . A^2_{13} - A^1_{13} . A^2_{23} & A^1_{13} . A^2_{12} - A^1_{12} . A^2_{13} & A^1_{12} . A^2_{23} - A^1_{23} . A^2_{12} \end{bmatrix}$$

Remark 3.6. *A first link between the matrix [A] and the matrix [T]*

Let calculate:

$$|A| . [A]^{-1} . [J]$$

$$=$$

$$\begin{bmatrix} A^2_{23} . A^3_{13} - A^2_{13} . A^3_{23} & A^2_{13} . A^3_{12} - A^2_{12} . A^3_{13} & A^2_{12} . A^3_{23} - A^2_{23} . A^3_{12} \\ A^1_{13} . A^3_{23} - A^1_{23} . A^3_{13} & A^1_{12} . A^3_{13} - A^1_{13} . A^3_{12} & A^1_{23} . A^3_{12} - A^1_{12} . A^3_{23} \\ A^1_{23} . A^2_{13} - A^1_{13} . A^2_{23} & A^1_{13} . A^2_{12} - A^1_{12} . A^2_{13} & A^1_{12} . A^2_{23} - A^1_{23} . A^2_{12} \end{bmatrix} . \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} A^2_{13} . A^3_{12} - A^2_{12} . A^3_{13} & A^2_{23} . A^3_{12} - A^2_{12} . A^3_{23} & A^2_{23} . A^3_{13} - A^2_{13} . A^3_{23} \\ A^1_{12} . A^3_{13} - A^1_{13} . A^3_{12} & A^1_{12} . A^3_{23} - A^1_{23} . A^3_{12} & A^1_{13} . A^3_{23} - A^1_{23} . A^3_{13} \\ A^1_{13} . A^2_{12} - A^1_{12} . A^2_{13} & A^1_{23} . A^2_{12} - A^1_{12} . A^2_{23} & A^1_{23} . A^2_{13} - A^1_{13} . A^2_{23} \end{bmatrix}$$

Let rewrite the matrix [T] in ordering the subscripts:

$$[T]$$

$$=$$

$$\begin{bmatrix} (A^2_{13}\,.\,A^3_{12} - A^2_{12}\,.\,A^3_{13}) & (A^1_{12}\,.\,A^3_{13} - A^1_{13}\,.\,A^3_{12}) & (A^1_{13}\,.\,A^2_{12} - A^1_{12}\,.\,A^2_{13}) \\ (A^2_{12}\,.\,A^3_{23} - A^2_{23}\,.\,A^3_{12}) & (A^1_{12}\,.\,A^3_{23} - A^1_{23}\,.\,A^3_{12}) & (A^1_{23}\,.\,A^2_{12} - A^1_{12}\,.\,A^2_{23}) \\ (A^2_{23}\,.\,A^3_{13} - A^2_{13}\,.\,A^3_{23}) & (A^1_{13}\,.\,A^3_{23} - A^1_{23}\,.\,A^3_{13}) & (A^1_{23}\,.\,A^2_{13} - A^1_{13}\,.\,A^2_{23}) \end{bmatrix}$$

And let observe that:

$$[T]^t = |A|\,.\,[A]^{-1}\,.\,[J]$$

This result brings two indications:

- The determinant of the matrix [T]:

$$|[T]^t| = |T| = |A|\,.\,\underbrace{|[A]^{-1}|}_{=\frac{1}{|A|}}\,.\,\underbrace{|J|}_{=-1} = -1 \neq 0$$

  This matrix is not degenerated and owns an inverse representation. Since the matrix [T] is not degenerated:

$$|T| \neq 0 \Rightarrow \exists\,[T]^{-1} : [T]\,.\,[T]^{-1} = [t_{ij}]\,.\,[T_{ij}] = Id_3 = [T]^{-1}\,.\,[T]$$

  From which one deduces that:

$$[T]^{-1} = |T|\,.\begin{bmatrix} T_{11} = (t_{22}\,.\,t_{33} - t_{32}\,.\,t_{23}) & T_{12} = (t_{13}\,.\,t_{32} - t_{12}\,.\,t_{33}) & T_{13} = (t_{12}\,.\,t_{23} - t_{22}.t_{13}) \\ T_{21} = (t_{31}\,.\,t_{23} - t_{33}\,.\,t_{21}) & T_{22} = (t_{11}\,.\,t_{33} - t_{31}\,.\,t_{13}) & T_{23} = (t_{21}\,.\,t_{13} - t_{11}\,.\,t_{23}) \\ T_{31} = (t_{21}\,.\,t_{32} - t_{31}\,.\,t_{22}) & T_{32} = (t_{12}\,.\,t_{31} - t_{11}\,.\,t_{32}) & T_{33} = (t_{22}\,.\,t_{11} - t_{12}\,.\,t_{21}) \end{bmatrix}$$

- The formalism of the matrix [T] - by transposition:

$$[T] = |A|\,.\,[J]^t\,.\,([A]^{-1})^t$$

Definition 3.4. *The golden rule*

A given and non-degenerated element [A] in $M(3,\mathbb{C})$ respects the golden rule when:

$$\{[A]^{-1}\}^t = \{[A]^t\}^{-1}$$

Recall that any element in $GL_3(\mathbb{C})$ respect this rule (basic knowledge).

Remark 3.7. *The exact formalism of the matrix* $[T]^{-1}$

Proposition 3.4. *For non-degenerated deforming matrices, some manipulations are proving that:*

$$[T]^{-1} = \frac{1}{|A|}\,.\,[A]^t\,.\,[J]$$

Proof. There are two paths to prove it. By hand but it is really annoying. Or in restricting the domain of validity of the proposition to a given subset of deforming matrices, precisely the ones respecting the golden rule. For these elements:

$$[T] = |A|\,.\,[J]^t\,.\,([A]^{-1})^t$$

$$\Downarrow$$

$$[T]^{-1} = \frac{1}{|A|}\,.\,\{([A]^{-1})^t\}^{-1}\,.\,\{[J]^t\}^{-1} = \frac{1}{|A|}\,.\,[A]^t\,.\,[J] = \frac{1}{|A|}\,.\,[A^*]$$

This seemingly unimportant relation will play a crucial role a little bit later. This is why it deserves a: □

Theorem 3.2. *The link between the intrinsic and the extrinsic method*

For non-degenerated deforming matrices respecting the golden rule, for example those in $GL_3(\mathbb{C})$, the effective deforming matrix is proportional to the inverse of the [T] matrix.

