Construction of M-theory(**the $\boldsymbol{\mathcal{M}}^!$-theory**)

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Abstract

The main process to solve the problem of quantum and gravity is

$$\begin{tikzcd}
\text{Mordern AG (global } \mathbf{GLC}) \arrow[r, "\boldsymbol{\mathcal{M}}\to\boldsymbol{\mathcal{M}}^!", "\bigcirc\text{-localizing}"'] \arrow[d, "\text{Grothendieck's dream comes in}"'] & \text{Derived AG (local } \mathbf{GLC}) \arrow[dl] \arrow[d, "\text{quantization of gravity}"] \\
\text{Complete Einstein's dream} & \text{Analytic stack (analytic AG)} \arrow[l, "\mathbf{UFT}"']
\end{tikzcd}$$

The Langlands duality from the M-theory is a part of U-duality,the M-theory and its dynamics give us the global geometric Langlands correspondence [15] simplified by combining math and physics.

$$\mathrm{D}^{\leq 11}\text{-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \xleftrightarrow{\mathbf{F}} \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})^{\leq 11}$$

where $\check{G}$ is the Langlands dual group acting on the F-dual part of M-brane.

Generally,construction of the M-theory is based on the modern algebraic geometry with category theory.Studying the dynamics of the M-theory is based on that with higher category theory (representation of underlying space).The former gives an unification of superstring theories

$$(\text{Superstring theories}^{\text{type}}) \to_{\text{retract}} (\mathcal{S}^{\text{type}}) \in \boldsymbol{\mathcal{M}}^{\text{pre}}$$

Retracted from fibered category of superstring theories to stack of Lie groupoids.And the latter gives us a well defined stackified flow which is retracted from a non-solvable theory.Combing them,we successfully establish an experiment-free theory with left evolution

$$\boldsymbol{\mathcal{M}}\text{-flow} \simeq \bigcirc\text{-flow}, \quad \bigcirc\text{-sense} \Leftarrow \mathbf{F}$$

where $\bigcirc$ is the 2-nonexistence.In the end,derived algebraic geometry gives us further support to summarize all things and we find the $\boldsymbol{\mathcal{M}}^!$-theory is the theory of everything originally proposed by Albert Einstein.

Preview. A pre M-theory is a geometric stack written by

$$\boldsymbol{\mathcal{M}}^{\text{pre}} = (\mathrm{ETSch}_{\text{eff}}^{\text{SupGen}}(\mathscr{M}), \mathrm{P}(\mathbf{T}), \mathscr{P}, \mathbf{\Pi})$$

The M-theory $\boldsymbol{\mathcal{M}}$ is a generalized super algebraic geometric stack

$$\Delta_{\mathbf{U}} = \mathrm{P}(\mathbf{U})^{-1} : \boldsymbol{\mathcal{M}} \to \boldsymbol{\mathcal{M}}^{\text{pre,rep}}_{\text{cons.}} \vee \boldsymbol{\mathcal{M}}^{\text{pre}}_{\text{cons.}}$$

We also put things into derived algebraic geometric n-stack and analytic derived stack to achieve our final goal completing **TOE** which is a category of no definition objects $\bigcirc$**-Sense** $\simeq \mathbf{Def}^{\Leftarrow^!}\mathbf{UFT}$.

Contents

1 Introduction

1.1 In this paper

In the journey to the M-theory,we answered several questions,(1) the underlying space for a theory with dualities (2) achieving to $D+1$ spacetime (3) D-brane in math (4) dualities (T-duality etc.) in math (5) property and geometry (6) general relativity with cosmological constant (7) unseen part of universe (dark energy etc.)(8) evolution of universe (9) wave-particle duality in math (10) what is an experiment-free theory (11) quantum collapse in math (12) unification of quantum and gravity etc.The settings we used are generalized super simplicial derived algebraic and sequential.

The first part is to review works about string theory based on the two volumes of books about string theory [2][3] written by Polchinski,the text about conformal field theory [1] and and a complete derivation of [2] that is [4] by Stany,including supersymmetry based on [7] and standard super algebraic generalized geometry based on [10].Purpose is to accumulate enough understandings and intuitions.The second part is to develop a new formal mathematical theory called modern super algebraic generalized geometry based on [12] written by Olsson.The section 8 lets us exclude the verification of experiments and establish a bridge between superstring theories and modern AG.Standing on the shoulders of Witten and Grothendieck,guided by our specific philosophy above [8.1] supported by our new formalism,we are able to construct M-theory as an experiment-free theory.Then,the dynamics of M-theory is closely connected to geometric Langlands programme and we based on [17][18][19][20] by Dennis Gaitsgory.Also,we find there can be only one type of strings called étale closed string [8.22],which is the key point to achieve unification of quantum and gravity.The open string is just a homotopy weak form of closed string [12.92] and derived algebraic geometry capturing the homotopic information naturally comes in and we based on [21].After developing the above abstract objects,we want to perform analytification which based on the theory of derived stack based on [22] by Peter scholze and we put things into the analytic derived stack [14.103] in the end.

1.2 Connection about fermions and bosons

Our discussion starts with connections between fermions and bosons.The first is an equivalence of fermionic and bosonic operator on OPE see section 3.1 on 2-d conformal field theory,called bosonisation.CFT is a quantum field theory with local conformal symmetry.2-d and conformal symmetry give strong constraints on theories that give many good properties.We can easily see an example by [1.21] in [4] and (1.31) in [2],for dimension D and weyl transformation that is conformal given in [2.11] with parameter ω,the metric $g'_{ab} \to e^{\omega} g_{ab}$ and $\sqrt{g} = \sqrt{|g_{ab}(\sigma)|}$ which combine with a Ricci tensor gives $\sqrt{g'}R' = \sqrt{g}[R-2(D-1)\nabla^2\omega-(D-2)(D-1)\partial\omega\cdot\partial\omega]$.If $D=2$,we have $\delta_{\omega}[\sqrt{g'}R'] = -2\sqrt{g}\nabla^2\omega$ and the term $\chi = 1/4\int \tau d\sigma\sqrt{g}R$ that is a topological term of string's world-sheet in the action gives $\delta_\omega \chi = -1/(2\pi)\int d\tau d\sigma \sqrt{g}\nabla_a \nabla^a \omega = -1/(2\pi)\int d\tau d\sigma \sqrt{g}\nabla_a \partial^a \omega = -1/(2\pi)\int d\tau d\sigma \partial_a(\sqrt{g}\partial^a \omega)$ where we used $\sqrt{g}\nabla_a v^a = \partial_a(\sqrt{g}v^a)$ we will see it in [3.16],that is zero because it is a total derivative, and we have this conformal invariant term in 2-d.The second is an equivalence between fermions and bosons that is supersymmetry that follows from a nontrivial term $\{Q^A_\alpha, \overline{Q}_{\dot{\alpha}B}\} = 2\sigma^m_{\alpha\dot{\alpha}}P_m \delta^A{}_B$ where Q^A_α is a anticomutative operator with spinor index α and inner space index A and P_m is a 4-momentum,of supersymmetry algebra (I) in [7].We will see they are similar in [6.31],bosonisation is a trivial supersymmetry on 2-d super CFT [3.48].

1.3 world-sheet and spacetime

We discuss all things under string frame work with $\eta^{\mu\nu}$=diag(-1,+1...),and we need to understand the position on world-sheet (a, b) living in spacetime (μ, ν).World-sheet is a 2-d surface with coordinates $\sigma^a, a = 1, 2$,We have a canonical embedding from world-sheet coordinate into spacetime.

$$(\sigma^1, \sigma^2) \to X(\sigma^1, \sigma^2) \hookrightarrow X^\mu(\sigma^1, \sigma^2) = (0, ..., X^\mu, ..., 0) \in \mathbb{R}^D \qquad [1.1]$$

In this case,we also call X^μ a spacetime point.Also,we have spacetime holomorphicity $\psi, \bar{\psi}$ and world-sheet holomorphicity $\psi, \tilde{\psi}$,see below [5.44].

1.4 Represented by geometry and its topology

In this paper,we will ignore calculation of amplitude and focus on algebra and geometry to some extent,because information of world-sheet just depends on the geometry,and quantum fluctuation is just a property from the topology of the boundary of the geometry (topological QFT).That is

$$\int [dXd\psi dg]e^{-S} \cong \mathscr{G} \subset M, \qquad \text{gauge fixed by} \quad \mathscr{G}/G_{\text{diff}\times\text{Weyl}} \qquad [1.2]$$

where M is a topological space and $\mathscr{G}$ is a classical super moduli space.We can see it is lengthy [4.79] with tiny information by using analytic approach,also this approach cannot be a foundation of a non-perturbative theory,thus we use functorial approach to replace the classical analytic approach.This should be started at two specific geometry we will get from physics,the first is about super setting (SUSY),the second is about generalized setting (T-duality).

2 Conformal symmetry

We base on chapter 4 of [1] and chapter 2 of [2] and give detailed calculations.First,we do not distinguish tensor with field operator transforms like a tensor and we define the conformal dimension h is the degree of covariance which means if h is lager the object with h is more likely transform covariantly.Conformal map(transformation) is an bi(anti)holomorphic function f that maps the coordinate $x \to x^{'} = f(x)$,such that it gives a conformal transformation that is a tensor transformation with x-dependence as a rescaling $\mathcal{O}^{'}(x') = (\partial_x x^{'})^{-h}\mathcal{O}(x) = \mathcal{A}(x)\mathcal{O}(x)$.For a 2-tensor $g_{\mu\nu}(x)\, h = 2$ with conformal map $x \to x' = x + \epsilon$ where ϵ is an infinitesimal parameter,gives conformal transformation

$$\begin{aligned} g^{'}_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \\ &= \frac{\partial(x^{'}-\epsilon)^\alpha}{\partial x'^\mu}\frac{\partial(x^{'}-\epsilon)^\beta}{\partial x'^\nu} g_{\alpha\beta} \\ &= (\delta^\alpha_\mu - \partial_\mu\epsilon^\alpha)(\delta^\beta_\nu - \partial_\nu\epsilon^\beta) g_{\alpha\beta} \\ &= \delta^\alpha_\mu\delta^\beta_\nu g_{\alpha\beta} - (\partial_\mu\epsilon^\alpha\delta^\beta_\nu + \delta^\alpha_\mu\partial_\nu\epsilon^\beta)g_{\alpha\beta} + \circ(\epsilon^2) \\ &= g_{\mu\nu} - (\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) \\ &= (1 - f(x)g^{-1}_{\mu\nu})g_{\mu\nu} = A(x)g_{\mu\nu} \end{aligned} \qquad [2.1]$$

It gives $\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = f(x)g_{\mu\nu}$ that gives for $g_{\mu\nu} = \eta_{\mu\nu} = I_n$

$$\begin{aligned} \eta^{\mu\nu}(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) &= f(x)\eta^{\mu\nu}\eta_{\mu\nu} \\ 2\partial_\rho\epsilon^\rho &= f(x)D \\ f(x) &= \frac{2}{D}\partial_\rho\epsilon^\rho \end{aligned} \qquad [2.2]$$

Then,we add a ∂_ρ to $f(x)$ and by permutations of indices we get

$$\begin{aligned} -\partial_\rho\partial_\rho\epsilon_\nu - \partial_\rho\partial_\nu\epsilon_\mu &= -\eta_{\mu\nu}\partial_\rho f \\ \partial_\nu\partial_\mu\epsilon_\rho + \partial_\nu\partial_\rho\epsilon_\mu &= \eta_{\mu\rho}\partial_\nu f \\ \partial_\mu\partial_\rho\epsilon_\nu + \partial_\mu\partial_\nu\epsilon_\rho &= \eta_{\nu\rho}\partial_\mu f \end{aligned} \qquad [2.3]$$

Add them together and contract with $\eta^{\mu\nu}$,we get

$$\begin{aligned} \eta^{\mu\nu}2\partial_\mu\partial_\nu\epsilon_\rho &= \eta^{\mu\nu}(\eta_{\mu\rho}\partial_\nu f + \eta_{\nu\rho}\partial_\mu f - \eta_{\mu\nu}\partial_\rho f) \\ 2\partial^2\epsilon_\rho &= \delta^\nu_\rho\partial_\nu f + \delta^\mu_\rho\partial_\mu f - D\partial_\rho f \\ 2\partial^2\epsilon_\nu &= (2-D)\partial_\nu f \end{aligned} \qquad [2.4]$$

And we act ∂^2 to $\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu$ and change the indices we get

$$\begin{aligned} &(\partial^2 f)\eta_{\mu\nu} = \partial^2(f\eta_{\mu\nu}) = \partial^2(\partial_\mu\epsilon_\nu + \partial_\mu\epsilon_\nu) = \partial_\mu(2\partial^2\epsilon_\nu) = (2-D)\partial_\mu\partial_\nu f \\ &\partial^2 f\eta_{\mu\nu} = (2-D)\frac{1}{D}\eta_{\mu\nu}\eta^{\mu\nu}\partial_\mu\partial_\nu f = (\frac{2}{D} - 1)\partial^2 f\eta_{\mu\nu} \\ &(D-1)\partial^2 f = 0 \end{aligned} \qquad [2.5]$$

Then we put f in,we get a constraint on the transformation parameter ϵ of the conformal map $x \to x' = x + \epsilon(x)$

$$(D-1)\partial^2\partial\cdot\epsilon(x) = 0 \qquad [2.6]$$

Also this constraint classifies different conformal transformations in different dimension and we discuss in the following.

For D=1, it is trivial case which means all smooth maps are conformal maps.For $D \geq 3$,we get $\partial^2 f = \partial_\mu \partial_\nu f = 0$ from the first line of [2.5].Then,for [2.6] we get a linear differential equation

$$\begin{aligned} &\partial_\rho \partial_\nu \partial^\mu \epsilon_\mu = \partial^\mu (\partial_\rho \partial_\nu \epsilon_\mu) = \partial^\mu \left(\frac{\partial}{\partial x^\rho} \frac{\partial}{\partial x^\nu} \epsilon_\mu \right) = 0^\mu 0_\mu \\ &\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + x^\nu x^\rho \qquad c_{\mu\nu\rho} = c_{\mu\rho\nu} \end{aligned} \tag{2.7}$$

When $\epsilon_\mu = a_\mu$,it shows a translation $x^\mu \to x'^\mu = x^\mu + a^\mu$ is conformal.And we put [2.7] into $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu}$ in [2.2] and focus on the b-dependent terms,we get

$$\begin{aligned} &\partial_\mu (b_{\nu\mu} x^\mu) + \partial_\nu (b_{\mu\nu} x^\nu) = (1/2) \partial^\rho (b^\rho_\nu x^\nu \eta_{\mu\nu}) \\ &b_{\mu\nu} + b_{\nu\mu} = \frac{2}{D} b^\rho_\nu \delta^\nu_\rho \eta_{\mu\nu} = \frac{2}{D} b^\rho_\rho \eta_{\mu\nu} \end{aligned} \tag{2.8}$$

For $\mu, \nu = 0, 1$ we have $b_{01} + b_{10} = \eta_{01} = 0$ which means that $b_{\mu\nu}$ is antisymmetric it is like a rotation matrix.Thus,it shows a rigid rotation $x \to x'^\mu = R^\mu_\nu x^\nu$ for $R^\mu_\nu \in$ SU(D) is conformal.And The trace b^ρ_ρ that is a number multiplying on x shows a dilation $x^\mu \to x'^\mu = b x^\mu$ is conformal.Then,we put [2.7] into the first line of [2.4] we get

$$\begin{aligned} &2\partial_\mu \partial_\nu (c_{\rho\sigma\tau} x^\sigma x^\tau) = (\eta_{\mu\rho} \partial_\nu + \eta_{\nu\rho} \partial_\mu - \eta_{\mu\nu} \partial_\rho) \frac{2}{D} \partial_\lambda c^\lambda_{\tau\sigma} x^\tau x^\sigma \\ &c_{\rho\mu\nu} = \frac{1}{D} (\eta_{\mu\rho} c^\lambda_{\lambda\nu} + \eta_{\nu\rho} c^\lambda_{\lambda\mu} - \eta_{\mu\nu} c^\lambda_{\lambda\rho}) \\ &c_{\mu\nu\rho} = \eta_{\mu\rho} b_\nu + \eta_{\mu\nu} b_\rho - \eta_{\nu\rho} b_\mu \quad \text{for} \quad b_\mu \equiv \frac{1}{D} c^\lambda_{\lambda\mu} \end{aligned} \tag{2.9}$$

Then,we put it back we get $\epsilon^{[c]}_\mu = c_{\mu\nu\rho} x^\nu x^\rho = b_\nu x^\nu x_\mu + b_\rho x^\rho x_\mu - b_\mu x^2$.And it shows the transformation $x \to x'^\mu = x^\mu + 2(b \cdot x) x^\mu - b^\mu x^2$ which called SCT is conformal.Because ϵ is infinitesimal for doing the expansion [2.1],the coefficients we discussed above are on the infinitesimal level, it means $b^\mu \to 0$ with $\circ(b^2)$ for SCT,we can adjust to

$$\begin{aligned} x'^\mu \to x^\mu + 2(b \cdot x) x^\mu - b^\mu x^2 &\approx x^\mu + 2(b \cdot x) x^\mu - b^\mu x^2 - 2(b \cdot x) b^\mu \\ &\approx (x^\mu - b^\mu x^2)(1 + 2(b \cdot x)) \\ &\approx \frac{x^\mu - b^\mu x^2}{1 - 2(b \cdot x)} \approx \frac{x^\mu - b^\mu x^2}{1 - 2(b \cdot x) + b^2 x^2} \end{aligned} \tag{2.10}$$

We use $\approx$ to represent a *reverse Taylor expansion*,and we end with a form finite SCT transformation for b^μ is finite.Next,we use this reverse expansion trick to see where is the Weyl transformation we mentioned above.We also start with

SCT on the infinitesimal level and we ignore the index

$$\begin{aligned}
\mathcal{A}(x) &= (\partial_x x')^{-2} = (\partial_x(x + 2(b\cdot x)x - bx^2))^{-2} = (1 + 4(b\cdot x) - 2(b\cdot x))^{-2} \\
&= \left(\frac{1}{1+2(b\cdot x)}\right)^2 \approx (1 + 2(-b\cdot x) + \circ(b^2, b^3...))^2 \\
&= (1 + (-2b)\cdot x + \sum_{n=2}^{\infty}\frac{(-2b\cdot x)^n}{n!})^2 \approx (e^{-2b\cdot x})^2 \\
&= e^{2\omega(x)} \quad \text{where} \quad \omega(x) = -2b\cdot x
\end{aligned} \tag{2.11}$$

For $\omega(x)$ is a local parameter of Weyl transformation.Thus,we find the Weyl transformation is a typical form of SCT transformation that is conformal.We have finished the discussion of (1.2.19)-(1.2.21) in [2].

Now,we observe that the conformal transformations we classified above form conformal groups and we set up category $\mathcal{C}_G$ with groups of conformal invariance as the objects $\mathrm{Ob}(\mathcal{C}_G)$.By Cayley's theorem,for $U \in X$ an affine scheme

$$\mathrm{Ob}(\mathcal{C}_G) \cong \mathrm{Sym}(\mathbb{C}^D) = \{f : \mathbb{C}^D \to \mathbb{C}^D | f \in C^{\infty}_{t=(\theta - tan^{-1}(x/y))} = \lim C^{\infty}(U)\} \tag{2.12}$$

where U is open set and C^{∞} is a sheaf of $\mathbb{C}$-algebra of holomorphic functions.t is an ideal for $t \in U$.Simply speaking,conformal symmetry is a symmetry that maintaining the angles.We can see definitions and details in section 7.Above all,conformal symmetry is a natural symmetry for a theory consider general relativity and gauge field theory,and it becomes a local symmetry after gauge fixing in string theory,to see more details about global scale in QFT and local scale in string theory and their meanings on theories around 3.26 in [4].Generally speaking,the local scale invariance contained in local conformal symmetry means the string theory is effective in the whole energy scale,but the QFT is effective under a typical effective energy scale.In this case,compared to QFT (effective theory),the string theory tells us how to understand somethings but not only describing somethings (QFT just gives us descriptions of quantum world).

3 Super Virasoro algebra

3.1 Operator product expansion

Before we discuss the affine lie algebra,we want to introduce superconformal algebra [3] for a consistent string theory that is an extension of Virasoro algebra to the level of superpartners which is also an example of simple Lie algebra.By definition from (13.1) in [1],a simple Lie algebra g $(V, [,])$ is a vector space V with commutator as the binary operation $[,] : g \times g \to g$ satisfying Jacobi identity which is equivalent to say the following diagram commutes

$$
\begin{array}{ccc}
g\times g\times g & \xrightarrow{\quad a\quad} & g\times g \\
\Big\downarrow b & & \Big\downarrow c \\
g\times g & \xrightarrow{\quad d\quad} & g
\end{array}
$$

with $a = 1 \times [,], b = [,] \times 1, c = [,], d = [,] \times 1 - ([,] \times 1) \circ (1 \times f)(g \times g \times g)$ for the flipping $f : g \times g \to g \times g$.And a vector in V is a generator of the algebra.

Follow from chapter 2 om [2],we first to see how we get Virasoro algebra from the normal field operator under 2-d CFT by complex analysis.On a world-sheet (2-d region),we have for spatial direction σ^1 and time direction σ^2

$$z = \sigma^1 + i\sigma^2 \quad \bar{z} = \sigma^1 - i\sigma^2 \quad \sigma^1 = \frac{z+\bar{z}}{2} \quad \sigma^2 = \frac{z-\bar{z}}{2i} \tag{3.1}$$

By the vector transformation $\partial_{z(\bar{z})} = \partial_{z(\bar{z})}\sigma^1\partial_1 + \partial_{z(\bar{z})}\sigma^2\partial_2$,we get

$$\partial = \frac{1}{2}(\partial_1 - i\partial_2) \quad \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2) \quad \partial_1 = \partial + \bar{\partial} \quad \partial_2 = i(\partial - \bar{\partial}) \tag{3.2}$$

where $\partial_z = \partial, \partial_{\bar{z}} = \bar{\partial}$,and by coordinates transformation we have

$$d^2z = dzd\bar{z} = \left|\det\begin{pmatrix}\partial_1 z & \partial_2 z\\ \partial_1\bar{z} & \partial_2\bar{z}\end{pmatrix}\right| d\sigma^1 d\sigma^2 = \left|\det\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}\right| d\sigma^1 d\sigma^2 = 2d^2\sigma \tag{3.3}$$

Because of uncertainty principle,when two fields get closed to each other,there will be a singularity on the correlation function,we can easily see for a scaler X

$$
\begin{aligned}
0 = &\delta\langle X\rangle = \frac{\delta}{\delta X_\mu(z,\bar{z})}\left(\int \mathcal{D}X e^{-S(z,\bar{z})} X^\nu(z',\bar{z}')\right)\\
&= \langle\delta S\rangle + \langle\eta^{\mu\nu}\delta^2(z-z',\bar{z}-\bar{z}')\rangle\\
&= \langle\left(\partial\frac{\partial}{\partial(\partial X_\mu)} + \bar{\partial}\frac{\partial}{\partial(\bar{\partial}X_\mu)}\right)\frac{1}{2\pi\alpha'}\partial X^\mu\bar{\partial}X_\mu X^\nu(z',\bar{z}')\rangle\\
&+ \eta^{\mu\nu}\langle\delta^2(z-z',\bar{z}-\bar{z}')\rangle\\
&\langle[\partial\bar{\partial}X^\mu(z,\bar{z})]X^\nu(z',\bar{z}')\rangle = -\pi\alpha'\eta^{\mu\nu}\langle\delta^2(z-z',\bar{z}-\bar{z}')\rangle\\
&\partial\bar{\partial}\overbrace{X^\mu(z,\bar{z})\,X^\nu}(z',\bar{z}') = -\pi\alpha'\eta^{\mu\nu}\delta^2(z-z',\bar{z}-\bar{z}')
\end{aligned}
\tag{3.4}
$$

where $S_X(z,\bar{z}) = (1/2\pi\alpha')\int d^2z\partial X^\mu\bar{\partial}X_\mu$.And this singularity emerges in form of delta function.Under the algebraically closed field $\mathbb{C}$,we can further analyze the delta function to a form that we can directly see the coordinates dependence.We start at divergence theorem for a 2-d closed region M and let $Q = 0$

$$
\begin{aligned}
\int_M d^2\sigma(\partial_1 Q - \partial_2 F) = \int_M d^2\sigma(-\partial_2 F) &= -\int_M d^2\sigma i(\partial-\bar{\partial})(F^z + F^{\bar{z}})\\
&= -\frac{1}{2}i\int_M dzd\bar{z}(\partial F^z - \bar{\partial}F^{\bar{z}})\\
&= -\frac{1}{2}i\oint_{\partial M}(e^{i\pi/2}dzF^{\bar{z}} + e^{i\pi/2}d\bar{z}F^z)\\
&= -\frac{1}{2}i\left(i\oint_{\partial M}dzF^{\bar{z}} - i\oint_{\bar{\partial}M}d\bar{z}F^z\right)
\end{aligned}
\tag{3.5}
$$

In the third line,we put a $\pi/2$ clockwise phase rotation because after applying divergence theorem the integration direction is outwards but we need a counter-clockwise direction to perform contour integral.Then from the second and the last equation of [3.5] we get two separate parts

$$\int_M d^2z\bar{\partial}F^{\bar{z}} = \frac{1}{i}\oint_{\partial M} dzF^{\bar{z}} \quad \int_M d^2z\partial F^z = \frac{1}{i}\oint_{\bar{\partial}M} d\bar{z}F^z \qquad [3.6]$$

then,we use [3.6] to solve the function $\int dz^2\delta^2(z,\bar{z}) = 1$

$$\int_M d^2z\bar{\partial}\frac{1}{2\pi z} = \frac{1}{2\pi i}\oint_{\partial M} dz\frac{1}{z} = \int_M d^2z\partial\frac{1}{2\pi\bar{z}} = \frac{1}{2\pi i}\oint_{\bar{\partial}M} d\bar{z}\frac{1}{\bar{z}} = 1 \qquad [3.7]$$

where we use $F^{\bar{z}} = 1/2\pi z, F^z = 1/2\pi\bar{z}$ and Cauchy theorem.Then we get

$$\delta^2(z-z',\bar{z}-\bar{z}') = \frac{1}{2\pi}\bar{\partial}\frac{1}{z-z'} = \frac{1}{2\pi}\partial\frac{1}{\bar{z}-\bar{z}'} \qquad [3.8]$$

Then,we transform a Cartesian integral to 2-d complex complex case $\sigma_1 \to z$,the subtile point is two operations integration and changing variable commute

$$(\sigma_1 \to z) \circ \left(\int\right)_{\sigma_1} = \left[\int \circ(\sigma_1 \to z)\right]_z \qquad [3.9]$$

Then,we use [3.9] to calculate the following integral

$$\begin{aligned} \left(\int d\sigma_1\frac{1}{\sigma_1}\right)_{\sigma_1\to z} &= \int \frac{\partial\sigma_1}{\partial z}dz\frac{1}{z} = \int \frac{1}{2}dz\frac{1}{z} \\ \ln z &= \frac{1}{2}\int dz\frac{1}{z} \qquad \text{where} \quad z \neq 0 \\ \ln|z|^2 &= \ln z + \ln\bar{z} = \int dz\frac{1}{z} \end{aligned} \qquad [3.10]$$

Next,we put $(1/2\pi)\partial\bar{\partial}$ on the two sides and perform [3.8]

$$\frac{1}{2\pi}\partial\bar{\partial}\ln|z-z'|^2 = \int \partial dz\bar{\partial}\frac{1}{2\pi(z-z')} = \delta^2(z-z',\bar{z}-\bar{z}') \qquad [3.11]$$

Finally,we sub [3.11] in [3.4] we get

$$\begin{aligned} \partial\bar{\partial}\overbracket{X^\mu(z,\bar{z})\,X}^\nu(z',\bar{z}') &= -\pi\alpha'\eta^{\mu\nu}\frac{1}{2\pi}\partial\bar{\partial}\ln|z-z'|^2 \\ \overbracket{X^\mu(z,\bar{z})\,X}^\nu(z',\bar{z}') &= -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-z'|^2 \\ \overbracket{X^\mu(z)\,X}^\nu(z') = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(z-z') &\quad \overbracket{X^\mu(\bar{z})\,X}^\nu(\bar{z}') = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(\bar{z}-\bar{z}') \end{aligned} \qquad [3.12]$$

Now,the normal wick contraction carrying the nontrivial information of contacting of fields in the correlation function is analyzed further to see directly its coordinate-dependence in 2-d CFT,we also call this operation product expansion OPE.Because we consider supersymmetry,we need to have periodic fermions in the 2-d CFT with $\mathscr{L}_\psi(z) = (1/2\pi)\psi^\mu\bar{\partial}\psi_\mu$ of holomorphic fermion and similar $\mathscr{L}_{\tilde{\psi}}(\bar{z}) = (1/2\pi)\tilde{\psi}^\mu\partial\tilde{\psi}_\mu$ of antiholomophic fermion we will see details later.And similarly,we get

$$\begin{aligned}
&\frac{\delta\mathscr{L}_\psi}{\delta\psi_\mu}(z)\psi^\nu(z') = -\eta^{\mu\nu}\delta^2(z-z') \\
&\frac{1}{4\pi}\left[\bar{\partial}\frac{\partial}{\partial(\bar{\partial}\psi_\mu)}(-\bar{\partial}\psi_\mu\psi^\mu) - \frac{\partial}{\partial\psi_\mu}(\psi_\mu\bar{\partial}\psi^\mu)\right](z,\bar{z})\psi^\nu(z',\bar{z}') = -\frac{1}{2\pi}\bar{\partial}\frac{\eta^{\mu\nu}}{z-z'} \\
&-\frac{1}{2\pi}\bar{\partial}(\psi^\mu(z)\psi^\nu(z')) = -\frac{1}{2\pi}\bar{\partial}\frac{\eta^{\mu\nu}}{z-z'} \\
&\overbrace{\psi^\mu(z)\,\psi}^{}{}^\nu(z') = \frac{\eta^{\mu\nu}}{z-z'} \quad \text{similarly} \quad \overbrace{\tilde{\psi}^\mu(\bar{z})\,\tilde{\psi}}^{}{}^\nu(\bar{z}') = \frac{\eta^{\mu\nu}}{\bar{z}-\bar{z}'}
\end{aligned} \tag{3.13}$$

3.2 Conformal invariance

After we analyses the wick contractions in $\mathbb{C}$,we want to analyze more things in normal QFT to support further calculations in 2-d CFT.First is Noether's theorem that claims a symmetry corresponds a conserved current and we follows from chapter 2.3 in [2] with detailed calculations.In QFT,a transformation $\phi'(\sigma) = \phi(\sigma) + \rho(\sigma)\epsilon(\sigma)$ with infinitesimal parameter ϵ appears as a symmetry in the field theory means it gives a total derivative and be invariant on the level of path integral.We can easily see this by setting $\rho(\sigma) = 1$,and for $[d\phi]e^{-S[\phi]}$

$$\begin{aligned}
[d\phi']e^{-S[\phi']} &= [d\phi + \frac{\partial\epsilon(\sigma)}{\partial\phi}d\phi]e^{-S[\phi+\epsilon(\sigma)]} = [d\phi]e^{-S[\phi]-\epsilon(\sigma)\partial S[\phi]+\circ(\epsilon^2)} \\
&= [d\phi]e^{-S[\phi]}e^{-\epsilon(\sigma)\partial S[\phi]} = [d\phi]e^{S[\phi]}(1-\epsilon(\sigma)\partial S[\phi]) \\
&= [d\phi]e^{-S[\phi]} - \partial([d\phi]e^{-S[\phi]}\epsilon(\sigma)S[\phi]) \\
&= [d\phi]e^{-S[\phi]}
\end{aligned} \tag{3.14}$$

Now,if $\rho(\sigma)$ is not constant and let $S[\phi] = (1/2\pi)\int d^{d-1}\sigma d(i\sigma)\sqrt{g}j^a(\sigma)$ where we did wick rotation,for preserving the symmetry on the field theory we need

$$0 = \langle -\frac{i}{2\pi}\int d^d\sigma\epsilon(\sigma)\rho(\sigma)\partial_a(\sqrt{g}j^a(\sigma))\rangle = \langle\frac{\epsilon(\sigma)}{2\pi i}\int d^d\sigma\rho(\sigma)\sqrt{g}\nabla_a j^a(\sigma)\rangle \tag{3.15}$$

which follows from the following equation and we use result $\ln(\det M) = \mathrm{tr}(\ln M)$ for a matrix M from linear algebra.

$$\begin{aligned}
\partial_a(\sqrt{g}j^a(\sigma)) &= \sqrt{g}\partial_a j^a(\sigma) + \partial_a\sqrt{g}j^a(\sigma) = \sqrt{g}\partial_a j^a(\sigma) + \partial_a\ln(\sqrt{g})\sqrt{g}j^a(\sigma) \\
&= [\sqrt{g}\partial_a + \frac{1}{2}\sqrt{g}\partial_a(\ln|g|)]j^a(\sigma) = \sqrt{g}[\partial_a + \frac{1}{2}\mathrm{tr}\partial_a(g_{cd})]j^a(\sigma) \\
&= \sqrt{g}[\partial_a + \frac{1}{2}g^{cd}\partial_a(g_{cd})]j^a(\sigma) + 0
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{g}[\partial_a + \frac{1}{2}g^{cd}\partial_a(g_{cd})]j^a(\sigma) + \frac{1}{2}[g^{cd}\partial_c g_{ad} - g^{cd}\partial_c g_{ad}] \\
&= \sqrt{g}[\partial_a + \frac{1}{2}g^{cd}\partial_a(g_{cd})]j^a(\sigma) + \frac{1}{2}[g^{cd}\partial_c g_{ad} - g^{cd}\partial_d g_{ca}] \\
&= \sqrt{g}[\partial_a + \frac{1}{2}g^{cd}(\partial_c g_{ad} + \partial_a g_{cd} - \partial_d g_{ca}]j^a(\sigma) \\
&= \sqrt{g}(\partial_a + \Gamma^c_{ca})j^a(\sigma) = \sqrt{g}\nabla_a j^a(\sigma)
\end{aligned}
\qquad [3.16]
$$

Then we get the Noether's theorem for a conserved current j^a that is

$$\nabla_a j^a = 0 \qquad [3.17]$$

For getting an expression of conserved current,we solve the exercise 2.5 in [2]

$$
\begin{aligned}
\delta\mathscr{L} = \epsilon\partial_a\mathscr{K}^a &= \frac{\partial\mathscr{L}}{\partial\phi}\delta\phi + \frac{\partial\mathscr{L}}{\partial(\partial_a\phi)}\delta\partial_a\phi \\
&= \frac{\partial\mathscr{L}}{\partial\phi}\delta\phi + \partial_a\left(\frac{\partial\mathscr{L}}{\partial(\partial_a\phi)}\delta\phi\right) - \partial_a\frac{\partial\mathscr{L}}{\partial(\partial_a\phi)}\delta\phi \\
&= \left(\frac{\partial\mathscr{L}}{\partial\phi} - \partial_a\frac{\partial\mathscr{L}}{\partial(\partial_a\phi)}\right)\delta\phi + \partial_a\left(\frac{\partial\mathscr{L}}{\partial(\partial_a\phi)}\delta\phi\right) = \partial_a\left(\frac{\partial\mathscr{L}}{\partial(\partial_a\phi)}\delta\phi\right)
\end{aligned}
\qquad [3.18]
$$

where we chose a simple case under a unit gauge (3.3.3) in [2] which is $\hat{g}_{ab} = \delta_{ab}$ that means $\partial\hat{g} = \Gamma = 0$ and $\delta\mathscr{L} = \epsilon\partial_a\mathscr{K}^a$ with $\mathscr{L}(\phi(\sigma), \partial^a\phi(\sigma))$,we also assume vanishing of equation of motion.Then,we put [3.17] in to get j^a

$$
\begin{aligned}
\nabla_a j^a = \partial_a j^a = 0 &= \partial_a\left(\frac{\partial\mathscr{L}}{\partial(\partial_a\phi)}\epsilon^{-1}\delta\phi\right) - \partial_a\mathscr{K}^a \\
j^a &= \frac{\partial\mathscr{L}}{\partial(\partial_a\phi)}\epsilon^{-1}\delta\phi - \mathscr{K}^a
\end{aligned}
\qquad [3.19]
$$

We derived [3.15] with no insertion,if we insert a operator to the path integral $\langle\mathscr{A}\rangle$,we can recalculate it like what we did in [3.4] and set $\rho(\sigma) = 1$

$$
\begin{aligned}
&0 = \delta\langle\mathscr{A}\rangle = \delta\langle\ \rangle(\sigma)\mathscr{A}(\sigma_0) + \langle\delta\mathscr{A}\rangle \\
&\frac{\epsilon}{2\pi i}\int d^d\sigma\rho(\sigma)\sqrt{g}\nabla_a j^a(\sigma\to\sigma_0)\mathscr{A}(\sigma_0) = -\delta\mathscr{A}(\sigma_0) \\
&\nabla_a j^a(\sigma\to\sigma_0)\mathscr{A}(\sigma_0) = g^{-1/2}\delta^d(\sigma-\sigma_0)\frac{2\pi}{i\epsilon}\delta\mathscr{A}(\sigma_0) \\
&\int_M d^2\sigma\partial_a j^a(\sigma\to\sigma_0)\mathscr{A}(\sigma_0) = \frac{2\pi}{i\epsilon}\delta\mathscr{A}(\sigma_0)
\end{aligned}
\qquad [3.20]
$$

we put a delta function two sides in the second equation and integrated the third equation where we used unit gauge gave $\sqrt{\hat{g}} = 1$.Then,we analyze the final equation of [3.20] in complex coordinates which is similar to [3.5]

$$
\begin{aligned}
\int_M d^2\sigma(\partial_1 j^1 + \partial_2 j^2)\mathscr{A}(\sigma_0) &= \int_M \frac{1}{2} d^2 z[(\partial+\bar{\partial})j^1 + i(\partial-\bar{\partial})j^2](\sigma)\mathscr{A}(\sigma_0) \\
&= \frac{1}{2}\int_M d^2 z[\partial(j^1+ij^2) + \bar{\partial}(j^1-ij^2)](z,\bar{z})\mathscr{A}(\sigma_0) \\
&= i\int_{\partial M}[dz\frac{1}{2}(j^1-ij^2) - d\bar{z}\frac{1}{2}(j^1+ij^2)]\mathscr{A}(z_0,\bar{z}_0) \\
&= i\left[\int_{\partial M} dz j(z) + \int_{\bar{\partial} M} d\bar{z}\tilde{\jmath}(\bar{z})\right]\mathscr{A}(z_0,\bar{z}_0)
\end{aligned}
\tag{3.21}
$$

Then we sub [3.21] in [3.20] with $z \to z_0, \bar{z} \to \bar{z}_0$ we get

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{\partial M} dz j(z)\mathscr{A}(z_0,\bar{z}_0) + \frac{1}{2\pi i}\int_{\bar{\partial} M} d\bar{z}\tilde{\jmath}(\bar{z})\mathscr{A}(z_0,\bar{z}_0) &= \frac{1}{i\epsilon}\delta\mathscr{A}(z_0,\bar{z}_0) \\
\mathrm{Res}_{z\to z_0} j(z)\mathscr{A}(z_0,\bar{z}_0) + \overline{\mathrm{Res}}_{\bar{z}\to\bar{z}_0}\tilde{\jmath}(\bar{z})\mathscr{A}(z_0,\bar{z}_0) &= \frac{1}{i\epsilon}\delta\mathscr{A}(z_0,\bar{z}_0)
\end{aligned}
\tag{3.22}
$$

Now,we get the complex version of Ward identity [3.22].

Then,we can talk about conserved current of symmetry and see how to get conformal invariance by using Ward identity in string theory.For a string theory,we know a 1-d string sweeps through a spacetime and gives a 2-d surface called world-sheet.Thus,we have translation of the whole world-sheet in spacetime called spacetime translation and translation in the world-sheet called world-sheet translation.Spacetime translation is simple reflecting property of spacetime around the motion of strings that is $\delta X^\mu(\sigma) = \epsilon\rho(\sigma)a^\mu$ and for $S_X(\sigma_1,\sigma_2) = (1/4\pi\alpha')\int d^2\sigma\partial^a X^\mu \partial_a X_\mu$we have

$$
\delta S_X(\sigma_1,\sigma_2) = \partial_a \frac{\partial S}{\partial(\partial_a X_\mu)}\delta X_\mu = \frac{2}{4\pi\alpha'}\int d^2\sigma\partial_a\partial^a X^\mu \epsilon\rho(\sigma)a_\mu \tag{3.23}
$$

And we perform [3.15] we get current $a_\mu j_a^\mu$ of spacetime translation invariance

$$
\begin{aligned}
\frac{\epsilon}{2\pi i}\int d^2\sigma\partial^a \frac{ia_\mu}{\alpha'}\partial_a X^\mu &= \frac{\epsilon a_\mu}{2\pi i}\int d^2\sigma\partial^a j_a^\mu \\
j_a^\mu &= \frac{i}{\alpha'}\partial_a X^\mu
\end{aligned}
\tag{3.24}
$$

Specific properties of string theory reflecting on that of 2-d world-sheets or 2-d world-sheets collect specific information about string theory.In this case,we want to discuss the current of world-sheet translation invariance.First,X is a scalar field that is $h = 0$,from the tensor transformation above [2.1] we get $X'^\mu(\sigma'^a) = X^\mu(\sigma^a)$,then for the world-sheet translation $\delta\sigma^a = \epsilon v^a$

$$
\begin{aligned}
X^\mu(\sigma^a) = X'^\mu(\sigma'^a) = X'^\mu(\sigma^a + \epsilon v^a) &= X'^\mu(\sigma^a) + \partial_a X'^\mu(\sigma^a)\epsilon v^a \\
\delta X^\mu = X'^\mu(\sigma^a) - X^\mu(\sigma^a) = -\partial_a X'^\mu(\sigma^a)\epsilon v^a &= -\epsilon v^a \partial_a X^\mu(\sigma^a)
\end{aligned}
\tag{3.25}
$$

where we used a trick that is for $\delta X \to 0, \partial\delta X = \partial(X' - X) \approx 0$ that gives $\partial X' \approx \partial X$.And we know the Lagrangian is also a scaler by Lorentz invariance,so we just change X^μ to $\mathscr{L}$ in [3.25] we get $\delta\mathscr{L} = -\epsilon v^a \partial_a \mathscr{L} = \epsilon\partial_a \mathscr{K}^a$ for [3.19] that gives $\mathscr{K}^a = -v^b \delta^a_b \mathscr{L}$,then we get the current of spacetime translation invariance

$$
\begin{aligned}
j_a &= \frac{\mathscr{L}}{\partial(\partial^a X_\mu)} \epsilon^{-1} \delta X_\mu - \mathscr{K}_a \\
&= \frac{2}{4\pi\alpha'} \partial_a X^\mu (-\epsilon v^b \partial_b X_\mu) - (-v^b \delta_{ab} \frac{1}{4\pi\alpha'} \partial_c X^\mu \partial^c X_\mu) \\
&= \frac{1}{2\pi\alpha'} v^b \left[-\frac{1}{\alpha'} \left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2} \delta_{ab} \partial_c X^\mu \partial^c X_\mu \right) \right]
\end{aligned}
\qquad [3.26]
$$

And this gives us the energy-momentum tensor if we extract the differentiable or finite term by using the normal ordering defined as $\langle : XX : \rangle = \langle XX \rangle - \langle \overbrace{XX} \rangle$

$$
T_{ab} = -\frac{1}{\alpha'} : \left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2} \delta_{ab} \partial_c X^\mu \partial^c X_\mu \right) : \qquad [3.27]
$$

for $\nabla^a : j_a := (1/2\pi\alpha') \nabla^a v^b T_{ab} = 0$.We see above (5.32) in [1],the tracelessness of the energy-momentum tensor shows the corresponding symmetry invariance,that is $\nabla^a v^b T_{ab} = \partial^a v^b \delta_{ab} (\delta^{ab} T_{ab}) = v^b \delta_{ab} (\delta^{ab} \partial^a T_{ab}) = 0$ for this case which gives us two equations

$$
\delta^{ab} T_{ab} = \delta^{ab} \partial^a T_{ab} = 0 \qquad [3.28]
$$

Now,we also want to analyze this tracelessness in 2-d complex case,we need the tensor transformation of δ^{ab} for $a, b = 1, 2$ to g^{ab} for $a, b = z, \bar{z}$ with $h_{\delta^{ab}} = -2$ we have

$$
\begin{aligned}
g^{zz} = g^{\bar{z}\bar{z}} &= \frac{\partial z \partial z}{\partial\sigma^a \partial\sigma^b} \delta^{ab} = \frac{\partial z \partial z}{\partial\sigma^1 \partial\sigma^1} \delta^{11} + \frac{\partial z \partial z}{\partial\sigma^2 \partial\sigma^2} \delta^{22} = 1^2 + i^2 = 0 \\
g^{z\bar{z}} = g^{\bar{z}z} &= \frac{\partial\sigma^a \partial\sigma^b}{\partial z \partial\bar{z}} \delta^{ab} = \frac{\partial z \partial\bar{z}}{\partial\sigma^1 \partial\sigma^1} \delta^{11} + \frac{\partial z \partial\bar{z}}{\partial\sigma^2 \partial\sigma^2} \delta^{22} = 1^2 - i^2 = 2
\end{aligned}
\qquad [3.29]
$$

Thus,the first term of [3.28] gives $\delta^{ab} T_{ab} \to g^{ab} T_{ab} = g^{z\bar{z}} T_{z\bar{z}} + g^{\bar{z}z} T_{\bar{z}z} = 0$ that is $T_{z\bar{z}} = T_{\bar{z}z} = 0$.The second term gives $\delta^{ab} \partial^a T_{ab} = \delta^{ab} \delta^{ac} \partial_c T_{ab} = [\partial_a T_{aa}](\sigma) \to \partial_a(\sigma)[T_{aa}(z, \bar{z})] = \partial_1 T_{zz} + \partial_2 T_{\bar{z}\bar{z}} = (\partial + \bar{\partial}) T_{zz} + i(\partial - \bar{\partial}) T_{\bar{z}\bar{z}} = 0$ that is $\bar{\partial} T_{zz} = -\partial T_{zz}, \partial T_{\bar{z}\bar{z}} = \bar{\partial} T_{\bar{z}\bar{z}}$ and we can also let $[\partial_a T_{aa}](\sigma) \to [\partial_a T_{aa}](z, \bar{z}) = 0$ that is $\partial T_{zz} = \bar{\partial} T_{\bar{z}\bar{z}} = 0$,combing two cases we get $\bar{\partial} T_{zz} = \partial T_{\bar{z}\bar{z}} = 0$.Then,we combine above solutions of [3.28] and we get the following properties of the energy-momentum tensor in 2-d CFT

$$
\begin{aligned}
&T_{z\bar{z}} = T_{\bar{z}z} = 0 \\
&T(z) = T_{zz}, \tilde{T}(\bar{z}) = T_{\bar{z}\bar{z}}
\end{aligned}
\qquad [3.30]
$$

where $T(z)$ is purely holomorphic and $\tilde{T}(\bar{z})$ is purely anti-holomorphic.Then,we want to use a trick to analyze [3.27] in complex case

$$\begin{aligned} T_{ab}(z,\bar{z}) &= g_{ab}(\delta^{ab}T_{ab})(\sigma) = g_{ab}(g^{ab}T'_{ab})(z,\bar{z}) = T'_{ab}(z,\bar{z}) \\ &= -\frac{1}{\alpha'} : \left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2} g_{ab} \partial_c X^\mu \partial^c X_\mu\right) : \end{aligned} \quad [3.31]$$

notice that $M_{ab} = ((1/2)\delta_{ab}\delta^{ab})M_{ab} \neq (1/2)\delta_{ab}(\delta^{ab}M_{ab})$ where associativity breaks for left $\delta^{ab}-$action.And we see [3.31] directly follows from [3.30] with

$$T(z) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : \quad \tilde{T}(\bar{z}) = -\frac{1}{\alpha'} : \bar{\partial} X^\mu \bar{\partial} X_\mu : \quad [3.32]$$

Now we ignore prefactor of the current under [3.27] and change the vector v^b to the biholomorphic function $v(z)$ with i from wick rotation,we have currents

$$j(z) = iv(z)T(z) \qquad \tilde{j}(\bar{z}) = iv(z)^*\tilde{T}(\bar{z}) \quad [3.33]$$

these are currents of conformal invariance in the corresponding free scalar field Lagrangian.Then,we can apply the Ward identity [3.22] to reproduce [3.25] for conformal transformation of the field X^μ

$$\begin{aligned} \delta_v X^\mu(w,\bar{w}) &= i\epsilon \mathrm{Res}_{z\to w} iv(z)\left(-\frac{1}{\alpha'}\partial X^\nu \partial X_\nu\right)(z)X^\mu(w) + h.c. \\ &= -\epsilon v(z)\mathrm{Res}_{z\to w}\partial X^\nu(z)\delta^\mu_\nu 2\partial_z\left(\frac{-1}{\alpha'}\frac{-\alpha'}{2}\ln(z-w)\right) + h.c. \\ &= -\epsilon v(z)\frac{1}{2\pi i}\int dz \frac{1}{z-w}\Big|_{z\to w}\partial X^\mu(z) + h.c. \\ &= -\epsilon v(w)\partial X^\mu(w) - \epsilon v(w)^*\bar{\partial}X^\mu(\bar{w}) \end{aligned} \quad [3.34]$$

And this is an infinitesimal conformal transformation follows from the conformal map $f(z) = z + \epsilon v(z)$ for the definition above [2.1].

Also,because of the bijective map $v(z)$,the currents [3.33] are also bijective,in this case the equation [3.22] gives us an correspondence of a conformal transformation of a operator with OPE of energy-momentum tensor $\delta\mathscr{A} \sim T\mathcal{A}$ which means conformal invariance gives a strong constraint on $T\mathscr{A}$ OPE along this correspondence.And we want to see how this constraint reflects on $T\mathscr{A}$ OPE for a general operator $\mathscr{A}(z,\bar{z})$ and similarly for antiholomorphic part.We notice that OPE is to collect all singular terms in contractions that is

$$\overbrace{j(z)}\mathscr{A}(0,0) = iv(z)\overbrace{T(z)}\mathscr{A}(0,0) = iv(z)\sum_{n=0}^{\infty}\frac{\mathscr{A}^{(n)}}{z^{n+1}} \quad [3.35]$$

and we put this in [3.22],we get conformal transformation for general operator

$$\begin{aligned} \delta\mathscr{A}(0,0) &= -\epsilon v(z)\mathrm{Res}_{z\to 0}\sum_{n=0}^{\infty}\frac{\mathscr{A}^{(n)}}{z^{n+1}} + h.c. \\ &= \epsilon v(z)\mathrm{Res}_{z\to 0}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n!}\partial^{(n)}\frac{\mathscr{A}^{(n)}}{z} + h.c. \end{aligned}$$

$$
\begin{aligned}
&= -\epsilon\frac{1}{2\pi i}\oint_C \frac{dz}{z}\sum_{n=0}^{\infty}\frac{1}{n!}\partial^{(n)}v(z)\mathscr{A}^{(n)} + h.c. \\
&= -\epsilon\sum_{n=0}^{\infty}\frac{1}{n!}[\partial^n v(z)\mathscr{A}^{(n)}(0,0) + \bar{\partial}^n v(z)^*\tilde{\mathscr{A}}^{(n)}(0,0)]
\end{aligned}
\qquad [3.36]
$$

where $\mathscr{A}^{(n)}$ is the coefficients of $1/z^{n+1}$ in [3.35].Now,we want to study the tensor operator $\mathcal{O}'(z',\overline{z}') = (\partial_z z')^{-h}(\partial_{\overline{z}}\overline{z}')^{-\tilde{h}}\mathcal{O}(z,\overline{z})$ we defined above [2.1],for $z' = z + \epsilon v(z)$ and we focus on the holomorphic part we have

$$
\begin{aligned}
\mathcal{O}'(z') = \partial_z(z+\epsilon v(z))^{-h}\mathcal{O}(z) &= \mathcal{O}'(z+\epsilon v(z)) = \mathcal{O}'(z) + \epsilon v(z)\partial\mathcal{O}'(z) \\
(1+\epsilon\partial v(z))^{-h}\mathcal{O}(z) &= \mathcal{O}'(z) + \epsilon v(z)\partial\mathcal{O}(z) \\
\mathcal{O}(z) - h\epsilon\partial v(z)\mathcal{O}(z) &= \mathcal{O}'(z) + \epsilon v(z)\partial\mathcal{O}(z) \\
\delta\mathcal{O}(z) &= -h\epsilon\partial v(z)\mathcal{O}(z) - \epsilon v(z)\partial\mathcal{O}(z)
\end{aligned}
\qquad [3.37]
$$

compare it with [3.36],we get $\mathcal{O}^{(1)} = h\mathcal{O}, \mathcal{O}^{(0)} = \partial\mathcal{O}$ which gives the $T\mathcal{O}$ OPE

$$
\overbrace{T(z)\mathcal{O}}(0,0) = \frac{h}{z^2}\mathcal{O}(0,0) + \frac{1}{z}\partial\mathcal{O}(0,0) \qquad [3.38]
$$

and this gives us clear expression about the meaning of an operator transforms like a tensor or a tensor operator will satisfy the $T\mathcal{O}$ OPE like [3.38].The conformal invariance preserve only for tensor operator,we can see [3.64] the T_B^X is not a tensor operator and for non-tensor transformation we need to quantify the degree of breaking conformal invariance by central charge c on $1/2z^4$.

3.3 Commutator expression

In math,a Lie bracket $L_Y X = [Y, X]$ for two vector fields $X, Y \in g$ quantifies the difference in differentiation order between these two differential operators by definition and it is also a vector field $\delta_Y X \in g$ quantifies how X transforms along the vector Y.Thus,we want to analyze our commutators to reflect clearly above information and support further calculations in SCFT.First,we set this order of commutators to time ordering T in physics and we indeed have a radial time order $t = e^{\sigma^2}$ after we perform a conformal map $z = e^{-iw} = e^{\sigma^2}e^{-i\sigma^1}$ where $w = \sigma^1 + i\sigma^2$,with a good property that is State-Operator Isomorphism in 2-d CFT from 2.64 in [4].We first put a combination of states with time t_k on a eigenstate that is $j_1(t_1)j_2(t_2) - j_1(t_3)j_2(t_2)|h\rangle = T[j_1, j_2](t)|h\rangle$ for $t_3 < t_2 < t_1$ with current j_i.The z-plane is a disc,we can set the isomorphism to be a contour map $\oint_{C_k} dz_i(1/2\pi i) : j_i(t_k) \mapsto Q_i(C_j)$ that is conserved charge with $C_3 < C_2 < C_1$

$$
\begin{aligned}
&T[j_1, j_2](t)|h\rangle_{z_1\to z_2} = [j_1, j_2](t_2)|h\rangle \cong [Q_1, Q_2]\{C_2\} = T[Q_1, Q_2](t)_{z_1\to z_2} \\
&= [Q_1(C_1)Q_2(C_2) - Q_1(C_3)Q_2(C_2)]_{z_1\to z_2} = [Q_1(C_1) - Q_1(C_3)]_{z_1\to z_2}Q_2(C_2) \\
&= \frac{1}{2\pi i}\left[\oint_{C_1} j_1(t_1) - \oint_{C_3} j_1(t_3)\right]_{z_1\to z_2} dz_1 Q_2(C_2)
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\sum_{C_1} \mathrm{Res} j_1(t_1) - \sum_{C_2} \mathrm{Res} j_1(t_3) \right]_{z_1 \to z_2} Q_2(C_2) \\
&= \left(\frac{1}{2\pi i} \oint_{(C_1 - C_3) \approx 0, z_2} dz_1 \right) \overbrace{j_1(t_1) Q}_2(C_2) = \mathrm{Res}_{z \to z_2} \overbrace{j_1(t_1) Q}_2(C_2)
\end{aligned} \qquad [3.39]
$$

$z_1 \to z_2$ means we let the first operator closed to the second one on time and position to open the contaction and we performed contour deformation at the fourth line by Residue theorem.Then we end with the expressions of commutators with a contour C_2

$$
[Q_1, Q_2]\{C_2\} = \{Q_1, Q_2\}\{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \mathrm{Res}_{z_1 \to z_2} j_1(z_1) j_2(z_2) \qquad [3.40]
$$

where $Q\{C\} = (1/2\pi i) \oint_C dz j$,and we also have similar anticommutator expression for fermionic operator.Also,we have similar expression for antiholomorphic part

$$
[\tilde{Q}_1, \tilde{Q}_2]\{C_2\} = \{\tilde{Q}_1, \tilde{Q}_2\}\{C_2\} = \oint_{C_2} \frac{d\bar{z}_2}{2\pi i} \overline{\mathrm{Res}}_{\bar{z}_1 \to \bar{z}_2} \tilde{\jmath}_1(\bar{z}_1) \tilde{\jmath}_2(\bar{z}_2) \qquad [3.41]
$$

where $\tilde{Q}\{C\} = -(1/2\pi i) \oint_C d\bar{z}\tilde{\jmath}$.Then we compare [3.40] and [3.41] with [3.22] without the contour C_2 we directly get

$$
\delta_Q \mathscr{A}(z_2, \bar{z}_2) = i\epsilon [Q, \mathscr{A}(z_2, \bar{z}_2)] \qquad \delta_{\tilde{Q}} \mathscr{A}(z_2, \bar{z}_2) = i\epsilon [\tilde{Q}, \mathscr{A}(z_2, \bar{z}_2)] \qquad [3.42]
$$

with obvious Lie bracket structures.

3.4 Superconformal algebra

Virasoro algebra is a simple Lie algebra about Laurant coefficients of bosonic energy-momentum tensor,fermions come in because of consideration of supersymmetry and in this case we need to extend previous algebra to a larger one containing two types of states which is Super Virasoro algebra based on superconformal algebra.First,for an operator $\mathcal{O}(z)$ with conformal dimension h there exists a laurant expansion or series in z-plane

$$
\mathcal{O}(z) = \sum_{m=-\infty}^{\infty} \frac{\mathcal{O}_m}{z^{m+h}} \qquad \mathcal{O}_m = \oint_C \frac{dz}{2\pi i z} z^{m+h} \mathcal{O}(z) \qquad [3.43]
$$

Next,we want to get bosonic and fermionic energy momentum tensor from the world-sheet superstring action $S = S_X + S_\psi$

$$
S = \frac{1}{2\pi} \int d^2 z \mathscr{L} = \frac{1}{4\pi} \int d^2 z \left(\frac{2}{\alpha'} \partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu \right) \qquad [3.44]
$$

with $h_S = h_X = 0, h_\partial = h_{\partial X} = 1, h_{\bar{\partial}} = h_{\bar{\partial} X} = -1, h_\psi = \frac{1}{2}, h_{\tilde{\psi}} = -\frac{1}{2}$.Then,we classify $T_B, \tilde{T}_B$ as the currents of conformal symmetry with same commutative multiplication and $T_F, \tilde{T}_F$ as that of superconformal symmetry with mixed commutative multiplication in $\mathbb{C}$,we get following expressions

$$\begin{aligned}T_B(z) &= \frac{\partial \mathscr{L}}{\partial(\bar{\partial}_a X^\mu)}\epsilon^{-1}v^{-1}\delta X^\mu + \frac{\partial \mathscr{L}}{\partial(\bar{\partial}_a \psi^\mu)}\epsilon^{-1}v^{-1}\delta\psi^\mu \\ T_B(z) &= -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu\partial\psi_\mu \quad \tilde{T}_B(\bar{z}) = -\frac{1}{\alpha'}\bar{\partial} X^\mu \bar{\partial} X_\mu - \frac{1}{2}\tilde{\psi}^\mu\bar{\partial}\tilde{\psi}_\mu\end{aligned} \tag{3.45}$$

where we used conformal variance [3.34] same for fermions and bosons

$$\begin{aligned}T_F(z) &= i\sqrt{2\alpha'}\frac{\partial \mathscr{L}}{\partial(\bar{\partial}_a \psi^\mu)}\epsilon^{-1}v^{-1}\delta X^\mu \\ T_F(z) &= i(1/\alpha')^{1/2}\psi^\mu\partial X_\mu \quad \tilde{T}_F(\bar{z}) = i(1/\alpha')^{1/2}\tilde{\psi}^\mu\bar{\partial}X_\mu\end{aligned} \tag{3.46}$$

where $\partial \in V_0$ for $V = V_0 \oplus V_1$ which is a super vector space in 1.1 [10] and adding prefactor for simplicity.we can set the currents to be

$$j^\eta = \eta(z)T_F(z) \qquad \tilde{j}^\eta(\bar{z}) = \eta^*(\bar{z})\tilde{T}_F(\bar{z}) \tag{3.47}$$

with anticommutative parameter η.Then we can get the superconformal transformation from [3.22]

$$\begin{aligned}\delta_\eta X^\mu(z_0,\bar{z_0}) &= i\epsilon \mathrm{Res}_{z\to z_0}\eta(z)i(2/\alpha')^{1/2}\psi^\mu(z)\partial \overbrace{X_\mu(z)X^\mu}(z_0) + h.c. \\ &= -\epsilon(2/\alpha')^{1/2}\mathrm{Res}_{z\to z_0}\eta(z)\psi^\mu(z)(-\alpha'/2)\frac{1}{z-z_0} + h.c. \\ &= (\alpha'/2)^{1/2}\epsilon[\eta(z_0)\psi^\mu(z_0) + \eta(z_0)^*\tilde{\psi}^\mu(\bar{z_0})]\end{aligned}$$

$$\begin{aligned}\delta_\eta\psi^\mu(z_0) &= i\epsilon \mathrm{Res}_{z\to z_0}\eta(z)i(2/\alpha')^{1/2}\partial X^\mu(z)\overbrace{\psi_\mu(z)\psi^\mu}(z_0) \\ &= -\epsilon(2/\alpha')^{1/2}\mathrm{Res}_{z\to z_0}\partial X^\mu(z)\frac{1}{z-z_0} \\ &= -(2/\alpha')^{1/2}\epsilon\eta(z_0)\partial X^\mu(z_0) \\ \delta_\eta\tilde{\psi}^\mu(\bar{z_0}) &= -(2/\alpha')^{1/2}\epsilon\eta(z_0)^*\bar{\partial}X^\mu(\bar{z_0})\end{aligned} \tag{3.48}$$

Now,we get conformal transformation δ_v [3.34] and superconformal transformation δ_η [3.48].Superconformal algebra is an algebra with these two transformations which means they closed under commutation relation.For instance

$$\begin{aligned}[\delta_{\eta_1},\delta_{\eta_2}]X^\mu(z) &= \delta_{\eta_1}\delta_{\eta_2}X^\mu(z) - \delta_{\eta_2}\delta_{\eta_1}X^\mu(z) \\ &= \epsilon_2(\alpha'/2)^{1/2}\eta_2(z)\delta_{\eta_1}\psi^\mu(z) - \epsilon_1(\alpha'/2)^{1/2}\eta_2(z)\delta_{\eta_2}\psi^\mu(z) \\ &= \epsilon_1\epsilon_2[-\eta_2\eta_1 + \eta_1\eta_2](z)\partial X^\mu(z) = \epsilon[2\eta_1(z)\eta_2(z)]\partial X^\mu(z) \\ &= \delta_v X^\mu(z) \qquad \text{where} \quad v(z) = -2\eta_1(z)\eta_2(z)\end{aligned} \tag{3.49}$$

Formally,An superconformal algebra is a $\mathbb{Z}_2$ graded set $\underline{\mathrm{Hom}}(A^{sc}, A^{sc})$ we will see in [7.8],for a super vector space $A^{sc} = A_0 \oplus A_1$,the element is tuple (δ_v, δ_η) with a binary operation $[,] : A^{sc} \otimes A^{sc} \to A^{sc}$ with $\mathbb{Z}_2$-grading $(\delta_{v_1}, \delta_{\eta_1}) \otimes (\delta_{v_2}, \delta_{\eta_2}) \mapsto (([\delta_{v_1}, \delta_{v_2}], [\delta_{\eta_1}, \delta_{\eta_2}]), ([\delta_{v_1}, \delta_{\eta_2}], [\delta_{\eta_1}, \delta_{v_2}])) \in A^{sc}$,which is a simple

Lie algebra.We want to claim that supersymmetry exists in a compactified dimension we will see details later,thus we want to study the $X^\mu\psi^\mu$ SCFT on a circle.We can set following boundary condition

$$w = \sigma^1 + i\sigma^2 \cong w + 2\pi = (\sigma^1 + 2\pi) + i\sigma^2 \tag{3.50}$$

which gives periodicity 2π on spatial dimension σ^1.And this classify fermions to R and NS that induce two distinct Hilbert spaces called sectors

$$\begin{aligned} \text{Ramond (R)} : \psi^\mu(w + 2\pi) = e^{2\pi i\nu}\psi^\mu(w) \quad , \nu = 0 \\ \text{Neveu-Schwarz (NS)} : \psi^\mu(w + 2\pi) = e^{2\pi i\nu}\psi^\mu(w) \quad , \nu = \frac{1}{2} \end{aligned} \tag{3.51}$$

similar to $\tilde{\psi}^\mu(\bar{w})$ with $-\tilde{\nu}$,because of the invariance of periodicity on $S_\psi(w)$,let us set $a\psi(w) = \psi(w + 2\pi)$ and we end with $a^2 = 1$ that is $a = \pm 1$

$$\int d^2(w + 2\pi)\psi^\mu(w + 2\pi)\partial_{\bar{w}}\psi_\mu(w + 2\pi) = a^2 \int d^2w\psi^\mu(w)\partial_{\bar{w}}\psi_\mu(w) \tag{3.52}$$

For fully remaining Poincaré invariance in action,$X^\mu(w) = X^\mu(w + 2\pi)$,we can easily see if we put antiperiodicity on X^μ and for an infinitesimal parameter ϵ

$$\begin{aligned} X^\mu(\sigma^1 + 2\pi - \epsilon) &= -X^\mu(\sigma^1 + 2\pi) \\ X^\mu(\sigma^1 + 2\pi) - \epsilon\partial X^\mu(\sigma^1 + 2\pi) &= X^\mu(\sigma^1) \\ -X^\mu(\sigma^1) + \epsilon\partial X^\mu(\sigma^1) &= X^\mu(\sigma^1) \\ \epsilon\partial X^\mu(\sigma^1) &= 2X^\mu(\sigma^1) \end{aligned} \tag{3.53}$$

which gives nontrivial translation $\delta X^\mu = -2X^\mu$ in [3.25] which is not a total derivative and breaks the translation invariance.Then,we put periodicity in [3.46]

$$T_F(w + 2\pi) = e^{2\pi i\nu}T_F(w) \qquad \tilde{T}_F(\bar{w} + 2\pi) = e^{-2\pi i\tilde{\nu}}\tilde{T}_F(\bar{w}) \tag{3.54}$$

Under the periodicity condition we can expand ψ^μ in exponential Fourier series

$$\psi^\mu(w) = i^{-1/2} \sum_{r\in\mathbb{Z}+\nu} \psi^\mu_r e^{irw}, \qquad \tilde{\psi}^\mu(\bar{w}) = i^{-1/2} \sum_{r\in\mathbb{Z}+\tilde{\nu}} \tilde{\psi}^\mu_r e^{-ir\bar{w}} \tag{3.55}$$

We need to transform w to z-plane that exists vertex operators we did in 3.3.

$$\psi^\mu(z) = (\partial_w z)^{-h_\psi}\psi^\mu(w) = \sqrt{\partial_z(i\ln z)}\psi^\mu(w) = i^{1/2}z^{-1/2}\psi^\mu(w) \tag{3.56}$$

then we put [3.55] in [3.56] and get Laurent expansions correspond to [3.43]

$$\psi^\mu(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\psi^\mu_r}{z^{r+1/2}}, \qquad \tilde{\psi}^\mu(\bar{z}) = \sum_{r\in\mathbb{Z}+\tilde{\nu}} \frac{\tilde{\psi}^\mu_r}{\bar{z}^{r+1/2}} \tag{3.57}$$

And we can now understand clearly [3.43] is actually an exponential Fourier expansion equipped with a tensor transformation of w-plane to z-plane.The expression of Laurent coefficients are

$$\psi^\mu_r = \oint_C \frac{dz}{2\pi iz} z^{r+1/2}\psi^\mu(z), \qquad \tilde{\psi}^\mu_r = -\oint_C \frac{d\bar{z}}{2\pi i\bar{z}}\bar{z}^{r+1/2}\tilde{\psi}^\mu(\bar{z}) \tag{3.58}$$

Also,for vector fields $\partial X, \bar{\partial} X$,we have following Laurent expansions with prefactor about string lenth scale $\sqrt{\alpha'}$

$$\partial X^\mu(z) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m=-\infty}^{\infty} \frac{\alpha_m^\mu}{z^{m+1}}, \quad \bar{\partial} X^\mu(\bar{z}) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m=-\infty}^{\infty} \frac{\tilde{\alpha}_m^\mu}{\bar{z}^{m+1}} \tag{3.59}$$

with Laurent coefficients

$$\alpha_m^\mu = i\left(\frac{2}{\alpha'}\right)^{1/2} \oint_C \frac{dz}{2\pi i z} z^{m+1} \partial X^\mu(z), \tilde{\alpha}_m^\mu = -i\left(\frac{2}{\alpha'}\right)^{1/2} \oint_C \frac{d\bar{z}}{2\pi i \bar{z}} \bar{z}^{m+1} \bar{\partial} X^\mu(\bar{z}) \tag{3.60}$$

By using [3.40] we can get commutation relations of Laurent coefficients in the $X^\mu \psi^\mu$ SCFT

$$\begin{aligned}
\{\psi_r^\mu, \psi_s^\nu\}\{C_2\} = \{\tilde{\psi}_r^\mu, \tilde{\psi}_s^\nu\}\{C_2\} &= \oint_{C_2} \frac{dz_2}{2\pi i} \mathrm{Res}_{z_1 \to z_2} z_1^{r-1/2} z_2^{s-1/2} \psi^\mu(z_1) \psi^\nu(z_2) \\
&= \oint_{C_2} \frac{dz_2}{2\pi i z_2} z_2^{r+s} \eta^{\mu\nu} = \eta^{\mu\nu} \delta_{r,-s}
\end{aligned}$$

$$\begin{aligned}
[\alpha_m^\mu, \alpha_n^\nu]\{C_2\} &= [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu]\{C_2\} \\
&= \left(\frac{-2}{\alpha'}\right) \oint_{C_2} \frac{dz_2}{2\pi i} \mathrm{Res}_{z_1 \to z_2} z_1^m z_2^n \partial_{z_1} X^\mu(z_1) \partial_{z_2} X^\nu(z_2) \\
&= \left(\frac{2}{\alpha'}\right) \oint_{C_2} \frac{dz_2}{2\pi i} \mathrm{Res}_{z_1 \to z_2} \partial_{z_1} z_1^m z_2^n \partial_{z_2} X^\mu(z_1) X^\nu(z_2) \qquad [3.61] \\
&= \left(\frac{2}{\alpha'}\right) \oint_{C_2} \frac{dz_2}{2\pi i} \mathrm{Res}_{z_1 \to z_2} m z_1^{m-1} z_2^n \left(\frac{\alpha'}{2} \frac{\eta^{\mu\nu}}{z_1 - z_2}\right) \\
&= m \eta^{\mu\nu} \delta_{m,-n}
\end{aligned}$$

Then,we can Laurent expand our energy momentum tensors [3.45] and [3.46] with $h_{T_B} = 2, h_{T_F} = 3/2$

$$\begin{aligned}
T_B(z) &= \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}, \qquad \tilde{T}_B(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\tilde{L}_m}{\bar{z}^{m+2}} \\
T_F(z) &= \sum_{r \in \mathbb{Z}+\nu} \frac{G_r}{z^{r+3/2}}, \qquad \tilde{T}_F(\bar{z}) = \sum_{r \in \mathbb{Z}+\tilde{\nu}} \frac{\tilde{G}_r}{\bar{z}^{r+3/2}}
\end{aligned} \tag{3.62}$$

Now we focus on the holomorphic part for simplicity and the reverse expansions

$$L_m = \oint_C \frac{dz}{2\pi i z} z^{m+2} T_B(z), \qquad G_r = \oint_C \frac{dz}{2\pi i z} z^{r+3/2} T_F(z) \tag{3.63}$$

The Laurent coefficients close under commutation relation to give an algebra which is super Virasoro algebra or Ramond and Neveu-Schwarz algebra.We first calculate OPEs of energy-momentum tensors which collect singular terms of single,double and higher contractions,for $T_B = T_B^X + T_B^\psi$

$$
\begin{aligned}
&T_B^X(z)T_B^X(w) = \frac{1}{\alpha'^2}\partial_z X^\mu(z)\partial_z X_\mu(z)\partial_w X^\nu(w)\partial_w X_\nu(w) \\
&= \frac{4\times 1}{\alpha'^2}\partial_z\partial_w X^\nu X_\nu \partial_z X^\mu(z)\partial_w X_\mu(w) + \frac{2\times 1}{\alpha'^2}\partial_z\partial_w X^\mu X^\mu \partial_z\partial_w X^\nu X^\nu \\
&= \frac{4}{\alpha'^2}\frac{-\alpha'}{2}\frac{1}{(z-w)^2}[\partial_w + (z-w)\partial_w^2]X^\mu(w)\partial_w X_\mu(w) + \frac{2}{\alpha'^2}\frac{\alpha'^2\eta^{\mu\nu}\eta_{\mu\nu}}{4(z-w)^4} \qquad [3.64] \\
&= \frac{D}{2(z-w)^4} + \frac{2T_B^X(w)}{(z-w)^2} + \frac{-2}{\alpha'}\frac{1}{2}(\partial_w^2 X^\mu \partial_w X_\mu + \partial_w X^\mu \partial_w^2 X_\mu)(w) \\
&= \frac{D}{2(z-w)^4} + \frac{2T_B^X(w)}{(z-w)^2} + \frac{\partial_w T_B^X(w)}{z-w}
\end{aligned}
$$

for fermionic part we need to notice the anticommutation relation

$$
\begin{aligned}
&T_B^\psi(z)T_B^\psi(w) = \frac{1}{4}\psi^\mu(z)\partial\psi_\mu(z)\psi^\nu(w)\partial\psi_\nu(w) \\
&= \frac{1}{4}[-\psi(z)\psi(w)\partial\psi(z)\partial\psi(w) + \psi(z)\partial\psi(w)\partial\psi(z)\psi(w) + \partial\psi(z)\psi(w)\psi(z)\partial\psi(w) \\
&- \partial\psi(z)\partial\psi(w)\psi(z)\psi(w) - \psi(z)\psi(w)\partial\psi(z)\partial\psi(w) + \psi(z)\partial\psi(w)\partial\psi(z)\psi(w)] \\
&= \frac{1}{4}\Big[-\frac{1}{z-w}\partial\psi(w)\partial\psi(w) + \frac{1}{(z-w)^2}(\partial + (z-w)\partial^2)\psi(w)\psi(w) \\
&- \frac{1}{(z-w)^2}(1+(z-w)\partial)\psi(w)\partial\psi(w) + \frac{2}{(z-w)^3}(1+(z-w)\partial + \frac{(z-w)^2}{2}\partial^2) \\
&\times \psi(w)\psi(w) + \frac{2}{z-w}\frac{\eta^{\mu\nu}\eta_{\mu\nu}}{(z-w)^3} - \frac{\eta^{\mu\nu}\eta_{\mu\nu}}{(z-w)^2}\frac{1}{(z-w)^2}\Big] \\
&= \frac{1}{4}\Big[\frac{\partial\psi(w)\psi(w)}{(z-w)^2} - \frac{\psi(w)\partial\psi(w)}{(z-w)^2} + \frac{2}{(z-w)^2}\partial\psi(w)\psi(w) - \frac{2}{z-w}\partial\psi(w)\partial\psi(w) \\
&+ \frac{1}{z-w}\partial^2\psi(w)\psi(w) + \frac{1}{z-w}\partial^2\psi(w)\psi(w) + \frac{D}{(z-w)^4}\Big] \\
&= \frac{D/2}{2(z-w)^4} - \frac{1}{(z-w)^2}\psi^\mu(w)\partial\psi_\mu(w) - \frac{1}{2(z-w)}(\partial\psi^\mu\partial\psi_\mu + \psi^\mu\partial^2\psi_\mu)(w) \\
&= \frac{D/2}{2(z-w)^4} + \frac{2T_B^\psi(w)}{(z-w)^2} + \frac{\partial_w T_B^\psi(w)}{z-w}
\end{aligned}
$$

[3.65]

where $\partial\psi(z) = \partial_z\psi(z)$ and $\psi(w)\psi(w) = 0, \partial^2\psi(w)\psi(w) = -\psi(w)\partial^2\psi(w)$.Also, we used tylor expansion.From the sentence below [3.38] and the above results,we find $c_{T_B^X} = D, c_{T_B^\psi} = D/2$ so the central charge of the whole energy momentum tensor $c = c_{T_B} = 3D/2$.Then

$$
\begin{aligned}
&[L_m, L_n]\{C_2\} = \oint_{C_2} \frac{dw}{2\pi i} \mathrm{Res}_{z\to w} z^{m+1} w^{n+1} T_B(z) T_B(w) \\
&= \oint_{C_2} \frac{dw}{2\pi i} \mathrm{Res}_{z\to w} z^{m+1} w^{n+1} \Big[\frac{c}{2(z-w)^4} + \frac{2T_B}{(z-w)^2} + \frac{\partial_w T_B}{z-w} \Big] \\
&= \oint_{C_2} \frac{dw}{2\pi i} \mathrm{Res}_{z\to w} z^{m+1} w^{n+1} \left[\frac{-c}{12} \partial_z^3 - 2T_B(w)\partial_z + \partial_w T_B(w) \right] \frac{1}{z-w} \\
&= \oint_{C_2} \frac{dw}{2\pi i} \mathrm{Res}_{z\to w} \frac{w^{n+1}}{z-w} \left[\frac{c}{12} \partial_z^3 z^{m+1} + 2\partial_z z^{m+1} T_B(w) + \partial_w T_B(w) z^{m+1} \right] \\
&= \oint_{C_2} \frac{dw}{2\pi i} \left[\frac{c}{12}(m^3 - m) w^{m+n-1} + (2(m+1)T_B(w) - T_B(w)\partial_w) w^{m+n+1} \right] \\
&= \oint_{C_2} \frac{dw}{2\pi i w} (m-n) w^{m+n+2} T_B(w) + \oint_{C_2} \frac{dw}{2\pi i w} w^{m-(-n)} \frac{c}{12}(m^3 - m) \\
&= (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}
\end{aligned}
\tag{3.66}
$$

where we reformulate [3.63] and we performed Cauchy theorem.Then

$$
\begin{aligned}
&T_F(z) T_F(w) = i^2 \frac{2}{\alpha'} \psi^\mu(z) \partial_z X_\mu(z) \psi^\nu(w) \partial_w X_\nu(w) \\
&= -\frac{2}{\alpha'} [\psi^\mu(z)\psi^\nu(w) \partial_z X_\mu(z) \partial_w X_\nu(w) + \psi^\mu(z)\psi^\nu(w) \partial_z X_\mu(z) \partial_w X_\nu(w) \\
&+ \partial_z X_\mu(z) \partial_w X_\nu(w) \psi^\mu(z)\psi^\nu(w)] \\
&= -\frac{2}{\alpha'} \Big[\frac{\eta^{\mu\nu}}{z-w} \frac{-\alpha'}{2} \eta_{\mu\nu} \partial_z \partial_w \ln(z-w) + \frac{\eta^{\mu\nu}}{z-w} (\partial_w + (z-w)\partial_w^2) \\
&\times X_\mu(w) \partial_w X_\nu(w) + \frac{-\alpha'}{2} \eta_{\mu\nu} \partial_z \partial_w \ln(z-w)(1 + (z-w)\partial_w) \psi^\mu(w)\psi^\nu(w) \Big] \\
&= -\frac{2}{\alpha'} \Big[\frac{-\alpha' \eta^{\mu\nu}\eta_{\mu\nu}}{2(z-w)^3} + \frac{1}{z-w} \partial_w X^\mu(w) \partial_w X_\mu(w) + \frac{\alpha'}{2(z-w)} \psi^\mu(w) \partial \psi_\mu(w) \Big] \\
&= \frac{2c}{3(z-w)^3} + \frac{2T_B(w)}{z-w}
\end{aligned}
\tag{3.67}
$$