3.6 The coefficients of degree two outside of the diagonal of [D]

Let now consider the three sums (for m $\neq$ n):

$$d_{mn} + d_{nm} = \{[_{mn}T] . [P]\}^{\oplus} + \{[_{nm}T] . [P]\}^{\oplus} = \{\{[_{mn}T] + [_{nm}T]\} . [P]\}^{\oplus}$$

Remark 3.8. *Looking for a useful correspondence*

The relation:

$$\forall m, n = 1, 2, 3 : d_{mn} = \{[_{mn}T] . [P]\}^{\oplus}$$

... suggests the existence of an underlying correspondence between the entries of [D] and [$_{mn}$T], whatever the unknown matrix [P] is:

$$\forall m, n = 1, 2, 3 : d_{mn} \overset{?}{\rightarrow} ?(d_{mn}) = {}_{mn}T$$

The relation:

$$Trace\{[D]\} = \{[T] . [P]\}^{\oplus}$$

... also suggests an interrogation: "Is there an equivalence between the matrix [D] and the product [T].[P]?"

$$[D] \equiv [T] . [P]?$$

Proposition 3.5. *The product [T].[P] is equivalent to the matrix [D]*

Proof. A simple way to prove the affirmation is to write:

$$[T]$$

$$=$$

$$\begin{bmatrix} (A^2_{13} . A^3_{12} - A^2_{12} . A^3_{13}) & (A^1_{12} . A^3_{13} - A^1_{13} . A^3_{12}) & (A^1_{13} . A^2_{12} - A^1_{12} . A^2_{13}) \\ (A^2_{12} . A^3_{23} - A^2_{23} . A^3_{12}) & (A^1_{12} . A^3_{23} - A^1_{23} . A^3_{12}) & (A^1_{23} . A^2_{12} - A^1_{12} . A^2_{23}) \\ (A^2_{23} . A^3_{13} - A^2_{13} . A^3_{23}) & (A^1_{13} . A^3_{23} - A^1_{23} . A^3_{13}) & (A^1_{23} . A^2_{13} - A^1_{13} . A^2_{23}) \end{bmatrix}$$

$$. \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

One can then directly verify that:

- Concerning the coefficients in the diagonal:
  It is easy to verify that the proposition is true for these coefficients in observing the results which have been obtained in sub-section 3.4, p.16.
- Concerning the coefficients which are not in the diagonal:
  Starting from the general formulation of the d_{mn} in remark 3.3 it is easy to directly verify that:

  $$d_{mn} + d_{nm} = \{[T] . [P]\}_{mn} + \{[T] . [P]\}_{nm}$$

  For a proof, see the example in the mathematical annexes at the end of this document.

These facts are sufficient to prove the equivalence:

$$[D] \equiv [T] . [P]$$

□

Theorem 3.3. *The reconstructed matrix*

The diverse parts of the puzzle can now be regrouped.

The intrinsic ingredients of the (E) question are some element **a** in V and the deforming matrix [A] which - up to now- will be supposed to be an element in $GL_3(\mathbb{C})$.

The unknown ingredients of the (E) question are an element [P] in $M(3, \mathbb{C})$ and a residual element **z** in V.

The intrinsic ingredients allow the definition of a whole set of deformed cross products. When any given deformed cross product accepts a decomposition such that:

$$|[\mathbf{a}, \mathbf{b}]_{[A]} >= [P] \,.\, |\mathbf{b} > + |\mathbf{z} >$$

Then:

1. There exists a polynomial $\Lambda(\mathbf{a})$ of degree at most equal to two; see the initial theorem.
2. Any deformed cross product can be related to its classical version through the relation:
$$|[\mathbf{a}, \mathbf{b}]_{[A]} >= \underbrace{[A]^t \,.\, [J]}_{[A^*]} \,.\, |\mathbf{a} \wedge \mathbf{b} >$$
3. As long as the coefficients of that polynomial $\Lambda(\mathbf{a})$ remains invariant, its Hessian writes:
$$[Hess_{(\mathbf{a},\,0)} \Lambda(a^1,\, a^2,\, a^3)] \;=\; [D] \,+\, [D]^t$$
... and the matrix [D] has the generic formalism:
$$[D] \;=\; \frac{1}{2} \,.\, [Hess_{(\mathbf{a},\,0)} \Lambda(a^1,\, a^2,\, a^3)] \,+\, [X]$$
Here the matrix [X] must be any element in su(3).
4. Since the deforming matrix is supposed to be non-degenerated, its inverse exists. The existence of $[A]^{-1}$ allows the construction of the matrix [T] because:
$$[T] \;=\; |A| \,.\, [J]^t \,.\, ([A]^{-1})^t$$
5. The matrix [T] can be inverted. For deforming matrices in $GL_3(\mathbb{C})$:
$$\{[A]^{-1}\}^t \;=\; \{[A]^t\}^{-1}$$
... the inverse writes:
$$[T]^{-1} \;=\; \frac{1}{|A|} \,.\, [A^*]$$
6. The matrix [D] is equivalent to the unknown product [T].[P]. If the equivalence is an exact equality, one gets a first visage for the unknown matrix [P]:
$$[P] \;=\; \frac{1}{|A|} \,.\, [A^*] \,.\, [D] \;=\; \frac{1}{|A|} \,.\, [A^*] \,.\, \{\frac{1}{2} \,.\, [Hess_{(\mathbf{a},\,0)} \Lambda(a^1,\, a^2,\, a^3)] \,+\, [X]\}$$
Therefore, knowing the deforming matrix [A] and the matrix [D] containing the coefficients of degree two of any polynomial, one can reconstruct a matrix [P] which may eventually be a solution of the (E) question.