And the commutation relation of corresponding Laurent coefficients

$$
\begin{aligned}
&\{G_r, G_s\}\{C_2\} = \oint_{C_2} \frac{dw}{2\pi i} \mathrm{Res}_{z\to w} z^{r+1/2} w^{s+1/2} T_F(z) T_F(w) \\
&= \oint_{C_2} \frac{dw}{2\pi i} \mathrm{Res}_{z\to w} z^{r+1/2} w^{s+1/2} \Big[\frac{2c}{3(z-w)^3} + \frac{2T_B(w)}{z-w} \Big] \\
&= \oint_{C_2} \frac{dw}{2\pi i} \mathrm{Res}_{z\to w} z^{r+1/2} w^{s+1/2} \Big[\frac{2c}{3} \frac{1}{2} \partial_z^2 + 2T_B(w) \Big] \frac{1}{z-w} \\
&= \oint_{C_2} \frac{dw}{2\pi i} \mathrm{Res}_{z\to w} \frac{1}{z-w} w^{s+1/2} \left[\frac{c}{3} \partial_z^2 + 2T_B(w) \right] z^{r+1/2}
\end{aligned}
$$

$$
\begin{aligned}
&= 2\oint_{C_2} \frac{dw}{2\pi i w} w^{r+s+2} T_B(w) + \oint_{C_2} \frac{dw}{2\pi i w} \frac{c}{3}\left(r^2 - \frac{1}{4}\right) w^{r-(-s)} \\
&= 2L_{r+s} + \frac{c}{12}(4r^2 - 1)\delta_{r,-s}
\end{aligned}
\qquad [3.68]
$$

Finally,the OPE of the cross terms is

$$
\begin{aligned}
&T_B(z)T_F(w) = i\left(\frac{1}{\alpha'}\right)^{1/2}\Big[-\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu \partial \psi_\mu\Big](z)[\psi^\nu \partial X_\nu](w) \\
&= -2\times i\left(\frac{1}{\alpha'}\right)^{3/2} \partial_z X^\mu(z)\partial_w X_\nu(w)\partial_z X_\mu(z)\psi^\nu(w) - i\frac{1}{2}\left(\frac{1}{\alpha'}\right)^{1/2}\times \\
&\Big[-\psi^\mu(z)\psi^\nu(w)\partial_z\psi_\mu(z)\partial_w X_\nu(w) + \partial_z\psi_\mu(z)\psi^\nu(w)\psi^\mu(z)\partial_w X_\nu(w)\Big] \\
&= -2\times i\left(\frac{1}{\alpha'}\right)^{3/2}\partial_z\partial_w \frac{-\alpha'}{2}\delta^\mu_\nu \ln(z-w)(\partial_w + (z-w)\partial_w^2)X_\mu(w)\psi^\nu(w) \\
&- i\frac{1}{2}\left(\frac{1}{\alpha'}\right)^{1/2}\Big[-\frac{\eta^{\mu\nu}}{z-w}\partial_w\psi_\mu(w)\partial_w X_\nu(w) + \partial_z\frac{\delta^\nu_\mu}{z-w}(1+(z-w)\partial_w) \\
&\times \psi^\mu(w)\partial_w X_\nu(w)\Big] \\
&= i\left(\frac{1}{\alpha'}\right)^{1/2}\frac{1}{(z-w)^2}\partial_w X_\nu(w)\psi^\nu(w) + i\left(\frac{1}{\alpha'}\right)^{1/2}\frac{1}{z-w}\partial_w^2 X_\nu(w)\psi^\nu(w) \\
&+ i\frac{1}{2}\left(\frac{1}{\alpha'}\right)^{1/2}\frac{1}{z-w}\partial_w\psi^\nu\partial_w X_\nu(w) + \frac{1}{2}i\left(\frac{1}{\alpha'}\right)^{1/2}\frac{1}{(z-w)^2}\psi^\nu(w)\partial_w X_\nu(w) \\
&+ \frac{1}{2}i\left(\frac{1}{\alpha'}\right)^{1/2}\frac{1}{z-w}\partial_w\psi^\nu(w)\partial_w X_\nu(w) \\
&= \frac{3}{2(z-w)^2}T_F(w) + \frac{1}{z-w}\partial_w T_F(w)
\end{aligned}
\qquad [3.69]
$$

and the commutation relation of corresponding Laurent coefficients

$$
\begin{aligned}
&[L_m, G_r]\{C_2\} = \oint_{C_2}\frac{dw}{2\pi i}\mathrm{Res}_{z\to w} z^{m+1} w^{r+1/2} T_B(z)T_F(w) \\
&= \oint_{C_2}\frac{dw}{2\pi i}\mathrm{Res}_{z\to w} z^{m+1} w^{r+1/2}\Big[\frac{3}{2(z-w)^2}T_F(w) + \frac{1}{z-w}\partial_w T_F(w)\Big] \\
&= \oint_{C_2}\frac{dw}{2\pi i}\mathrm{Res}_{z\to w} z^{m+1} w^{r+1/2}\Big[-\frac{3T_F(w)}{2}\partial_z + \partial_w T_F(w)\Big]\frac{1}{z-w} \\
&= \oint_{C_2}\frac{dw}{2\pi i}\mathrm{Res}_{z\to w} w^{r+1/2}\frac{1}{z-w}\Big[\frac{3T_F(w)}{2}\partial_z + \partial_w T_F(w)\Big] z^{m+1} \\
&= \oint_{C_2}\frac{dw}{2\pi i}\Big[(m+1)w^{m+r+1/2}\frac{3T_F(w)}{2} - \partial_w w^{m+r+3/2} T_F(w)\Big] \\
&= \oint_{C_2}\frac{dw}{2\pi i w}\frac{m-2r}{2}T_F(w) w^{m+r+3/2} = \frac{m-2r}{2}G_{m+r}
\end{aligned}
\qquad [3.70]
$$

Now,we finish the commutation relations of the Laurent coefficients of energy-momentum tensor of $X^\mu\psi^\mu$ SCFT and they indeed close.For integer r,s the algebra is called Ramond algebra and for half-integer r,s the algebra is called Neveu-Schwarz algebra.Then we want to verify the Jacobi identity $[X,[Y,Z]]+[Z,[X,Y]]+[Y,[Z,X]]=0$ for $X,Y,Z\in g$.For parity $|X|=|Y|=|Z|=0\in\mathbb{Z}_2$

$$\begin{aligned}
0 &= [L_m,[L_n,L_p]]+[L_p,[L_m,L_n]]+[L_n,[L_p,L_n]]\\
&= (n-p)[L_m,L_{n+p}]+(m-n)[L_p,L_{m+n}]+(p-m)[L_n,L_{p+m}]\\
&= [(n-p)(m-n-p)+(m-n)(p-m-n)+(p-m)(n-p-m)]L_{m+n+p}\\
&+\frac{c}{12}[(n-p)(m^3-m)+(m-n)(p^3-p)+(p-m)(n^3+n)]\delta_{m,-(n+p)}\\
&= \frac{c}{12}[(n-p)(-(n+p)^3+n+p)+(-2n-p)(p^3-p)+(2p+n)(n^3+n)]
\end{aligned} \tag{3.71}$$

Thus,L_m individually forms a algebra and gives a Lie algebra structure called Virasoro algebra.Also,if we let $v=1/2$ for antiperiodic fermions that gives half integer r,s in [3.63] and gives a form that the T_F transform like T_B.

To see the last point above clearly,we want to perform only supersymmetry transformation on our theory for avoiding the holomorphic parameter in [3.48].Actually,we want to find a low energy effective field theory (LEE) of corresponding string theory [3.44] which will be in a similar form of (3.11-3.12) in [7] .Also,we open the interaction of strings for completeness.And we based on the chapter 3 in [2] and chapter 10 in [3] in the following.

We will see opening the interaction in string theory corresponds a perturbation in path integral of a curved spacetime metric.The way to put interacting objects on world-sheet in w-plane is to put vertex operators in z-plane.So,we find the vertex operators for bosons and fermions,we used [3.58] and [3.60]

$$\begin{aligned}
&\alpha^\mu_{-m}=i\left(\frac{2}{\alpha'}\right)^{1/2}\oint_C\frac{dz}{2\pi iz}z^{-(m-1)}\partial X^\mu(z)=i\left(\frac{2}{\alpha'}\right)^{1/2}\partial^m X^\mu(0)\\
&-i\partial|0;k\rangle=k|0;k\rangle\Rightarrow|0;k\rangle\equiv\mathbb{1}=:e^{ik\cdot X(0,0)}:
\end{aligned} \tag{3.72}$$

where $m\geq 0$ and k is world-sheet momentum.For fermionic states,we need first analyze the spectrum of NS and R.NS ground state where $r=\mathbb{Z}+v, v=1/2$ is

$$\psi^\mu_r|0\rangle_{\mathrm{NS}}=0,\qquad r>0 \tag{3.73}$$

For R ground state,ψ^μ_o forms a Clifford algebra,we can set $\Gamma^\mu\cong 2^{1/2}\psi^\mu_0$

$$\{\Gamma^\mu,\Gamma^\nu\}=\{\sqrt{2}\psi^\mu_0,\sqrt{2}\psi^\nu_0\}=2\{\psi^\mu_0,\psi^\nu_0\}=2\eta^{\mu\nu}I \tag{3.74}$$

$\psi^\mu_r\psi^\nu_0|0\rangle_{\mathrm{R}}=(\{\psi^\mu_r,\psi^\nu_0\}-\psi^\nu_0\psi^\mu_r)|0\rangle_{\mathrm{R}}=0$ for $r>0$ means $\psi^\mu_0\times:V_{R_0}\to V_{R_0}$ which gives us a representation of Clifford algebra $\Gamma\times V_{R_0}\to V_{R_0}$.We want to give details of the spin representation in various spacetime dimension $\mu=0...d$.First we need to know several concepts,we represent elements of Clifford algebra by linear transformations that are Dirac matrices of corresponding vector space and a spinor is an object that transforms under the corresponding spin representation.For even dimension $d = 2k + 2$ we can form linear combinations

$$\Gamma^{0\pm} = \frac{1}{2}(\pm\Gamma^0 + \Gamma^1), \quad \Gamma^{a\pm} = \frac{1}{2}(\Gamma^{2a} \pm i\Gamma^{2a+1}), \quad a = 1, ...k \qquad [3.75]$$

The only nontrivial commutation relation is

$$\{\Gamma^{a+}, \Gamma^{b-}\} = \frac{1}{4}\left(\{\Gamma^{2a}, \Gamma^{2b}\} + \{\Gamma^{2a+1}, \Gamma^{2b+1}\}\right) = \delta^{ab} \qquad [3.76]$$

Also we can find the property

$$(\Gamma^{0-})^2 = (\Gamma^{a+})^2 = (\Gamma^{a-})^2 = \frac{1}{4}[(\Gamma^{2a})^2 - (\Gamma^{2a+1})^2 + i\{\Gamma^{2a}, \Gamma^{2a+1}\}] = 0 \quad [3.77]$$

We can form a matrix $\zeta = \prod_{a=0}^{k+1} \Gamma^{a-}$ up to constant,then

$$\Gamma^{a-}\zeta = 0, \qquad \text{for all}\, a = 0, ...1 \qquad [3.78]$$

makes ζ to be a ground state spinor and Γ^{a-} to be annihilation operator which gives a representation ρ for $\zeta \in V_\zeta$ satisfy [4.7] for $s_a = \pm 1/2, \mathbf{s} \equiv (s_0, ...s_k)$

$$\mathrm{Mat}_{\mathrm{Dirac}} \times V_\zeta \to V_\zeta, \quad (\Gamma^{k+})^{s_k+1/2}..(\Gamma^{0+})^{s_0+1/2} \times \zeta \mapsto \zeta^{(\mathbf{s})} \in V_\zeta \qquad [3.79]$$

with the Dirac representation $\rho : g_{SO(d)} \to \mathrm{End}(V_\zeta) = \mathrm{Mat}_{\mathrm{Dirac}}$ that is group of gamma matrices,then we get $\dim\rho(g_{SO(d)}) = \dim(\mathrm{Mat}_{\mathrm{Dirac}}) = 2^{k+1}$.We can view $\zeta^{(\mathbf{s})}$ as generators of representation of Clifford algebra [3.74].For seeing more connections,we want to derive supersymmetry algebra in the following section based on the text [7] and show that the representation we got above is actually isomorphic to a representation of supersymmetry algebra by regarding [3.75] as a compactification of dimension $2a$ and $2a + 1$ which follows from normalization which means the radial lenth is 1 and we have a compact space.

4 Supersymmetry algebra

4.1 Representation of Lorentz group SO(1,3)

We start at the representation of SO(4) which we start from $U(n)$ with $\dim(U(n)) = n^2$ for $n \times n$ unitary matrices.And for $U \in U(n), U^{-1} = U^*$

$$|\mathrm{det}U|^2 = \mathrm{det}U(\mathrm{det}U)^* = \mathrm{det}U\mathrm{det}U(U^*) = \mathrm{det}U\mathrm{det}(U^{-1}) = \mathrm{det}U/\mathrm{det}(U) = 1 \qquad [4.1]$$

Thus,we can write $\mathrm{det}U = e^{i\theta} = 1$ for $SU(n)$,this equation fixes the polar coordinates and total degree of freedom loose one which means $\dim(\mathrm{SU}(n)) = n^2 - 1$.Then we get $\dim(SU(2)) = 2^2 - 1 = 3$.In this case we know the Lie group $SU(2)$ is a manifold unit S^3.On the S^3,we have symmetry group $SO(4)$, and elements are rotations that can connect two points on the three sphere. Thus, we get a natural map $SU(2) \times SU(2) \to SO(4), (p_1, p_2) \mapsto M$ which sends a pair of points on sphere to a rotational transformation,which is not injective.For $\mathbb{Z}_2$ symmetry which sends (p_1, p_2) to $(-p_1, -p_2)$, we have short exact sequence $0 \to \mathbb{Z}_2 \to SU(2) \times SU(2) \to SO(4) \to 0$.But the surjection lets us claim that their Lie algebras are isomorphism, $g_{SO(4)} \cong g_{SU(2)} \oplus g_{SU(2)}$.To see this we follow [5], we can originally choose J_i, K_i for $SO(4)$ with indices $i, j = 1, 2, 3$, totally six dimensions for 3 rotations and 3 translations.

$$[J_i, J_j] = i\sum_{k=1}^{3} \epsilon_{ijk} J_k, [J_i, K_j] = i\sum_{k=1}^{3} \epsilon_{ijk} K_k, [K_i, K_j] = i\sum_{k=1}^{3} \epsilon_{ijk} J_k \qquad [4.2]$$

Based on these,we can form a linear combination $J_{\pm,i} = \frac{1}{2}(J_i \pm K_i)$,with

$$\begin{aligned}
&[J_{+,i}, J_{-,j}] = [\frac{1}{2}(J_i + K_i), \frac{1}{2}(J_i - K_i)] = \frac{1}{4}\left\{[J_i, (J_i - K_j))] + [K_i, (J_i - K_j)]\right\} \\
&= \frac{1}{4}\left\{[J_i, J_j] - [J_i, K_j] + [K_i, J_j] - [K_i, K_j]\right\} \\
&= \frac{1}{4}\left\{[J_i, J_j] + [J_j, K_i] + [K_i, J_j] - [K_i, K_j]\right\} \\
&= \frac{1}{4}\left\{i\sum_{k=1}^{3} \epsilon_{ijk} J_k + [J_j, K_i] - [J_j, K_i] - i\sum_{k=1}^{3} \epsilon_{ijk} J_k\right\} = 0 \\
&[J_{+,i}, J_{+,j}] = [\frac{1}{2}(J_i + K_i), \frac{1}{2}(J_j + K_j)] = \frac{1}{4}\left\{[J_i, (J_j + K_j)] + [K_i, (J_j + K_j)]\right\} \\
&= \frac{1}{4}\left\{[J_i, J_j] + [J_i, K_j] + [K_i, J_j] + [K_i, K_j]\right\} \\
&= \frac{1}{4}\left\{2[J_i, J_j] + [J_i, K_j] - [K_j, J_i]\right\} = \frac{1}{4}2\left\{[J_i, J_j] + [J_i, K_j]\right\} \\
&= \frac{1}{2}\left\{i\sum_{k=1}^{3} \epsilon_{ijk} J_k + i\sum_{k=1}^{3} \epsilon_{ijk} K_k\right\} = \frac{1}{2}i\sum_{k=1}^{3} \epsilon_{ijk}[J_k + K_k] = i\sum_{k=1}^{3} \epsilon_{ijk} J_{+,k} \\
&[J_{-,i}, J_{-,j}] = [\frac{1}{2}(J_i - K_i), \frac{1}{2}(J_j - K_j)] \\
&= \frac{1}{4}\left\{[J_i, (J_j - K_j)] + [-K_i, (J_j - K_j)]\right\} \\
&= \frac{1}{4}\left\{[J_i, J_j] - [J_i, K_j] - [K_i, J_j] + [K_i, K_j]\right\} \\
&= \frac{1}{4}\left\{2[J_i, J_j] - [J_i, K_j] + [K_J, J_i]\right\} = \frac{1}{4}\left\{2[J_i, J_j] - [J_i, K_j] + [J_i, K_j]\right\} \\
&= \frac{1}{4}2\left\{[J_i, J_j] - [J_i, K_j]\right\} = \frac{1}{2}i\sum_{k=1}^{3} \epsilon_{ijk}(J_k - K_k) = i\sum_{k=1}^{3} \epsilon_{ijk} J_{-,k}
\end{aligned}$$

$$[4.3]$$

Again,Lie bracket describes the differences in the order of differentiation between two differential operators on manifolds.Translate it to physics,the commutator describes if there is a contact term when two fields are closed to each other.In this case,the 2nd and 3rd commutator in [4.3] give closure of Lie algebra with dimension 3 separately,so $J_{+,i}, J_{-,i}$give two copies of $g_{SU(2)}$,and the 1st commutator means there is no contact between these copies that means they are individual to each other.Thus,$g_{SO(4)}$indeed splits and is isomorphic to $g_{SU(2)} \oplus g_{SU(2)}$.

And one copy has irreducible representation indexed by j and have dimension $(2j+1)$.Recall that we have a 2-1 map $SU(2)\times SU(2) \to SO(4)$, thus we need to know what type of representation does the $SU(2) \times SU(2)$ descent to.Theorem 5.7.4 in [5] told us $SU(2) \times SU(2)$ is indexed by non negative half-integer j_1, j_2 have dimension $(2j_1+1)(2J_2+1)$ when $J_1 + J_2$ is integer it descents to ordinary representation,otherwise it descents to spin representation.Thus we have lowest spin representation that is $(j_1, j_2) = (1/2, 0)$ or $(0, 1/2)$.And these give us two objects that transformed under the spin representation called spinors.

For $SO(1,3)$,there is a Wick rotation $K_i \to iK_i$ $[K_i, K_j] = -i\sum_{k=1}^{3} \epsilon_{ijk} J_k$ in [4.2],so their Lie algebras are same over $\mathbb{C}$.We use the above isomorphism below, and use $\mathbb{C}$ denote complexification map.

$$\begin{aligned} \mathbb{C}g_{SO(1,3)} &\cong \mathbb{C}g_{SO(4)} \cong \mathbb{C}g_{SU(2)} \oplus \mathbb{C}g_{SU(2)} \\ &\cong g_{SL(2,\mathbb{C})} \oplus g_{SL(2,\mathbb{C})} \cong g_{SL(2,\mathbb{C})} \oplus i g_{SL(2,\mathbb{C})} \\ &\cong \mathbb{C}g_{SL(2,\mathbb{C})} \end{aligned} \qquad [4.4]$$

we have used $\mathbb{C}g_{SU(2)} \cong g_{SL(2,\mathbb{C})}$ and perform contour rotation again.Then after restriction to real, we get $g_{SL(2,\mathbb{C})} \cong g_{SO(1,3)}$, in this case we find the spin group of Lorentz group that is $SL(2,\mathbb{C}) \to SO(1,3)$ and along the map the representation of SO(1,3) descents from the representation of $SL(2,\mathbb{C})$,which is same as that of SO(4).

$$\begin{array}{ccc} SL(2,\mathbb{C}) \times V_1 & \longrightarrow & V_1 \\ \downarrow & & \downarrow \\ SO(1,3) \times V_2 & \longrightarrow & V_2 \end{array}$$

4.2 Spinors and Pauli matrices

Because of above,we know that the spinors of $SO(1,3)$ can be viewed as objects transformed under representation of $SL(2,\mathbb{C})$.We focus on the lowest spin representation $(1/2, 0)(0, 1/2)$ and each has dimension 2.Thus,we get [7](A.1) for $M \in SL(2,\mathbb{C})$

$$\begin{aligned} \psi'_\alpha &= {M_\alpha}^\beta \psi_\beta & \qquad \overline{\psi}'_{\dot\alpha} &= {M^*}_{\dot\alpha}{}^{\dot\beta} \overline{\psi}_{\dot\beta} \\ \psi'^\alpha &= M^{-1\alpha}_{\beta} \psi^\beta & \qquad \overline{\psi}'^{\dot\alpha} &= (M^*)^{-1\dot\alpha}_{\dot\beta} \overline{\psi}^{\dot\beta} \end{aligned} \qquad [4.5]$$

The spinors with undotted indices transform under the (1/2,0), and with dotted indices transform under the (0,1/2),and the dimension of the spinor is equal to the dimension of corresponding representation is 2 here that explain the two-components indices $\alpha, \beta = 1, 2$.And the transformation matrices need to be

Hermitian,that can be Pauli matrices with $(\sigma^m)^2 = I$

$$\begin{aligned} \sigma^0 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{4.6}$$

4.3 Derivation of SUSY algebra

We can construct two operator Q^A_α and $\overline{Q}_{\dot{\alpha}B}$,A,B are indices for internal space.The first indexed by undotted 2-components index transforms under (1/2,0), and the second indexed by dotted 2-components index transforms under (0,1/2), so the product of these two operator transform under $(1/2, 0) \oplus (0, 1/2) = (1/2, 1/2)$.As we see above, (1/2,1/2) has dimension 2×2, thus we conclude that $Q^A_\alpha \overline{Q}_{\dot{\alpha}B} \propto P$ with P a 2×2 matrix which has dimension 4.Then we need to find the expression of P and this is the case we need to use Pauli matrices.We consider a combination $\sigma^m P_m$ with P_m a 4-vector

$$\begin{aligned} \sigma^m P_m &= \sigma^0 P_0 + \sigma^1 P_1 + \sigma^2 P_2 + \sigma^3 P_3 \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} P_0 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P_1 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} P_2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P_3 \\ &= \begin{pmatrix} -P_0 + P_3 & P_1 - iP_2 \\ P_1 + iP_2 & -P_0 - P_3 \end{pmatrix} \end{aligned} \tag{4.7}$$

In this case,we find properties of the combination.Firstly,$\sigma^m P_m$is an 2×2 complex matrix.Secondly,$(\sigma^m P_m)_{12} = (\sigma^m P_m)^*_{21}$, this gives $\sigma^m P_m = (\sigma^m P_m)^\dagger$ which shows it is Hermitian and by any choice of real P_m we can express any Hermitian matrix in the form $\sigma^m P_m$.Now,we can guess $Q^A_\alpha \overline{Q}_{\dot{\alpha}B} \stackrel{?}{=} \sigma^m P_m C^A{}_B$.Our aim is to express the anticommutator $\{Q^A_\alpha, \overline{Q}_{\dot{\alpha}B}\}$,we only have two questions left,the first one is does $\sigma^m P_m$ has same spinor indices as the product?and the second one is if the anticommutator is hermitian or not.For the first one,we know given a 2×2 hermitian matrix we can obtain others by $SL(2, \mathbb{C})$,we let $P = \sigma^m P_m$, and $M \in SL(2, \mathbb{C})$, this is $P' = MPM^\dagger$,then we use [4.5] we get

$$\begin{aligned} P' = \sigma^m P'_m &= MPM^\dagger = M\sigma^m P_m M^\dagger = M^\alpha_\beta \sigma^m P_m (M^\alpha_\beta)^\dagger \\ &= M^\alpha_\beta (\sigma^m P_m)^\dagger (M^\alpha_\beta)^\dagger = M^\alpha_\beta [M^\alpha_\beta (\sigma^m P_m)]^\dagger \end{aligned} \tag{4.8}$$

We can regard [4.8] as a test equation,if we plug the spinor indices that are same as that of $\{Q^A_\alpha, \overline{Q}_{\dot{\alpha}B}\}$,the equation maintains,then we can conclude that $\sigma^m P_m$ indeed agree on spinor indices.That is we do $\sigma^m \to \sigma^m_{\alpha\dot{\alpha}}$,then from [4.8] we get for the left hand side $\sigma^m_{\alpha\dot{\alpha}} P'_m$ and for the right hand side

$$\begin{aligned} M^\alpha_\beta [M^\alpha_\beta (\sigma^m_{\alpha\dot{\alpha}} P_m)]^\dagger &= M^\alpha_\beta (\sigma^m_{\beta\dot{\alpha}} P_m)^\dagger = M^\alpha_\beta (P_m)^\dagger (\sigma^m_{\beta\dot{\alpha}})^\dagger = M^\alpha_\beta P_m (\sigma^m_{\beta\dot{\alpha}})^\dagger \\ &= P'_m M^\alpha_\beta (\sigma^m_{\beta\dot{\alpha}})^\dagger = P'_m M^\alpha_\beta (\sigma^m_{\dot{\beta}\alpha})^{\mathrm{T}} = P'_m M^\alpha_\beta \sigma^m_{\alpha\dot{\beta}} \\ &= P'_m \sigma^m_{\beta\dot{\beta}} = (P'_m)^\dagger (\sigma^m_{\beta\dot{\beta}})^\dagger = [(\sigma^m_{\beta\dot{\beta}}) P'_m]^\dagger = \sigma^m_{\beta\dot{\beta}} P'_m \end{aligned} \tag{4.9}$$

Then,by changing index β to α,the right hand side indeed agree with the left hand side.Therefore,we conclude that σ^m in P can be equipped with lower indices $\sigma^m_{\alpha\dot{\alpha}}$.Thus,we can express $Q^A_\alpha\overline{Q}_{\dot{\alpha}B} = \sigma^m_{\alpha\dot{\alpha}}P_m C^A{}_B$.In this case,we get

$$\begin{aligned}\{Q^A_\alpha, \overline{Q}_{\dot{\alpha}B}\} &= Q^A_\alpha\overline{Q}_{\dot{\alpha}B} + \overline{Q}_{\dot{\alpha}B}Q^A_\alpha = \sigma^m_{\alpha\dot{\alpha}}P_m C^A{}_B + \sigma^m_{\dot{\alpha}\alpha}P_m C^B{}_A \\ &= \sigma^m_{\alpha\dot{\alpha}}P_m(C^A{}_B + C^B{}_A)\end{aligned} \tag{4.10}$$

And the verification of the anticommutator is Hermitian

$$\begin{aligned}&\{Q^A_\alpha, \overline{Q}_{\dot{\alpha}B}\}^\dagger = (Q^A_\alpha\overline{Q}_{\dot{\alpha}B} + \overline{Q}_{\dot{\alpha}B}Q^A_\alpha)^\dagger = (Q^A_\alpha\overline{Q}_{\dot{\alpha}B})^\dagger + (\overline{Q}_{\dot{\alpha}B}Q^A_\alpha)^\dagger \\ &= ((Q_\alpha\overline{Q}_{\dot{\alpha}})^A{}_B^{\mathrm{T}})^* + ((\overline{Q}_{\dot{\alpha}}Q_\alpha)_B{}^{A\mathrm{T}})^* = [((Q_\alpha\overline{Q}_{\dot{\alpha}})^{\mathrm{T}})^*]^A{}_B + [((\overline{Q}_{\dot{\alpha}}Q_\alpha)^{\mathrm{T}})^*]_B{}^A \\ &= [(\overline{Q}_{\dot{\alpha}})^{\mathrm{T}}((Q_\alpha)^{\mathrm{T}}]^{*A}{}_B + [(Q_\alpha)^{\mathrm{T}}(\overline{Q}_{\dot{\alpha}})^{\mathrm{T}}]^*_B{}^A = (\overline{Q}_{\dot{\alpha}}Q_\alpha)^{*A}{}_B + (Q_\alpha\overline{Q}_{\dot{\alpha}})^*_B{}^A \\ &= [(\overline{Q}_{\dot{\alpha}})^*(Q_\alpha)^*]^A{}_B + [(Q_\alpha)^*(\overline{Q}_{\dot{\alpha}})^*]_B{}^A = (Q_\alpha\overline{Q}_{\dot{\alpha}})^A{}_B + (\overline{Q}_{\dot{\alpha}}Q_\alpha)_B{}^A \\ &= Q^A_\alpha\overline{Q}_{\dot{\alpha}B} + \overline{Q}_{\dot{\alpha}B}Q^A_\alpha = \{Q^A_\alpha, \overline{Q}_{\dot{\alpha}B}\}\end{aligned} \tag{4.11}$$

Also,each product is Hermitian.Thus, $C^A{}_B, C^B{}_A$ in [4.10] are all Hermitian.And a theorem told us any hermitian matrix can be diagonalized by a unitary matrix.Then,we use an unitary transformation U to diagonalize $C^A{}_B, C^B{}_A$,that is $[\{Q^A_\alpha, \overline{Q}_{\dot{\alpha}B}\}, U] = 0$ which is $\{Q^A_\alpha, \overline{Q}_{\dot{\alpha}B}\}U = U\{Q^A_\alpha, \overline{Q}_{\dot{\alpha}B}\}$,then

$$\begin{aligned}\{Q^A_\alpha, \overline{Q}_{\dot{\alpha}B}\} &= U\{Q^A_\alpha, \overline{Q}_{\dot{\alpha}B}\}U^{-1} = U\sigma^m_{\alpha\dot{\alpha}}P_m(C^A{}_B + C^B{}_A)U^{-1} \\ &= \sigma^m_{\alpha\dot{\alpha}}P_m(UC^A{}_BU^{-1} + UC^B{}_AU^{-1}) \\ &= \sigma^m_{\alpha\dot{\alpha}}P_m(\delta^A{}_B + \delta^A{}_B) \\ &= 2\sigma^m_{\alpha\dot{\alpha}}P_m\delta^A{}_B\end{aligned} \tag{4.12}$$

Now,[4.12] gives the only nontrivial term in Supersymmetry algebra.

4.4 Properties of SUSY

One property is that equal number of fermions and bosons are contained in the supersymmetry representation.We have a fact that a non-vanishing correlator need to have even number of fermions and the first loop amplitude we consider in string theory is the torus that will give periodicity of boundary.We know on world-sheet we have one direction for time,anticommutative fields on the periodic time direction will give antiperiodic time-ordering in path integral.Thus,an operator $(-1)^F$ with F the world-sheet spinor number is needed to correct the time-ordering,gives -1 for each fermionic operator.Thus,we put $\{Q^A_\alpha, \overline{Q}_{\dot{\alpha}B}\}$ on a periodic world-sheet or world-sheet in compactified dimensions with A, B counts for the dimensions with periodicity and $(-1)^F$ must be acted on for correct calculation of path integral.In this case we apply trace

$$
\begin{aligned}
&\mathrm{tr}[(-1)^F\{Q^A_\alpha,\overline{Q}_{\dot{\alpha}B}\}]=\mathrm{tr}[(-1)^F(Q^A_\alpha\overline{Q}_{\dot{\alpha}B}+\overline{Q}_{\dot{\alpha}B}Q^A_\alpha)]\\
&\;=\mathrm{tr}[-1Q^A_\alpha(-1)^F\overline{Q}_{\dot{\alpha}B}+(-1)^F\overline{Q}_{\dot{\alpha}B}Q^A_\alpha)]\\
&\;=\mathrm{tr}[-1Q^A_\alpha(-1)^F\overline{Q}_{\dot{\alpha}B}]+\mathrm{tr}[(-1)^F\overline{Q}_{\dot{\alpha}B}Q^A_\alpha)]\\
&\;=\mathrm{tr}[-1Q^A_\alpha(-1)^F\overline{Q}_{\dot{\alpha}B}]+\mathrm{tr}[Q^A_\alpha(-1)^F\overline{Q}_{\dot{\alpha}B})]\\
&\;=\mathrm{tr}[-1Q^A_\alpha(-1)^F\overline{Q}_{\dot{\alpha}B}+Q^A_\alpha(-1)^F\overline{Q}_{\dot{\alpha}B})]=0\\
&0=\mathrm{tr}[(-1)^F\{Q^A_\alpha,\overline{Q}_{\dot{\alpha}B}\}]=\mathrm{tr}[(-1)^F2\sigma^m_{\alpha\dot{\alpha}}P_m\delta^A{}_B]=2\sigma^m_{\alpha\dot{\alpha}}P_m\delta^A{}_B\mathrm{tr}[(-1)^F]
\end{aligned}
\tag{4.13}
$$

We performed [4.12] in the second equation.For non-vanishing momentum,[4.13] reduces to $\mathrm{tr}[(-1)^F]=0$,this gives us that the number of fermions and number of bosons are equal in compact space.