3.7 The coefficients of degree one

The coefficients of degree one can be grouped within an element in V:

$$\mathbf{d}^* \equiv (d_1,\, d_2,\, d_3) \in \mathbb{C}^3$$

As a result of the calculations:

$$\begin{aligned}
&d_m \\
&= \\
&+ A^1_{m1} \,.\, (p_{22} \,.\, p_{33} \,-\, p_{32} \,.\, p_{23}) \\
&- A^1_{m2} \,.\, (p_{21} \,.\, p_{33} \,-\, p_{31} \,.\, p_{23}) \\
&+ A^1_{m3} \,.\, (p_{21} \,.\, p_{32} \,-\, p_{22} \,.\, p_{31}) \\
&- p_{11} \,.\, [(p_{23} \,.\, A^3_{m2} \,+\, p_{32} \,.\, A^2_{m3}) \,-\, (p_{22} \,.\, A^3_{m3} \,+\, p_{33} \,.\, A^2_{m2})] \\
&+ p_{12} \,.\, [(p_{23} \,.\, A^3_{m1} \,+\, p_{31} \,.\, A^2_{m3}) \,-\, (p_{21} \,.\, A^3_{m3} \,+\, p_{33} \,.\, A^2_{m1})] \\
&- p_{13} \,.\, [(p_{22} \,.\, A^3_{m1} \,+\, p_{31} \,.\, A^2_{m2}) \,-\, (p_{21} \,.\, A^3_{m2} \,+\, p_{32} \,.\, A^2_{m1})] \\
&= \\
&+ A^1_{m1} \,.\, (p_{22} \,.\, p_{33} \,-\, p_{32} \,.\, p_{23}) \\
&+ A^1_{m2} \,.\, (p_{31} \,.\, p_{23} \,-\, p_{21} \,.\, p_{33}) \\
&+ A^1_{m3} \,.\, (p_{21} \,.\, p_{32} \,-\, p_{22} \,.\, p_{31}) \\
&+ A^2_{m1} \,.\, (p_{13} \,.\, p_{32} \,-\, p_{12} \,.\, p_{33}) \\
&+ A^2_{m2} \,.\, (p_{11} \,.\, p_{33} \,-\, p_{22} \,.\, p_{31}) \\
&+ A^2_{m3} \,.\, (p_{12} \,.\, p_{31} \,-\, p_{11} \,.\, p_{32}) \\
&+ A^3_{m1} \,.\, (p_{12} \,.\, p_{23} \,-\, p_{13} \,.\, p_{22}) \\
&+ A^3_{m2} \,.\, (p_{13} \,.\, p_{21} \,-\, p_{11} \,.\, p_{23}) \\
&+ A^3_{m3} \,.\, (p_{11} \,.\, p_{22} \,-\, p_{12} \,.\, p_{21})
\end{aligned}$$

At this stage, the matrix [P] is unknown because it is one of the object one is looking for. But let suppose that it is a non degenerated matrix, then:

$$|P| \neq 0 \Rightarrow \exists\, [P]^{-1} \,:\, [P]\,.\,[P]^{-1} \,=\, [p_{ij}]\,.\,[P_{jk}] \,=\, Id_3 \,=\, [\delta^k_i] \,=\, [P]^{-1}\,.\,[P]$$

And its inverse representation is:

$$\begin{gathered}
|P|\,.\,[P]^{-1} \\
= \\
\begin{bmatrix}
(p_{22} \,.\, p_{33} - p_{32} \,.\, p_{23}) & (p_{13} \,.\, p_{32} - p_{12} \,.\, p_{33}) & (p_{12} \,.\, p_{23} - p_{22} . p_{13}) \\
(p_{31} \,.\, p_{23} - p_{33} \,.\, p_{21}) & (p_{11} \,.\, p_{33} - p_{31} \,.\, p_{13}) & (p_{21} \,.\, p_{13} - p_{11} \,.\, p_{23}) \\
(p_{21} \,.\, p_{32} - p_{31} \,.\, p_{22}) & (p_{12} \,.\, p_{31} - p_{11} \,.\, p_{32}) & (p_{22} \,.\, p_{11} - p_{12} \,.\, p_{21})
\end{bmatrix}
\end{gathered}$$

Let consider the three matrices:

$$\forall\, m \,=\, 1,\, 2,\, 3 \,:\, [{}_m A] \,=\, \begin{bmatrix} A^1_{m1} & A^1_{m2} & A^1_{m3} \\ A^2_{m1} & A^2_{m2} & A^2_{m3} \\ A^3_{m1} & A^3_{m2} & A^3_{m3} \end{bmatrix} \in M(3,\, \mathbb{C})$$

More precisely:

$$[{}_1 A] \,=\, \begin{bmatrix} 0 & A^1_{12} & A^1_{13} \\ 0 & A^2_{12} & A^2_{13} \\ 0 & A^3_{12} & A^3_{13} \end{bmatrix} \in M(3,\, \mathbb{C})$$

$$[{}_2A] = \begin{bmatrix} A^1_{21} & 0 & A^1_{23} \\ A^2_{21} & 0 & A^2_{23} \\ A^3_{21} & 0 & A^3_{23} \end{bmatrix} \in M(3, \mathbb{C})$$

$$[{}_3A] = \begin{bmatrix} A^1_{31} & A^1_{32} & 0 \\ A^2_{31} & A^2_{32} & 0 \\ A^3_{31} & A^3_{32} & 0 \end{bmatrix} \in M(3, \mathbb{C})$$

And state that:

$$\forall m = 1, 2, 3 : d_m = \{|P| . [{}_mA] . [P]^{-1}\}^{\oplus}$$

In details:

$$d_1$$
$$=$$
$$A^1_{12} . (p_{31} . p_{23} - p_{21} . p_{33}) + A^2_{12} . (p_{33} . p_{11} - p_{31} . p_{13}) + A^3_{12} . (p_{21} . p_{13} - p_{11} . p_{23})$$
$$+$$
$$A^1_{13} . (p_{21} . p_{32} - p_{22} . p_{31}) + A^2_{13} . (p_{31} . p_{12} - p_{11} . p_{32}) + A^3_{13} . (p_{11} . p_{22} - p_{21} . p_{12})$$

$$d_2$$
$$=$$
$$A^1_{12} . (p_{32} . p_{23} - p_{22} . p_{33}) + A^2_{12} . (p_{33} . p_{12} - p_{32} . p_{13}) + A^3_{12} . (p_{22} . p_{13} - p_{12} . p_{23})$$
$$+$$
$$A^1_{23} . (p_{21} . p_{32} - p_{22} . p_{31}) + A^2_{23} . (p_{31} . p_{12} - p_{11} . p_{32}) + A^3_{23} . (p_{11} . p_{22} - p_{21} . p_{12})$$

$$d_3$$
$$=$$
$$A^1_{13} . (p_{32} . p_{23} - p_{22} . p_{33}) + A^2_{13} . (p_{33} . p_{12} - p_{32} . p_{13}) + A^3_{13} . (p_{22} . p_{13} - p_{12} . p_{23})$$
$$+$$
$$A^1_{23} . (p_{21} . p_{33} - p_{23} . p_{31}) + A^2_{23} . (p_{31} . p_{13} - p_{11} . p_{33}) + A^3_{23} . (p_{11} . p_{23} - p_{21} . p_{13})$$