The second property is states of representation SUSY algebra.For seeing clearly,we need to boost the momentum to rest frame that is $P_m\to P_m=(-M,0,0,0)$,thus

$$
\begin{aligned}
\sigma^m_{\alpha\dot{\alpha}}P_m=\sigma^0_{\alpha\dot{\alpha}}P_0&=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}_{\alpha\dot{\alpha}}P_0=-1\begin{pmatrix}1&0\\0&1\end{pmatrix}_{\alpha\dot{\alpha}}(-M)\\
&=M\begin{pmatrix}1&0\\0&1\end{pmatrix}_{\alpha\dot{\alpha}}=M\delta_{\alpha\dot{\alpha}}
\end{aligned}
\tag{4.14}
$$

In this frame,SUSY algebra [4.12] becomes

$$
\{\frac{1}{\sqrt{2M}}Q^A_\alpha,\frac{1}{\sqrt{2M}}\overline{Q}_{\dot{\alpha}B}\}=\delta_{\alpha\dot{\alpha}}\delta^A{}_B,\qquad\{Q^A_\alpha,Q^B_\beta\}=\{\overline{Q}_{\dot{\alpha}B},\overline{Q}_{\dot{\beta}B}\}=0 \tag{4.15}
$$

we can define $a^A_\alpha=1/\sqrt{2M}Q^A_\alpha,(a^A_\alpha)^\dagger=1/\sqrt{2M}\overline{Q}_{\dot{\alpha}A}$ to get a rescaled algebra

$$
\{a^A_\alpha,(a^B_\beta)^\dagger\}=\delta_{\alpha\beta}\delta^A{}_B \tag{4.16}
$$

Similar to [3.78] we introduce Clifford vacuum for supersymmetric field theory

$$
a^A_\alpha\Omega^{(n)}=0,\quad\text{for all}\,A,\alpha \tag{4.17}
$$

with $\Omega^{(n)}=a^{A_1}_{\alpha_1}...a^{A_n}_{\alpha_n}|\psi_{RQFT}\rangle$,each distinct operator at most has one copy and $\langle\psi_{RQFT}|\psi_{RQFT}\rangle=1,\overline{\psi}|\psi_{RQFT}\rangle=\overline{Q}_{\dot{\alpha}A}|\psi_{RQFT}\rangle=(a^A_\alpha)^\dagger|\psi_{RQFT}\rangle=0$.$n$ can have $2N$ choices for $A=1...N$.For the normalization $\Omega^{(n)}_{\text{norm}}=C\Omega^{(n)}$,the subtile thing for getting C is that we need to write [4.16] into a $2N\times2N$ matrix

$$
a^A_\alpha=\begin{pmatrix}a^A_1&0\\0&a^A_2\end{pmatrix}\qquad(a^A_\alpha)^\dagger=\begin{pmatrix}(a^A_1)^\dagger&0\\0&(a^A_2)^\dagger\end{pmatrix} \tag{4.18}
$$

This gives us diagonal commutation relation

$$\begin{aligned}
\{a^A_\alpha,(a^B_\beta)^\dagger\}_{\text{diag}} &= \{\begin{pmatrix} a^A_1 & 0 \\ 0 & a^A_2 \end{pmatrix},\begin{pmatrix} (a^B_1)^\dagger & 0 \\ 0 & (a^B_2)^\dagger \end{pmatrix}\} \\
&= \begin{pmatrix} a^A_1 & 0 \\ 0 & a^A_2 \end{pmatrix}\begin{pmatrix} (a^B_1)^\dagger & 0 \\ 0 & (a^B_2)^\dagger \end{pmatrix} + \begin{pmatrix} (a^B_1)^\dagger & 0 \\ 0 & (a^B_2)^\dagger \end{pmatrix}\begin{pmatrix} a^A_1 & 0 \\ 0 & a^A_2 \end{pmatrix} \\
&= \begin{pmatrix} a^A_1(a^B_1)^\dagger & 0 \\ 0 & a^A_2(a^B_2)^\dagger \end{pmatrix} + \begin{pmatrix} (a^B_1)^\dagger a^A_1 & 0 \\ 0 & (a^B_2)^\dagger a^A_2 \end{pmatrix} \\
&= \begin{pmatrix} \{a^A_1,(a^B_1)^\dagger\} & 0 \\ 0 & \{a^A_2,(a^B_2)^\dagger\} \end{pmatrix} = \begin{pmatrix} \delta_{11}\delta^A_B & 0 \\ 0 & \delta_{22}\delta^A_B \end{pmatrix} \\
&= \begin{pmatrix} \delta^A_B & 0 \\ 0 & \delta^A_B \end{pmatrix}
\end{aligned} \qquad [4.19]$$

And we call it *diagonal supersymmetry algebra*.And clearly from the bijection [4.18],$\{a^A_\alpha,(a^B_\beta)^\dagger\}_{\text{diag}} \cong \{a^A_\alpha,(a^B_\beta)^\dagger\}$.Now,we can use this form to calculate C for $A=B$,we get $\{a^A_\alpha,(a^A_\beta)^\dagger\}_{\text{diag}} = \delta^{2A}_{2A}$,for $n=2A$ and by induction we get

$$\begin{aligned}
1 &= (\Omega^{(n)}_{\text{norm}})^\dagger\Omega^{(n)}_{\text{norm}} = (C\Omega^{(n)})^\dagger C\Omega^{(n)} = C^2(\Omega^{(n)})^\dagger\Omega^{(n)} \\
&= C^2\langle\psi_{RQFT}|(a^{A_1}_{\alpha_1}...a^{A_n}_{\alpha_n})^\dagger a^{A_1}_{\alpha_1}...a^{A_n}_{\alpha_n}|\psi_{RQFT}\rangle \\
&= C^2\langle\psi_{RQFT}|(a^{A_n}_{\alpha_n})^\dagger...(a^{A_1}_{\alpha_1})^\dagger a^{A_1}_{\alpha_1}...a^{A_n}_{\alpha_n}|\psi_{RQFT}\rangle \\
&= C^2\langle\psi_{RQFT}|(a^{A_n}_{\alpha_n})^\dagger...(a^{A_2}_{\alpha_2})^\dagger(\delta^{2A}_{2A} - a^{A_1=2A}_{\alpha_1}(a^{A_1=2A}_{\alpha_1})^\dagger)a^{A_2}_{\alpha_2}...a^{A_n}_{\alpha_n}|\psi_{RQFT}\rangle \\
&= C^2\langle\psi_{RQFT}|(a^{A_n}_{\alpha_n})^\dagger...(a^{A_2}_{\alpha_2})^\dagger(\delta^{2A}_{2A})a^{A_2}_{\alpha_2}...a^{A_n}_{\alpha_n}|\psi_{RQFT}\rangle \\
&- \langle\psi_{RQFT}|(a^{A_n}_{\alpha_n})^\dagger...(a^{A_2}_{\alpha_2})^\dagger a^{A_2}_{\alpha_2}...a^{A_n}_{\alpha_n}(a^{A_1=2A}_{\alpha_1}(a^{A_1=2A}_{\alpha_1})^\dagger)|\psi_{RQFT}\rangle \\
&= C^2(\delta^{2A}_{2A})\langle\psi_{RQFT}|(a^{A_n}_{\alpha_n})^\dagger...(a^{A_2}_{\alpha_2})^\dagger a^{A_2}_{\alpha_2}...a^{A_n}_{\alpha_n}|\psi_{RQFT}\rangle \\
&= C^2(\delta^{2A}_{2A})\langle\psi_{RQFT}|(a^{A_n}_{\alpha_n})^\dagger...(a^{A_3}_{\alpha_3})^\dagger(\delta^{2A-1}_{2A-1} - a^{A_2=2A-1}_{\alpha_2}(a^{A_2=2A-1}_{\alpha_2})^\dagger) \\
&\times a^{A_3}_{\alpha_3}...a^{A_n}_{\alpha_n}|\psi_{RQFT}\rangle \\
&= C^2(\delta^{2A}_{2A})(\delta^{2A-1}_{2A-1})\langle\psi_{RQFT}|(a^{A_n}_{\alpha_n})^\dagger...(a^{A_3}_{\alpha_3})^\dagger a^{A_3}_{\alpha_3}...a^{A_n}_{\alpha_n}|\psi_{RQFT}\rangle \\
&= C^2(\delta^{2A}_{2A})(\delta^{2A-1}_{2A-1})...(\delta^1_1)\langle\psi_{RQFT}|1|\psi_{RQFT}\rangle \\
&= C^2(2A)(2A-1)...(1) \\
&= C^2(2A)!
\end{aligned} \qquad [4.20]$$

Therefore, we get normalisation constant $C = \frac{1}{\sqrt{(2A)!}} = \frac{1}{\sqrt{n!}}$.And now we can build consistent states based on the Clifford vacuum.

$$\Omega^{(n)\lambda_0+\frac{n}{2}}_{(\alpha_1,A_1)...(\alpha_n,A_n)} = \frac{1}{\sqrt{n!}}(a^{A_1}_{\alpha_1})^\dagger...(a^{A_n}_{\alpha_n})^\dagger\Omega^{(n)\lambda_0} \qquad [4.21]$$

where λ_0 is the spin.And states [4.21] generate the representation of supersymmetry algebra,called supermultiplet.Above all,we can see a representation of supersymmetry algebra is that of Clifford algebra [3.79] on compactified dimensions.For each n,we have $2N$ choices,thus statistically for each n we have C_n^{2N} choices with $n = 0, ..., 2N$.Thus,we get the dimension (number of states) of the supersymmetry representation

$$\dim\Omega = \sum_{n=0}^{2N} \binom{2N}{n} = 2^{2N} = (\dim(\mathbb{C}\Omega))^2 = 2^{2k} = (\dim\rho(g_{SO(d)})/2)^2 \quad [4.22]$$

where $\mathbb{C}\Omega$ is defined by $a^A = a_1^A + ia_2^A$.We see that the k compactified part of representation [3.79] is actually a supersymmetry representation with N internal dimensions for $N = k$ in [3.75].With 2^{2N-1} integer spin states and 2^{2N-1} half-integer spin states.And we list below several cases of supermultiplet below, each tuple shows the corresponding states in the representation.

$$\begin{aligned}
&N = 1, \quad \dim(\mathbb{C}\Omega) = 2^1, \quad (\lambda_0, \lambda_0 + \frac{1}{2}) \\
&N = 2, \quad \dim(\mathbb{C}\Omega) = 2^2, \quad (\lambda_0, \lambda_0 + \frac{1}{2}, \lambda_0 + \frac{1}{2}, \lambda_0 + 1) \\
&N = 4, \quad \dim(\mathbb{C}\Omega) = 2^4, \quad (\lambda_0, {}^4\lambda_0 + \frac{1}{2}, {}^6\lambda_0 + 1, {}^4\lambda_0 + \frac{3}{2}, \lambda_0 + 2)
\end{aligned} \quad [4.23]$$

4.5 Component fields

We want to change the supersymmetry algebra to a version with fields for constructing supersymmetric field theory.Firstly,we introduce the anticommuting parameters ξ, η.., and satisfy

$$\{\xi^\alpha, \xi^\beta\} = \{\xi^\alpha, Q_\beta\} = ... = \{\xi^\alpha, P_m\} = 0 \quad \xi Q = \xi^\alpha Q_\alpha, \overline{\xi Q} = \bar{\xi}_{\dot{\alpha}} \overline{Q}^{\dot{\alpha}} \quad [4.24]$$

In this case,the nontrivial term becomes

$$\begin{aligned}
[\xi Q, \overline{\xi Q}] &= [\xi^\alpha Q_\alpha, \bar{\xi}_{\dot{\alpha}} \overline{Q}^{\dot{\alpha}}] = \xi Q \overline{\xi Q} - \overline{\xi Q} \xi Q = \xi \bar{\xi} Q \overline{Q} - \bar{\xi} \xi \overline{Q} Q \\
&= \xi \bar{\xi} (Q \overline{Q} + \overline{Q} Q) = \xi \bar{\xi} \{Q, \overline{Q}\} = \xi \bar{\xi} 2\sigma^m P_m \\
&= 2\xi^\alpha \sigma^m_{\alpha\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} P_m
\end{aligned} \quad [4.25]$$

Notice that we need to be vary careful about the spinor indices,but sometimes we ignore indices for simplicity.Then,we introduce a component multiplet that is a set of fields(A,ψ ...) with the infinitesimal supersymmetry transformation

$$\delta_\xi A = (\xi Q + \overline{\xi Q}) \times A, \quad \delta_\xi \psi = (\xi Q + \overline{\xi Q}) \times \psi \quad [4.26]$$

The $\times$ means undefined multiplication.And satisfy

$$\begin{aligned}
&[\delta_\eta, \delta_\xi]A = \delta_\eta\delta_\xi - \delta_\xi\delta_\eta = \delta_\eta(\xi Q + \overline{\xi Q})A - \delta_\xi(\eta Q + \overline{\eta}\overline{Q})A \\
&= (\eta Q + \overline{\eta}\overline{Q})(\xi Q + \overline{\xi Q})A - (\xi Q + \overline{\xi Q})(\eta Q + \overline{\eta}\overline{Q})A \\
&= \eta Q(\xi Q + \overline{\xi Q})A + \overline{\eta}\overline{Q}(\xi Q + \overline{\xi Q})A - \xi Q(\eta Q + \overline{\eta}\overline{Q})A - \overline{\xi Q}(\eta Q + \overline{\eta}\overline{Q})A \\
&= \eta Q\xi QA + \eta Q\overline{\xi Q}A + \overline{\eta}\overline{Q}\xi QA + \overline{\eta}\overline{Q}\overline{\xi Q}A \\
&- \xi Q\eta QA - \xi Q\overline{\eta}\overline{Q}A - \overline{\xi Q}\eta QA - \overline{\xi Q}\overline{\eta}\overline{Q}A \\
&= [\eta Q, \xi Q]A + [\eta Q, \overline{\xi Q}]A + [\overline{\eta}\overline{Q}, \xi Q]A + [\overline{\eta}\overline{Q}, \overline{\xi Q}] \\
&= 0 + 2\eta\sigma^m\overline{\xi}P_mA - 2\xi\sigma^m\overline{\eta}P_mA + 0 = 2(\eta\sigma^m\overline{\xi} - \xi\sigma^m\overline{\eta})P_mA \\
&= 2(\eta\sigma^m\overline{\xi} - \xi\sigma^m\overline{\eta})(-i\partial_m)A = -2i(\eta^\alpha\sigma^m_{\alpha\dot{\alpha}}\overline{\xi}^{\dot{\alpha}} - \xi^\alpha\sigma^m_{\alpha\dot{\alpha}}\overline{\eta}^{\dot{\alpha}})\partial_mA
\end{aligned} \qquad [4.27]$$

The last line is for scalar field,$A \propto e^{iP_mx^m}$ and $\partial_m = \frac{\partial}{\partial^m}$, $-i\frac{\partial}{\partial^m}A = -i(ip_m)A = P_mA$.This means the supersymmetry transformation closes.And supersymmetry transformation needs to transform tensor field to spinor field and vice versa.For tracking the field produced we need to do dimension analysis,we have

$$\begin{aligned}
&Q\psi \propto F + \partial^nA, \qquad [Q] = \frac{1}{2} \quad [\psi] = \frac{k}{2} \quad [\partial^n] = n[\partial] = n \quad k \in \mathbb{Z}_{\text{odd}} \\
&\text{with} \quad [F] = \frac{1}{2} + \frac{k}{2} \quad n = \frac{1}{2} + \frac{k}{2} - l \quad [A] = l \quad l \in \mathbb{Z}
\end{aligned} \qquad [4.28]$$

[ ] is for mass dimension and F is an auxiliary field.By the guidance of [4.28],we can set

$$\delta_\xi A = \sqrt{2}\xi^\alpha\psi_\alpha, \qquad \delta_\xi\psi_\alpha = i\sqrt{2}\sigma^m_{\alpha\dot{\alpha}}\overline{\xi}^{\dot{\alpha}}\partial_mA + \sqrt{2}\xi_\alpha F \qquad [4.29]$$

we can verify closure similarly to [4.27] for closure on ψ

$$\begin{aligned}
[\delta_\eta, \delta_\xi]\psi_\alpha &= i\sqrt{2}\sigma^m\overline{\xi}\partial_m(\delta_\eta A) + \sqrt{2}\xi(\delta_\eta F) - i\sqrt{2}\sigma^m\overline{\eta}\partial_m(\delta_\xi A) - \sqrt{2}\eta(\delta_\xi F) \\
&= i(-2\partial_m\psi)(\eta\sigma^m\overline{\xi} - \xi\sigma^n\overline{\eta}) + \sqrt{2}(\xi\delta_\eta F - \eta\delta_\xi F) \\
&= i(\sigma^m_{\alpha\dot{\alpha}}\overline{\sigma}^{n\dot{\alpha}\beta}\partial_n\psi_\beta)(\eta^\beta\sigma^m_{\beta\dot{\gamma}}\overline{\xi}^{\dot{\gamma}} - \xi^\beta\sigma^m_{\beta\dot{\gamma}}\overline{\eta}^{\dot{\gamma}}) + \sqrt{2}(\xi\delta_\eta F - \eta\delta_\xi F) \\
&= i(\delta_{\beta\alpha}\sigma^{m\alpha}_{\dot{\alpha}}\overline{\sigma}^{n\dot{\alpha}\beta}\partial_n\psi_\beta)(\eta^\beta\sigma^{m\dot{\gamma}}_\beta\overline{\xi}_{\dot{\gamma}} - \xi^\beta\sigma^{m\dot{\gamma}}_\beta\overline{\eta}_{\dot{\gamma}}) + \sqrt{2}(\xi\delta_\eta F - \eta\delta_\xi F) \\
&= -i(\sigma^{m\alpha}_{\dot{\alpha}}\sigma^{m\dot{\gamma}}_\beta\overline{\sigma}^{n\dot{\alpha}\beta}\partial_n\psi_\beta)(\eta^\beta\delta_{\beta\alpha}\overline{\xi}_{\dot{\gamma}} - \xi^\beta\delta_{\beta\alpha}\overline{\eta}_{\dot{\gamma}}) + \sqrt{2}(\xi\delta_\eta F - \eta\delta_\xi F) \\
&= -i(\sigma^{m\alpha}_{\dot{\alpha}}\sigma^{m\dot{\gamma}}_\beta)(\overline{\xi}_{\dot{\gamma}}\overline{\sigma}^{n\dot{\alpha}\beta}\eta^\beta\delta_{\beta\alpha} - \overline{\eta}_{\dot{\gamma}}\overline{\sigma}^{n\dot{\alpha}\beta}\xi^\beta\delta_{\beta\alpha})\partial_n\psi_\beta \\
&+ \sqrt{2}(\xi\delta_\eta F - \eta\delta_\xi F) \\
&= -i(\delta^\alpha_\beta\delta^{\dot{\gamma}}_{\dot{\alpha}})(\overline{\xi}_{\dot{\gamma}}\overline{\sigma}^{n\dot{\alpha}\beta}\eta_\alpha - \overline{\eta}_{\dot{\gamma}}\overline{\sigma}^{n\dot{\alpha}\beta}\xi_\alpha)\partial_n\psi_\beta + \sqrt{2}(\xi\delta_\eta F - \eta\delta_\xi F) \\
&= -i(\overline{\xi}_{\dot{\alpha}}\overline{\sigma}^{n\dot{\alpha}\alpha}\eta_\alpha - \overline{\eta}_{\dot{\gamma}}\overline{\sigma}^{n\dot{\alpha}\alpha}\xi_\alpha)\partial_n\psi_\alpha + \sqrt{2}(\xi\delta_\eta F - \eta\delta_\xi F) \\
&= -i(\overline{\xi}\overline{\sigma}^m\eta - \overline{\eta}\overline{\sigma}^m\xi)\partial_m\psi_\alpha + \sqrt{2}(\xi\delta_\eta F - \eta\delta_\xi F)
\end{aligned} \qquad [4.30]$$

And for closure on A

$$\begin{aligned}[\delta_\eta, \delta_\xi]A &= \delta_\eta\delta_\xi A - \delta_\xi\delta_\eta A = \sqrt{2}\xi\delta_\eta\psi - \sqrt{2}\eta\delta_\xi\psi \\ &= \sqrt{2}\xi(i\sqrt{2}\sigma^m\overline{\eta}\partial_m A + \sqrt{2}\eta F) - \sqrt{2}\eta(i\sqrt{2}\sigma^m\overline{\xi}\partial_m A + \sqrt{2}\xi F) \\ &= -2i(\eta\sigma^m\overline{\xi} - \xi\sigma^m\overline{\eta})\partial_m A + 4\xi\eta F\end{aligned} \quad [4.31]$$

For closure and maintaining the similar form of 1st term in [4.30],we have

$$\delta_\xi F = i\sqrt{2}\overline{\xi}\overline{\sigma}^m\partial_m\psi \quad [4.32]$$

Then,from Dirac equation (3.39) in [8] we get below for $i = 1, 2, 3$

$$-m\psi_L = i(-\partial_0 - \sigma^i\partial_i)\psi_R = i(\sigma^0\partial_0 - \sigma^i\partial_i)\psi_R = i\overline{\sigma}^m\partial_m\psi_R \quad [4.33]$$

we can let $F = -mA^*$,and [4.32] becomes

$$i\sqrt{2}\overline{\xi}\overline{\sigma}^m\partial_m\psi = -m\delta_\xi A^* = -m\sqrt{2}\overline{\xi\psi} \quad [4.34]$$

we exactly get the Dirac equation [4.32] in periodic dimensions $\psi_L = \overline{\psi}, \psi_R = \psi$ in this case.Which means we get a right form of [4.32] and we can view the closure has guaranteed by the field equation [4.32].And we call A, ψ, F with supersymmetry transformations $\delta_\xi A, \delta_\xi\psi, \delta_\xi F$ closing the chiral or scalar multiplet.

Then,we can use above to construct the following supersymmetrically invariant action $\mathscr{L}_{SUSY} = \mathscr{L} + m\mathscr{L}_m$

$$\begin{aligned}\mathscr{L}_0 &= i\partial_n\overline{\psi}\overline{\sigma}^n\psi + A^*\Box A + F^*F \\ \mathscr{L}_m &= AF + A^*F^* - \frac{1}{2}\psi\psi - \frac{1}{2}\overline{\psi\psi}\end{aligned} \quad [4.35]$$

We can see the variation

$$\begin{aligned}\delta_\xi\mathscr{L}_0 &= i\partial_n\delta_\xi\overline{\psi}\overline{\sigma}^n\psi + i\partial_n\overline{\psi}\overline{\sigma}^n\delta_\xi\psi + \delta_\xi A^*\Box A + A^*\Box\delta_\xi A + \delta_\xi F^*F + F^*\delta_\xi F \\ &= i\partial_n(-i\sqrt{2}\xi\sigma^m\partial_m A^* + \sqrt{2}\overline{\xi}F^*)\overline{\sigma}^n\psi + i\partial_n\overline{\psi}\overline{\sigma}^n(i\sqrt{2}\sigma^m\overline{\xi}\partial_m A + \sqrt{2}\xi F) \\ &+ \sqrt{2}\overline{\xi\psi}\Box A + A^*\Box\sqrt{2}\xi\psi - i\sqrt{2}\partial_m\overline{\psi}\overline{\sigma}^m\xi F + F^*i\sqrt{2}\overline{\xi}\overline{\sigma}^m\partial_m\psi \\ &= -\sqrt{2}(-\sigma^m\overline{\sigma}^n\partial_m\partial_n)A^*\xi\psi - \sqrt{2}(-\overline{\xi\psi})\sigma^m\overline{\sigma}^n\partial_m\partial_n A \\ &+ \sqrt{2}\overline{\xi\psi}\Box A + \sqrt{2}A^*\xi\Box\psi \\ &+ i\sqrt{2}\overline{\xi}\partial_n F^*\overline{\sigma}^n\psi + i\sqrt{2}\partial_n\overline{\psi}\overline{\sigma}^n\xi F + i\sqrt{2}\xi\sigma^m\partial_m\overline{\psi}F + i\sqrt{2}F^*\overline{\xi}\overline{\sigma}^m\partial_m\psi \\ &= -\sqrt{2}(--\eta^{mn}\partial_m\partial_n)A^*\xi\psi - \sqrt{2}\overline{\xi\psi}(--\eta^{mn}\partial_m\partial_n)A \\ &+ \sqrt{2}\overline{\xi\psi}\Box A + \sqrt{2}A^*\xi\Box\psi \\ &- i\sqrt{2}\overline{\xi}F^*\overline{\sigma}^n\partial_n\psi + i\sqrt{2}\partial_m\overline{\psi}\overline{\sigma}^m\xi F - i\sqrt{2}\partial_m\overline{\psi}\overline{\sigma}^m\xi F + i\sqrt{2}\overline{\xi}F^*\overline{\sigma}^m\partial_m\psi \\ &= -\sqrt{2}A^*\xi\Box\psi - \sqrt{2}\overline{\xi\psi}\Box A + \sqrt{2}\overline{\xi\psi}\Box A + \sqrt{2}A^*\xi\Box\psi + 0 = 0\end{aligned}$$

[4.36]

We have performed partial derivative and $\xi\sigma^n\overline{\psi} = -\overline{\psi}\overline{\sigma}^n\xi$.And for unbar part

$$\begin{aligned}
&\delta_\xi \mathscr{L}_m(A,F,\psi) = \delta_\xi A F + A\delta_\xi F - \psi^\alpha \delta_\xi \psi_\alpha \\
&= \sqrt{2}\xi^\alpha\psi_\alpha F + Ai\sqrt{2}\overline{\xi}\overline{\sigma}^m\partial_m\psi - \psi^\alpha(i\sqrt{2}\sigma^m_{\alpha\dot{\alpha}}\overline{\xi}^{\dot{\alpha}}\partial_m A + \sqrt{2}\xi_\alpha F) = 0
\end{aligned} \tag{4.37}$$

Then,easily we can find the field equations

$$\begin{aligned}
0 &= \partial_n \frac{\partial \mathscr{L}}{\partial(\partial_n\overline{\psi})} - \frac{\partial \mathscr{L}}{\partial\overline{\psi}} = \partial_n \frac{\partial}{\partial(\partial_n\overline{\psi})}(i\partial_n\overline{\psi}\overline{\sigma}^n\psi) - \frac{\partial}{\partial\overline{\psi}}(-m\frac{1}{2}\overline{\psi\psi}) \\
&= i\overline{\sigma}^n\partial_n\psi - \frac{1}{2}(-m\overline{\psi} - m\overline{\psi}) = i\overline{\sigma}^n\partial_n\psi + m\overline{\psi} \\
0 &= F + mA^* \\
0 &= \partial_n \frac{\partial \mathscr{L}}{\partial(\partial_n A^*)} + \frac{\partial \mathscr{L}}{\partial A^*} = \partial_n \frac{\partial}{\partial(\partial_n A^*)}(A^*\Box A) - \frac{\partial}{\partial A^*}(mA^*F^*) \\
&= \partial_n \frac{\partial}{\partial(\partial_n A^*)}(A^*\eta^{mn}\partial_m\partial_n A) - mF^* = \partial_n \frac{\partial}{\partial(\partial_n A^*)}(-\partial_n A^*\eta^{mn}\partial_m A) - mF^* \\
&= -\eta^{mn}\partial_m\partial_n A - mF^* = \Box A + mF^*
\end{aligned} \tag{4.38}$$

Next,we want to proof a good property $\mathscr{L}_{SUSY} =: \mathscr{L}_{SUSY}:$ which means the path integral of the interacting Lagrangian is regular,the interactions counteract with each other,the interactions of bosons counteract that of fermions exactly.And we perform Dyson-Schwinger equation

$$\begin{aligned}
0 &= \int [d\overline{\psi}(z,\overline{z})]\frac{\delta}{\delta\overline{\psi}}[exp(-S)\overline{\psi}(z',\overline{z}')] \\
&= \int [d\overline{\psi}(z,\overline{z})]\{exp(-S)\frac{\delta S}{\delta\overline{\psi}}\overline{\psi}(z',\overline{z}') + exp(-S)\delta^2(z-z',\overline{z}-\overline{z}')\} \\
&= \int [d\overline{\psi}(z,\overline{z})]exp(-S)\{[i\overline{\sigma}^n\partial_n\psi + m\overline{\psi}](z,\overline{z})\overline{\psi}(z',\overline{z}') + \delta^2(z-z',\overline{z}-\overline{z}')\} \\
&= \left\langle i\partial_n\overline{\psi}(z',\overline{z}')\overline{\sigma}^n\psi(z,\overline{z})\right\rangle + \left\langle m\overline{\psi}(z,\overline{z})\overline{\psi}(z',\overline{z}')\right\rangle + \left\langle \delta^2(z-z',\overline{z}-\overline{z}')\right\rangle
\end{aligned} \tag{4.39}$$

Following the same method of [4.39] and use the field equations [4.38],We finally get

$$\begin{aligned}
&\left\langle i\partial_n\overline{\psi}(z',\overline{z}')\overline{\sigma}^n\psi(z,\overline{z})\right\rangle + \left\langle m\overline{\psi}(z,\overline{z})\overline{\psi}(z',\overline{z}')\right\rangle = -\left\langle \delta^2(z-z',\overline{z}-\overline{z}')\right\rangle \\
&\left\langle i\partial_n\overline{\psi}(z,\overline{z})\overline{\sigma}^n\psi(z',\overline{z}')\right\rangle + \left\langle m\psi(z,\overline{z})\psi(z',\overline{z}')\right\rangle = -\left\langle \delta^2(z-z',\overline{z}-\overline{z}')\right\rangle \\
&\left\langle F^*(z,\overline{z})F(z',\overline{z}')\right\rangle + \left\langle mA(z,\overline{z})F(z',\overline{z}')\right\rangle = -\left\langle \delta^2(z-z',\overline{z}-\overline{z}')\right\rangle \\
&\left\langle A^*(z,\overline{z})\Box A(z',\overline{z}')\right\rangle + \left\langle mA^*(z,\overline{z})F^*(z',\overline{z}')\right\rangle = -\left\langle \delta^2(z-z',\overline{z}-\overline{z}')\right\rangle
\end{aligned} \tag{4.40}$$

And for the 2nd term in [4.40]

$$\begin{aligned}
\left\langle i\partial_n\overline{\psi}(z,\overline{z})\overline{\sigma}^n\psi(z',\overline{z}')\right\rangle^* + \langle m\psi(z,\overline{z})\psi(z',\overline{z}')\rangle^* &= -\left\langle \delta^2(z-z',\overline{z}-\overline{z}')\right\rangle^* \\
\left\langle -i\partial_n\psi(z,\overline{z})\overline{\sigma}^n\overline{\psi}(z',\overline{z}')\right\rangle + \left\langle m\overline{\psi}(z,\overline{z})\overline{\psi}(z',\overline{z}')\right\rangle &= -\left\langle \delta\overline{\delta}(z-z',\overline{z}-\overline{z}')\right\rangle^* \\
\left\langle i\overline{\psi}(z',\overline{z}')\overline{\sigma}^n\partial_n\psi(z,\overline{z})\right\rangle + \left\langle m\overline{\psi}(z,\overline{z})\overline{\psi}(z',\overline{z}')\right\rangle &= -\left\langle \overline{\delta}\delta(z-z',\overline{z}-\overline{z}')\right\rangle \\
\left\langle -i\partial_n\overline{\psi}(z',\overline{z}')\overline{\sigma}^n\psi(z,\overline{z})\right\rangle + \left\langle m\overline{\psi}(z,\overline{z})\overline{\psi}(z',\overline{z}')\right\rangle &= -\left\langle \delta^2(z-z',\overline{z}-\overline{z}')\right\rangle \\
\left\langle i\partial_n\overline{\psi}(z',\overline{z}')\overline{\sigma}^n\psi(z,\overline{z})\right\rangle - \left\langle m\overline{\psi}(z,\overline{z})\overline{\psi}(z',\overline{z}')\right\rangle &= \left\langle \delta^2(z-z',\overline{z}-\overline{z}')\right\rangle
\end{aligned} \qquad [4.41]$$

Then use 1st term in [4.40] minus final equation in [4.41],we find

$$\langle m\psi(z,\overline{z})\psi(z',\overline{z}')\rangle = -\left\langle \delta^2(z-z',\overline{z}-\overline{z}')\right\rangle \qquad [4.42]$$

Same procedure above for consider all field equations,we get OPE

$$\begin{aligned}
\left\langle m\overline{\psi}(z,\overline{z})\overline{\psi}(z',\overline{z}')\right\rangle = \langle m\psi(z,\overline{z})\psi(z',\overline{z}')\rangle &= -\left\langle \delta^2(z-z',\overline{z}-\overline{z}')\right\rangle \\
\langle mA(z,\overline{z})F(z',\overline{z}')\rangle = \langle mA^*(z,\overline{z})F^*(z',\overline{z}')\rangle &= -\left\langle \delta^2(z-z',\overline{z}-\overline{z}')\right\rangle
\end{aligned} \qquad [4.43]$$

And other OPEs vanish.The subtile point is that a spinor field has 2-component index that means it has 2 degree of freedom compared to a bosonic field which has 1 degree of freedom thus $\langle m\psi\psi\rangle_{\mathscr{L}} = 2\langle m\psi(z,\overline{z})\psi(z',\overline{z}')\rangle$.Thus,

$$\begin{aligned}
:\mathscr{L}_{SUSY}: &= \mathscr{L}_{SUSY} - \langle\mathscr{L}\rangle_{\mathscr{L}} \\
&= \mathscr{L}_{SUSY} - \langle AF\rangle_{\mathscr{L}} - \langle A^*F^*\rangle_{\mathscr{L}} + \frac{1}{2}\langle\psi\psi\rangle_{\mathscr{L}} + \frac{1}{2}\left\langle\overline{\psi\psi}\right\rangle_{\mathscr{L}} \\
&= \mathscr{L}_{SUSY} - \delta^2 - \delta^2 + \frac{1}{2}2\delta^2 + \frac{1}{2}2\delta^2 = \mathscr{L}_{SUSY}
\end{aligned} \qquad [4.44]$$

Also,[4.44] tells us that the number of degree of freedom of bosons is indeed equal to that of fermions in a supersymmetry invariant action.

4.6 Superspace and Superfields

The supersymmetry algebra with anticommuting parameters [4.24] is a Lie algebra,we can verify the axioms

$$\begin{aligned}
&[a\xi Q + b\xi Q, \overline{\xi Q}] = a[\xi Q, \overline{\xi Q}] + b[\xi Q, \overline{\xi Q}] \\
&[\xi Q, \xi Q] = [\overline{\xi Q}, \overline{\xi Q}] = [P_m, P_m] = 0 \\
&[Pm, [\xi Q, \overline{\xi Q}]] + [\xi Q, [\overline{\xi Q}, Pm]] + [\overline{\xi Q}, [Pm, \xi Q]] = 0 \\
&[\xi Q, \overline{\xi Q}] = -[\overline{\xi Q}, \xi Q]
\end{aligned} \qquad [4.45]$$

we can define a corresponding group element by using the linear combination of basis $(P_m, Q, \overline{Q})$ that expand the parameter space $(x, \theta, \overline{\theta})$ of a multiplicative group with element

$$G(x,\theta,\overline{\theta}) = e^{i\{-x^m P_m + \theta Q + \overline{\theta Q}\}} \qquad [4.46]$$

Because of Jacobi identity,we use Hausdorff's formula $e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$ to multiplication of two group elements.We find

$$
\begin{aligned}
G(0,\xi,\overline{\xi})G(x^m,\theta,\overline{\theta}) &= e^{i\{\xi Q+\overline{\xi Q}\}}e^{i\{-x^m P_m+\theta Q+\overline{\theta Q}\}} \\
&= e^{i\{-x^m P_m+(\xi+\theta)Q+(\overline{\xi}+\overline{\theta})\overline{Q}\}+i^2\frac{1}{2}[\xi Q,\overline{\theta Q}]+i^2\frac{1}{2}[\overline{\xi Q},\theta Q]} \\
&= e^{i\{-x^m P_m+(\xi+\theta)Q+(\overline{\xi}+\overline{\theta})\overline{Q}-(-i\frac{1}{2}2\xi\sigma^m\overline{\theta})P_m+(-i\frac{1}{2}2\theta\sigma^m\overline{\xi})P_m\}} \\
&= G(x^m+i\theta\sigma^m\overline{\xi}-i\xi\sigma^m\overline{\theta},\theta+\xi,\overline{\theta}+\overline{\xi})
\end{aligned}
\tag{4,47}
$$

Notice that $G(0,\xi,\overline{\xi})$ is the only nontrivial multiplication element from the observation $G(x_1^m,0,0)G(x_2^m,\theta,\overline{\theta}) = G(0,0,0)G((x_1+x_2)^m,\theta,\overline{\theta})$.This multiplicative group can naturally induce an additive group with element $(x^m,\theta,\overline{\theta})$ with multiplication $(x_1^m,\theta_1,\overline{\theta}_1)(x_2^m,\theta_2,\overline{\theta}_2) = (x_1^m+x_2^m,\theta_1+\theta_2,\overline{\theta}_1+\overline{\theta}_2)$ which makes $((x^m,\theta,\overline{\theta}),+) \cong (\mathbb{R}\times\mathbb{R}\times\mathbb{R},+)$,we can find a subgroup of it descents from the group [4.46] based on super Lie algebra [4.45] by

$$
\begin{tikzcd}
G(0,\xi,\overline{\xi})\times G(x^m,\theta,\overline{\theta}) \arrow[r, "a"] \arrow[d] \arrow[dd, bend right, "(0,\xi,\overline{\xi})"', "\simeq"] & G(x^m+i\theta\sigma^m\overline{\xi}-i\xi\sigma^m\overline{\theta},\theta+\xi,\overline{\theta}+\overline{\xi}) \arrow[dd] \arrow[dd, bend right, "\simeq"] \\
\times\,(x^m,\theta,\overline{\theta}) \arrow[d] & \\
\mathscr{D}\times(x^m,\theta,\overline{\theta})\times(x^m,\theta,\overline{\theta}) \arrow[r, "b"] & (x^m+i\theta\sigma^m\overline{\xi}-i\xi\sigma^m\overline{\theta},\theta+\xi,\overline{\theta}+\overline{\xi})
\end{tikzcd}
$$

Because elements $G(0,\xi,\overline{\xi})$ forms a subgroup,the first line in the diagram induce a group action.If group action is a property,the natural isomorphism in the diagram descents the group action of a to that of b,that means we get a natural group action induced by the last line in diagram from [4.46] over [4.45].

$$
\begin{aligned}
&\left[\mathscr{D}\times(x^m,\theta,\overline{\theta})\right]\times(x^m,\theta,\overline{\theta}) \to (x^m,\theta,\overline{\theta}) \\
&d\times(x_1,x_2,x_3)\times(x^m,\theta,\overline{\theta}) \mapsto (x^m+dx_1,\theta+dx_2,\overline{\theta}+dx_3)
\end{aligned}
\tag{4.48}
$$

which means $G_{\mathscr{D}} = (\left[\mathscr{D}\times(x^m,\theta,\overline{\theta})\right],+) \subset ((x^m,\theta,\overline{\theta}),+)$,is a subgroup.Then,the group axiom of inverse gives,for $d\in\mathscr{D},x\in(x^m,\theta,\overline{\theta})$ the inverse $(dx)^{-1} = x^{-1}d^{-1}$ exists which means for any $x\in G_{\mathscr{D}}$,d^{-1} exists to make $d^{-1}x\in(x^m,\theta,\overline{\theta})$. Next,we want to find out the elements in $\mathscr{D}$.By the following calculation

$$
\begin{aligned}
&\left[\xi^\alpha\left(\frac{\partial}{\partial\theta^\alpha}-i\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^m}\right)+\overline{\xi}_{\dot{\alpha}}\left(\frac{\partial}{\partial\overline{\theta}_{\dot{\alpha}}}-i\theta^\alpha\sigma^m_{\alpha\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\alpha}}\frac{\partial}{\partial x^m}\right)\right](x^m,\theta,\overline{\theta}) \\
&=\left(-\xi^\alpha i\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^m}x^m-\overline{\xi}_{\dot{\alpha}}i\theta^\alpha\sigma^m_{\alpha\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\alpha}}\frac{\partial}{\partial x^m}x^m,\xi^\alpha\frac{\partial}{\partial\theta^\alpha}\theta^\alpha,\overline{\xi}_{\dot{\alpha}}\frac{\partial}{\partial\overline{\theta}_{\dot{\alpha}}}\overline{\theta}_{\dot{\alpha}}\right) \\
&=\left(-i\xi^\alpha\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}-(-i\theta^\alpha\sigma^m_{\alpha\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\alpha}}\overline{\xi}_{\dot{\alpha}}),\xi^\alpha,\overline{\xi}_{\dot{\alpha}}\right) \\
&=\left(-i\xi^\alpha\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}+i\theta^\alpha\sigma^m_{\alpha\dot{\beta}}\overline{\xi}^{\dot{\beta}},\xi^\alpha,\overline{\xi}_{\dot{\alpha}}\right)
\end{aligned}
\tag{4.49}
$$

and compare it with [4.48] we get $d=\xi Q+\overline{\xi Q}\in\mathscr{D}$,which are also differential operators.

$$\xi Q+\overline{\xi Q}=\xi^{\alpha}\left(\frac{\partial}{\partial\theta^{\alpha}}-i\sigma^{m}_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_m\right)+\bar{\xi}_{\dot{\alpha}}\left(\frac{\partial}{\partial\overline{\theta}_{\dot{\alpha}}}-i\theta^{\alpha}\sigma^{m}_{\alpha\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\alpha}}\partial_m\right) \qquad [4.50]$$

Thus,if $x,y\in(x^m,\theta,\overline{\theta})$ for $x\neq 0$ that means x is nontrivial,we have

$$d^{-1}x=y\Rightarrow\int x...=y\Rightarrow dy=x\neq 0\quad\forall y \qquad [4.51]$$

In this case,[4.51] tells us the group $G_{\mathscr{D}}$ is a Lie group which is a differentiable manifold and the space parametrized by $(x^m,\theta,\overline{\theta})\in G_{\mathscr{D}}$ is also a differential manifold,we call this superspace which is a *supermanifold* here.For completion,we need to verify the commutation relation of operators in [4.50] indeed agree with the superalgebra [4.12].And we can verify the supersymmetry algebra

$$\begin{aligned}
&\{Q_\alpha,\overline{Q}_{\dot{\alpha}}\}=\left(\frac{\partial}{\partial\theta^{\alpha}}-i\sigma^{m}_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_m\right)\left(\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}}-i\theta_{\alpha}\sigma^{\alpha\dot{\beta}m}\varepsilon_{\dot{\beta}\dot{\alpha}}\partial_m\right)\\
&+\left(\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}}-i\theta_{\alpha}\sigma^{\alpha\dot{\beta}m}\varepsilon_{\dot{\beta}\dot{\alpha}}\partial_m\right)\left(\frac{\partial}{\partial\theta^{\alpha}}-i\sigma^{m}_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_m\right)\\
&=\frac{\partial}{\partial\theta^{\alpha}}\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}}-i\frac{\partial}{\partial\theta^{\alpha}}\theta_{\alpha}\sigma^{\alpha\dot{\beta}m}\varepsilon_{\dot{\beta}\dot{\alpha}}\partial_m-i\sigma^{m}_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_m\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}}-\sigma^{m}_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_m\theta_{\alpha}\sigma^{\alpha\dot{\beta}m}\varepsilon_{\dot{\beta}\dot{\alpha}}\partial_m\\
&+\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}}\frac{\partial}{\partial\theta^{\alpha}}-i\sigma^{m}_{\alpha\dot{\alpha}}\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}}\overline{\theta}^{\dot{\alpha}}\partial_m-i\theta_{\alpha}\sigma^{\alpha\dot{\beta}m}\varepsilon_{\dot{\beta}\dot{\alpha}}\partial_m\frac{\partial}{\partial\theta^{\alpha}}-\theta_{\alpha}\sigma^{\alpha\dot{\beta}m}\varepsilon_{\dot{\beta}\dot{\alpha}}\partial_m\sigma^{m}_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_m\\
&=-i\frac{\partial}{\partial\theta^{\alpha}}\theta_{\alpha}\varepsilon^{\alpha\beta}\sigma^{\dot{\beta}m}_{\beta}\varepsilon_{\dot{\beta}\dot{\alpha}}\partial_m-i\sigma^{m}_{\alpha\dot{\alpha}}\partial_m+i\sigma^{m}_{\alpha\dot{\alpha}}\partial_m\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}}\overline{\theta}^{\dot{\alpha}}+i\varepsilon^{\alpha\beta}\sigma^{\dot{\beta}m}_{\alpha}\varepsilon_{\dot{\beta}\dot{\alpha}}\partial_m\frac{\partial}{\partial\theta^{\alpha}}\theta_{\alpha}\\
&=-i\frac{\partial}{\partial\theta^{\alpha}}\varepsilon^{\alpha\beta}\theta_{\alpha}\sigma^{\dot{\beta}m}_{\beta}\varepsilon_{\dot{\beta}\dot{\alpha}}\partial_m-i\sigma^{m}_{\alpha\dot{\alpha}}\partial_m+i\sigma^{m}_{\alpha\dot{\alpha}}\partial_m+i\sigma^{\dot{\beta}m}_{\alpha}\varepsilon_{\dot{\beta}\dot{\alpha}}\varepsilon^{\alpha\beta}\partial_m\frac{\partial}{\partial\theta^{\alpha}}\theta_{\alpha}\\
&=i\frac{\partial}{\partial\theta_{\beta}}\theta_{\alpha}\sigma^{\dot{\beta}m}_{\beta}\varepsilon_{\dot{\beta}\dot{\alpha}}\partial_m-i\sigma^{\dot{\beta}m}_{\alpha}\varepsilon_{\dot{\beta}\dot{\alpha}}\varepsilon^{\beta\alpha}\partial_m\frac{\partial}{\partial\theta^{\alpha}}\theta_{\alpha}\\
&=i\delta^{\beta}_{\alpha}\sigma^{\dot{\beta}m}_{\beta}\varepsilon_{\dot{\beta}\dot{\alpha}}\partial_m-i\sigma^{\dot{\beta}m}_{\alpha}\varepsilon_{\dot{\beta}\dot{\alpha}}\partial_m\varepsilon^{\beta\alpha}\frac{\partial}{\partial\theta_{\alpha}}\theta^{\alpha}=i\sigma^{m}_{\alpha\dot{\alpha}}\partial_m+i\sigma^{\dot{\beta}m}_{\alpha}\varepsilon_{\dot{\beta}\dot{\alpha}}\partial_m\frac{\partial}{\partial\theta^{\beta}}\theta^{\alpha}\\
&=2i\sigma^{m}_{\alpha\dot{\alpha}}\partial_m=-2\sigma^{m}_{\alpha\dot{\alpha}}P_m
\end{aligned} \qquad [4.52]$$

for $P_m=-i\partial_m$.We can see in the diagram above [4.48],the [4.50] is a left multiplication,we want to shift it to right multiplication $D_\alpha,\overline{D}_{\dot{\alpha}}$

$$D_\alpha(\leftarrow)=\frac{\partial}{\partial\theta^{\alpha}}+i\sigma^{m}_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_m \qquad \overline{D}_{\dot{\alpha}}(\leftarrow)=-\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}}-i\theta^{\alpha}\sigma^{m}_{\alpha\dot{\alpha}}\partial_m \qquad [4.53]$$

with anticommutation relations that are right multiplicative version of [4.52]

$$\{D_\alpha,\overline{D}_{\dot{\alpha}}\}=-2i\sigma^{m}_{\alpha\dot{\alpha}}\partial_m\quad\{D_\alpha,Q_\beta\}=0 \qquad [4.54]$$

other anticommutators vanish.Now,we can introduce superfields living in the superspace.For component fields $(f...\chi_{\dot{\alpha}}...d)(x)$ in a theory,a superfield is a function of degree 0 of sipinor index in superspace that can be expanded in a power series over all $\theta, \overline{\theta}$ index-contractions with the component fields with certain indices.

$$\begin{aligned} F(x,\theta,\overline{\theta})_{(\alpha,\dot{\alpha})} =& f(x) + \theta^{\alpha}\phi_{\alpha}(x) + \overline{\theta}^{\dot{\alpha}}\overline{\chi}_{\dot{\alpha}}(x) \\ &+ \theta^{\alpha}\theta_{\alpha}m(x) + \overline{\theta}^{\dot{\alpha}}\overline{\theta}_{\dot{\alpha}}n(x) + \theta^{\alpha}\sigma^{m}_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\nu_m(x) \\ &+ \theta^{\alpha}\theta_{\alpha}\overline{\theta}^{\dot{\alpha}}\lambda_{\dot{\alpha}}(x) + \overline{\theta}^{\dot{\alpha}}\overline{\theta}_{\dot{\alpha}}\theta^{\alpha}\psi_{\alpha}(x) + \theta^{\alpha}\theta_{\alpha}\overline{\theta}^{\dot{\alpha}}\overline{\theta}_{\dot{\alpha}}d(x) \end{aligned} \qquad [4.55]$$

Higher powers of $\theta, \overline{\theta}$ vanish,because it will contain repreated $\theta^{\alpha}, \overline{\theta}^{\dot{\alpha}}$.This construction gives us that the supersymmery transformation of superfield is that of component field [5.3].

$$\begin{aligned} \delta_{\xi}F(x,\theta,\overline{\theta}) =& \delta_{\xi}f(x) + \theta^{\alpha}\delta_{\xi}\phi_{\alpha}(x) + \overline{\theta}^{\dot{\alpha}}\delta_{\xi}\overline{\chi}_{\dot{\alpha}}(x) \\ &+ \theta^{\alpha}\theta_{\alpha}\delta_{\xi}m(x) + \overline{\theta}^{\dot{\alpha}}\overline{\theta}_{\dot{\alpha}}\delta_{\xi}n(x) + \theta^{\alpha}\sigma^{m}_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\delta_{\xi}\nu_m(x) \\ &+ \theta^{\alpha}\theta_{\alpha}\overline{\theta}^{\dot{\alpha}}\delta_{\xi}\lambda_{\dot{\alpha}}(x) + \overline{\theta}^{\dot{\alpha}}\overline{\theta}_{\dot{\alpha}}\theta^{\alpha}\delta_{\xi}\psi_{\alpha}(x) + \theta^{\alpha}\theta_{\alpha}\overline{\theta}^{\dot{\alpha}}\overline{\theta}_{\dot{\alpha}}\delta_{\xi}d(x) \\ \equiv& (\xi Q + \overline{\xi Q})F(x,\theta,\overline{\theta}) \end{aligned} \qquad [4.56]$$

In this case,we get the following properties for a constant a.

$$\begin{aligned} &(\xi Q + \overline{\xi Q})(F_1 + F_2) = (\xi Q + \overline{\xi Q})F_1 + (\xi Q + \overline{\xi Q})F_2 \\ &(\xi Q + \overline{\xi Q})(aF) = a(\xi Q + \overline{\xi Q})F \end{aligned} \qquad [4.57]$$

that matches the definition of linear transformation,which means we can represent elements of supersymmetry algebra as linear transformations of superfields.Thus,superfields form a linear representation of the supersymmetry algebra.But the representation space consisting of superfields is highly reducible.For this,the problem of studding supersymmetry representation to that of finding solution space of vanishing of differential equations of superfields.

$$\begin{aligned} &\overline{D}\Phi = D\Phi^{\dagger} = 0 \qquad \text{chiral or scalar superfields} \\ &F = F^{\dagger} \qquad \text{vector superfields} \end{aligned} \qquad [4.58]$$

Another way to construct a superfield is applying $\exp(\theta Q + \overline{\theta Q})\times$ to a component multiplet A,with undefined multiplication $\times$.

$$\begin{aligned} F(x,\theta,\overline{\theta}) &= e^{(\theta Q + \overline{\theta Q})} \times A = \sum_{n=0}^{\infty} \frac{((\xi Q + \overline{\xi Q})\times)^n}{n!} A \\ &= A + (\xi Q + \overline{\xi Q}) \times A + \frac{1}{2}((\xi Q + \overline{\xi Q})\times)^2 A + ... \end{aligned} \qquad [4.59]$$

with transformations $\delta_{\xi}F(x,\theta,\overline{\theta}) = (\xi Q + \overline{\xi Q})F \neq (\xi Q + \overline{\xi Q}) \times F$ with the first equation from [4.50].We can find out the undefined multiplication by

$$
\begin{aligned}
&(\xi Q+\overline{\xi Q})e^{(\theta Q+\overline{\theta Q})}\times \\
&=\left[\xi^\alpha\left(\frac{\partial}{\partial\theta^\alpha}-i\sigma^m_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\partial_m\right)+\overline{\xi}_{\dot\alpha}\left(\frac{\partial}{\partial\overline{\theta}_{\dot\alpha}}-i\theta^\alpha\sigma^m_{\alpha\dot\beta}\varepsilon^{\dot\beta\dot\alpha}\partial_m\right)\right]e^{(\theta Q\times+\overline{\theta Q}\times)} \\
&=\left[\xi^\alpha\left(\frac{\partial}{\partial\theta^\alpha}-i\sigma^m_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\partial_m\right)+\overline{\xi}_{\dot\alpha}\left(\frac{\partial}{\partial\overline{\theta}_{\dot\alpha}}-i\theta^\alpha\sigma^m_{\alpha\dot\beta}\varepsilon^{\dot\beta\dot\alpha}\partial_m\right)\right]e^{\theta Q\times}e^{\overline{\theta Q}\times}e^{-\theta\sigma^m\overline{\theta}P_m} \\
&=\left[\xi Q\times-\xi\sigma^m\overline{\theta}P_m+\overline{\xi Q}\times+\theta\sigma^m\overline{\xi}P_m\right]e^{\theta Q\times}e^{\overline{\theta Q}\times}e^{-\theta\sigma^m\overline{\theta}P_m} \\
&=\left(\xi Q\times+\overline{\xi Q}\times-\xi\sigma^m\overline{\theta}P_m+\theta\sigma^m\overline{\xi}P_m\right)e^{(\theta Q+\overline{\theta Q})}\times
\end{aligned}
\tag{4.60}
$$

the expression for the undefined multiplication is

$$
(\xi Q+\overline{\xi Q})\times=(\xi Q+\overline{\xi Q})+\xi\sigma^m\overline{\theta}P_m-\theta\sigma^m\overline{\xi}P_m \tag{4.61}
$$

4.7 Chiral superfields

Chiral superfields are characterized by the condition [4.58],they correspond to chiral multiplets [4.23] for $N=1$,with $\Phi\in(-1/2,0),\Phi^\dagger\in(0,1/2)$.For Φ with coordinate-dependence,the solution on coordinate space is

$$
y^m=x^m+i\theta\sigma^m\overline{\theta}\quad\text{and}\quad\theta \tag{4.62}
$$

Thus,the solution of the superfield is $\Phi(y^m,\theta)$ with

$$
\Phi(y^m,\theta)=A(y^m)+\sqrt{2}\theta\psi(y^m)+\theta\theta F(y^m) \tag{4.63}
$$

We give spinor field ψ a $\sqrt{2}$ for convenience.And tylor expansion gives

$$
\begin{aligned}
\Phi&=A(x^m+i\theta\sigma^m\overline{\theta})+\sqrt{2}\theta\psi(x^m+i\theta\sigma^m\overline{\theta})+\theta\theta F(x^m+i\theta\sigma^m\overline{\theta}) \\
&=A(x)+i\theta\sigma^m\overline{\theta}\partial_mA(x)-\frac{1}{2}\theta\sigma^m\overline{\theta}\theta\sigma^n\overline{\theta}\partial_m\partial_nA(x) \\
&\quad+\sqrt{2}\theta\psi-\sqrt{2}i\theta^\beta\sigma^m_{\beta\dot\beta}\overline{\theta}^{\dot\beta}\theta^\alpha\partial_m\psi_\alpha+\theta\theta F(x) \\
&=A(x)+i\theta\sigma^m\overline{\theta}\partial_mA(x)--\frac{1}{2}\frac{1}{2}\theta\theta\overline{\theta\theta}\eta^{mn}\partial_m\partial_nA(x) \\
&\quad+\sqrt{2}\theta\psi-\sqrt{2}i\frac{1}{2}\varepsilon^{\alpha\beta}\theta\theta\sigma^m_{\beta\dot\beta}\overline{\theta}^{\dot\beta}\partial_m\psi_\alpha+\theta\theta F(x) \\
&=A(x)+i\theta\sigma^m\overline{\theta}\partial_mA(x)+\frac{1}{4}\theta\theta\overline{\theta\theta}\Box A(x) \\
&\quad+\sqrt{2}\overline{\theta\psi}-\frac{i}{\sqrt{2}}\theta\theta\partial_m\psi\sigma^m\overline{\theta}+\theta\theta F(x)
\end{aligned}
\tag{4.64}
$$

We have used [7](B.13)(B.14).The right multiplications [4.53] are x-fields,we want express it in y-space,and notice that they are first-order derivative operators.

$$\begin{aligned}
D_\alpha(x^m) &= D_\alpha(y^m - i\theta\sigma^m\overline{\theta}) = D_\alpha(y^m) - i\theta\sigma^m\overline{\theta}\frac{\partial}{\partial y^m}D_\alpha(y^m) \\
&= D_\alpha(y^m) - i\theta\sigma^m\overline{\theta}\frac{\partial}{\partial y^m}\left(\frac{\partial}{\partial\theta^\alpha} + i\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\frac{\partial}{\partial y^m}\right) \\
&= D_\alpha(y^m) + i\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\theta^\alpha\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial y^m} + \circ\left(\frac{\partial}{\partial y^m}\right)^2 \\
&= D_\alpha(y^m) + i\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\frac{\partial}{\partial y^m} + \circ\left(\frac{\partial}{\partial y^m}\right)^2 \\
&= \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\frac{\partial}{\partial y^m}
\end{aligned}$$

$$\begin{aligned}
\overline{D}_{\dot{\alpha}}(x^m) &= \overline{D}_{\dot{\alpha}}(y^m - i\theta\sigma^m\overline{\theta}) = \overline{D}_{\dot{\alpha}}(y^m) - i\theta\sigma^m\overline{\theta}\frac{\partial}{\partial y^m}\overline{D}_{\dot{\alpha}}(y^m) \\
&= \overline{D}_{\dot{\alpha}}(y^m) - i\theta\sigma^m\overline{\theta}\frac{\partial}{\partial y^m}\left(-\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^m}\right) \\
&= \overline{D}_{\dot{\alpha}}(y^m) + i\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^m} + \circ\left(\frac{\partial}{\partial y^m}\right)^2 \\
&= -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}}
\end{aligned} \tag{4.65}$$

Same method for $D\Phi^\dagger = 0$ gives us things about conjugation.And we list them below.

$$y^\dagger = x^m - i\theta\sigma^m\overline{\theta} \quad \text{and} \quad \overline{\theta} \tag{4.66}$$

which is solution in coordinate space.And the corresponding superfield is

$$\begin{aligned}
\Phi^\dagger &= A^*(y^\dagger) + \sqrt{2}\overline{\theta}\overline{\psi}(y^\dagger) + \overline{\theta}\overline{\theta}F^*(y^\dagger) \\
&= A^*(x) - i\theta\sigma^m\overline{\theta}\partial_m A^*(x) + \frac{1}{4}\theta\theta\overline{\theta}\overline{\theta}\Box A^*(x) \\
&\quad + \sqrt{2}\overline{\theta}\overline{\psi}(x) + \frac{i}{\sqrt{2}}\overline{\theta}\overline{\theta}\theta\sigma^m\partial_m\overline{\psi}(x) + \overline{\theta}\overline{\theta}F^*(x)
\end{aligned} \tag{4.67}$$

And differential operators expressed in $y^\dagger$-space

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} \qquad \overline{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} - 2i\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^{\dagger m}} \tag{4.68}$$

Products of superfields are always superfields,products of chiral superfields Φ are always chiral superfields

$$\overline{D}_{\dot{\alpha}}(\Phi_i\Phi_j...) = \overline{D}_{\dot{\alpha}}\Phi_i\Phi_j... + \Phi_i\overline{D}_{\dot{\alpha}}\Phi_j... + ... = 0 \tag{4.69}$$

But products of superfields with conjugations are not chiral superfields.

$$\overline{D}_{\dot{\alpha}}(\Phi_i\Phi_j^\dagger...) = \overline{D}_{\dot{\alpha}}\Phi_i\Phi_j^\dagger... + \Phi_i\overline{D}_{\dot{\alpha}}\Phi_j^\dagger... \neq 0 \tag{4.70}$$

Thus,we can construct following product chiral superfields.

$$
\begin{aligned}
\Phi_i\Phi_j &= [A_i(y) + \sqrt{2}\theta\psi_i(y) + \theta\theta F_i(y)][A_j(y) + \sqrt{2}\theta\psi_j(y) + \theta\theta F_j(y)] \\
&= A_i(y)A_j(y) + \sqrt{2}\theta A_i(y)\psi_j(y) + \theta\theta A_i(y)F_j(y) + \sqrt{2}\theta\psi_i(y)A_j(y) \\
&\quad + \sqrt{2}\theta^\alpha\psi_{i\alpha}(y)\sqrt{2}\theta^\beta\psi_{j\beta}(y) + \sqrt{2}\theta^\alpha\psi_{i\alpha}(y)\theta^\alpha\theta_\alpha F_j(y) \\
&\quad + \theta\theta F_i(y)A_j(y) + \sqrt{2}\theta^\alpha\theta_\alpha\theta^\alpha\psi_{j\alpha}(y)F_i(y) + \theta^\alpha\theta_\alpha\theta^\alpha\theta_\alpha F_i(y)F_j(y) \\
&= A_i(y)A_j(y) + \sqrt{2}\theta[\psi_i(y)A_j(y) + A_i(y)\psi_j(y)] \\
&\quad + \theta\theta[A_i(y)F_j(y) + A_j(y)F_i(y)] + 2(-\frac{1}{2}\theta\theta\varepsilon^{\alpha\beta})\psi_{i\alpha}\psi_{j\beta} \\
&= A_i(y)A_j(y) + \sqrt{2}\theta[\psi_i(y)A_j(y) + A_i(y)\psi_j(y)] \\
&\quad + \theta\theta[A_i(y)F_j(y) + A_j(y)F_i(y) - \psi_i\psi_j]
\end{aligned}
\tag{4.71}
$$

$$
\begin{aligned}
\Phi_i\Phi_j\Phi_k &= \{A_i(y)A_j(y) + \sqrt{2}\theta[\psi_i(y)A_j(y) + A_i(y)\psi_j(y)] \\
&+\theta\theta[A_i(y)F_j(y) + A_j(y)F_i(y) - \psi_i\psi_j]\}[A_k(y) + \sqrt{2}\theta\psi_k(y) + \theta\theta F_k(y)] \\
&= A_i(y)A_j(y) + \sqrt{2}\theta[\psi_i A_j A_k + \psi_j + A_k A_i + \psi_k A_i A_j](y) \\
&+\theta\theta[F_i A_j A_k + F_j A_k A_i + F_k A_i A_j - \varepsilon^{\alpha\beta}(\psi_{i\alpha}\psi_{j\beta}A_k + \psi_{j\alpha}\psi_{k\beta}A_i + \psi_{k\alpha}\psi_{i\beta}A_j)] \\
&= A_i(y)A_j(y) + \sqrt{2}\theta[\psi_i A_j A_k + \psi_j + A_k A_i + \psi_k A_i A_j](y) \\
&+\theta\theta[F_i A_j A_k + F_j A_k A_i + F_k A_i A_j - \psi_\alpha\psi_\beta A_k - \psi_\alpha\psi_\beta A_i - \psi_{k\alpha}\psi_\beta A_j]
\end{aligned}
\tag{4.72}
$$

$$
\begin{aligned}
\Phi_i^\dagger\Phi_j|_{\theta\theta\overline{\theta\theta}} &= \theta\theta\overline{\theta\theta}[F_i^*F_j + \frac{1}{4}A_i^*\Box A_j + \frac{1}{4}\Box A_i^*A_j] + (-i\theta\sigma^n\overline{\theta}\partial_n A_i^*)i\theta\sigma^m\overline{\theta}\partial_m A_j \\
&\quad - i\theta\theta\overline{\theta}^{\dot\alpha}\overline{\psi}_{i\dot\alpha}(-\overline{\theta}^{\dot\beta}\overline{\sigma}^{m\beta}_{\dot\beta}\partial_m\psi_{j\beta}) + i\overline{\theta\theta}(-\partial_m\overline{\psi}_{i\dot\alpha}\overline{\sigma}^{m\dot\alpha}_{\alpha}\theta^\alpha)\theta^\beta\psi_{j\beta} \\
&= \theta\theta\overline{\theta\theta}[F_i^*F_j + \frac{1}{4}A_i^*\Box A_j + \frac{1}{4}\Box A_i^*A_j] - \frac{1}{2}\theta\theta\overline{\theta\theta}\partial_n A_i^*\partial_m A_j \\
&\quad + i\theta\theta(-\overline{\psi}_{i\dot\alpha})(\frac{1}{2}\varepsilon^{\dot\alpha\dot\beta})\overline{\sigma}^{m\beta}_{\dot\beta}\partial_m\psi_{j\beta} + i\overline{\theta\theta}(-\partial_m\overline{\psi}_{i\dot\alpha}\overline{\sigma}^{m\dot\alpha}_{\alpha})(-\frac{1}{2}\varepsilon^{\alpha\beta})\psi_{j\beta} \\
&= \theta\theta\overline{\theta\theta}[F_i^*F_j + \frac{1}{4}A_i^*\Box A_j + \frac{1}{4}\Box A_i^*A_j - \frac{1}{2}\partial_n A_i^*\partial_m A_j \\
&\quad - \frac{i}{2}\overline{\psi}_i\overline{\sigma}^m\partial_m\psi_j + \frac{i}{2}\partial_m\overline{\psi}_i\overline{\sigma}^m\psi_j]
\end{aligned}
\tag{4.73}
$$

Notice that $\Phi_i^\dagger\Phi_i|_{\theta\theta\overline{\theta\theta}}$ is chiral.

$$
\begin{aligned}
D_\alpha\left(\Phi_i^\dagger\Phi_i|_{\theta\theta\overline{\theta\theta}}\right) &= \frac{\partial}{\partial\theta^\alpha}\left(\theta\theta\overline{\theta\theta}[F_i^*F_i + \frac{1}{4}A_i^*\Box A_i + \frac{1}{4}\Box A_i^*A_i - \frac{1}{2}\partial_m A_i^*\partial_m A_i]\right) \\
&= \overline{\theta\theta}\theta^\alpha[F_i^*F_{i\alpha} + \frac{1}{4}A_i^*\Box A_{i\alpha} + \frac{1}{4}\Box A_i^*A_{i\alpha} - \frac{1}{2}\partial_m A_i^*\partial_m A_i\alpha] \\
&= 0
\end{aligned}
\tag{4.74}
$$

Because the symmetric vectors with antisymmetric indices vanish.Now,we are ready to build supersymmetric Lagrangian of chiral superfields.

First,recall that calculating path integral is about summing over all Feynman diagrams.And superficial degree of divergence D is a quantity about UV divergence that for $D > 0$.

$$D_i = [\text{diagram}_i] - \sum_{n=3}^{\infty} V_n[g_n] \tag{4.75}$$

V_n is number of vertices and $[g_n]$ is the mass dimension of the vertex in the diagram$_i$.And this leads to the definition of renormalizability.A theory is renormalisable means there are finite diagrams with $D > 0 \Leftrightarrow [g_n] \geq 0.[\mathscr{L}] = 4$ for 4-dimensional case tells us the mass dimension of product fields in the Lagragian must be equal or less 4 to give a renormalisable theory.

Now we can use the terms with appropriate mass dimensions in above chiral superfields to build the most general renormalizable Lagrangian.

$$\begin{aligned}
\mathscr{L}_{G_{\mathscr{D}}} &= \sum_{[\text{superfield}]=0}^{4} \mathscr{L}_{[\text{superfield}]} = \sum_{d=[\text{superfield}]=0}^{4} (c_{4-d}) \times \text{superfield}_d \\
&= \Phi_i^\dagger \Phi_i|_{\theta\theta\overline{\theta\theta}.c} + \left[\left(\frac{1}{2} m_{ij}\Phi_i\Phi_j + \frac{1}{3} g_{ijk}\Phi_i\Phi_j\Phi_k + \lambda_i \Phi_i \right)\Bigg|_{\theta\theta.c} + h.c. \right] \\
&[\Phi_i^\dagger \Phi_i|_{\theta\theta\overline{\theta\theta}}] = 0,1 \quad [\Phi_i|_{\theta\theta}] = 2 \quad [\Phi_i\Phi_j|_{\theta\theta}] = 3 \quad [\Phi_i\Phi_j\Phi_k|_{\theta\theta}] = 4
\end{aligned} \tag{4.76}$$

where we use $\mathscr{L}_{G_{\mathscr{D}}}$ to show it is a Lagrangian on the supermanifold and the $.c$ means the component of the restriction and the fraction and symmetric-index coupling are for symmetrization to cancel double counting.For $\Phi_i\Phi_j$ case

$$\frac{1}{2} m[A_i F_j + A_j F_i] = \frac{1}{2} m_{ij}[A_i F_j + A_j F_i] = m_{ij} A_i F_j \tag{4.77}$$

And use [4.64][4.67][4.73],changing basis from y to x does not change Lagrangian

$$\begin{aligned}
\Phi(x) \setminus (\overline{\theta}.c) &= A(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) \\
&\Rightarrow \Phi_i(y)...|_{\theta\theta.c} = \Phi_i(x)...|_{\theta\theta.c} \\
\Phi^\dagger \Phi_j|_{\theta\theta\overline{\theta\theta}}(x) &= \Phi^\dagger \Phi_j|_{\theta\theta\overline{\theta\theta}}(y) - \theta\sigma^m\overline{\theta}\partial[\Phi^\dagger \Phi_j|_{\theta\theta\overline{\theta\theta}}(y)]... \\
&= \Phi^\dagger \Phi_j|_{\theta\theta\overline{\theta\theta}}(y) + \circ(\theta^\alpha\theta^\alpha, \overline{\theta}^{\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}...) \\
&= \Phi^\dagger \Phi_j|_{\theta\theta\overline{\theta\theta}}(y)
\end{aligned} \tag{4.78}$$

The two sides are same on form,we can do a reparametrisation $x \leftrightarrow y$.Thus,we get $\mathscr{L}_{G_{\mathscr{D}}}(x) = \mathscr{L}_{G_{\mathscr{D}}}(y)$. In terms of component fields

$$\begin{aligned}
\mathscr{L}_{G_{\mathscr{D}}} &= F_i^* F_i + \frac{1}{4}(A_i^* \Box A_i + \Box A_i^* A_i) - \frac{1}{2}\partial_n A_i^* \partial_m A_i \\
&\quad + \frac{i}{2}(-\overline{\psi}_i \overline{\sigma}^m \partial_m \psi_i + \partial_m \overline{\psi}_i \overline{\sigma}^m \psi_i) + \Big[\frac{1}{2} m_{ij}(A_i F_j + A_j F_i - \psi_i \psi_j) \\
&\quad + \frac{1}{3} g_{ijk}(F_i A_j A_k + F_j A_k A_i + F_k A_i A_j) \\
&\quad + \frac{1}{3} g_{ijk}(-\psi_i \psi_j A_k - \psi_j \psi_k A_k - \psi_k \psi_i A_j) + \lambda_i F_i + \text{h.c.}\Big] \\
&= F_i^* F_i + \frac{1}{4}(A_i^* \Box A_i + A_i^* \Box A_i) + \frac{1}{2} A_i^* \Box A_i \\
&\quad + \frac{i}{2}(\partial_m \overline{\psi}_i \overline{\sigma}^m \psi_i + \partial_m \overline{\psi}_i \overline{\sigma}^m \psi_i) + \Big[m_{ij}(\frac{1}{2} 2 A_i F_j - \frac{1}{2}\psi_i \psi_j) \\
&\quad + g_{ijk}(\frac{1}{3} 3 F_i A_j A_k - \frac{1}{3} 3 \psi_i \psi_j A_k) + \lambda_i F_i + \text{h.c.}\Big] \\
&= i\partial_m \overline{\psi}_i \overline{\sigma}^m \psi_i + A_i^* \Box A_i + F_i^* F_i + \Big[m_{ij}\Big(A_i F_j - \frac{1}{2}\psi_i \psi_j\Big) \\
&\quad + g_{ijk}(F_i A_j A_k - \psi_i \psi_j A_k) + \lambda_i F_i + h.c.\Big]
\end{aligned} \qquad [4.79]$$

We can use $\frac{\delta \mathscr{L}}{\delta F}$ and $\frac{\delta \mathscr{L}}{\delta F^*}$ to find equations of auxiliary fields.

$$\begin{aligned}
0 = \partial_m \frac{\partial \mathscr{L}_{G_{\mathscr{D}}}}{\partial(\partial_m F)} = \frac{\partial \mathscr{L}_{G_{\mathscr{D}}}}{\partial F} &= F_i^* \delta_i^k + \lambda_i \delta_i^k + m_{ij} A_i \delta_j^k + g_{ijk} A_i A_j \\
&= F_k^* + \lambda_k + m_{ik} A_i + g_{ijk} A_i A_j \\
0 = \partial_m \frac{\partial \mathscr{L}_{G_{\mathscr{D}}}}{\partial(\partial_m F^*)} = \frac{\partial \mathscr{L}_{G_{\mathscr{D}}}}{\partial F^*} &= F_k + \lambda_k^* + m_{ik}^* A_i^* + g_{ijk}^* A_i^* A_j^*
\end{aligned} \qquad [4.80]$$

And we put [4.80] in [4.79]

$$\begin{aligned}
\mathscr{L}_{G_{\mathscr{D}}}|_{F_k} &= F_k^* F_k + m_{ik} A_i F_k + g_{ijk} A_i A_j F_k + \lambda_k F_k \\
&\quad + m_{ik}^* A_i^* F_k^* + g_{ijk}^* A_i^* A_j^* F_k^* + \lambda_k^* F_k^* \\
&= F_k^* F_k - (-m_{ik} A_i - g_{ijk} A_i A_j - \lambda_k) F_k - (-m_{ik}^* A_i^* - g_{ijk}^* A_i^* A_j^* - \lambda_k^*) F^* \\
&= F_k^* F_k - F_k^* F_k - F_k F_k^* = -F_k^* F_k = -\mathscr{V}(A_i, A_j^*) \\
\mathscr{L}_{G_{\mathscr{D}}} &= \ i\partial_m \overline{\psi}_i \overline{\sigma}^m \psi_i + A_i^* \Box A_i - \frac{1}{2} m_{ik} \psi_i \psi_k - \frac{1}{2} m_{ik}^* \psi_i \overline{\psi}_k \\
&\quad - g_{ijk} \psi_i \psi_j A_k - g_{ijk}^* \overline{\psi}_i \overline{\psi}_j A_k^* - \mathscr{V}(A_i, A_j^*)
\end{aligned} \qquad [4.81]$$

$\mathscr{V}(A_i, A_j^*) = F_k^* F_k$ is the potential term.The reason of expressing in terms of component fields is that [4.76] is clear on chiral superfields but now we can clearly see the kinetic,mass and potential terms in [4.81].

Note that constant a is the superfield $F(x,0,0) = f(x)$ in [4.55] with $f(x) = a$ and satisfy $\overline{D}_{\dot{\alpha}}a = D_\alpha a = 0$,so it is a chiral superfield.Thus $\Phi_i + a$ is chiral superfield and we can get another Lagrangian made of chiral superfields by shift $\Phi_i \to \Phi_i + a$.The shift on Lagrangian [4.76] of terms on $\theta\theta.c$ is

$$\begin{aligned}
&\frac{1}{2}m_{ij}(\Phi_i + a_i)(\Phi_j + a_j) + \frac{1}{3}g_{ijk}(\Phi_i + a_i)(\Phi_j + a_j)(\Phi_k + a_k) + \lambda_i(\Phi_i + a_i) \\
&= \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{2}m_{ij}(a_i\Phi_j + a_j\Phi_i) + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k + \frac{1}{3}g_{ijk}(a_i\Phi_j\Phi_k + a_j\Phi_k\Phi_i \\
&\quad + a_k\Phi_i\Phi_j) + \frac{1}{3}g_{ijk}(a_ia_j\Phi_k + a_ja_k\Phi_i + a_ka_i\Phi_j) + \lambda_i\Phi_i + \circ(a, a^2, a^3) \\
&= \frac{1}{2}m_{ij}\Phi_i\Phi_j + g_{ijk}a_k\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k + \lambda_i\Phi_i + m_{ij}a_j\Phi_i + g_{ijk}a_ja_k\Phi_i
\end{aligned} \quad [4.82]$$

And then we get following coupling constants in the shifted Lagrangian

$$\begin{aligned}
g'_{ijk} &= g_{ijk} \\
m'_{ij} &= m_{ij} + 2g_{ijk} \\
\lambda'_i &= \lambda_i + m_{ij}a_j + g_{ijk}a_ja_k
\end{aligned} \quad [4.83]$$

This is a good property,if the previous potential had a minimum at $\Phi_i = -a_i$ then we can shift it to the origin $\Phi'_i = -a_i + a_i = 0$ and with shifted couplings calculated by [4.83].

In addition,we find supersymmetry algebra is invariant under multiplication of supercharge by a phase factor $Q' = e^{-i\alpha}Q$,thus [4.25] gives

$$[\theta Q, \overline{\theta Q}]' = e^{-i\alpha}e^{i\alpha}2\theta\sigma^m\overline{\theta}P_m = [\theta Q, \overline{\theta Q}] \quad [4.84]$$

We see that it is invariant on algebra,and we call it R-invariance.In the algebra [4.25],we can let the anticommutative parameter absorb the phase factor

$$[\theta Q, \overline{\theta Q}]' = [\theta(e^{-i\alpha}Q), \overline{\theta}(e^{i\alpha}\overline{Q})] = [(e^{-i\alpha}\theta)Q, (e^{i\alpha}\overline{\theta})\overline{Q}] \quad [4.85]$$

Then we get R-transformation on the anticommuting parameter $\mathbf{R} : \theta \to e^{-i\alpha}\theta$. Notice that the R-transformation is invariant on supersymmetry algebra but not necessarily on $\theta.c$ in Lagrangian.thus we get a constraint on renormalizable Lagrangian [4.76] that is being R-invariant,this needs to let us define a unify quantity to superfields called R-character n to capture and cancel the effect of R-transformation on the parameters in the chiral superfields.In this case R-transformation acts on chiral superfields is

$$\begin{aligned}
\mathbf{R}\Phi(\theta, x) &= e^{2in\alpha}\Phi(e^{-i\alpha}\theta, x) \\
\mathbf{R}\Phi^\dagger(\overline{\theta}, x) &= e^{-2in\alpha}\Phi^\dagger(e^{i\alpha}\overline{\theta}, x)
\end{aligned} \quad [4.86]$$

We can put it on the component fields in [4.78] as this is the case we have used in building the Lagrangian.Now,$\Phi(\theta, x) = \Phi(x)/(\overline{\theta}.c, x)$

$$\mathbf{R}\Phi(\theta, x) = e^{2in\alpha}A(x) + \sqrt{2}e^{2in\alpha}e^{-i\alpha}\theta\psi + e^{2in\alpha}e^{-i\alpha}\theta e^{-i\alpha}\theta F(x) \quad [4.87]$$

Then we get

$$
\begin{aligned}
\mathbf{R}: A &\rightarrow e^{2in\alpha}A \\
\psi &\rightarrow e^{2i(n-\frac{1}{2})\alpha} \\
F &\rightarrow e^{2i(n-1)\alpha}F
\end{aligned} \tag{4.88}
$$

In this case,we shift the effects of R-transformation to the R-character of each superfield and assign the R-character to each component of the superfield.And in our Lagrangian [4.81],the phase factors of kinetic terms cancel on the conjugate pair.Thus,the R-invariance is a constraint on the mass term and potential term.Thus the Lagrangian has R-invariance only if the total R-characters of products superfields of mass and potential terms need to be integer.For instance,we use $\psi_i\psi_j A_k$ in a mass term of [4.81],the R transformation gives $e^{2i(n_i-\frac{1}{2})}\psi_i e^{2i(n_j-\frac{1}{2})}\psi_j e^{2in_k\alpha}A_k$ as [4.88].This term is R-invariance only if $n_i - \frac{1}{2} + n_j - \frac{1}{2} + n_k = n_i + n_j + n_k - 1 \in \mathbb{Z}$,that is just $n_i + n_j + n_k \in \mathbb{Z}$.