3.8 Another formalism for the coefficients of degree one

Let write for convenience:

$$|P| . [P]^{-1}$$
$$=$$
$$\begin{bmatrix} P_{11} = (p_{22} . p_{33} - p_{32} . p_{23}) & P_{12} = (p_{13} . p_{32} - p_{12} . p_{33}) & P_{13} = (p_{12} . p_{23} - p_{22} . p_{13}) \\ P_{21} = (p_{31} . p_{23} - p_{33} . p_{21}) & P_{22} = (p_{11} . p_{33} - p_{31} . p_{13}) & P_{23} = (p_{21} . p_{13} - p_{11} . p_{23}) \\ P_{31} = (p_{21} . p_{32} - p_{31} . p_{22}) & P_{32} = (p_{12} . p_{31} - p_{11} . p_{32}) & P_{33} = (p_{22} . p_{11} - p_{12} . p_{21}) \end{bmatrix}$$

Then state that:

$$d_1 = |P| . (A^1_{12} . P_{21} + A^2_{12} . P_{22} + A^3_{12} . P_{23} + A^1_{13} . P_{31} + A^2_{13} . P_{32} + A^3_{13} . P_{33})$$

$$d_2 = |P| . (-A^1_{12} . P_{11} - A^2_{12} . P_{12} - A^3_{12} . P_{13} + A^1_{23} . P_{31} + A^2_{23} . P_{32} + A^3_{23} . P_{33})$$

$$d_3 = |P| . (-A^1_{13} . P_{11} - A^2_{13} . P_{12} + A^3_{13} . P_{13} - A^1_{23} . P_{21} - A^2_{23} . P_{22} + A^3_{23} . P_{23})$$

Let introduce the matrix [L] for future purpose:

$$[L] = |P| . [A] . ([P]^{-1})^t$$

In details:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{23} & L_{33} \end{bmatrix} = |P| . \begin{bmatrix} A^1_{12} & A^2_{12} & A^3_{12} \\ A^1_{23} & A^2_{23} & A^3_{23} \\ A^1_{13} & A^2_{13} & A^3_{13} \end{bmatrix} . \begin{bmatrix} P_{11} & P_{21} & P_{31} \\ P_{12} & P_{22} & P_{32} \\ P_{13} & P_{23} & P_{33} \end{bmatrix}$$

Let state that:

$$d_1 = L_{12} + L_{33}$$

$$d_2 = L_{23} - L_{11}$$

$$d_3 = -L_{31} - L_{22}$$

3.9 Approaching the formalism of [D] with the intrinsic ingredients

Since it has been proven that:

$$[D] \equiv [T] . [P] = [N]$$

- There exists an element (n_{23}, n_{13}, n_{12}) in $\mathbb{C}^3$ such that:

$$[N]$$

$$=$$

$$\begin{bmatrix} d_{11} & n_{12} & n_{13} \\ d_{12} + d_{21} - n_{12} & d_{22} & n_{23} \\ d_{13} + d_{31} - n_{13} & d_{23} + d_{32} - n_{23} & d_{33} \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ d_{12} + d_{21} & 0 & 0 \\ d_{13} + d_{31} & d_{23} + d_{32} & 0 \end{bmatrix} + \begin{bmatrix} 0 & n_{12} & n_{13} \\ -n_{12} & 0 & n_{23} \\ -n_{13} & -n_{23} & 0 \end{bmatrix}$$

Therefore, it is technically possible to give a visage to $[N]^{-1}$. Let define:

$$\forall m < n : D_{mn} = d_{mn} + d_{nm}$$

$$\forall m = n : D_{mm} = 2 . d_{mm}$$

Usual calculations give:

$$|N| . [N]^{-1}$$

$$=$$

$$\begin{bmatrix} \frac{D_{22} . D_{33}}{4} - n_{23} . (D_{23} - n_{23}) & n_{13} . (D_{23} - n_{23}) - \frac{D_{33}}{2} . n_{12} & n_{12} . n_{23} - \frac{D_{22}}{2} . n_{13} \\ n_{23} . (D_{13} - n_{13}) - \frac{D_{33}}{2} . (D_{12} - n_{12}) & \frac{D_{11} . D_{33}}{4} - n_{13} . (D_{13} - n_{13}) & (D_{12} - n_{12}) . n_{13} - \frac{D_{11}}{2} . n_{23} \\ (D_{12} - n_{12}) . (D_{23} - n_{23}) - \frac{D_{22}}{2} . (D_{13} - n_{13}) & n_{12} . (D_{13} - n_{13}) - \frac{D_{11}}{2} . (D_{23} - n_{23}) & \frac{D_{11} . D_{22}}{4} - n_{12} . (D_{12} - n_{12}) \end{bmatrix}$$

By the way, it must be remarked that:

$$[N]$$

$$=$$

$$\frac{1}{2} . \{[N] + [N]^t\} + \frac{1}{2} . \{[N] - [N]^t\}$$

$$=$$

$$\frac{1}{2} \, . \, [Hess_{(\mathbf{a},\,0)} \Lambda(a^1,\, a^2,\, a^3)] \; - \; {}_{[J]}\Phi(n_{23},\, -n_{13},\, n_{12})$$

This remark will have some importance a little bit later in the document. It can also be noted that the ${}_{[J]}\Phi$ matrix is an anti-symmetric matrix. Because of this fact, it is a suitable representation for the matrix [X] which has been introduced at a former step in this progression.

- The equivalence between the matrix [D] and the product [T].[P] has another consequence:

$$[D]^{-1} \equiv [P]^{-1} \, . \, [T]^{-1} = [N]^{-1}$$

$$\downarrow$$

$$\{[D]^{-1}\}^t \equiv \{[P]^{-1} \, . \, [T]^{-1}\}^t = \{[T]^{-1}\}^t \, . \, \{[P]^{-1}\}^t = \{[N]^{-1}\}^t$$

Since (recall):

$$[T]^{-1} = \frac{1}{|A|} \, . \, [A]^t \, . \, [J]$$

One gets:

$$\{[T]^{-1}\}^t = \frac{1}{|A|} \, . \, [J]^t \, . \, [A]$$

And:

$$\{[D]^t\}^{-1} \equiv \frac{1}{|A|} \, . \, [J]^t \, . \, [A] \, . \, \{[P]^t\}^{-1} = \{[N]^t\}^{-1}$$

Re-introducing the [L] matrix here:

$$\{[D]^t\}^{-1} \equiv \frac{1}{|A| \, . \, |P|} \, . \, [J]^t \, . \, [L] = \{[N]^t\}^{-1}$$

Hence one may induce that:

$$[L] = |A| \, . \, |P| \, . \, [J] \, . \, \{[N]^t\}^{-1}$$

One must now calculate:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$. \begin{bmatrix} \frac{D_{22} \, . \, D_{33}}{4} - n_{23} \, . \, (D_{23} - n_{23}) & n_{13} \, . \, (D_{23} - n_{23}) - \frac{D_{33}}{2} \, . \, n_{12} & n_{12} \, . \, n_{23} - \frac{D_{22}}{2} \, . \, n_{13} \\ n_{23} \, . \, (D_{13} - n_{13}) - \frac{D_{33}}{2} \, . \, (D_{12} - n_{12}) & \frac{D_{11} \, . \, D_{33}}{4} - n_{13} \, . \, (D_{13} - n_{13}) & (D_{12} - n_{12}) \, . \, n_{13} - \frac{D_{11}}{2} \, . \, n_{23} \\ (D_{12} - n_{12}) \, . \, (D_{23} - n_{23}) - \frac{D_{22}}{2} \, . \, (D_{13} - n_{13}) & n_{12} \, . \, (D_{13} - n_{13}) - \frac{D_{11}}{2} \, . \, (D_{23} - n_{23}) & \frac{D_{11} \, . \, D_{22}}{4} - n_{12} \, . \, (D_{12} - n_{12}) \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} (D_{12} - n_{12}) \, . \, (D_{23} - n_{23}) - \frac{D_{22}}{2} \, . \, (D_{13} - n_{13}) & n_{12} \, . \, (D_{13} - n_{13}) - \frac{D_{11}}{2} \, . \, (D_{23} - n_{23}) & \frac{D_{11} \, . \, D_{22}}{4} - n_{12} \, . \, (D_{12} - n_{12}) \\ \frac{D_{22} \, . \, D_{33}}{4} - n_{23} \, . \, (D_{23} - n_{23}) & n_{13} \, . \, (D_{23} - n_{23}) - \frac{D_{33}}{2} \, . \, n_{12} & n_{12} \, . \, n_{23} - \frac{D_{22}}{2} \, . \, n_{13} \\ -n_{23} \, . \, (D_{13} - n_{13}) + \frac{D_{33}}{2} \, . \, (D_{12} - n_{12}) & -\frac{D_{11} \, . \, D_{33}}{4} + n_{13} \, . \, (D_{13} - n_{13}) & -(D_{12} - n_{12}) \, . \, n_{13} + \frac{D_{11}}{2} \, . \, n_{23} \end{bmatrix}$$

Recalling that (see above):

$$d_1 = L_{12} + L_{33}$$

$$d_2 = L_{23} - L_{11}$$

$$d_3 = -L_{31} - L_{22}$$

One gets:

$$d_1 = \frac{|A| \, . \, |P|}{|N|} \, . \, \{n_{12} \, . \, (D_{13} - n_{13}) - \frac{D_{11}}{2} \, . \, (D_{23} - n_{23}) - (D_{12} - n_{12}) \, . \, n_{13} + \frac{D_{11}}{2} \, . \, n_{23}\}$$

...

$$d_2 = \frac{|A| . |P|}{|N|} . \{n_{12} . n_{23} - \frac{D_{22}}{2} . n_{13} - (D_{12} - n_{12}) . (D_{23} - n_{23}) + \frac{D_{22}}{2} . (D_{13} - n_{13})\}$$

$$d_3 = \frac{|A| . |P|}{|N|} . \{n_{23} . (D_{13} - n_{13}) - \frac{D_{33}}{2} . (D_{12} - n_{12}) - n_{13} . (D_{23} - n_{23}) + \frac{D_{33}}{2} . n_{12}\}$$

Since:

$$[N] = [T] . [P] \Rightarrow |N| = |T| . |P|$$

And since (see above):

$$|T| = -1$$

It is evident that:

$$\frac{|A| . |P|}{|N|} = -|A|$$

This system can now be reorganized and simplified:

$$D_{11} . n_{23} - D_{12} . n_{13} + D_{13} . n_{12} = -\frac{d_1}{|A|} + \frac{1}{2} . D_{11} . D_{23}$$

$$D_{12} . n_{23} - D_{22} . n_{13} + D_{23} . n_{12} = -\frac{d_2}{|A|} + (D_{12} . D_{23} - \frac{1}{2} . D_{22} . D_{13})$$

$$D_{13} . n_{23} - D_{23} . n_{13} + D_{33} . n_{12} = -\frac{d_3}{|A|} + \frac{1}{2} . D_{33} . D_{12}$$

Remark 3.9. *A useful coincidence*

The discriminant of this system coincides with the determinant of the Hessian of the polynomial Λ when (n_{23}, -n_{13}, n_{12}) are the unknown data

$$\begin{vmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{vmatrix} = |[D] + [D]^t| = |Hess_{(\mathbf{a},0)}\Lambda(a^1, a^2, a^3)|$$

$$=$$

$$D_{11} . \{D_{22} . D_{33} - (D_{23})^2\}$$
$$- D_{12} . \{D_{12} . D_{33} - D_{13} . D_{23}\}$$
$$+ D_{13} . \{D_{12} . D_{23} - D_{13} . D_{22}\}$$

$$=$$

$$D_{11} . D_{22} . D_{33} - D_{11} . (D_{23})^2$$
$$- (D_{12})^2 . D_{33} + D_{12} . D_{13} . D_{23}$$
$$+ D_{13} . D_{12} . D_{23} - (D_{13})^2 . D_{22}$$

$$=$$

$$=$$

The system at hand can be summarized with:

$$\begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} . \left| \begin{matrix} n_{23} \\ -n_{13} \\ n_{12} \end{matrix} \right\rangle = \left| \begin{matrix} -\frac{d_1}{|A|} + \frac{1}{2} . D_{11} . D_{23} \\ -\frac{d_2}{|A|} + (D_{12} . D_{23} - \frac{1}{2} . D_{22} . D_{13}) \\ -\frac{d_3}{|A|} + \frac{1}{2} . D_{33} . D_{12} \end{matrix} \right\rangle$$

This formulation resembles that one which is resulting from the derivation of the polynomial $\Lambda(\mathbf{a})$:

$$\begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} . \left| \begin{matrix} a^1 \\ a^2 \\ a^3 \end{matrix} \right\rangle = \left| \begin{matrix} -d_1 - \frac{\partial\Lambda(a^1,a^2,a^3)}{\partial a^1} \\ -d_2 - \frac{\partial\Lambda(a^1,a^2,a^3)}{\partial a^2} \\ -d_3 - \frac{\partial\Lambda(a^1,a^2,a^3)}{\partial a^3} \end{matrix} \right\rangle$$

The important point here is the natural appearance of a classification related to the degeneracy of the Hessian. If the latter is not degenerated, prior system has a unique solution.