4.8 Vector superfields

Vector superfield satisfies [4.58]

$$V = V^\dagger \tag{4.89}$$

which means a vector superfield needs to contain conjugate pair.Also,it should be understood as the power serious expansion of $\theta, \overline{\theta}$ over all $\theta, \overline{\theta}$ index-contractions with component fields with certain indices.

$$
\begin{aligned}
V(x,\theta,\overline{\theta}) = {} & C(x) + i\theta^\alpha\chi_\alpha - i\overline{\theta}^{\dot{\alpha}}\overline{\chi}_{\dot{\alpha}}(x) \\
& + \frac{i}{2}\theta^\alpha\theta_\alpha[M(x)+N(x)] - \frac{i}{2}\overline{\theta}^{\dot{\alpha}}\overline{\theta}_{\dot{\alpha}}[M(x)-iN(x)] - \theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\nu_m(x) \\
& + i\theta\theta\overline{\theta}^{\dot{\alpha}}\left[\overline{\lambda}_{\dot{\alpha}} + \frac{i}{2}\overline{\sigma}^{m\alpha}_{\dot{\alpha}}\partial_m\chi_\alpha(x)\right] - i\overline{\theta\theta}\theta^\alpha\left[\lambda_\alpha(x) + \frac{i}{2}\sigma^{m\dot{\alpha}}_{\alpha}\partial_m\overline{\chi}_{\dot{\alpha}}(x)\right] \\
& + \frac{1}{2}\theta^\alpha\theta_\alpha\overline{\theta}^{\dot{\alpha}}\overline{\theta}_{\dot{\alpha}}\left[D(x) + \frac{1}{2}\Box C(x)\right]
\end{aligned} \tag{4.90}
$$

The component fields C, D, M, N and ν_m must all be real for forming conjugation pairs.The vector field ν_m can be entire multiplet.The particular combinations of components fields in $\theta\theta\overline{\theta}.c, \overline{\theta\theta}\theta.c$ and $\theta\theta\overline{\theta\theta}.c$ follow from the chiral superfield $\Phi + \Phi^\dagger$ from [4.64],[4.67] and it satisfy Hermitian condition [4.89].

$$(\Phi + \Phi^\dagger) = (\Phi^\dagger + \Phi) = (\Phi + \Phi^\dagger)^\dagger \tag{4.91}$$

So it is a vector superfield of addition of a Hermitian pair of a chiral superfield.

$$
\begin{aligned}
\Phi + \Phi^\dagger = {} & A + A^* + \sqrt{2}(\theta\psi + \overline{\theta\psi}) + \theta\theta F + \overline{\theta\theta}F^* + i\theta\sigma^m\overline{\theta}\partial_m(A - A^*) \\
& + \frac{i}{\sqrt{2}}\theta\theta\overline{\theta}\overline{\sigma}^m\partial_m\psi + \frac{i}{\sqrt{2}}\overline{\theta\theta}\theta\sigma^m\partial_m\overline{\psi} + \frac{1}{4}\theta\theta\overline{\theta\theta}\Box(A + A^*)
\end{aligned} \tag{4.92}
$$

we have used$-\frac{i}{\sqrt{2}}\theta\theta\partial_m\psi\sigma^m\overline{\theta} = -\frac{i}{\sqrt{2}}\theta\theta(-\overline{\theta}\overline{\sigma}^m\partial_m\psi)$.In [4.92],there is a gradient $i\partial_m(A - A^*)$ as coefficient of $\theta\sigma^m\overline{\theta}$.This leads us define a supersymmetic generalisation of a gauge transformation(or phase transformation) on [4.90].

$$\begin{aligned}
V \to \exp&\left\{\frac{i}{V}\left[(A - A^*) - i\int(A + A^* + \theta\theta F + \overline{\theta\theta}F^*)dz^2\right.\right. \\
&\left.\left.+ \frac{1}{\sqrt{2}}\theta\theta\overline{\theta}\overline{\sigma}^m\psi + \frac{1}{\sqrt{2}}\overline{\theta\theta}\theta\sigma^m\overline{\psi} + \frac{1}{4}\theta\theta\overline{\theta\theta}\partial^m(A + A^*)\right]\right\}V \\
= V + i\partial_m&\left[(A - A^*) - i\int(A + A^* + \theta\theta F + \overline{\theta\theta}F^*)dz^2\right. \\
&\left.+ \frac{1}{\sqrt{2}}\theta\theta\overline{\theta}\overline{\sigma}^m\psi + \frac{1}{\sqrt{2}}\overline{\theta\theta}\theta\sigma^m\overline{\psi} + \frac{1}{4}\theta\theta\overline{\theta\theta}\partial^m(A + A^*)\right]\frac{1}{V}V \\
= V + \Phi &+ \Phi^\dagger
\end{aligned} \tag{4.93}$$

Indeed this is gauge transformation and the $V+\Phi+\Phi^\dagger$ is also a vector superfield by [4.91] as V,thus this is a supersymmetric generalisation.And from [4.90] in that particular combinations with [4.92] formed from [4.93],we clearly get

$$\begin{aligned}
C &\to C + A + A^* \\
\chi &\to \chi - i\sqrt{2}\psi \\
M + iN &\to M + iN - 2iF \\
\nu_m &\to \nu_m - i\partial_m(A - A^*) \\
\lambda &\to \lambda \\
D &\to D
\end{aligned} \tag{4.94}$$

under this supersymmetric gauge transformation.

5 Classification of superstring theories

5.1 Spinors in various dimensions

Now,we have used a whole section for basics of SUSY,because we need to accumulate enough physics intuition for further study.We continue our discussion at the last of section 3.For generalized case,the generator of Lorentz algebra

[4.2] in various dimension d with signature $(d-1,1)$ is $\Sigma^{\mu\nu} = -i/4[\Gamma^\mu, \Gamma^\nu]$ with

$$\begin{aligned}
& i[\Sigma^{\mu\nu}, \Sigma^{\sigma\rho}] = \frac{-i}{16}\big[[\Gamma^\mu, \Gamma^\nu], [\Gamma^\sigma, \Gamma^\rho]\big] = \frac{i}{16}\big[[\Gamma^\mu, \Gamma^\nu], \Gamma^\sigma\Gamma^\rho\big] - \frac{i}{16}\big[[\Gamma^\mu, \Gamma^\nu], \Gamma^\rho\Gamma^\sigma\big] \\
&= \frac{-i}{16}[\Gamma^{4[\mu\nu\sigma\rho]-[\rho\sigma\nu\mu]} + \Gamma^{4[\nu\mu\rho\sigma]-[\sigma\rho\mu\nu]} + \Gamma^{4[\rho\sigma\mu\nu]-[\nu\mu\sigma\rho]} + \Gamma^{4[\sigma\rho\nu\mu]-[\mu\nu\rho\sigma]}] \\
&= \frac{-i}{16}[(2\eta^{\mu\nu} - \Gamma^\nu\Gamma^\mu)\Gamma^\sigma\Gamma^\rho - (2\eta^{\rho\sigma} - \Gamma^\sigma\Gamma^\rho)\Gamma^\nu\Gamma^\mu] + ... \\
&= \frac{-i}{16}[2\eta^{\mu\nu}\Gamma^\sigma\Gamma^\rho - 2\eta^{\rho\sigma}\Gamma^\nu\Gamma^\mu - \Gamma^\nu(2\eta^{\mu\sigma} - \Gamma^\sigma\Gamma^\mu)\Gamma^\rho + \Gamma^\sigma(2\eta^{\rho\nu} - \Gamma^\nu\Gamma^\rho)\Gamma^\mu]... \\
&= \frac{-i}{16}[2\eta^{\mu\nu}\Gamma^{2[\sigma\rho]} - 2\eta^{\rho\sigma}\Gamma^{2[\nu\mu]} - \Gamma^\nu 2\eta^{\mu\sigma}\Gamma^\rho + \Gamma^\sigma 2\eta^{\rho\nu}\Gamma^\mu + \Gamma^{4[\nu\sigma\mu\rho]-[\sigma\nu\rho\mu]}]... \\
&= \frac{-i}{16}\Big\{2\Big([\eta\Gamma^2]^{\mu\nu\sigma\rho} - [\eta\Gamma^2]^{\rho\sigma\nu\mu} - [\Gamma\eta\Gamma]^{\nu\mu\sigma\rho} + [\Gamma\eta\Gamma]^{\sigma\rho\nu\mu}\Big) \\
&\quad + (2\eta^{\nu\sigma} - \Gamma^{2[\sigma\nu]})\Gamma^{2[\mu\rho]} - (2\eta^{\sigma\nu} - \Gamma^{2[\nu\sigma]})\Gamma^{2[\rho\mu]}\Big\} + ... \\
&= \frac{-i}{16}\Big\{... + 2\eta^{\nu\sigma}(4i\Sigma^{\mu\rho} + \Gamma^{2[\rho\mu]}) - 2\eta^{\sigma\nu}(4i\Sigma^{\rho\mu} + \Gamma^{2[\mu\rho]})\Big\} + ... \\
&= \frac{-i}{16}\Big\{16i\eta^{\nu\sigma}\Sigma^{\mu\rho} + \Gamma^{4\{\mu,\nu\}\sigma\rho} - \Gamma^{4\{\rho,\sigma\}\nu\mu} - \Gamma^{4\nu\{\mu,\sigma\}\rho} + \Gamma^{4\sigma\{\rho,\nu\}\mu} \\
&\quad + \Gamma^{4\{\nu,\sigma\}\rho\mu} - \Gamma^{4\{\sigma,\nu\}\mu\rho}\Big\} = [\eta\Sigma]^{\nu\sigma\mu\rho} + [\eta\Sigma]^{\mu\rho\nu\sigma} - [\eta\Sigma]^{\nu\rho\mu\sigma} - [\eta\Sigma]^{\mu\sigma\nu\rho} \\
&\quad + \circ(\Gamma^4) \propto \Gamma^{4[\mu\nu,\sigma\rho]+[\sigma\rho,\mu\nu]} + \Gamma^{4[\mu\sigma,\nu\rho]+[\nu\rho,\mu\sigma]} + \Gamma^{4[\mu\rho,\sigma\nu]+[\sigma\nu,\mu\rho]} \\
&= [\eta\Sigma]^{\nu\sigma\mu\rho} + [\eta\Sigma]^{\mu\rho\nu\sigma} - [\eta\Sigma]^{\nu\rho\mu\sigma} - [\eta\Sigma]^{\mu\sigma\nu\rho}
\end{aligned} \qquad [5.1]$$

with the observation from the commutator decomposition

$$[f_1 f_2, f_3 f_4] = f_1\{f_2, f_3\}f_4 - f_1 f_3\{f_2, f_4\} + \{f_1, f_3\}f_2 f_4 - f_3\{f_1, f_4\}f_2 \quad [5.2]$$

We showed the details [5.1] for completeness without explanation for simplicity.And exactly we see the algebra closes and represents Lie algebra $g_{SO(d-1,1)}$. From [3.75] and apply [5.1] we can see the generators $\Sigma^{2a,2a+1}$ commute,and we can use it to define an operator

$$S_a = i^{\delta_{a,0}}\Sigma^{2a,2a+1} = i^{\delta_{a,0}}\frac{-i}{4}(\Gamma^{2a}\Gamma^{2a+1} - \Gamma^{2a+1}\Gamma^{2a} - 2) - \frac{1}{2} = \Gamma^{a+}\Gamma^{a-} - \frac{1}{2} \quad [5.3]$$

which let $\zeta^{(\mathbf{s})}$ in [3.79] be its eigenstate with eigen value s_a,for instance

$$S_a\Gamma^{a+}\zeta = (\Gamma^{a+}\Gamma^{a-} - \frac{1}{2})\Gamma^{a+}\zeta = (\Gamma^{a+}(1 - \Gamma^{a+}\Gamma^{a-}) - \frac{1}{2}\Gamma^{a+})\zeta = \frac{1}{2}\Gamma^{a+}\zeta \quad [5.4]$$

We want to generalize Γ_5 to various dimension which leads us to define $\Gamma = i^{-k}\prod_{\mu=0}^{d-1}\Gamma^\mu$ that is

$$\Gamma = i^{-k}\prod_{a=0}^{k}\Gamma^{2a}\Gamma^{2a+1} = -\prod_{a=0}^{k}(-2\Gamma^{a+}\Gamma^{a-} + 1) = 2^{k+1}\prod_{a=0}^{k}S_a \qquad [5.5]$$

with eigenvalue $2^k \prod_{a=0}^{k} s_a = \pm 1$,we can diagonalize Γ to $\Gamma_{\mathbf{ss'}} = \sum_{\mathbf{ss'}} \zeta^{(\mathbf{s})\dagger}\Gamma\zeta^{(\mathbf{s'})}$ by the eigenvalues that are $+1$ for even $s_a = -1/2$ and -1 for odd $s_a = -1/2$.And the two states with eigenvalue as chirality split representation of Lorentz algebra to two Weyl representations over $\mathbb{C}$ we discussed in [4.4].For superstring case $d = 10$ and for the representation ρ.

$$\mathbf{32}_{\text{Dirac}} \in \rho(g_{SO(10)}), = \mathbf{16} + \mathbf{16'} \in \rho(g_{SU(5)}) \oplus \rho(g_{SU(5)}) \qquad [5.6]$$

denote as their dimensions 2^{k+1} and 2^k for $d = 2k + 2$ and subscript Dirac denotes the Dirac representation [3.79].Also,for the matrix form below [5.5],we get $\Gamma^*_{\mathbf{ss'}} = \Gamma_{\mathbf{ss'}}$ and $\Gamma = (\sum_{\mathbf{s}} \zeta^{(\mathbf{s})\dagger})^{-1}\Gamma_{\mathbf{ss'}}(\sum_{\mathbf{s'}} \zeta^{(\mathbf{s'})})^{-1}$ that means $\Gamma^* = \Gamma$ up to changing basis,and we get S_a is real from [5.5],then $\Gamma^{a\pm}$ is real that means Γ^{2a+1} is imaginary in [3.75] which differ from the remainders that are real,thus we collect them to form a subgroup $\mathcal{B}$ with $B_1 = \prod_{2a+1=3}^{d-1} \Gamma^{2a+1}, B_2 = \Gamma B_1$

$$\mathcal{B} = \{B_1, B_2, \Gamma ...\}, \quad B_1^2 = (-1)^{\sum_{2a+1=3}^{d-1}(a-1)}, B_2^2 = \{\pm 1\}, B_2B_1 = \{\pm\Gamma\} \quad [5.7]$$

and for anticommutation $[\Gamma^{\mu *}, \Gamma^{\nu}] = 0$,we find the following conjugacy classes

$$\Gamma^{\mu} \sim_{B_1} (-1)^k\Gamma^{\mu *}, \quad \Gamma^{\mu} \sim_{B_2} (-1)^{k+1}\Gamma^{\mu *}, \quad \Sigma^{\mu\nu} \sim_{B=\{B_1,B_2\}} -\Sigma^{\mu\nu *} \qquad [5.8]$$

dividing the gamma matrices group $\text{Mat}_{\text{Dirac}} \subset \text{Mat}(\mathbb{C})$,satisfying Clifford algebra [3.74].We can reformulate [5.8] and we get

$$B\Sigma^{\mu\nu} = \Sigma^{\mu\nu}B \Rightarrow [\Sigma^{\mu\nu}, B] = 0 \qquad [5.9]$$

Thus,for Dirac representation $\rho : g_{SO(2^{d/2})} \to \text{Mat}_{\text{Dirac}}$ [3.79] with injective ρ

$$\rho^{-1}(B) \subset Z(g_{SO(d)}) = \{z \in g_{SO(d)} | zy = yz, \forall y \in g_{SO(d)}\} \qquad [5.10]$$

the center of the Clifford algebra.And we want to use a theorem 7.20 in [11] that is ρ is injective if and only if $Z(G) = \text{Ker}(\rho) = \{e\}$,which means $\rho^{-1}(B) = \{e_{g_{SO(d)}}\}\{\rho^{-1}(B_1), \rho^{-1}(B_2)\}$,then we get $B = \{e_{\text{Mat}_{\text{Dirac}}}\}$ by the injectivity.In this case,we get the property that the Dirac representation is self-dual.

$$\text{Mat}^*_{\text{Dirac}} = \text{Mat}_{\text{Dirac}}/B = \text{Mat}_{\text{Dirac}}/\{e\} = \text{Mat}_{\text{Dirac}} \qquad [5.11]$$

Also,we can put conjugate action on chirality matrix Γ

$$\Gamma \sim_B (-1)^k\Gamma^*, \qquad k^{\text{even}} \quad \text{self-dual}, k^{\text{odd}} \text{ dual to other} \qquad [5.12]$$

which give property of Weyl representation.In this case,we have an notation

$$(d = 4, k = 1), \mathbf{4}_{\text{Dirac}} = \mathbf{2} + \bar{\mathbf{2}}(= \mathbf{2'}) \quad (d = 10, k = 4), \mathbf{32}_{\text{Dirac}} = \mathbf{16}(= \bar{\mathbf{16}}) + \mathbf{16'} \qquad [5.13]$$

B need to consistent with Lorentz transformation see example [4.5],which means it contains the spinor index and preserve index contraction and notations are same with section 4.From [5.9] in explicit index form,we have

$$B_{\dot{\alpha}}{}^{\alpha}\Sigma^{\mu\nu\dot{\alpha}}_{\alpha} = \Sigma^{\mu\nu\dot{\alpha}}_{\alpha}B_{\dot{\alpha}}{}^{\alpha} \quad \Rightarrow \quad B_{\dot{\alpha}}{}^{\alpha} = \zeta^*_{\dot{\alpha}}(\zeta_{\alpha})^{-1} \quad \Rightarrow \quad \zeta^*_{\dot{\alpha}} = B_{\dot{\alpha}}{}^{\alpha}\zeta_{\alpha} \qquad [5.14]$$

And we end with a Majorana condition.Following from [5.7],we have

$$B_1^*B_1 = (-1)^k B_1^2 = (-1)^{\frac{k(k+1)}{2}}, B_2^*B_2 = (-1)^{k+1}(-1)^{\frac{d(d+1)}{2}} B_1^2 = (-1)^{\frac{k(k-1)}{2}} \quad [5.15]$$

Condition [5.14] can be translated to $(\zeta^*)^* = B^*B\zeta \Rightarrow B^*B = 1$.In this case,if $k \bmod 4 = 0,3$ B_1 is open,if $k \bmod 4 = 0,1$ B_2 is open in the Majorana condition.B_1, B_2 are physically equivalent,we can see the equivalence above [5.11] in mathematical structure.If we regard B as a map,[5.14] induces a self-dual representation.Majorana condition is open on Weyl spinor only if $k \bmod 4 = 0, d \bmod 8 = 2$ which is in self-dual Weyl representation [5.12].In this case,this Majorana-Weyl condition is open on the spinors in the spacetime (D=10) and world-sheet (d=2) of superstring theory.There is a duality between Majorana and Weyl,in superstring theory the two sides of the duality preserve,and one or two is closed for other case.We can see the duality by the chirality projection operators $\mathrm{P}_\pm = (1 \pm \Gamma)/2$

$$\zeta'_{\mathrm{Weyl}} = \mathrm{P}_+\chi = \frac{1+\Gamma}{2}\chi, \quad (\frac{\chi}{2})'_{\mathrm{Maj}} = \frac{1}{2}(\zeta + B^*\zeta^*) = \frac{1+B^*B}{2}\zeta = \zeta'_{\mathrm{Weyl}} \quad [5.16]$$

where we used a fact that a Weyl or Majorana spinor is the object transformed under Weyl or Majorana representation.The $-\Gamma^{\mu T}$ satisfy Clifford algebra,we can consider charge conjugation in various dimension in Dirac representation

$$C^\mu = \tau^\mu_{2^k \times 2^k} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{with} \quad \tau^\nu \gamma^\mu_{2^k\times 2^k}(\tau^\nu)^{-1} = \gamma^{\mu T} \quad [5.17]$$

for $\mu, \nu = 0...d-3$,then the charge conjugation on Γ^μ gives

$$\begin{aligned} C^\nu\Gamma^\mu(C^\nu)^{-1} &= C\gamma^\mu \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} C^{-1} \\ &= (\tau\gamma\tau^{-1})^\mu \otimes \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \\ &= \gamma^{\mu T} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma^{\mu T} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^T = -\Gamma^{\mu T} \end{aligned} \quad [5.18]$$

by induction we get a C for various dimension with $C\Gamma^\mu C^{-1} = -\Gamma^{\mu T}$ the antihermiticity and hermiticity of Γ^μ in various dimension

$$\begin{aligned} &\Gamma^{\mu\dagger} = \Gamma^\nu\eta_{\mu\nu} = -\Gamma^0\Gamma^\mu(\Gamma^0)^{-1} = \gamma^0\gamma^\mu(\gamma^0)^{-1} \otimes [\sigma^3\sigma^3(\sigma^3)^{-1}] \\ &= \gamma^0\gamma^\mu(\gamma^0)^{-1} \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{with}\, \gamma^0(\gamma^0, \gamma^{\mu,\neq 0})(\gamma^0)^{-1} = (-\gamma^0, \gamma^{\mu,\neq 0}) \end{aligned} \quad [5.19]$$

where we showed details with [4.6] and we do not explain for simplicity.And we combine [5.18] and [5.19],we get

$$\begin{aligned} &\Gamma^{\mu *} = (-\Gamma^0\Gamma^\mu(\Gamma^0)^{-1})^T = (\Gamma^0)^{-1^T}(-\Gamma^{\mu T})\Gamma^{0^T} = \Gamma^0 C\Gamma^\mu C^{-1}(\Gamma^0)^{-1} \\ &= -C\Gamma^0\Gamma^\mu(\Gamma^0 C)^{-1} = C\Gamma^0\Gamma^\mu(C\Gamma^0)^{-1} \end{aligned} \quad [5.20]$$

where we used $\Gamma^0 C = -C\Gamma^0, \Gamma^0 = (\Gamma^0)^{-1} = {\Gamma^0}^T$.We can compare this with [5.8] and use $(\Gamma^0)^2 = 1$,we find clear expression for [5.17]

$$C = B_1\Gamma^0, k = 2n, \quad C = B_2\Gamma^0, k = 2n+1, \quad n \in \mathbb{Z} \tag{5.21}$$

And we easily to calculate the conjugacy of Lorentz generator with [3.74]

$$\begin{aligned} C\Sigma^{\mu\nu}C^{-1} &= \frac{-i}{4}B\Gamma^0(\Gamma^\mu\Gamma^\nu - \Gamma^\nu\Gamma^\mu)(\Gamma^0)^{-1}B^{-1} \\ &= \frac{-i}{4}B[(-\Gamma^\mu\Gamma^0 + 1)\Gamma^\nu - (-\Gamma^\nu\Gamma^0 + 1)\Gamma^\mu](\Gamma^0)^{-1}B^{-1} \\ &= B\Sigma^{\mu\nu}B^{-1} = -\Sigma^{\mu\nu *} \end{aligned} \tag{5.22}$$

5.2 Spinor product decomposition

There is a natural antisymmetrization from wedge algebra to tensor algebra

$$\bigwedge(V)^p = T(V)/\mathfrak{b} \to T(V)^\mu, \quad A_p \wedge B_q \mapsto \frac{(p+q)}{p!q!}A_{[\mu_1\ldots\mu_p}B_{\mu_{p+1}\ldots\mu_{p+q}]} = \sum_{n\geq 0}[n] \tag{5.23}$$

where T defined in [7.17] with a normal vector space V here,and $\mathfrak{b}$ is graded ideal generated from anticommutative relation.$[n]$ means a set of antisymmetric n-tensor.Subscript p,q for forms and μ for degree of tensors.This antisymmetrization makes the wedge algebra isomorphic to a tensor subalgegra.And isomorphism induces an isomorphism on basis,which means we can use a basis in tensor subalgebra to generate the wedge algebra,which is spinor product decomposition.A spin presentation is a wedge algebra which means we can construct product wedge algebra by [3.79] and all elements in this product algebra can be decomposed in tensors which is a [0]-module for [0] be a field

$$\begin{aligned} \bigwedge(V_{\zeta^*}) \times \bigwedge(V_\chi) &= \mathrm{tr}\left(\left\{\prod_{a=0}^{k}(\Gamma^{a+})^{s_a+1/2}\zeta\right\} \times \left\{\prod_{b=0}^{k}(\Gamma^{a+})^{s_b+1/2}\chi\right\}\right) \\ &= \mathrm{tr}\left\{\sum_{s_a,s_b}\left(\prod_{a=0}^{k}(\Gamma^{a+})^{s_a+1/2}\prod_{b=0}^{k}(\Gamma^{a+})^{s_b+1/2}\right)_{\dot\alpha\alpha}\zeta^{*\dot\alpha}\chi^\alpha\right\} \\ &= \mathrm{tr}\left\{\sum_{s_a}\left(\prod_{a=0}^{k}(\Gamma^{a+})^{s_a+1/2}\right)_{\dot\alpha\alpha}B^{\dot\beta}_{\beta}\Gamma^{0\beta}_{\dot\beta}\zeta^{*\dot\alpha}\chi^\alpha\right\} \\ &\cong \mathrm{Span}(\left\{\zeta^{*\dot\alpha}C(\Gamma^{[a_1+}\Gamma^{a_2+}\ldots\Gamma^{a_{k+1}+]})_{\dot\alpha\alpha}\chi^\alpha\right\}) \\ &= \mathrm{Span}(\left\{\zeta C\Gamma^{[\mu_1}\Gamma^{\mu_2}\ldots\Gamma^{\mu_\mu]}\chi\right\}) = \sum_{n=0}^{\mu}[n] \end{aligned} \tag{5.24}$$

where the upper [ ] means fully antisymmetrized product as in [5.23] and $C \in [0]$ also is charge conjugation [5.21].And we applied opinion in [7.18] and isomorphism [5.23],and the prefractor for canceling double counting was abosorbed in constant C,also we changed of basis from $a = 0, ..., k$ to $\mu = 0, ..., d$.This finishes (B.1.32) in [3],and we do not explain more for simplicity.Because Poincare duality on homology inherits to Hodge duality on tensors,and the tensor representation reduces.We can clearly see Hodge star from [5.5]

$$\begin{aligned}
&i^{-k}(\epsilon^{\mu_1\cdots\mu_d}\Gamma_{\mu_1\cdots\mu_d}) = d! i^{-k}\Gamma^0\Gamma^1...\Gamma^{d-1} = d!\Gamma \\
&i^{-k}(-1)^{\frac{s(s-1)}{2}}\epsilon^{\mu_1\cdots\mu_d}\Gamma_{\mu_{s+1}\cdots\mu_d}\Gamma_{\mu_1\cdots\mu_s} = d!\Gamma \\
&-i^{-k}(-1)^{\frac{s(s-1)}{2}}\Gamma_{\mu_1\cdots\mu_s}(\epsilon^{\mu_1\cdots\mu_d}\Gamma_{\mu_{s+1}\cdots\mu_d}) = \frac{d!}{A_s^d}\Gamma \\
&(\Gamma_{\mu_1\cdots\mu_s})^{-1}\Gamma = -\sqrt{-1}^{s(s-1)}i^{-k}\frac{d!/(d-s)!}{d!}\epsilon^{\mu_1\cdots\mu_d}\Gamma_{\mu_{s+1}\cdots\mu_d} \\
&\Gamma^{\mu_1\cdots\mu_s}\Gamma = -\frac{i^{-k+s(s+1)}}{(d-s)!}\epsilon^{\mu_1\cdots\mu_d}\Gamma_{\mu_{s+1}\cdots\mu_d}
\end{aligned} \qquad [5.25]$$

where we set $\Gamma^{\mu_1,\dots,\mu_m} = \Gamma^{[\mu_1}\Gamma^{\mu_2}...\Gamma^{\mu_m]}$,and we get Hodge star $* = \times\Gamma$.For even dimension $d = 2k+2$,Γ is a non zero constant,thus s-tensors are Hodge dual to a new copy of $(d-s)$-tensors.But in odd dimension $d = 2k+3$,we set $\Gamma^d = \pm\Gamma$ that means they are Hodge dual to same copy because they are linearly dependent in [5.25] now.In this case,we have decompositioin

$$\begin{aligned}
&d = 2k+3 \quad \mathbf{2^{k+1}} \times \mathbf{2^{k+1}} = \sum_{n=0}^{k+1}[n] + *\sum_{n=k+2}^{d}[n] = [0]+[1]+...+[k+1] \\
&d = 2k+2 \quad \mathbf{2}^{\mathbf{k+1}}_{\text{Dirac}} \times \mathbf{2}^{\mathbf{k+1}}_{\text{Dirac}} = \sum_{n=0}^{d}[n] = \sum_{n=0}^{k+1}[n] + \left(*\sum_{n=k+2}^{d}[n]\right)' \\
&\qquad\qquad = [0]^2+[1]^2+...+[k]^2+[k+1]
\end{aligned} \qquad [5.26]$$

An observation is the setting $\Gamma^d = \pm\Gamma$ is actually a dimension reduction condition and a d-dim spin representation naturally lives in that of higher dimension

$$r: \quad \Gamma^0...\Gamma^d = \Gamma^0...\Gamma^d\Gamma = \Gamma^0...\Gamma^d(\Gamma^{d+1} = \Gamma) \quad \in \quad \Gamma^0...\Gamma^d\Gamma^{d+1} \qquad [5.27]$$

By using the dimension reduction we construct a reduction r consists of chirality matrix [5.5] and projection [5.16],$r \sim \Gamma^2$,which means we can reduce 2 dimensions to form this combination in representation.the [5.24] separates to

$$\begin{aligned}
&\bigwedge_d(V_{\zeta^*}) \times \bigwedge_d(V_\chi) = \mathbf{2^{k+1}} \times \mathbf{2^{k+1}} \overset{r}{=} \bigwedge_{d-2}(\mathrm{P}_\pm V_{\zeta^*}(\Gamma)) \times \bigwedge_{d-2}(\mathrm{P}_\pm V_\chi(\Gamma)) \\
&= \mathrm{P}^{\mathrm{r}}_{\mathrm{r}}\left[\bigwedge_{d-2}(V_{\zeta^*}) \times \bigwedge_{d-2}(V_\chi)\right]_{\mathrm{rr}'}(C\Gamma\Gamma) = \mathrm{tr}(\mathrm{P}(\bigwedge ... \times \bigwedge ...))((-1)^{k+n+1}\Gamma C\Gamma) \\
&= (-1)^{k+n+1}\Gamma^{\mathrm{r}}_{\mathrm{P}r}\left[\bigwedge_{d-2}(V_{\Gamma\zeta^*}) \times \bigwedge_{d-2}(V_{\Gamma\chi})\right]_{\mathrm{rr}'} = \mathbf{2^k} \times \mathbf{2^k} + \mathbf{2^{k'}} \times \mathbf{2^{k'}} + 2\mathbf{2^k} \times \mathbf{2^{k'}}
\end{aligned} \qquad [5.28]$$

where we define matrix projection $\mathrm{P}^{\mathrm{rr}'}$ with $r, r' = \pm$ and $\mathrm{P}^{++} = \mathrm{P}_+\mathrm{P}_+$ which means the first part in product has chirality 1 and the second has also 1 and we used [5.18] n for n-tensor.And $\Gamma_{\mathrm{P}}^{+-} = 1 \times (-1)$.Clearly,the dimensionality agree in [5.28].The orthogonality gives constraint on [5.28] that is $(-1)^{k+n+1}\Gamma_{\mathrm{P}}^{\mathrm{rr}'} = 1$.Which gives a constraint to classify the decomposition in tensors [5.26].

$$\begin{aligned} \mathbf{2^k} \times \mathbf{2^k} &= \begin{cases} [1] + [3] + ... + [k+1]_{++},\, k\,\text{even} \\ [0] + [2] + ... + [k+1]_{++},\, k\,\text{odd} \end{cases} \\ \mathbf{2^{k'}} \times \mathbf{2^{k'}} &= \begin{cases} [1] + [3] + ... + [k+1]_{--},\, k\,\text{even} \\ [0] + [2] + ... + [k+1]_{--},\, k\,\text{odd} \end{cases} \\ \mathbf{2^k} \times \mathbf{2^{k'}} &= \begin{cases} [0] + [2] + ... + [k]_{+-},\, k\,\text{even} \\ [1] + [3] + ... + [k]_{+-},\, k\,\text{odd} \end{cases} \end{aligned} \qquad [5.29]$$

5.3 Decomposition under subgroups

Notice that for [4.4] in various dimension that is $SO(2l) \to SU(l) \times SU(l)$,the decomposition of representation of the right product is to set $2l = 2k+2, k = l-1$ in [5.29].Above case is trivial,the representation of two sides are same.But,we can use that to study decomposition under typical subgroups

$$SO(2k+2) \to SO(2l+2) \times SO(2k-2l) \cong_\rho (SU(l+1) \times SU(k-l))^2 \qquad [5.30]$$

We can only focus on one sector of product,and set $k+1 \to k$ in [5.28]

$$\begin{aligned} (\mathbf{2^k})^2 &= (2^{\mathbf{k\text{-}1}} + 2^{\mathbf{k\text{-}1}'})^2 \Rightarrow \mathbf{2^k} = 2^{\mathbf{k\text{-}1}} + 2^{\mathbf{k\text{-}1}'} \\ &= \mathrm{P}_{\mathrm{r}}^{\mathrm{r}}\rho(g_{SU(l+1-1)\times SU(k-l-1)}) \\ \mathbf{2^k} &= \mathbf{2^l} \times \mathbf{2^{k\text{-}l\text{-}1}} + \mathbf{2^{l'}} \times \mathbf{2^{k\text{-}l\text{-}1'}}, \qquad \mathbf{2^{k'}} = \mathbf{2^{l'}} \times \mathbf{2^{k\text{-}l\text{-}1}} + \mathbf{2^l} \times \mathbf{2^{k\text{-}l\text{-}1'}} \end{aligned} \qquad [5.31]$$

Next,we want to study $SO(2n)/U(1) \to SU(n)$,we know $\rho(g_{SU(n)})$.Recall we have a ground state condition [3.78],we can regard it as a local conserved current [3.17] by a map $\Gamma^{a-} \to \partial$,and it is invariant under $U(n)$ rotation,$M\partial\zeta = M0 = 0$ for $M \in U(n)$.In this case,it does gives a conserved quantity,that is global on the orbit of $U(n)$.Also,conserved law gives $\zeta \in [0]$.For this,we can define a $U(1)$ charge by just $(1/2\pi i)\oint \zeta dz = -n$.From [3.75] over $\mathbb{C}$,$|\Gamma^{a+}|^2 = 1/2$,we can assign $\det\Gamma^{a+} = 1$ and make them into $SU(n)$,In this case,a decomposition under $SO(2n) \to SU(n) \times U(1)$ is just a charge decomposition on the original $SU(n)$,and from [5.24],$[n+1]$ is based on $[n]$ added by a Γ^μ with charge of $SU(n)$.And we know expression of ζ above [3.78],we get charge of Γ^{a-} is $-n/n = -1$ which means charge of Γ^{a+} is $+1$.In this case,we have enough reasons to assign charge of Γ^μ $+2$ from $\Gamma^{a+} = (1/2)(\Gamma^{2a} + i\Gamma^{2a+1})$.Then we get

$$\mathbf{2^n} = [0]_{-n} + [1]_{2-n} + ... + [n]_n, \quad \text{under} \quad (\Gamma^\mu, \zeta) \in SU(n) \times U(1) \qquad [5.32]$$

In this case we have $\mathbf{2^3} = [0]_{-3} + [1]_{-1} + [2]_1 + [3]_3$,this tensor with charge need to have a modified Hodge duality that is ordinary version with charge conjugation

$$*_C : \rho(g_{SU(n)\times 1}) \to \rho(g_{1\times SU(n)}), \qquad [n-1]_{n-2} \mapsto \overline{[1]}_{n-2} \qquad [5.33]$$

Notice the subscript.By using [5.33] we have further classification

$$\mathbf{2^3} = [0]_{-3} + \overline{[0]}_3 + [1]_{-1} + \overline{[1]}_1, \quad \mathbf{2^2} = [0]_3 + [1]_{-1}, \, \bar{\mathbf{2}}^{\mathbf{2}} = [0]_{-3} + \overline{[1]}_1 \quad [5.34]$$

where we use reality $[0] = \overline{[0]}$ and $\mathbf{4}, \bar{\mathbf{4}}$ should have opposite total charge.

5.4 Fermionic state with bosonisation

Now,we can continue from [3.74].Dirac representation [3.79] can be used to construct R ground state in $D = 10$,that is an eigenstate of S_a [5.3]

$$|s_0, s_1, ..., s_4\rangle_{\mathrm{R}} = \prod_{a=0}^{4} (\Gamma^{a+})^{s_a + 1/2} \zeta = |\mathbf{s}\rangle_{\mathrm{R}}, \quad s_a = \pm\frac{1}{2} \quad [5.35]$$

Because of the discussion above [4.13],we need an operator $e^{\pi i F}$ and the world sheet fermion number is defined mod 2.We can see it anticommute with ψ

$$e^{\pi i F}\psi = \sum_n \frac{(\pi i)^n F^n}{n!}\psi = \sum_n \frac{(\pi i)^n}{n!}\psi(F+1)^n = \psi e^{\pi i(F+1)} = -\psi e^{\pi i F} \quad [5.36]$$

For [3.61] we have Clifford algebra $\{\sqrt{2}\psi_r^\mu, \sqrt{2}\psi_{-r}^\lambda\}\{C_2\} = 2\eta^{\mu\lambda}$ apply to the original Lorentz generator above [5.1],we get $\Sigma^{\mu\lambda} = -i/2\sum_{r\in\mathbb{Z}+v}[\psi^{\mu_r}, \psi_{-r}^\lambda]$.In this case we can define $F = \sum_{a=0}^4 S_a$,we can see with [5.3]

$$2S_1\psi_r^{1\pm} = S_1(\psi_r^2 \pm i\psi_r^3) = \frac{-i}{2}\sum_s (\psi_s^2\psi_{-s}^3 - \psi_{-s}^3\psi_s^2)(\psi_r^2 \pm i\psi_r^3)$$

$$= \frac{-i}{2}\sum_s [-\psi_s^2\psi_r^2\psi_{-s}^3 - \psi_r^3 + \psi_{-s}^3\psi_r^2\psi_s^2 \pm i\psi_r^2 \mp i\psi_s^2\psi_r^3\psi_{-s}^3 \pm i\psi_{-s}^3\psi_r^3\psi_s^2] = \frac{-i}{2}$$

$$\sum_s [-\psi_r^3 + \psi_r^2\psi_{-s}^2\psi_{-s}^3 - \psi_r^3 - \psi_r^2\psi_{-s}^3\psi_s^2 \pm i\psi_r^2 \pm i\psi_r^3\psi_s^2\psi_{-s}^3 \pm i\psi_r^2 \mp i\psi_r^3\psi_{-s}^3\psi_s^2]$$

$$= \frac{-i}{2}[(\psi_r^2 \pm i\psi_r^3)(\psi_s^2\psi_{-s}^3 - \psi_{-s}^3\psi_s^2)] \pm (\psi_r^2 \pm i\psi_r^3)$$

$$= 2\psi_r^{1\pm}S_1 \pm 2\psi_r^{1\pm} = 2\psi_r^{1\pm}(S_1 \pm 1)$$

$$F\psi_r^{1\pm} = \sum_{a=0}^4 S_a\psi_r^{1\pm} = S_1\psi_r^{1\pm} + \psi_r^{1\pm}\sum_{a=0,\neq 1}^4 S_a = \psi_r^{1\pm}\sum_{a=0}^4 S_a \pm \psi_r^{1\pm} \quad [5.37]$$

$$= \psi_r^{1\pm}(F+1)$$

where we use this notation in [3.74] and [3.75] and the subtile point is we need to individually treat the case $\delta_{r,s}, \delta_{r,-s} \neq 0$ in [3.61].And indeed the world sheet fermion number operator counts the number.For closed string from [3.58],NS-NS states have integer spin,R-R states the two half-integers add to a integer,R-NS and NS-R have half-integer spin.

We can apply state-operator isomorphism [3.72] for NS fermionic state [3.73]

$$\psi^\mu_{-r} = \oint_C \frac{dz(-1)^{r-1/2}}{(r-1/2)2\pi i}\partial^{r-1/2}z^{-1}\psi^\mu(z) = \frac{1}{(r-1/2)!}\partial^{r-1/2}\psi^\mu(0) \qquad [5.38]$$

where we used [3.58] and partial integral.And we now we need a version of [3.36] for superconformal variance.We start with [3.62]

$$T_F(z)\mathscr{A} = \sum_r z^{-r-3/2}\oint_C \frac{dz}{2\pi i}z^{r+1/2}T_F(z)\mathscr{A} = \sum_r z^{-r-3/2}G_r\cdot\mathscr{A}(0,0) \quad [5.39]$$

and compare it with [3.35] we get $r = n-1/2$ and $\mathscr{A}^{(n)} = G_{n-1/2}\cdot\mathscr{A}(0,0)$ and

$$\delta_\eta\mathscr{A}(z,\bar z) = -\epsilon\sum_{n=0}^{\infty}\frac{1}{n!}\Big[\partial^n\eta(z)G_{n-1/2} + (\partial^n\eta(z))^*\tilde G_{n-1/2}\Big]\cdot\mathscr{A}(z,\bar z) \qquad [5.40]$$

And for R sector vertex operators,because $\psi^\mu(z)\propto z^{-1/2}$ in [3.57] there is a brunch cut.It is complicate to solve in bosonic case,when we make orbifold twist state that also gives a brunch cut in 8.5 of [2],this is an inspiration to [7.31] that orbifold can give effect like bunch cut.But now,we can use bosonisation to simplify all things.