In former case (the Hessian is not degenerated), the polynomial $\Lambda(\mathbf{a})$ has a unique singular vector which will be denoted:

$$|_{\Lambda}\mathbf{s} >= -[Hess_{(\mathbf{a},\,0)}\Lambda(\mathbf{a})]^{-1} . |\mathbf{d}^* >$$

3.10 The logical path to get the solutions

Looking for the solution of the system, one get the temptation to envisage what will appear later to be:

Remark 3.10. *A misleading set of situations*

Let suppose that:

$$(n_{23},\, -n_{13},\, n_{12}) \;=\; \frac{1}{|A|} . ({}_{\Lambda}s^1,\, {}_{\Lambda}s^2,\, {}_{\Lambda}s^3)$$

And:

$$D_{12} \;=\; D_{23} \;=\; D_{13} \;=\; 0$$
$$|Hess_{(\mathbf{a},\,0)}\Lambda(a^1,\, a^2,\, a^3)| \;=\; D_{11} . D_{22} . D_{33} \;\neq\; 0$$

Then, the system is reduced to:

$$\frac{1}{|A|} . \begin{bmatrix} D_{11} & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix} . \left| \begin{matrix} {}_{\Lambda}s^1 \\ {}_{\Lambda}s_2 \\ {}_{\Lambda}s_3 \end{matrix} \right\rangle = \left| \begin{matrix} -\frac{d_1}{|A|} \\ -\frac{d_2}{|A|} \\ -\frac{d_3}{|A|} \end{matrix} \right\rangle$$

Whilst in general:

$$\begin{bmatrix} D_{11} & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix} . \left| \begin{matrix} {}_{\Lambda}s^1 \\ {}_{\Lambda}s^2 \\ {}_{\Lambda}s^3 \end{matrix} \right\rangle = \left| \begin{matrix} -d_1 \\ -d_2 \\ -d_3 \end{matrix} \right\rangle$$

Hence, in that case:

- The Hessian is a diagonal matrix.
- Both systems coincide.
- The matrix [N] can be now written more precisely:

$$[N]$$
$$=$$
$$\frac{1}{2} . [Hess_{(\mathbf{a},\,0)}\Lambda(\mathbf{a})] \;-\; \frac{1}{|A|} . {}_{[J]}\Phi({}_{\Lambda}\mathbf{s})$$
$$=$$
$$\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} + \frac{1}{2 . |A|} . {}_{[J]}\Phi(\begin{bmatrix} \frac{1}{d_{11}} & 0 & 0 \\ 0 & \frac{1}{d_{22}} & 0 \\ 0 & 0 & \frac{1}{d_{33}} \end{bmatrix} . \left| \begin{matrix} d_1 \\ d_2 \\ d_3 \end{matrix} \right\rangle)$$

A deeper analysis of the way of proceeding is needed before going any further. The procedure gives a first concrete formulation of the matrix [P]:

$$[P]$$
$$=$$
$$[T]^{-1} . [N]$$
$$=$$
$$\frac{1}{|A|} . [A]^t . [J] . \{\frac{1}{2} . [Hess_{(\mathbf{a},\,0)} \Lambda(\mathbf{a})] - \frac{1}{|A|} . {}_{[J]}\Phi({}_{\Lambda}\mathbf{s})\}$$
$$=$$
$$\frac{1}{|A|} . [A]^t . [J] . \{\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} + \frac{1}{2 . |A|} . {}_{[J]}\Phi(\begin{bmatrix} \frac{1}{d_{11}} & 0 & 0 \\ 0 & \frac{1}{d_{22}} & 0 \\ 0 & 0 & \frac{1}{d_{33}} \end{bmatrix} . \left| \begin{matrix} d_1 \\ d_2 \\ d_3 \end{matrix} \right\rangle)\}$$

But this formulation is only useful if the coefficients and the nature of the polynomial are known. Unfortunately, these coefficients depend not only on the deforming matrix [A] (this is not a problem since this matrix is an intrinsic ingredient of the (E) question) but also on the unknown matrix [P].

Furthermore and in general, any condition which is imposed on the coefficients generates constraints on the entries of the deforming matrices and on the unknown matrix [P] or on both.

Since one is looking for situations/solutions of the (E) question where the constraints only depend on the deforming matrix or on its inverse, one must:

- go back to the generic formulations obtained in subsection 3.3 and in subsection 3.7 and
- then exclusively transpose the constraints concerning the d_{mn} on the $[_{mn}T]$ matrices and those concerning the d_m on the $[_{m}A]$ matrices.
- After that, one must verify that the transpositions are not furnishing incoherent or impossible results.

There is yet a long way to go.

Example 3.1. *The pair (m, n) = (1, 2)*

Let transpose the constraint the $d_{mn} + d_{nm} = 0$.

$$[_{12}T]$$
$$=$$
$$\begin{bmatrix} (A^2_{13} . A^3_{22} - A^2_{12} . A^3_{23}) & (A^1_{12} . A^3_{23} - A^1_{13} . A^3_{22}) & (A^1_{13} . A^2_{22} - A^1_{12} . A^2_{23}) \\ (A^2_{13} . A^3_{21} - A^2_{11} . A^3_{23}) & (A^1_{13} . A^3_{21} - A^1_{11} . A^3_{23}) & (A^1_{11} . A^2_{23} - A^1_{13} . A^2_{21}) \\ (A^2_{12} . A^3_{21} - A^2_{11} . A^3_{22}) & (A^1_{11} . A^3_{22} - A^1_{12} . A^3_{21}) & (A^1_{12} . A^2_{21} - A^1_{11} . A^2_{22}) \end{bmatrix}$$
$$=$$
$$\begin{bmatrix} -A^2_{12} . A^3_{23} & A^1_{12} . A^3_{23} & -A^1_{12} . A^2_{23} \\ -A^2_{13} . A^3_{12} & -A^1_{13} . A^3_{12} & A^1_{13} . A^2_{12} \\ -A^2_{12} . A^3_{12} & A^1_{12} . A^3_{12} & -A^1_{12} . A^2_{12} \end{bmatrix}$$
$$[_{21}T]$$
$$=$$

$$\begin{bmatrix} (A^2_{23}\,.\,A^3_{12} - A^2_{22}\,.\,A^3_{13}) & (A^1_{22}\,.\,A^3_{13} - A^1_{23}\,.\,A^3_{12}) & (A^1_{23}\,.\,A^2_{12} - A^1_{22}\,.\,A^2_{13}) \\ (A^2_{23}\,.\,A^3_{11} - A^2_{21}\,.\,A^3_{13}) & (A^1_{23}\,.\,A^3_{11} - A^1_{21}\,.\,A^3_{13}) & (A^1_{21}\,.\,A^2_{13} - A^1_{23}\,.\,A^2_{11}) \\ (A^2_{22}\,.\,A^3_{11} - A^2_{21}\,.\,A^3_{12}) & (A^1_{21}\,.\,A^3_{12} - A^1_{22}\,.\,A^3_{11}) & (A^1_{22}\,.\,A^2_{11} - A^1_{21}\,.\,A^2_{12}) \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} A^2_{23}\,.\,A^3_{12} & -A^1_{23}\,.\,A^3_{12} & A^1_{23}\,.\,A^2_{12} \\ A^2_{12}\,.\,A^3_{13} & A^1_{12}\,.\,A^3_{13} & -A^1_{12}\,.\,A^2_{13} \\ A^2_{12}\,.\,A^3_{12} & -A^1_{12}\,.\,A^3_{12} & A^1_{12}\,.\,A^2_{12} \end{bmatrix}$$

Hence:

$$[{}_{12}T] + [{}_{21}T]$$

$$=$$

$$\begin{bmatrix} A^2_{23}\,.\,A^3_{12} - A^2_{12}\,.\,A^3_{23} & A^1_{12}\,.\,A^3_{23} - A^1_{23}\,.\,A^3_{12} & A^1_{23}\,.\,A^2_{12} - A^1_{12}\,.\,A^2_{23} \\ A^2_{12}\,.\,A^3_{13} - A^2_{13}\,.\,A^3_{12} & A^1_{12}\,.\,A^3_{13} - A^1_{13}\,.\,A^3_{12} & A^1_{13}\,.\,A^2_{12} - A^1_{12}\,.\,A^2_{13} \\ 0 & 0 & 0 \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} -a_{13} & -a_{23} & -a_{33} \\ -a_{12} & a_{22} & a_{32} \\ 0 & 0 & 0 \end{bmatrix}$$

Recall: the a_{ij} are the entries in $[A]^{-1}$, the inverse of the deforming matrix [A]; see the details of the calculations in the subsection titled *Some useful identities.*

If one decides to transpose the constraint $d_{12} + d_{21} = 0$ in writing $[{}_{12}T] + [{}_{21}T] = [0]$ whatever the matrix [P] is, then one indirectly imposes the nullity of six entries in $[A]^{-1}$ out of nine. The re-iteration of the maneuver for the vanishing sums $d_{23} + d_{32}$ and $d_{13} + d_{31}$ impose the complete vanishing of $[A]^{-1}$.

This result is incompatible with the existence of the situations which have been envisaged in remark 3.10.

3.11 Class I

Let us look for the generic solutions of the non-degenerated system in starting with:

$$|Hess_{(\mathbf{a},\,0)}\Lambda(\mathbf{a})| \neq 0$$

$$n_{23} = \frac{\begin{vmatrix} -\frac{d_1}{|A|} + \frac{1}{2}\,.\,D_{11}\,.\,D_{23} & D_{12} & D_{13} \\ -\frac{d_2}{|A|} + (D_{12}\,.\,D_{23} - \frac{1}{2}\,.\,D_{22}\,.\,D_{13}) & D_{22} & D_{23} \\ -\frac{d_3}{|A|} + \frac{1}{2}\,.\,D_{33}\,.\,D_{12} & D_{23} & D_{33} \end{vmatrix}}{|Hess_{(\mathbf{a},\,0)}\Lambda(\mathbf{a})|}$$

Hence:

$$|Hess_{(\mathbf{a},\,0)}\Lambda(\mathbf{a})|\,.\,n_{23}$$

$$=$$

$$\{-\frac{d_1}{|A|} + \frac{1}{2}\,.\,D_{11}\,.\,D_{23}\}\,.\,\{D_{22}\,.\,D_{33} - (D_{23})^2\}$$

$$-D_{12}\,.\,\{\{-\frac{d_2}{|A|} + (D_{12}\,.\,D_{23} - \frac{1}{2}\,.\,D_{22}\,.\,D_{13})\}\,.\,D_{33} - \{-\frac{d_3}{|A|} + \frac{1}{2}\,.\,D_{33}\,.\,D_{12}\}\,.\,D_{23}\}$$

$$+D_{13}\,.\,\{\{-\frac{d_2}{|A|} + (D_{12}\,.\,D_{23} - \frac{1}{2}\,.\,D_{22}\,.\,D_{13})\}\,.\,D_{23} - \{-\frac{d_3}{|A|} + \frac{1}{2}\,.\,D_{33}\,.\,D_{12}\}\,.\,D_{22}\}$$

$$=$$

$$= -\{D_{22}\,.\,D_{33}\ -\ (D_{23})^2\}\,.\,\frac{d_1}{|A|}\ +\ \frac{1}{2}\,.\,D_{11}\,.\,D_{23}\,.\,\{D_{22}\,.\,D_{33}\ -\ (D_{23})^2\}$$

$$+\{D_{12}\,.\,D_{33}-D_{13}\,.\,D_{23}\}\,.\,\frac{d_2}{|A|}-D_{12}\,.\,D_{33}\,.\,\{D_{12}\,.\,D_{23}-\frac{1}{2}\,.\,D_{22}\,.\,D_{13}\}-\frac{1}{2}\,.\,D_{13}\,.\,D_{22}\,.\,D_{33}\,.\,D_{12}$$

$$\{D_{13}\,.\,D_{22}-D_{12}\,.\,D_{23}\}\,.\,\frac{d_3}{|A|}+\frac{1}{2}\,.\,(D_{12})^2\,.\,D_{23}\,.\,D_{33}+D_{13}\,.\,D_{23}\,.\,\{D_{12}\,.\,D_{23}-\frac{1}{2}\,.\,D_{22}\,.\,D_{13}\}$$