For setting $\alpha' = 2$ in [3.12] we get $\overbrace{H(z)H}(0) = \ln z$ for scalar $H(z)$.We have

$$\begin{aligned}
:e^{iH(z)}::e^{-iH(0)}: &= \exp(\frac{\alpha'}{2}k_1k_2\ln z):e^{ik_1\cdot H(z)}e^{ik_2H(0)}: \\
&= e^{-\ln z}:(1+z\partial)e^{iH(0)}e^{-iH(0)}:= \frac{1}{z} \\
:e^{iH(z)}::e^{iH(0)}: =&: e^{-iH(z)}::e^{-iH(0)}:= \circ(z)
\end{aligned} \qquad [5.41]$$

where $k_1 = 1, k_2 = -1$ and we used Taylor expansion.The matter contractions without self-contractions in scattering amplitude of S_2 (6.2.17) in [2] gives us expectation value of such exponentials in general

$$\Big\langle\prod_i e^{ik_iH(z_i)}\Big\rangle = \prod_{i<j}(z_i-z_j)^{k_ik_j}\delta^d(\sum_i k_i) \qquad [5.42]$$

The delta function is momentum conservation that gives constraint on k_1, k_2 for [5.41].And for fermionic part,we form two Majorana-Weyl fermions by linear combination of $\psi^{1,2}(z)$ in $D=10$ spacetime we discussed below [5.15].

$$\psi = 2^{-1/2}(\psi^1 + i\psi^2), \quad \bar\psi = 2^{-1/2}(\psi^1 - i\psi^2) \qquad [5.43]$$

And the OPEs are

$$\begin{aligned}
\overbrace{\psi(z)\bar\psi}(0) &= \frac{1}{2}(\overbrace{\psi^1(z)\psi^1}(0) - \overbrace{\psi^1(z)i\psi^2}(0) + i\overbrace{\psi^2(z)\psi^1}(0) + \overbrace{\psi^2(z)\psi^2}(0)) \\
&= \frac{1}{2}(\frac{\eta^{11}}{z} - i\frac{\eta^{12}}{z} + i\frac{\eta^{21}}{z} + \frac{\eta^{22}}{z}) = \frac{1}{z} \\
\overbrace{\psi(z)\psi}(0) &= \overbrace{\bar\psi(z)\bar\psi}(0) = \circ(z)
\end{aligned} \qquad [5.44]$$

Now from the equivalence of OPE,we claim that $\psi(z)\cong e^{iH(z)}, \bar\psi(z)\cong e^{-iH(z)}$. and for world-sheet antiholomphic part,$\tilde\psi(\bar z)\cong e^{i\tilde H(\bar z)}, \tilde{\bar\psi}(\bar z) = e^{-i\tilde H(\bar z)}$

6 D-brane and algebraic generalized geometry

6.1 A physics intuition to D-brane

We want to develop details of a special geometric object in string theory which is Dp-brane,for further application.And we based on chapter 8 in [2],a note [4] and chapter 13 in [3].For a pure gauge $\Lambda(x^{25}) = e^{iAx^{25}} \cong e^{iAx^{25}} e^{i2\pi RA}$

$$A_{25} \to e^{i\theta} A_{25} = e^{i\theta}(-i\partial \ln\Lambda) \cong e^{i\theta} e^{i2\pi RA}(-i\partial \ln\Lambda) = e^{i(\theta+2\pi RA)} A_{25} = A_{25} \tag{6.1}$$

with periodicity of toroidal compactification $x^{25} \cong x^{25} + 2\pi R$ and [6.1] is for fixing the global gauge from the toroidal setting by a constant θ.And toroidal setting induces a loop path of $U(1)$ charge,which gives Wilson line to the gauge field living in the spacetime,a measurable quantity about magnetic field

$$W_q = e^{iq\oint dx^{25} A_{25}} = e^{-iq(\theta/2\pi R)\oint dx^{25}} = e^{-iq\theta} \tag{6.2}$$

which is invariant under a local gauge transformation $A \to e^{i\alpha(x^{25})} A$

$$W_q = W_q e^{i\oint \sum_{n=1}^{\infty} i^n \alpha(x^{25})^n dx^{25}} = W_q e^{i\int_0^{2\pi R} f(x^{25}) dx^{25}} = W_q e^{i[g(x^{25})]_0^{2\pi R}} \tag{6.3}$$

where we used toroidal setting for $x^{25} = 0, 0 \cong 2\pi R$.A field operator is isomorphic to a state made from path integral see details in section 3.3.In this case,adding a $U(1)$ gauge field gives a modification on path integral,and we end with non-linear sigma model coupled to a $U(1)$ gauge field A_M

$$\begin{aligned} &\int [dX] e^{-\int d\tau \left(\frac{1}{2}\dot{X}^M \dot{X}_M + \frac{m^2}{2}\right)} \delta\left(\frac{-q\partial_t A_M}{2\pi}\right) \\ &= \int [dX] e^{-S_\sigma} \int [dX] e^{-i\int d\tau q\partial_t A_M X^M} = \int [dX] e^{-(S_\sigma - \int d\tau iqA_M \dot{X}^M)} \end{aligned} \tag{6.4}$$

where t denotes Minkowski time.The contour rotation $t \to it$ and the vector $\partial_{it} = \partial_{it} t \partial_t = -i\partial_t$.In this case the canonical momentum $P_M = \partial_{\partial_{it} X^M} \mathscr{L} = \partial_{-i\partial_t X^M} \mathscr{L} = -\partial_{i\dot{X}^M} \mathscr{L} = \partial_{v^M}(-\mathscr{L})$ with Minkowski velocity $v^M = i\dot{X}^M$.And the Minkowski Hamiltonian is $H = v^M p_M + \mathscr{L}$,we get

$$\begin{aligned} &p_M = v_M + qA_M = v_\mu + v_d + qA_d, \qquad A_\mu = 0, A_d = \frac{-\theta}{2\pi R} \\ &H = v^M v_M + v^M qA_M - \frac{1}{2}(v^M v_M + m^2) - qA_M v^M = \frac{1}{2}(p_\mu p^\mu + v_{25}^2 + m^2) \end{aligned} \tag{6.5}$$

where we used μ for non-compact space and d for compact space and now we discuss $d = 25$.See [3.55],field operator can be expressed in Fourier serious,so periodicity on field gives that on Fourier serious,$e^{ip_d x^d} \cong e^{ip_d(x^d+2\pi R)}$in momentum space,which gives quantization on compact space $p_d = l/R, l \in \mathbb{Z}$,and we have $v_{25} = (2\pi l + q\theta)/2\pi R$.In BRST quantization in section 4.4 in [2],H vanishes physical states and gives mass-shell that means mass $\tilde{m}^2 = -p^\mu p_\mu$

shifted by θ from initial mass m^2 in [6.5].In strong interaction,we have a conserved quantity that is color which is extra degree of freedom for $SU(n)$ gauge theory means the representation must forms a closed loop along the world-sheet boundary to match the degrees for gluing free string world-sheets to a whole interaction world-sheet,this let us introduce Chan-paton factor on open string boundary and the representation matrix is $\lambda_{ij}^a \in U(n)$ with constraint $\mathrm{tr}(\lambda^a\lambda^b) = \delta^{ab}$.The gauge boson A_d can be generated by open string vertex operator $\partial X e^{ik\cdot X}\lambda_{ij}$ from [3.72],and we naturally have a diagonalization of A_d from that of $\lambda_{ij} \in U(n)$.Which means

$$A_d = -\frac{\theta_{ij}}{2\pi R} = -\frac{U\theta_{ij}U^{-1}}{2\pi R} = -\frac{1}{2\pi R}\mathrm{diag}(\theta_1, \theta_2, ..., \theta_n)_{ii} \qquad [6.6]$$

In BRST,the initial mass $m^2 = (1/\alpha')(N-1)$ for N the total level of oscillator

$$(\tilde{m}^2)_i^i = \frac{(2\pi l + \mathrm{tr}(\mathrm{diag}(q_1\theta_1, ..., q_n\theta_n)_{ii})^2}{4\pi^2R^2} + m^2 = \frac{(2\pi l + \sum_i q_i\theta_i)}{4\pi^2R^2} + \frac{1}{\alpha'}(N-1) \qquad [6.7]$$

we want to study $\Delta(\tilde{m}^2)_{ji} = (\theta_j - \theta_i)^2/(4\pi^2R^2)$ when $q = 1, l = 0, N = 1$,counting difference of energy between different color degree of freedom.We see if $\theta_i \neq \theta_j$ we have discrete dynamics that induces breaking of gauge group

$$U(n) \to U(r_1) \times ... \times U(r_s), \qquad \sum_{i=1}^{s} r_i = n \qquad [6.8]$$

for r_i equal θ.This gauge group breaking spontaneously breaks the vacuum expectation value.Which directly means the vacuum which is the underlying geometry that the open strings boundaries attaching on can be spitted to different geometric objects letting open strings attach,with corresponding dynamics.Which is an abstract explanation to D-brane.To see concretely,we need to apply T-duality,and this directly gives us meaning of D-brane with underlying generalized geometry.And we will see the non-trivial meaning of the insertion of delta function in [6.4],which actually a Dirichlet boundary condition.

6.2 T-duality with algebraic generalized geometry

The mass formulae for d compactified dimension for closed string is

$$m^2 = \frac{n_d n^d}{R_d R^d} + \frac{w_d w^d R_d R^d}{\alpha'} + \frac{2}{\alpha'}(N + \tilde{N} - 2), \quad K = (n, w) \qquad [6.9]$$

then a T-duality $\mathbf{T}$ for an ordered pair K is

$$\mathbf{T}K = (w, n), \quad \mathbf{T}R = \frac{\alpha'}{R} = R', \quad \mathbf{T}(p_L, p_R) = (p_L, -p_R) \qquad [6.10]$$

where the winding number w is easily calculated from [3.59]

$$\begin{aligned}
2\pi R^d w^d &= \oint_{\text{oriented}} (dX^d(z) + dX^d(\bar{z})) = \oint (dz\partial X^d + d\bar{z}\bar{\partial} X^d) \\
&= -i\left(\frac{\alpha'}{2}\right)^{1/2} \frac{2\pi}{2\pi} \oint (dz \sum_{m=-\infty}^{\infty} \frac{\alpha_m^d}{z^{m+1}} + d\bar{z} \sum_{m=-\infty}^{\infty} \frac{\tilde{\alpha}_m^d}{\bar{z}^{m+1}}) \\
&= 2\pi(\alpha'/2)^{1/2}(\alpha_0^d + \tilde{\alpha}_0^d)
\end{aligned} \quad [6.11]$$

The spacetime momentum p^d is calculated below [3.41] from current [3.24]

$$p^d = \frac{1}{2\pi i}\oint (dz j^d - d\bar{z}\tilde{j}^d) = \frac{1}{2\pi\alpha'}\oint (dz\partial X - d\bar{z}\bar{\partial}X) = (2\alpha')^{-1/2}(\alpha_0^d - \tilde{\alpha}_0^d) \quad [6.12]$$

where we have $p^d = n^d/R^d$ below [6.5] and combine two equations we get

$$\begin{aligned}
p_L^d &= (2/\alpha')^{1/2}\alpha_0^d = \frac{n^d}{R^d} + \frac{w^d R^d}{\alpha'} \\
p_R^d &= (2/\alpha')^{1/2}\tilde{\alpha}_0^d = \frac{n^d}{R^d} - \frac{w^d R^d}{\alpha'}, \qquad (n, w) \in (\mathbb{Z}, \mathbb{Z})
\end{aligned} \quad [6.13]$$

And $m^2 = (1/2)(p_L^d p_{Ld} + p_R^d p_{Rd}) + (2/\alpha')(N + \tilde{N} - 2))$ in [6.9].The discrete pair is to describe properties(energy etc.) generated by underlying geometry(vacuum).We need a *reverse quantization* to study underlying geometry,that is $\{(n, w)\} = (\mathbb{Z}, \mathbb{Z}) \to (\mathbb{R}, \mathbb{R})$.In this case

$$\{(n^d, w^d)\} = \mathbb{Z}^{(d,d)} \to \mathbb{R}^{(d,d)} = \mathbb{R}^d \oplus (\mathbb{R}^d)^* \quad [6.14]$$

Actually,generalized geometry is a field in differential geometry which is a mathematical structure describing LEE with $O(d, d, \mathbb{R})$ of string theory with T-duality group $O(d, d, \mathbb{Z})$ [9].But we want to use this idea of Double field theory and develop on our algebraic language.

A generalized module $\hat{M}$ is a graded module over a field k,$\hat{M} = M \oplus M^*$,with a structure map $\mathbf{T} \in O(d, d, k)$.A free generalized module is a generalized module $\hat{M} \cong k^{(d,d)}$ with $k^{(d,d)}$ is a k-module generated by $(x^1, ..., x^d, x'^1, ..., x'^d)$.

$$\mathbf{T} = \begin{pmatrix} 0 & A_{d\times d} \\ B_{d\times d} & 0 \end{pmatrix}_{2d\times 2d}, \qquad \mathbf{T} : M \oplus M^* \to M^* \oplus M \quad [6.15]$$

A generalized ringed space $\hat{X}$,see AG basics in section 7.2,is a topological space X with a structure sheaf $\hat{\mathcal{O}}_X$,which is

$$\hat{X} = (X, \hat{\mathcal{O}}_X), \qquad \hat{\mathcal{O}}_X(\hat{X}) = k[x^1, ..., x^d, x'^1, ..., x'^d] \quad [6.16]$$

A generalized function $\hat{f} = f \oplus f^*$ is a element in the section of generalized ringed space,with $f \in k[x^1, ..., x^d], f^* \in k[x'^1, ..., x'^d]$.Now,we want to find the coordinates in[6.16].By using T-duality [6.10] on spacetime point $X^d(z,\bar{z})$ with expression (2.7.4) in [2],we have

$$\begin{aligned}\mathbf{T}X^d(z,\bar{z}) &= \mathbf{T}(X_L^d(z)+X_R^d(\bar{z})), \qquad x_0^d = x_L^d + x_R^d \cong_{\text{OPE}} x_L^d - x_R^d \\ &= \mathbf{T}\left(x_0^d - i\frac{\alpha'}{2}(p_L^d \ln z + p_R^d \ln\bar{z}) + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{m\neq 0}\left(\frac{\alpha_m^d}{mz^m}+\frac{\tilde{\alpha}_m^d}{m\bar{z}^m}\right)\right) \\ &= X_L^d(z) - X_R^d(\bar{z}) = X'^d(z,\bar{z}), \qquad m\to -m\end{aligned} \tag{6.17}$$

In this case,we get the coordinates in [6.16] if we $X^d \cong x^d, X'^d \cong x'^d$.Now we need to study the property of this coordinates,we use [3.2]

$$\begin{aligned}&\partial_{d^\perp}X^d(z,\bar{z}) = \partial_1 X^d(z,\bar{z}) = (\partial+\bar{\partial})(X_L^d(z)+X_R^d(\bar{z})) \\ &= (\partial-\bar{\partial})(X_L^d(z)-X_R^d(\bar{z})) = -i\partial_2 X'^d(z,\bar{z}) = -i\partial_d X'^d(z,\bar{z})\end{aligned} \tag{6.18}$$

where we described the whole dimension with the d dimension compacitfied as a tube with $(d, d^\perp)$ similar to a world-sheet made from an open string wipe around with $(\sigma^2, \sigma^{2\perp} = \sigma^1)$.The [6.18] tells us a Neumann boundary condition which is trivial in ordinary coordinates is T-dual to a Dirichlet boundary condition which is non-trivial on dual coordinates.And this Dirichlet condition fixes a position on X'^d and reduce one degree of freedom in T-dual space,leaving with dim $(D-1)$ hyperplane called D$(D-1)$-brane.For getting an expression for X',we need

$$\begin{aligned}&\Delta X'^d_{ij} = [X'^d(z,\bar{z})]^{\sigma^1=\pi,i}_{\sigma^1=0,j} = \int_0^\pi dX'^d = \int_0^\pi d\sigma^1\partial_1 X'^d = -i\int_0^\pi d\sigma^1\partial_2 X^d \\ &= -i\oint \frac{i}{2}(\frac{1}{z}dz - \frac{1}{\bar{z}}d\bar{z})(z\partial - \bar{z}\bar{\partial})X^d(z,\bar{z}) \\ &= -\frac{1}{2}\oint_{\text{oriented}}(dz\partial X^d + d\bar{z}\bar{\partial}X^d + \frac{\bar{z}}{z}dz\bar{\partial}X^d - \frac{z}{\bar{z}}d\bar{z}\partial X^d) \\ &= -\frac{1}{2}2\pi(\alpha'/2)^{1/2}2(2\alpha')^{1/2}p^d = -2\pi\alpha'\Delta v^d = -2\pi\alpha'\frac{2\pi\Delta l_{ij}^d - \theta_j^d + \theta_i^d}{2\pi R^d} \\ &= -(2\pi\triangle l_{ij}^d + \theta_i - \theta_j)R'\end{aligned} \tag{6.19}$$

This is difference of two endpoints of open strings on D-brane,where we used [3.59] in [1] with $z = e^{-i(\sigma^1+i\sigma^2)}$and $\alpha_m = \tilde{\alpha}_m$ and $\alpha_0^d = (2\alpha')^{1/2}p^d$ for open string attaching on.Also,we applied [6.11] and quantization of momentum in the sense of [6.7].And we get the expression of dual coordinates for $\Delta l_{ij}^d = 0$

$$X'^d_{ii} = \theta_{ii}R' = -2\pi\alpha' A_{d,ii} \tag{6.20}$$

where we used [6.6].Which directly gives understanding about the delta function in [6.4].That is $\partial_t A_d = 0 \Rightarrow \partial_t X'^d = 0$ when t towards along d,which is only a Dirichlet condition in that field theory,but in string theory we have more properties based on this.

An interesting thing is in [6.16],because of Dirichlet boundary condition we have $\partial_d {x'}^d = 0$ gives a point $x' = (c^1, ..., c^d)$ with integration constant c,which corresponds to a prime ideal $(x' - (c^1, ..., c^d))$.This motivates us to develop generalized scheme.Also,from [6.20],actually a point in dual space corresponds a family of D-brane in the undual space.And we want to enlarge the dimension $d = D$ to the whole theory.

An affine complex pre generalized scheme is a generalized locally ringed space

$$(\mathrm{Spec}(\mathbb{C}[x'^1, ..., x'^D]), (C^\infty[x^{d+1}, ..., x^D] \oplus C^\infty[x'^1, ..., x'^d])_{\mathbb{C}^{(D-d,d)}}) \qquad [6.21]$$

where $C^\infty[x^{d+1}...x^D]$ is a sheaf of complex smooth functions on the D-brane and $C^\infty[x'^1, ..., x'^d]$ is sheaf of complex smoothe functions on the corresponding T-dual fixed space to the D-brane.

And a pre M-brane $\mathscr{M}_{\mathbf{T}}$ is the above pre scheme with typical limit $d = D$

$$\begin{aligned}
(\mathscr{M}_{\mathbf{T}}, \mathscr{P}) &= ((\mathrm{Spec}(\mathbb{C}^{(0,D)}), \cdot \oplus C^\infty[x'^1, ..., x'^D]), \mathscr{P}) \\
&= ((\mathrm{Spec}(\mathbb{C}^{(0;D,*)}), \cdot \oplus C^\infty[(x^* = 0), x'^1, ..., x'^D]), \mathscr{P}) \\
&\cong (((\mathrm{Spec}(\mathbb{C}^{(D,*)}) \oplus_{(0,D+D)} (\mathrm{Spec}(\mathbb{C}^{(*,D)})), \mathscr{P}) \qquad [6.22] \\
&\cong_{\mathrm{P}(\mathbf{T})} (\mathrm{Spec}(\mathbb{C}^{(D,*)}) \times \mathrm{Spec}(\mathbb{C}^{(*,D)}), \mathscr{P}) \\
&= (\mathrm{Proj}(\mathbb{R}^{D+1}), \mathscr{P})
\end{aligned}$$

where $*$ is an enhancing dimension and $\mathrm{P}(\mathbf{T})$ is for fusing the two D dimensional affine schemes to a $D + 1$ dimensional projective scheme,and we call this operation ***T**-fusion*,which should be a natural property of generalized geometry.And **T**-fusion works similarly in $(D + D, 0)$.With a highly nontrivial presheaf $\mathscr{P}$ on it now and we will see later.See definition below [12.12] explaining [6.22].

Then,we want to discuss orientifold.We start with unoriented string theory which is a collection of unoriented world-sheets with a gauging world-sheet parity $\Omega \in PSL(2, \mathbb{C})$ acting on,we can do it because the Möbius transformation is conformal.This collection with Chan-paton factors reduces to that of operator $\Omega = +1$ on initial states.And $\Omega : \sigma^1 \to (2)\pi - \sigma^1$ for (closed) open string

$$\begin{aligned}
&\Omega z = \Omega e^{-i(\sigma^1 + i\sigma^2)} = e^{\sigma^2} e^{+i\sigma} e^{-i2\pi} = \bar{z} \\
&\Omega X_L^d(z) = x_L^d - i\frac{\alpha'}{2} p_L^d \ln z + C \sum_{m \neq 0} \frac{\oint dz z^m \partial_z X^d}{mz} = X_L^d(\bar{z})
\end{aligned} \qquad [6.23]$$

where we put trivial constant to C and used [3.60] and [6.11].In this case,we get

$$\begin{aligned}
\Omega X^d(z, \bar{z}) &= X_L^d(\bar{z}) + X_R^d(z) = X^d(\bar{z}, z) \\
\Omega X'^d(z, \bar{z}) &= (\Omega X^d(z, \bar{z}))' = (X_R^d(z) + X_L^d(\bar{z}))' \qquad [6.24] \\
&= X_R^d(z) - X_L^d(\bar{z}) = -X'^d(\bar{z}, z)
\end{aligned}$$

where we used [6.17].And we see an unoriented string theory in the dual space is living on an orientifold $(X \cong -X) \times (z \leftrightarrow \bar{z})$,which is just a unoriented world-sheet embedded into an orbifold spacetime.

6.3 D-brane in superstring theory

Now,we can enter the case of consistent string theory.T-duality reflects right moving parts both of bosons and fermions as supersymmetry.if we let $d = 9$

$$\mathbf{T}(X_R'^9(\bar{z}), \tilde{\psi}_R'^9(\bar{z})) = (-X_R^9(\bar{z}), -\tilde{\psi}_R^9(\bar{z})), \quad \mathbf{T}(\Gamma^{4+}, \Gamma\zeta^{(\mathbf{s})}) = (\Gamma^{4-}, -\Gamma\mathbf{T}\zeta^{(\mathbf{s})}) \tag{6.25}$$

we applied first tuple in [3.75] and [5.5].For the R-R field translating into the product representation [5.24] guided by [6.23]

$$\begin{aligned}&\mathbf{T}(\bar{\mathscr{V}}\Gamma^{\mu_1\cdots\mu_p}\tilde{\mathscr{V}}) = \bar{\mathscr{V}}\Gamma^{\mu_1\cdots\mu_p}\Gamma^9\Gamma\tilde{\mathscr{V}} = \begin{cases} \exists & \mu_p = 9 \\ \forall & \mu_p \neq 9 \end{cases} \\ &= \begin{cases} \bar{\mathscr{V}}\Gamma^{\mu_1\cdots 9\cdots\mu_{p-1}}\Gamma^9\Gamma\tilde{\mathscr{V}} = \bar{\mathscr{V}}(-\Gamma^{\mu_1\cdots\mu_{p-1}})\Gamma\tilde{\mathscr{V}} = \bar{\mathscr{V}}(-\Gamma\Gamma^{\mu_1\cdots\mu_{p-1}})^\dagger\tilde{\mathscr{V}} \\ \bar{\mathscr{V}}\Gamma^{\mu_1\cdots\mu_p}\Gamma^9\Gamma\tilde{\mathscr{V}} = \bar{\mathscr{V}}\Gamma^{\mu_1\cdots\mu_p 9}\Gamma\tilde{\mathscr{V}} = \bar{\mathscr{V}}(-\Gamma\Gamma^{\mu_1\cdots\mu_p 9})^\dagger\tilde{\mathscr{V}} \end{cases}\end{aligned} \tag{6.26}$$

And we quotient an equivalence relation of Hodge duality [5.25],we get

$$\mathbf{T}(C_9, C_\mu, C_{\mu\nu 9}, C_{\mu\nu\lambda}) = (C, C_{\mu 9}, C_{\mu\nu}, C_{\mu\nu\lambda 9}) \tag{6.27}$$

which are non trivial T-duality on antysymmetric tensor in IIA to those in IIB.And for general case with T-duality on m dimensions on R-R state

$$\mathbf{T}_{R-R} = \prod_m \beta^m, \quad \beta^m = \Gamma\Gamma^m, \quad \beta^m\beta^n = e^{\pi i\tilde{\mathbf{F}}}\beta^n\beta^m \tag{6.28}$$

with spacetime fermion number $\tilde{\mathbf{F}}$ of R-states and an observation,$m = n, \tilde{\mathbf{F}} = 0$ and we have for $m \neq n, \Gamma\Gamma^m\Gamma\Gamma^n = \Gamma\Gamma^m(-\Gamma^n\Gamma) = -\Gamma\Gamma^n\Gamma\Gamma^m$.Where

$$\tilde{\mathbf{F}} = 0, \Gamma^m\Gamma^m = -1 \qquad \tilde{\mathbf{F}} = 3 \bmod 2 = 1, \Gamma^m, \Gamma^n, \Gamma^m\Gamma^n \tag{6.29}$$

And $\mathbf{T}^2\tilde{\mathscr{V}}_\alpha = e^{\pi i\tilde{\mathbf{F}}}\tilde{\mathscr{V}}_\alpha$ with an observation $\beta^m\beta^m\tilde{\mathscr{V}}_\alpha = e^{\pi i(\tilde{\mathbf{F}}+1)}(-1)\tilde{\mathscr{V}}_\alpha$ where

$$\tilde{\mathbf{F}} = 1; \Gamma^m\Gamma^m\tilde{\mathscr{V}}_\alpha = -\tilde{\mathscr{V}}_\alpha \qquad \tilde{\mathbf{F}} = 4 \bmod 2 = 0; \tilde{\mathscr{V}}_\alpha, \Gamma^m\tilde{\mathscr{V}}_\alpha, \Gamma^n\tilde{\mathscr{V}}_\alpha, \Gamma^m\Gamma^n\tilde{\mathscr{V}}_\alpha \tag{6.30}$$

And IIA and IIB superstring theories are T-dual.

The type I unoriented string theory made from acting [6.23] on type II oriented theory.The interesting thing is take $R \to 0$ in [6.10] let us focus on the phenomenon in the bulk of T-dual space away from the orientifold boundary [6.24].The existing state is $NS+, R+$ in type I theory,which means T-dual on one dimension gives $NS+, R+, \tilde{R}-$ by [6.25],that form type IIA theory in the local bulk.IIA theory is also dual to IIB,thus locally in the bulk of non-compact like dual space,it is a type II theory.Also,in this bulk,for a D-brane away from unoriented or reflected boundary,the superpartners are

$$\partial X^\mu, A_{d,ii} \cong_{SUSY} \psi^\mu, \psi^d \qquad e^{-\phi/2}\Theta_\alpha \cong_{Bos} \mathscr{V}_\alpha \tag{6.31}$$

on the level of vertex operator.Where Bos denotes bosonisation,which we can see it is a way to compensate the bosonic and fermionic degrees of freedom for

[4.13],a way to construct superpartner.And for a theory with one compactified dimension d around [4.22],we have operators $Q^d_\alpha, \tilde{Q}^d_{\dot\alpha}$ with internal index d corresponds to $d = 10, N = 1$ supersymmetry.But the conserved supercharge is the total charge $Q^d_\alpha + \tilde{Q}^d_{\dot\alpha}$,we can easily see by method around [3.41]

$$\begin{aligned} Q^d_\alpha + \tilde{Q}^d_{\dot\alpha} &= \frac{1}{2\pi i}\left(\oint_C dz j^d_\alpha - \oint_{\bar{C}} d\bar{z}\tilde{j}^d_{\dot\alpha}\right) = \frac{1}{2\pi i}\left(\oint_C dz j^d_\alpha + \oint_C d\bar{z}\tilde{j}^d_{\dot\alpha}\right) \\ &= \frac{1}{2\pi i}\oint_C (dz j^d_\alpha + \Omega^{-1} d\bar{z}\Omega\tilde{j}^d_{\dot\alpha}) = \frac{1}{2\pi i}\oint dz(j^d_\alpha + \Omega\tilde{j}^d_{\dot\alpha}) \end{aligned} \quad [6.31]$$

where we used corformal invariance [6.23],which exactly obeys [3.17]

$$\partial(j^d_\alpha(z) + \Omega\tilde{j}^d_{\dot\alpha}(\bar{z})) = \partial(j^d_\alpha + \tilde{j}^d_{\dot\alpha})(z) = \partial j^d_\alpha - \partial j^d_\alpha = 0 \quad [6.32]$$

because,the directions of the flows are opposite.And in T-dual space by [6.28],it is $Q'^d_\alpha + (\beta^d\tilde{Q}'^d)_{\dot\alpha}$ which acts on T-dual spacetime point X'^d and makes the scattering amplitudes of type II closed strings from D-brane invariant under supersymmetry.From another version of [6.18] from [3.2]

$$\begin{aligned} \partial_d X^d(z,\bar{z}) &= \partial_2(X^d(z) + X^d(\bar{z})) = i(\partial - \bar\partial)(X^d(z) + X^d(\bar{z})) \\ &= i(\partial + \bar\partial)(X^d(z) - X^d(\bar{z})) = i\partial_1 X'^d(z,\bar{z}) = i\partial_{d^\perp} X'^d(z,\bar{z}) \end{aligned} \quad [6.33]$$

In this case,we can discuss translation invariance for T-dual space that is

$$X^d(\sigma+\epsilon) = X^d(\sigma) + \epsilon\partial_d X^d(\sigma) \Rightarrow_{\mathbf{T}} X'^d(\sigma) + i\epsilon\partial_{d^\perp} X'^d(\sigma) \quad [6.34]$$

Local translation invariance breaks because the current $\partial_{d^\perp} X'^d(\sigma)$ is not a total derivative,it cannot be integrated out.To quantify the total nonconservation of momentum,we can put [6.34] back to T-dual bosonic part of action [3.44]

$$\begin{aligned} \Delta P^{\text{total}}_d(z) &= \delta_\epsilon \frac{\partial S'^d_X}{\partial(\partial X'^d)}\Big|_z = \partial_{\partial X'^d}\left\{\frac{1}{2\pi\alpha'}\int_M d^2z\partial X'^d\bar\partial(X'_d + i\epsilon\partial_{d^\perp}X'_d - X'_d)\right\} \\ &= \frac{i\epsilon}{2\pi\alpha'}\int d^2z\bar\partial\partial_{d^\perp}X'_d = \frac{1}{2\pi\alpha'}\int_{\partial M} dz\partial_{d^\perp}X'_d\mathbb{1} \end{aligned} \quad [6.35]$$

where we used split Stoke's theorem [3.6].The explanation is subtle,around the T-dual spacetime point on the D-brane or along the closed boundary ∂M,the original global momentum splits to infinite many local pieces with interval between them $(1/2\pi\alpha')\partial_{d^\perp}X'_d$.This is a spontaneously breaking that the global translation invariance splits into infinite many local degenerate states.Which means local Feynman diagram has a leg of emitting a infrared Goldstone boson with [6.35] as its vertex operator.And this should have a superpartner by supersymmetry [4.29] in T-dual space,with degeneracy

$$\int_{\partial M} dz\partial_{d^\perp}\psi'_d = \int_{\partial M} dz\mathscr{V}'_{d\alpha} = \int_{\bar\partial M} d\bar{z}\beta_d\tilde{\mathscr{V}}'_{d\alpha} = 2\pi i(\beta^d\tilde{Q}'_d)_{\dot\alpha} \quad [6.36]$$

which is global goldstino.Over all,Goldstone with goldstino correspond to spontaneously broken supersymmetry of dimension d below [6.32] to degenerate states,one corresponds to a half.A point is translation invariance indeed be involved in supersymmetry.We can see from [4.29]

$$\delta_\xi^2 X^\mu = \delta_\xi \xi^\alpha \psi_\alpha^\mu = \xi^\alpha \sigma^a_{\alpha\dot\alpha} \bar\xi^{\dot\alpha} \partial_a X^\mu \quad \Rightarrow \quad \delta_\xi^2 = \delta_\epsilon, \quad \epsilon = \xi^\alpha \sigma_{\alpha\dot\alpha} \bar\xi^{\dot\alpha} \qquad [6.37]$$

with ignorance of pointless prefactor,which is similar to [3.49].Recall the generalized module below [6.14] and global translation invariance maintained in ordinary space [6.34],the broken supersymmetry is just a half of supersymmetry totally in the generalized module.Which means we have $Q_\alpha^d + \tilde{Q}_{\dot\alpha}^d$ is unbroken and assigned to one state with $Q_\alpha'^d + (\beta^d \tilde{Q}'^d)_{\dot\alpha}$ in T-dual space,which is vacuum

$$X'^d = X'^d|_{z,\bar z}, \qquad \partial_{d^\perp} X'^d = \beta_d \tilde{\mathscr{V}}'_{d\alpha} = 0 \quad \text{with}\, \partial_d X'^d = 0 \qquad [6.38]$$

And we call this vacuum in T-dual space the BPS state,which is a point on the D-brane with the Dirichlet condition,carring the conserved charge.It also means D-brane is not the real vacuum,it carries charges.

7 Standard super algebraic geometry

Motivation

One problem that prevent us to achieve unification is the problem of vacuum,string theory also cannot describe the phenomenon of vacuum.We claim that this is because we use analytic approach to study quantum gravity or unification problem.To solve this thing,we need to use functorial approach instead and follow the Grothendieck's philosophy.We can simply consider a scheme $W = \mathrm{Spec}(\mathbb{C}[z, \bar z])$ with a sheaf $\mathcal{X}$ of $\mathbb{C}$-algebra of C^∞ functions that is a sheaf of scalar fields.A derivation of degree 1 is following for specific open set $U \in W$

$$\mathcal{D} \equiv \mathcal{D}^1/\mathcal{D}^2 : \mathcal{X}(U) \to \mathcal{X}(U), \quad X(z, \bar z) \mapsto \partial X(z, \bar z) \qquad [7.1]$$

Based on this we can form spectrum of kinetic terms as

$$\mathcal{W} = \mathrm{Spec}_U(\mathcal{D}(\mathcal{X}(W))), \quad \mathcal{I}_{\mathscr{L}} \in \mathcal{W} \quad \text{for } \mathscr{L} \in \mathcal{X}(W) \qquad [7.2]$$

where the ideal of Lagrangian is $\mathcal{I}_{\mathscr{L}} = (\mathscr{L} - \mathrm{tr}(\mathcal{D}X \otimes \bar{\mathcal{D}} X^T))$ for existence of trace.Then,we can define a presheaf of world-sheets $\mathscr{F}$ that is

$$\begin{aligned} &\mathscr{F} : \mathcal{U}_{\mathscr{L}} \mapsto \mathscr{F}(\mathcal{U}_{\mathscr{L}}), && \mathcal{U}_{\mathscr{L}} = \mathcal{W} \setminus V(X) \\ &F : \mathcal{I}_{\mathscr{L}} \mapsto \int [dX] e^{-\int d^2 z \mathscr{L}}, && F \in \mathscr{F}(\mathcal{U}_{\mathscr{L}}) \end{aligned} \qquad [7.3]$$

And for a diff$\times$weyl group action $G_{d\times w} \times \mathcal{W} \to \mathcal{W}$.The free bosonic string theory is a category of $(\mathcal{W}/G_{d\times w}, \mathscr{F})$ which is non-trivial for string theory because 1-d object has geometric property.In this case,we can reconstruct string theory by using algebraic geometry with functorial approach.Thus,we indeed need super algebraic geometry to develop such structure for consistent string theory.And we based on the text [10].Also,we need to consider T-duality,which means the geometry needs to be equipped with generalized setting.

7.1 Super linear algebra

A super Lie algebra $\mathfrak{g}$ is an object of category of super vector spaces with *contaction* $[\,,\,] : L \otimes L \to L$ and (super)*commutation* $c_{L,M} : L \otimes M \to M \otimes L$ with $ab \mapsto (-1)^{|a||b|}ba, a \in L, b \in M$.And super bracket defined by $a, b \in L$ $[a,b] = (-1)[\,,\,] \circ c_{L,L}(ab)$ with Jacobi identity defined on $L \otimes L \otimes L$

$$\begin{aligned} 0 &= [\,,\,]^2 \circ [(xyz) + (123) \circ (xyz) + (123)^2 \circ (xyz)] \\ &= [\,,\,]^2 \circ [(xyz) + (-1)^{|y||z|+|x||z|}(zxy) + (-1)^{|x||y|+|x||z|} \circ (xyz)] \end{aligned} \qquad [7.4]$$

with $(123)^2 = (132) \in S_3$.For a super algebra A we can form a left A-module M by $A \otimes M \to M$,which is a super vector space,the morphism of A-modules M, N that is $\phi : M \to N, am \mapsto a\phi(m)$ let us have category of A-modules.And the tensor product for a commutative A defined by $\otimes : M \times N \to M \otimes N$ which is universal by the unique map $M \otimes N \to A, (ma) \otimes (bn) \mapsto ab \in A$ with M a right A-module and N a left A-module.Then we can define $A^{p|q} = A \otimes k^{p|q}$ with the $\mathbb{Z}_2$-grading,

$$\begin{aligned} (A^{p|q})_0 &= (A_0 \oplus A_1) \otimes (k_0^{p|q} \otimes k_1^{p|q}) \\ (A^{p|q})_1 &= (A_q \oplus A_0) \otimes (k_0^{p|q} \otimes k_1^{p|q}) \end{aligned} \qquad [7.5]$$

with $k^{p|q}$ is a free k-module generated by the basis $\{e_1, ..., e_p, \epsilon_1, ..., \epsilon_q\}$.And a *free A-module M* is an A-module such that $M \cong A^{p|q}$ that is

$$\begin{aligned} & A \otimes M \cong A \otimes A \otimes k^{p|q} \Rightarrow A \otimes k^{p|q} \xrightarrow{\sim} M \\ & M_0 = \operatorname{span}_{A_0}\{e_1, ..., e_p\} \oplus \operatorname{span}_{A_1}\{\epsilon_1, ..., \epsilon_q\} \\ & M_1 = \operatorname{span}_{A_1}\{e_1, ..., e_p\} \oplus \operatorname{span}_{A_0}\{\epsilon_1, ..., \epsilon_q\} \end{aligned} \qquad [7.6]$$

And morphism $T : A^{p|q} \to A^{r|s}$ of free A-modules,follows from the linear transformation $T : k^{p|q} \to k^{r|s}$,for $\epsilon_1 = e_{p+1}...$,we have $T(e_i) = t_i^j e_j$ for $j = 1, ..., r+s$ and for $i = 1, ..., p+q$,Then the matrix of the transformation in $\mathrm{M}(A^{p|q})$ is

$$T_{(r+s)\times(p+q)} \in \mathrm{Hom}(A^{p|q}, A^{r|s}), = t_{ij} = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \qquad [7.7]$$

With sub matrices $r \times p$ even T_1,$s \times q$ even T_4 and $r \times q$ odd T_2,$s \times p$ odd T_3.And T transforms coordinates.In category of A-modules,Hom set is set of parity preserving maps.For setting grading,we need $\underline{\mathrm{Hom}} = \underline{\mathrm{Hom}}_0 \oplus \underline{\mathrm{Hom}}_1$ set with even map as T above,odd map is parity reversing on each sub matrices.For a simple case,$\underline{\mathrm{Hom}}(A^{p|q}, A^{p|q})$ for maps T in the set

$$T^{\mathrm{even}} \in \mathrm{M}(A^{p|q}) = \underline{\mathrm{Hom}}_0(A^{p|q}, A^{p|q}), \qquad T^{\mathrm{odd}} = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \qquad [7.8]$$

with all sub matrices formed by even elements.And we have a super Lie algebra $\mathrm{Mat}(A^{p|q})$ which is a super vector space $\underline{\mathrm{Hom}}(A^{p|q}, A^{p|q})$ with commutator defined by $[T,S] = TS - (-1)^{|T||S|}ST$ for $S,T \in \mathrm{Mat}(A^{p|q})$.And we can define super trace to be

$$\mathrm{str}(T^{\mathrm{even}}) = \mathrm{tr}(T_1) - \mathrm{tr}(T_4), \quad \mathrm{str}(T^{\mathrm{odd}}) = \mathrm{tr}(T_1) + \mathrm{tr}(T_4) \qquad [7.9]$$

with $\mathrm{str}(TS) = (-1)^{|T||S|}\mathrm{str}(ST)$.And we can collect the automorphisms in $\mathrm{Hom}(M,M)$ to form a group $\mathrm{GL}(M)$ called special GL group of automorphisms of M.