$$=$$

$$\{(D_{23})^2-D_{22}\,.\,D_{33}\}\,.\,\frac{d_1}{|A|}+(D_{12}\,.\,D_{33}-D_{13}\,.\,D_{23})\,.\,\frac{d_2}{|A|}+(D_{13}\,.\,D_{22}-D_{12}\,.\,D_{23})\,.\,\frac{d_3}{|A|}$$

$$+\frac{1}{2}\,.\,D_{11}\,.\,D_{23}\,.\,D_{22}\,.\,D_{33}\ -\ \frac{1}{2}\,.\,D_{11}\,.\,(D_{23})^3\ -\ (D_{12})^2\,.\,D_{33}\,.\,D_{23}$$

$$+\frac{1}{2}\,.\,(D_{12})^2\,.\,D_{23}\,.\,D_{33}\ +\ D_{13}\,.\,(D_{23})^2\,.\,D_{12}\ -\ \frac{1}{2}\,.\,(D_{13})^2\,.\,D_{23}\,.\,D_{22}$$

$$=$$

$$\{(D_{23})^2-D_{22}\,.\,D_{33}\}\,.\,\frac{d_1}{|A|}+(D_{12}\,.\,D_{33}-D_{13}\,.\,D_{23})\,.\,\frac{d_2}{|A|}+(D_{13}\,.\,D_{22}-D_{12}\,.\,D_{23})\,.\,\frac{d_3}{|A|}$$

$$+D_{23}\,.\,\{\frac{1}{2}\,.\,D_{11}\,.\,D_{22}\,.\,D_{33}\ -\ \frac{1}{2}\,.\,D_{11}\,.\,(D_{23})^2\ -\ (D_{12})^2\,.\,D_{33}\}$$

$$+D_{23}\,.\,\{\frac{1}{2}\,.\,(D_{12})^2\,.\,D_{33}\ +\ D_{13}\,.\,D_{23}\,.\,D_{12}\ -\ \frac{1}{2}\,.\,(D_{13})^2\,.\,D_{22}\}$$

$$=$$

$$\{(D_{23})^2-D_{22}\,.\,D_{33}\}\,.\,\frac{d_1}{|A|}+(D_{12}\,.\,D_{33}-D_{13}\,.\,D_{23})\,.\,\frac{d_2}{|A|}+(D_{13}\,.\,D_{22}-D_{12}\,.\,D_{23})\,.\,\frac{d_3}{|A|}$$

$$+|Hess_{(\mathbf{a},\,0)}\Lambda(\mathbf{a})|\,.\,\frac{D_{23}}{2}$$

One gets the simplified but important relation involving the first component of the singular vector:

$$n_{23}\ =\ -\frac{1}{|A|}\,.\,\underbrace{\{[Hess_{(\mathbf{a},\,0)}\Lambda(\mathbf{a})]^{-1}\,.\,|\mathbf{d}^*>\}^1}_{=-_{\Lambda}s^1}\ +\ \frac{D_{23}}{2}$$

I leave the rest of the calculations to the readers:

$$n_{13}\ =\ -\frac{1}{|A|}\,.\,_{\Lambda}s^2\ +\ \frac{D_{13}}{2}$$

$$n_{12}\ =\ \frac{1}{|A|}\,.\,_{\Lambda}s^3\ +\ \frac{D_{12}}{2}$$

These calculations justify why the writings below are correct for any non-degenerated Hessian and for any non-degenerated deforming matrix:

$$[N]\ =\ \frac{1}{2}\,.\,[Hess_{(\mathbf{a},\,0)}\Lambda(\mathbf{a})]\ -\ \frac{1}{|A|}\,.\,_{[J]}\Phi(_{\Lambda}\mathbf{s})$$

And:

$$[P]\ =\ \frac{1}{|A|}\,.\,[A]^t\,.\,[J]\,.\,\{\frac{1}{2}\,.\,[Hess_{(\mathbf{a},\,0)}\Lambda(\mathbf{a})]\ -\ \frac{1}{|A|}\,.\,_{[J]}\Phi(_{\Lambda}\mathbf{s})\}$$

Within this set of solutions, the existence of a decomposition is characterized by the relation:

$$|[\mathbf{a},...]_{[A]}>=\frac{1}{|A|}\,.\,[A]^t\,.\,[J]\,.\,\{\frac{1}{2}\,.\,[Hess_{(\mathbf{a},\,0)}\Lambda(\mathbf{a})]\ -\ \frac{1}{|A|}\,.\,_{[J]}\Phi(_{\Lambda}\mathbf{s})\}\,.\,|...>\ +|\mathbf{z}>$$

...

Example 3.2. $[A] = [J]$

In that case, |A| = -1 and:

$$\mathbf{a} \wedge ... = -\{\frac{1}{2} . [Hess_{(\mathbf{a},\,0)}\Lambda(\mathbf{a})] + {}_{[J]}\Phi({}_{\Lambda}\mathbf{s})\} . |... > + |\mathbf{z} >$$

Within this theory, even a classical cross product may have a representation with a non-vanishing residual part. This affirmation sounds counter-intuitive. Or it carries an important information: the decomposition depends on a background parameter; precisely: the non-degenerated polynomial $\Lambda(\mathbf{a})$. This mathematical result makes only sense when it is involved in a physical context where one is certain of the existence of such a polynomial. Fortunately, there are many real physical situations where it happens.

3.12 Class II

The topic is explained in [b].

4 Physical applications

These items are addressed in diverse explorations concerning, e.g.: the Riemann's element of length, fields of which the gradient has a $1/\mathrm{r}^2$ dependence and their link with quadrupolar radiations, the Einstein-Rosen proposition (1935) and the Lichnerowicz-York-Bowen solutions for the initial data problem in general relativity [d], the electromagnetic duality in vacuum [e], a.s.a.

Any person which is interested by this thematic is free to counter check the validity of the intrinsic method and to apply it. Its complementary method (the extrinsic one [c]) gives the possibility to get the pairs ([P], $\mathbf{z}$).

References

5 Bibliography

5.1 My contributions

[a] Maxwell's Voids and cosmic strings; vixra.org/abs/2405.0053, 13 pages.

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[c] The so-called Extrinsic Method; viXra:2406.0127, 12 pages.

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