An observation is $\mathrm{M}(A^{p|q})$ is a super linear algebra that cannot define normal det,we need generalize it to *Berezinian*.First we have a natural map $A \to \bar{A} = A/J_A$ for an ideal $J_A \subset A$.And we got a induced map

$$\begin{aligned}\mathrm{M}(A^{p|q}) \to \mathrm{M}(\bar{A}^{p|q}) &= \underline{\mathrm{Hom}}(A^{p|q}, A^{p|q}) \to \underline{\mathrm{Hom}}(\bar{A}^{p|q}, \bar{A}^{p|q}) \\ &= A \otimes \underline{\mathrm{Hom}}(k^{p|q}, k^{p|q}) \to \bar{A} \otimes \underline{\mathrm{Hom}}(k^{p|q}, k^{p|q})\end{aligned} \qquad [7.10]$$

an element of A corresponds to an matrix in the form of [7.7] in $\mathrm{M}(A^{p|q})$.If we have an invertible matrix $T \in \mathrm{M}(A^{p|q})$,it corresponds to an invertible element $a \in A$ which means $ab = I_A$ for a $b \in A$ and above map send ab to $(a+J_A)(b+J_A) = ab + J = I_A + J_A = I_{\bar{A}}$,which means if T is invertible then we get $\bar{T} \in \mathrm{M}(\bar{A}^{p|q})$ is invertible.And for the reverse,if a $\bar{T}$ is invertible,we get $(a+J_A)(b+J_A) = I_{\bar{A}}$ that is $ab = I_{\bar{A}} - J_A$,on the level of matrices is $TS = I+N$ for $S, N \in \mathrm{M}(A^{p|q})$ and N corresponds to the ideal J_A.Then,$TS = I$ means $TSN^r = IN^r + N^{r+1}$ for $N^{r+1} = 0$ and N is indeed nilpotent when r is enough for all times if and only if J_A is an ideal generated by odd elements by anti-commutativity.Thus,we get T is invertible if and only if $\bar{T}$ is invertible for odd J_A.Also,$\bar{T}$ is invertible means T_1, T_2 with even elemtents are invertible.Which gives us a proposition1.5.1[10]that is for $T \in \mathrm{M}(A^{p|q})$,then T is invertible if and only if T_1, T_4 is invertible.In this case,we define Berezinian on an invertible T

$$\mathrm{Ber}(T) = \det(T_1 - T_2T_4^{-1}T_3)\det(T_4)^{-1} = \det(T_4 - T_3T_1^{-1}T_2)^{-1}\det(T_1) \quad [7.11]$$

Now,we want to study properties of $\mathrm{Ber}(T)$.First,we have a set G with elements $S,T \in \mathrm{GL}(A^{p|q})$ such that $\mathrm{Ber}(ST) = \mathrm{Ber}(S)\mathrm{Ber}(T)$ for all T,because of the uniqueness of inverse,it is equivalent for all T^{-1},clearly $I, T^{-1} \in G$ and for all $S, P \in G$,we can find a $T = P, \mathrm{Ber}(SP^{-1}P) = \mathrm{Ber}(S)\mathrm{Ber}(P^{-1})\mathrm{Ber}(P) = \mathrm{Ber}(SP^{-1})\mathrm{Ber}(P)$ if $P \in \mathrm{GL}(A^{p|q})$ that means $SP^{-1} \in G$ and G is a subgroup of $\mathrm{GL}(A^{p|q})$,but now $P = G$.Thus,G is a subgroup of $\mathrm{GL}(A^{p|q})$ if and only if $G = \{p \in G | p \in \mathrm{GL}(A^{p|q})\} = \mathrm{GL}(A^{p|q})$.At the same time,$G$ must be a subgroup because for all T is equivalent for all ST,so $S^{-1} \in G$ for $S \in G$ which is the only one we need to check in group axioms.Thus,Ber is multiplicative

$$\mathrm{Ber}(ST) = \mathrm{Ber}(S)\mathrm{Ber}(T), \qquad \forall\, S,T \in \mathrm{GL}(A^{p|q}) \qquad [7.12]$$

And [7.12] let Ber become a homomorphism $\mathrm{GL}(A^{p|q}) \to \mathrm{GL}.(A^{1|0})$.For $T \in \mathrm{GL}(A^{p|q})$,there exists a decomposition of T

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}\begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} H_1 & H_1S \\ TH_1 & TH_1S + H_2 \end{pmatrix} \quad [7.13]$$

induces a decomposition $\mathrm{GL}(A^{p|q}) = UHV$,for subgroups $U, H, V \subset \mathrm{GL}(A^{p|q})$

$$U = \left\{ \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix} \right\}, \quad H = \left\{ \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \right\}, \quad V = \left\{ \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \right\} \qquad [7.14]$$

And we see that there is a relation between determinant and trace in [3.16].Now there is also a relation between Ber and str.We want to study this relation around the identity,in this case we need to set up an algebraic infinitesimal system,ϵ is infinitesimal means a localization $A \to A[\epsilon]/\epsilon^2$,for $I + \epsilon T \in \mathrm{GL}((A[\epsilon]/\epsilon^2)^{p|q})$,the inverse $(I + \epsilon T)^{-1} = (I - \epsilon T)$ which is an *algebraic Taylor expansion* denotes $\simeq$

$$\begin{aligned} \mathrm{Ber}(I + \epsilon T) &= \det(1 + \epsilon T_1 + \circ(\epsilon^2))\det(1 + \epsilon T_4 + \circ(\epsilon^2))^{-1} \\ &\simeq \det(1 + \epsilon T_1)\det(1 - \epsilon T_4) \approx \det(e^{T_1})\det(e^{-T_4}) \\ &= e^{\mathrm{tr}(T_1)} e^{\mathrm{tr}(T_4)} \simeq (1 + \epsilon \mathrm{tr}(T_1))(1 - \epsilon \mathrm{tr}(T_4)) = 1 + \epsilon \mathrm{str}(T) \end{aligned} \qquad [7.15]$$

where we used reverse Taylor expansion above [2.11] and normal relation above [3.16].Based on this we can get cyclic property

$$\begin{aligned} &\mathrm{Ber}(1 + \epsilon STP) \simeq (1 + \epsilon \mathrm{tr}((STP)_1))(1 - \epsilon \mathrm{tr}((STP)_4)) \\ &= (1 + \epsilon \mathrm{tr}(S_1T_1P_1 + S_2T_3P_1 + S_1T_2P_3 + S_2T_4P_3)) \\ &\quad \times (1 - \epsilon \mathrm{tr}(S_3T_1P_2 + S_4T_3P_2 + S_3T_2P_4 + S_4T_4P_4)) \\ &= 1 + \epsilon \mathrm{tr}(P_1T_1S_1 + P_2T_1S_3 + P_2T_3S_4 + P_4T_2S_3) \\ &\quad - \epsilon \mathrm{tr}(P_1T_3S_2 + P_3T_2S_1 + P_3T_4S_2 + P_4T_4S_4) \\ &= 1 + \mathrm{str}PTS = 1 + \epsilon \mathrm{str}(STP) \\ &\Rightarrow \mathrm{str}(PTS) = \mathrm{str}(STP) \end{aligned} \qquad [7.16]$$

For a super vector space V,we naturally define a tensor superalgebra

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n}, \quad T(V)_0 = \bigoplus_{n\,\mathrm{even}} V^{\otimes n}, \quad T(V)_1 = \bigoplus_{n\,\mathrm{odd}} V^{\otimes n} \qquad [7.17]$$

induces an universal enveloping superalgebra (UESA) of a super Lie algebra g

$$\mathfrak{U}(\mathfrak{g}) = T(\mathfrak{g})/I, \quad I = (i(X) \otimes i(Y) - (-1)^{|X||Y|} i(Y) \otimes i(X) - i([X, Y])) \quad [7.18]$$

for $X, Y \in \mathfrak{g}$ with immersion $i : \mathfrak{g} \to T(\mathfrak{g})$.And we define $\pi : T(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$,for $i^{-1}(I) = 0 \in \mathfrak{g}$.We can find morphisms $\xi : \mathfrak{g} \to A$ such that $\xi(i^{-1}(I)) = 0_A$,in this case,$\xi' = \xi^{-1} \circ i : T(\mathfrak{g}) \to A$,then we get $\sigma = \xi^{-1} \circ i \circ \pi^{-1} : \mathfrak{U}(\mathfrak{g}) \to A$.And the universal property follows from that $i(\mathfrak{g})$ is uniquely generate $T(\mathfrak{g})$ which means σ is unique.By the universality,for a representation of super Lie algebra $\mathfrak{g}$ that is $\rho : \mathfrak{g} \to \mathrm{End}(V)$,it uniquely extends to $\rho' : \mathfrak{U}(\mathfrak{g}) \to \mathrm{End}(V)$ follows from $\xi(i^{-1}(I)) = 0_{V \otimes V^*}$.And because of the commutation in [7.18],a basis of $\mathfrak{U}(\mathfrak{g})$ is

$$\left\{ 1, \prod_{i=1}^{r} j(X_{k_i}) \prod_{j=r+1}^{r+s} j(X_{k_j}) \middle| k_i \in K \text{ordered} \quad X_{k_i} \text{even}, X_{k_j} \text{odd}, X \in \mathfrak{g} \right\} \quad [7.18]$$

where $j = \pi \circ i$ we defined above.And we have a similar corollary to ordinary case above [5.10],that is $\rho : \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$ is injective because of the immersion i.and easily,we can split [7.18] by $\mathbb{Z}_2$ grading,$I = I_0 \oplus I_1 = I^{|X||Y|=1} \oplus I^{|X||Y|=0}$

$$\mathfrak{U}(\mathfrak{g}) \cong \mathfrak{U}(\mathfrak{g}_0) \otimes \bigwedge(\mathfrak{g}_1) = (T(\mathfrak{g}_0)/I_0) \otimes (T(\mathfrak{g}_1)/I_1) \qquad [7.19]$$

directly see from [7.18].

Next,we want to use graded and filtered superalgebra that are defined in definition 1.6.8 in [10] which are just ordinary case with grading.For a filtered superalgebra $A = \bigcup_{n\geq 0} A^n$ for $A^{n-1} \subset A^n$,we have associate graded superalgebra $\mathrm{Gr}(A) = \bigoplus_{n\geq 0} A^n/A^{n-1}$.Also,for a graded superalgebra $A = \bigoplus_{n\geq 0} A_n$,we have associative filtered superalgebra $\mathrm{Fi}(A) = \bigcup_{n\geq 0}(\bigoplus_{n\geq 0} A_n/A_{n-1})$.because we cannot define the grade of the monomial tensors of I in [7.18],in $\mathfrak{U}(\mathfrak{g})$,we need to start with filtration.By the definition we have for $n = r + s$ in [7.18]

$$\mathrm{Gr}(\mathfrak{U}(\mathfrak{g})) = \bigoplus_{0\leq m\leq n} \Big((T(\mathfrak{g})^m/T(\mathfrak{g})^{m-1})/I\Big) \subset T_0 \oplus \bigoplus_{1\leq m\leq n} \Big(T_m \cap \prod_{k_i\in K} j(X_{K_i})\Big) \qquad [7.20]$$

where T_m is span of monomial tensors with degree m.$\mathrm{Gr}(\mathfrak{U}(\mathfrak{g}))$ is commutative because the nontrivial term which expressed in Lie bracket we explained above [3.39] is also a vector field of degree $m + p - 1$ for X, Y with degree m, p,that is modded by the filtration setting in [7.20].In this case,the operation Gr for Lie algebra can be called as *free collection* or closing the contaction that is

$$\mathrm{Gr}(I) = I^{\mathrm{free}} = I(i(X), i(Y), i([X, Y]) = 0), \qquad X, Y \in \mathfrak{g} \qquad [7.21]$$

Next,we define a symmetric algebra to be $\mathrm{Sym}(V) = T(V)/I_V^{\mathrm{free}}$.When $m = 1$ in [7.20],we find $\mathrm{Gr}(\mathfrak{U}(\mathfrak{g}))^1 = (T_0 \oplus T_0\mathfrak{g})/T_0 = T_0\mathfrak{g}$ and also by definition $\mathrm{Sym}(\mathfrak{g})^1 = (T_0 \oplus T_0\mathfrak{g})/I_1^{\mathrm{free}}$,we have $I^{\mathrm{free}} = T_0$,in this case,we find an injective homormorphism $\mathrm{Sym}(\mathfrak{g}) \to \mathrm{Gr}(\mathfrak{U}(\mathfrak{g}))$,and the basis generating them are based on $\prod_{k_i}(X_{K_i})$ which means this need to be surjective.Thus,it is an isomorphism

$$\mathrm{Gr}(\mathfrak{U}(\mathfrak{g})) \cong \mathrm{Sym}(\mathfrak{g}) \qquad [7.22]$$

which gives the graded UESA geometric property,the corresponding physical understanding is the geometry was broken down at the contaction point.And,we can also see that the (anti)commutation as information collected in I indeed be the property of underlying space but not on the fields living in the space.

7.2 Standard super algebraic gen. geometry

Now,we can apply the super linear algebra to standard algebraic geometry.A sheaf $\mathcal{F}$ is a functor from the opposite category of open sets or topological spaces X satisfying gluing property.The stalk $\mathcal{F}_x$ of the sheaf $\mathcal{F}$ for a point $x \in X$ is

$$\mathcal{F}_x = \lim \mathcal{F}(U) = (\prod_i s_i, \bigcup U_i)/\cong_x, \qquad \forall\, x \in U_i \subset X \qquad [7.23]$$

$U_i \sim_x U_j \Leftrightarrow U_i \cap U_j$ and $\cong_x \equiv U_i|_V \cong U_j|_V, s_i|_V = s_j|_V, V \subset U_i \cap U_j$.A point in stalk is an equivalent class of sections around the fiber of x and $U_i \setminus \{x\}$ is open.And $\mathcal{O}_X$ denotes the sheaf of algebra of regular functions.A ringed space $M = (|M|, \mathcal{F})$ with topological space $|M|$ and $\mathcal{F}$ the sheaf of commutative ring.A morphism of ringed space uniquely induce a fiber product which means $\phi : M \to N, = (|\phi|, \phi^*)$,where $\phi^* : \mathcal{O}_N \to \phi_* \mathcal{O}_M$ and $(\phi^* \mathcal{O}_N)(U) = \mathcal{O}_N(\phi(U))$ for an open set $U \subset |M|$.A locally ringed space is a pair $(|M|, \mathcal{F})$ such that $\forall x \in |M|, \mathcal{F}_x$ is a local ring.Local ring has a good property with unique maximal ideal like irreducibility.In this case,$\mathcal{F}_x$ is local everywhere means a point on base corresponds a point on the section.For any commutative ring,we define Spec$A$ to be the spectrum of the ring,that consists of all prime ideals in A.A closed point in this spectrum is a prime ideal,a closed set $V(S) = \{\mathfrak{p} \in \mathrm{Spec}A | S \subset \mathfrak{p}\}$ consists of closed points.An open set $U_f = \mathrm{Spec}A \backslash V(f) = \mathrm{Spec}A_f = \mathrm{Spec}A[f^{-1}], f \in A$.In this case,we define Zariski topology on Spec$A$ and it is an topological space.For a commutative ring A,we define a $\mathscr{B}$-sheaf $U_f \mapsto \mathcal{O}_A(U_f) = A_f$.And now we use definition 2.2.10 and proposition 2.2.11 in [10],a $\mathscr{B}$-sheaf with $\mathscr{B}$ a base of open sets in a topological space,is an assignment $U \mapsto \mathcal{F}(U), \forall U \in \mathscr{B}$ satisfying sheaf axioms.And a $\mathscr{B}$-sheaf uniquely extends to a sheaf $\mathcal{F}$ on the topological space.For extending to a sheaf $\mathcal{O}_A$,we only need to check gluing property

$$\frac{h}{f}\Big|_{U_f \cap U_g} = \frac{h}{f}\frac{1}{g} = \frac{h}{g}\frac{1}{f} = \frac{h}{g}\Big|_{U_f \cap U_g}, \qquad \frac{h}{f} \in \mathcal{O}_A(U_f), \frac{h}{g} \in \mathcal{O}_A(U_g) \qquad [7.24]$$

In this case,it indeed extends to $\mathcal{O}_A$ called structure sheaf on Spec$A$.And the unique maximal ideal is (f) in the stalk $\mathcal{O}_{A,(f)}$ which means we have a locally ringed space $\underline{\mathrm{Spec}}A = (\mathrm{Spec}A, \mathcal{O}_A)$.

An affine scheme is a locally ringed space isomorphic to $\underline{\mathrm{Spec}}A$.A scheme X is a locally ringed space,locally isomorphic to an affine scheme.An observation is a scheme is a solution space.For instance,a prime ideal in a polynomial ring express a solution in $D+1$ dimensions of a polynomial in D dimensions just from linear algebra.On the other hand,we have a *function-solution correspondence*

$$\begin{array}{ccc} \text{a function in } D\text{-dim space} & \longleftrightarrow & \text{a solution in } (D+1)\text{-dim scheme} \\ y = k[x^1, ..., x^D] & \longleftrightarrow & (0) = (k[y, x^1, ..., x^D]) \end{array} \qquad [7.25]$$

Next,we want to introduce projective scheme with dimension d,which covered by $(d-1)$-dim affine schemes.Let $M = \bigoplus_{i \geq 0} M_i$ be graded commutative k-algebra over a field k and M_i is homogeneous elements of degree i.We define $\mathrm{Proj}\, M$ to be set of all homogeneous prime ideals.Closed that $V(I) = \{\mathfrak{p} \in \mathrm{Proj}\, M | I \subset \mathfrak{p}\}$.The open set is $W_f = \mathrm{Proj}\, M - V(f) = \mathrm{Spec}M(f^{-1})_0$,the subscript means the set of homogeneous primes in degree 0.For a basis $x^0, ..., x^D$ with degree 1 elements generating ideal in M,then the projective scheme is covered by $W_{x^0}, ..., W_{x^D}$.

We define an assignment $U_f \to \widetilde{M}(U_f) = M_f$ for M_f an A_f-module we want it to be a $\mathscr{B}$-sheaf,we only need to check gluing property based on [7.24]

$$\frac{h}{f}m\Big|_{U_f \cap U_g} = \frac{h}{f}m\frac{1}{g} = \frac{h}{g}m\frac{1}{f} = \frac{h}{g}m\Big|_{U_f \cap U_g} \in A_{fg}M, \quad m \in M \qquad [7.26]$$

actually M can be a cotangent space easily generated by Grassmann coordinates

$$\begin{aligned} & M \cong \bigwedge(\theta) = \mathrm{Span}((\theta^1, ..., \theta^D)), \quad \theta^i = dx^i \\ & \text{with} \quad d(\mathrm{ln} f) = (\partial_{x^0} \mathrm{ln} f) dx^0 + ... + (\partial_{x^D} \mathrm{ln} f) dx^D, \, \forall \, f \in A \\ & d(\mathrm{ln} f) \in A_f M, \, d(\mathrm{ln} g) \in A_g M, \, d(\mathrm{ln} f) + d(\mathrm{ln} g) = d(\mathrm{ln} fg) \in A_{fg} M \end{aligned} \tag{7.27}$$

And the above $\mathscr{B}$-sheaf uniquely extends to a sheaf $\tilde{M}$ on $\underline{\mathrm{Spec}}A$,which is an $\mathcal{O}_A$-module that is a sheaf of A-modules.For a sheaf $\mathcal{F}$ on a scheme X,of $\mathcal{O}_X$-module is quasi-coherent,if $\mathcal{F}|_{U_i} \cong \tilde{M}_i$ that is a $\mathcal{O}_{A_i}$-module for $\{U_i = \mathrm{Spec}A_i\}_{i \in I}$ covering X.In the topological space,we can naturally retract $\mathcal{O}_X(U)$-modules to $\mathcal{O}_X(U)$ and to U on the U_{Zar} [9.79].

A super ringed space $S = (|S|, \mathcal{O}_S)$ is a $\mathbb{Z}_2$ graded ringed space with a graded structure sheaf,$\mathcal{O}_{S,0}$ is ordinary sheaf of algebra of regular function on $|S|$ and $\mathcal{O}_{s,1}$ is an $\mathcal{O}_{s,0}$-module.A super space is a super ringed space S with the property that the stalk $\mathcal{O}_{s,x}$ is local ring for all $x \in |S|$.A supermanifold $M = (|M|, \mathcal{O}_M)$ of dimension $p|q$ is a superspace that is locally isomorphic to $\mathbb{R}^{p|q}$ which is just a superspace with smoothness setting.Exactly,$\mathbb{R}^{p|q} = (\mathbb{R}^p, C^\infty_{\mathbb{R}^p}[\theta^1, ..., \theta^q])$

$$\begin{aligned} & C^\infty_{\mathbb{R}^p}[\theta^1, ..., \theta^q](\mathbb{R}^p)_0 = \Big\{ f_0 + \sum_{\text{even } |I|} f_I \theta_I \big| I = \{i_1 < \cdots < i_m\} \Big\} \\ & C^\infty_{\mathbb{R}^p}[\theta^1, ..., \theta^q](\mathbb{R}^p)_1 = \Big\{ \sum_{\text{odd } J} f_J \theta_J \big| J = \{j_1 < \cdots < i_r\} \Big\} \end{aligned} \tag{7.28}$$

where $\theta_I = \theta^{i_1} \theta^{i_2} ... \theta^{i_m}, i = 1, ..., q$ and $|I|$ counting for the total parity and regular functions $f_0, f_I \in k[x^1, ..., x^p]$.Which gives a formal definition for the supermanifold descents from the supersymmetry algebra below [4.51].The smoothness means we descents the information of free collection into a geometric object by our opinion around [7.22] and now we see the commutation relation is a property of the supermanifold and fields living on this geometric object naturally follow its property.A superscheme is a superspace S with $(|S|, \mathcal{O}_{S,0})$ is ordinary scheme and $\mathcal{O}_{S,1}$ is a quasi-coherent $\mathcal{O}_{S,0}$-module.

An abelian category is a category of Abelian groups,with the zero object being the identity and binary biproduct as the binary operation,morphisms are homomorphisms with kernel and cokernel.The commutativity based on the comutativity of Abelian groups reflecting by normality of all monomorphisms and epmorphisms.Clearly,we can form an Abelian category of R-modules with a ring R,denote by Mod_R.For a module $M \in \mathrm{Mod}_R$,it is flat means the functor of tensor product of modules $(-) \otimes_R M : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is an exact functor.M is called faithfully flat means faithfulness of homomorphism as a property preserve by $(-) \otimes_R M$ and fit with corresponding commutative square of category

$$g^{-1} : \mathrm{Hom}_R(N, N') \to \mathrm{Hom}_R(N \otimes_R M, N' \otimes_R M) \tag{7.29}$$

where $g : M \to M \otimes M^* \otimes N \in N$,$M$ is faithfully flat means the M is flat and g is injective.The flatness is just like regularity.A morphism of schemes $f : X \to Y$

is flat means for all point $x \in X$,the map of the stalks $\mathcal{O}(f)_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat which means $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module.And f is faithfully flat means it is also surjective which means injectivity on morphsims on the level of sections by the structure of morphsm of ringed spaces below [7.23].An remark is morphisms of affine schemes $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ are (faithfully)flat if and only if R' is a (faithfully)flat R-module.From a R-module M,we have an observation that is a map $M \to R^s, m = \sum_s r_s a_s \mapsto (r_1, ..., r_s)$,property of finite generating reflects on s is an integer,because infinity is not integer.Which is a surjection reflecting on exact sequence is $R^r \to R^s \to M \to 0$ and if this exact sequence exists,we call M is of finite presentation(of R^s).If $A \to B$ is a ring homomorphism,we call B is of finite presentation over A if there exists a surjection $\pi : A[x^1, ..., x^s] \to B$ with $\ker(\pi)$ is a finitely generated ideal in $A[x^1, ..., x^s]$.We can understand by first isomorphism theorem,$B = \mathrm{im}(\pi) \cong A[x^1, ..., x^s]/\ker(\pi)$,the set of cosets is finitely generated if and only if $\ker(\pi)$ is finitely generated.A quasi-coherent sheaf $\mathcal{F}$ on a scheme is called locally finitely presented if for every open subset $U = \mathrm{Spec}(A) \subset X$,the section $\mathcal{F}(\mathrm{Spec}(A))$ is a finitely presented A-module.If X is locally noetherian,then the quasi-coherent sheaf is locally finitely presented,if and only if it is coherent.A morphism of schemes $f : X \to Y$ is locally of finite presentation if for every $U = \mathrm{Spec}(B) \subset Y$ and $\mathrm{Spec}(A) \subset f^{-1}(\mathrm{Spec}(B))$,$A$ is of finite presentation over B.A morphism $f : X \to Y$ is of finite presentation if f is locally of finite presentation and quasi-compact and quasi-seperated.A morphism of schemes is quasi-seperated if $X \times_Y X \to X$ is quasi-compact.Easily speaking,we have finiteness setting on the number of fibers and each fiber and the pullback along fibers.

Now,we are able to define étale morphism of schemes which has very good property like locally isomorphism also see below [9.89] for a motivation.Let $f : X \to Y$ be a morphism of schemes.We call f formally étale(formally smooth,formally unramified) if for every affine Y-schemes $Y' \to Y$ and every closed embadding $i : Y'_0 \hookrightarrow Y'$ defined by nilpotent ideal J_A,the following map is bijective(smooth,injective).

$$\circ i : \mathrm{Hom}_Y(Y', X) \to \mathrm{Hom}_Y(Y'_0, X) \qquad [7.30]$$

The f is étale(smooth,unramified) if it is also locally of finte presentation.To understand above,we need an observation that is if $x^2 = 0, x \neq 0, d(x^2) = d0 = 0 = 2xdx \neq 0$,which is a contradiction,that means if $\Omega_{X/S} = 0$ for a S-scheme,and $df = 0 \in \Omega_{X/S}$ must give f is a constant and $f^s \neq 0$ for some integer s,this gives a corollary that $\Omega_{X/S} = 0$ if and only if the functions on the scheme X cannot be nilpotent.And $\Omega_{X/S} = 0$ just for excluding the relative ramification over $\mathbb{C}$.An interesting thing is we focus on nilpotent ideal because the brunch cut which forms discrete space,but if we put superscheme in we find nilpotent ideal can be generated by all odd elements.Intuitively,this nilpotence should also correspond to a discrete space structure over $\mathbb{C}$.In (8.5.1) in [2],we see an orbifold strucure which is a reflection of space $X^\mu \cong -X^\mu$,and by discussion around [7.22],if we just absorb the anticommutation to the space

$$(\triangle \wedge \square) \circ (\theta_1\theta_2) = (-\square \wedge \triangle) \circ (\theta_1\theta_2) \quad \text{and} \quad \theta_1\theta_2 \cong X^\mu \qquad [7.31]$$

This is exactly an orbifold structure which forms a discrete space,this inspire us to regard the orbifold as the correspondence to the nilpotence from the odd elements.We want to use definition in [13] to define orbifold as a groupoid associating with the next chapter.A groupoid is constructed by a pair (G_0, G_1, s, t) with G_0 a set of objects and G_1 a set of arrow,with structure maps $s, t : G_1 \rightrightarrows G_0$,for $f \in G_1, f : x \to y, s(f) = x, t(f) = y$ and all these are categorized into category structure.A Lie groupoid is a groupoid with G_0, G_1 are smooth manifolds and structure maps are smooth and submersions.The intersection of source and target fiber at a point $x \in G_0$,$(G_1)_x = s^{-1}(x) \cap t^{-1}(x)$ is a Lie group called isotropy of G_1 at x.A Lie groupoid is proper if the diagonal map $(s, t) : G_1 \to G_0 \times G_0$ is proper.It is étale if the structure maps are local diffeomorphisms.An orbifold groupoid G^{orb} is a proper étale Lie groupoid which is equivalent to be a proper Lie groupoid with all isotropies are discrete spaces.In this way,a category of orbifolds is a differentiable stack denoting as Ψ

$$\Psi \cong |G_0^{\text{orb}}/G_1^{\text{orb}}|, \qquad x \cong y \Leftrightarrow s(g) = x, t(g) = y \qquad [7.32]$$

where $x, y \in G_0^{\text{orb}}, g \in G_1^{\text{orb}}$ and we will see the verification in the next section.[7.32] means it is an orbit space from quotient of the equivalence relation.And we can reformulate [7.31] just for $f : X^\mu \to -X^\mu \in G_1^{\text{orb}}, s(f) \cong t(y)$.And we claim that for $\mathbb{C}$-superscheme S over $\mathbb{C}$,$S_1 = (|S|, \mathcal{O}_{S,1}) \in \Psi$.We define a $\mathbb{C}$-superscheme to be a complex superscheme locally generated by

$$\mathbb{C}^{p|q} = (\mathbb{C}^p, C^\infty_{\mathbb{C}^p}[\theta^1, .., \theta^q, \bar{\theta}^1, ..., \bar{\theta}^q]), \quad \Gamma(\mathbb{C}^{p|q}) = \mathbb{C}[z^\mu, \bar{z}^\mu, \theta^\sigma, \bar{\theta}^q] \qquad [7.33]$$

where $z^1 = z^1_1 + iz^1_2, \theta^1 = \theta^1_1 + i\theta^1_2, (\theta^1)^2 = \theta^1\bar{\theta}^1 = 0$.Notice that this bar θ for conjugation is not overline θ in section 4 which has overline for denoting another Weyl copy,because we want to keep notation same as in section 3.And [7.33] are just complex case of [7.28].

Notice that algebraic supergeometry is for super setting for both considering fermions and bosons without the constraint on their number.Supersymmetry directly reflects by agreement on number counting see [4.13].In this case we use the terminology *super algebraic geometry* denoting the algebraic geometry with SUSY and generalized setting.Actually,we can call it supersymmetric geometry,but we want to be same with superstring theory.For develop a supersymmetric AG,we can start at the work by Renaud Gauthier [14],he extends $\mathbb{Z}_2$-module to $\mathbb{Z}_2$-bigraded-module to consider number counting on the string world-sheet,that is $M = M_{i0} \oplus M_{i1} = (M_{00} \oplus M_{01}) \oplus (M_{10} \oplus M_{11})$ and i for denoting the coordinates we used for σ^i.And σ^i corresponds to anticommutative coordinates θ^i because of supersymmetry gives a correspondence between $X(\sigma)$ with $\psi(\theta)$.But we see we can add more dimensions in [7.33],in this case we want to generalized 1-d string to Dp-brane.

A symmetric superscheme of dimension $d = 2n + 1$ is a superscheme generated by $\mathbb{R}^{2n-1|2n-1}$ with supermap δ^A.A symmetric $\mathbb{C}$-superscheme of dimension $d = 2n$ is a $\mathbb{C}$-superscheme generated by $\mathbb{C}^{n|n}$ for $n \geq 0, \in \mathbb{Z}$ with supermap on the underlining super ring R on the compact dimension indexed by A.

$$\delta^A = \delta^A_0 \oplus \delta^A_1 : (z^A, \theta^A) \to (\theta^A, z^A), \qquad z \in R_0, \theta \in R_1 \qquad [7.34]$$

Notice that we have a $\mathbb{C}$ winding [3.79] which becomes a compactification by normalization of physical states and we see this gives an equivalence of supersymmmetry representation [4.22].In this case we use geometric part of compactification geo^{c} for only $\mathbb{C}$ winding and physical part of compactification phy^{c} for physics consideration.And we define a termiology that is a *supersymmetry exact compactification* is $susy^{\mathrm{c}} = phy^{\mathrm{c}} \circ geo^{\mathrm{c}}$,which is not toroidal compactification etc..This is a setting for naturalness.Thus,for symmetric $\mathbb{C}$-scheme of even dimension,we naturally have complete exact compactifications.But for symmetric superscheme of odd dimension we leave with one dimension do not be compactified exactly which means we do not have full supermaps for a odd dimension symmetric superscheme.We want to apply the fact below [7.25] that is a projective scheme of dimension d is covered by $d-1$ dim affine schemes,and for considering the odd dim brane exists in superstring theory,we need a constraint that all symmetric superschemes are isomorphic to projective schemes or symmetric exists in universe as a projective scheme.And another constraint that is all symmetric $\mathbb{C}$-superschemes are isomorphic to affine schemes or a symmetric $\mathbb{C}$-superscheme lives in universe as an affine scheme.In this case,an projective symmetric superscheme of dimension d naturally has supermaps from the affine symmetric $\mathbb{C}$-superschemes covering it of dimension $d-1$.And exact supersymmetry compactification also restrict their dimension need to be in superstring theory $D = 10$,that is $0 \leq d \leq 10$.

Now we want to combing these super settings with the former generalized settings for T-duality in section 6.A M-brane $(\mathscr{M}, \mathscr{P})$ is a pre M-brane [6.22] from the **T**-fusion of two $D = 10$ dimensional affine generalized symmetric $\mathbb{C}$-superschemes,which is a 11 dimensional projective symmetric superscheme,with a highly nontrivial sheaf $\mathscr{P}$ of *properties* with a clear form [9.15].And we want to use a section to discuss a theory with properties,which should be the correct way to the unification.

8 Experiment-free programme

Introduction and motivation

Every thing is actually about property,every property is actually about geometry.we can regard generating of quarks pair by vacuum as a process of displaying properties from the underlying geometry.

We claim that we can construct a theory with complete elimination of experiments.And this follows from a philosophy that is non-existence must exist which means we cannot understand a thing exists but we can understand a thing does not exist without explanation.Thus,achieving an experiment-free theory is to create a non-existing theory,for a theory does not exist we can explain the existence of such a theory without any doubts and experiments.This leads us to construct a theory of properties that for quantifying the extent of existence.And we call the process following from the philosophy and leads to such a theory,the *experiment-free programme* which is based on modern algebraic geometry which we developed in section 9.

8.1 Fundamental settings

Definition1.1 A *property* $\mathcal{P}$ is a relation $\mathcal{P} \subseteq \{(x,y)|x,y \in X\}$,a point or a *thing* is a trivial relation $x=y, x,y \in X$.In this case,a thing is a property.

Definition1.2 A category of property $\mathscr{C}_{\mathcal{P}}$ over C is a fibered category,with objects are P-properties for $P \in C$,and a morphism is a fiber product

$$\begin{array}{ccc} \{(x_1,x_2)\times(x_2,y_2)\} & \longrightarrow & \{(x_2,y_2)\} \\ \downarrow & & \downarrow \\ \{(x_1,y_1)\} & \longrightarrow & P \end{array} \qquad [8.1]$$

where $(x_1,y_1)\times(x_2,y_2)=(x_1x_2,y_1y_2)$.For instance,for two properties $>,=$,we have the fiber product $\{(x_1x_2,y_1y_2)|x_1x_2 \geq y_1y_2\}$ over a field P.Morphisms of categories $\mathscr{C}_{\mathcal{P}}, \mathscr{C}_{\mathcal{R}}$ of properties over C are functors.a morphism of functors is a base preserving natural transformation.In this case,we have a 2-category $HOM_C(\mathscr{C}_{\mathcal{P}},\mathscr{C}_{\mathcal{R}})$,for functors g,g',the natural transformation $\alpha : g \to g'$,the morphism $\alpha_{\mathcal{P}} : g(\mathcal{P}) \to g'(\mathcal{P})$ is a identity morphism in $\mathscr{C}_{\mathcal{R}}$.

Corollary1.3 A normal commutation is a property $\{(x,y)|xy=yx\}$.An anticommutation is a property $\{(x,y)|xy=-yx\}$.A (super)commutation above [7.4] is a property of a $\mathbb{Z}_2$-graded super ring A

$$\{(x,y)|xy=(-1)^{|x||y|}yx, \quad \forall\, x,y \in A\} \qquad [8.2]$$

The **T**-duality [6.10] is a property on the generalized module

$$\{(X^d,X'^d)|X'^d=\mathbf{T}X^d, \quad \forall\, d \in D\} \qquad [8.3]$$

Definition1.4 A *geometry* is a singular simplex $\sigma^n : \triangle^n \to X$,where X is a topological space.And $\triangle^n$ is an algebraic normalized n-simplex

$$\triangle^n = \mathrm{Spec}(R[\Delta^n]), \quad R[\Delta^n]=R[x^1,..,x^{n+1}]/\Big(1-\sum_i x^i\Big) \qquad [8.4]$$

which is an n-dim affine scheme.The simplex $\sigma^0 : \triangle^0 \to X$ is a point in X.

Definition1.5 A category of geometries $\mathscr{G}$ over C that is a category of topological spaces,is a fibered category.The objects are simplexes and a morphism is fiber product,for $X \in C$ and $\triangle^n \times \triangle^m = \triangle^{n+m}$

$$\begin{array}{ccc} \triangle^n \times \tilde{\triangle}^m & \longrightarrow & \tilde{\triangle}^m \\ \downarrow & & \downarrow{\scriptstyle \tilde{\sigma}^m} \\ \triangle^n & \xrightarrow{\sigma^n} & X \end{array} \qquad [8.5]$$

Corollary1.6 A family of open strings s^o in X is a geometry $[s^o] : \triangle^1 \to X$ and a family of closed string s^c in X is a geometry $[s^c] : \partial\triangle^2 \to X$ with boundary operator from $\partial^2\Delta^n = 0$

$$\partial\Delta^n = \sum_i (-1)^i [x^1, ..., x^{i-1}, x^{i+1}, ..., x^{n+1}] \qquad [8.6]$$

Then,we can formalize open and closed string by algebraic topology.As Δ^n forms a basis for $\mathbb{R}^n$,we have a simplicial group $S_n(X) = (\mathbb{Z}\Delta_n \to X, +)$ which is free Abelian.And the subscript means it forms chain complex

$$\cdots \longrightarrow S_{n+1}(X) \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \longrightarrow \cdots \qquad [8.7]$$

Then,by taking long exact sequence we get simplicial homology group

$$H_n(X) = \frac{\mathcal{Z}_n(X)}{\mathcal{B}_n(X)}, \quad \mathcal{Z}_n(X) = \ker(\partial_n)(X), \, \mathcal{B}_n(X) = \mathrm{im}(\partial_{n+1})(X) \qquad [8.8]$$

In this case we can see strings s in topological space X are elements of H_1 with closed string $s^c \in \mathcal{Z}_1(X)$ and open string $s^o \in \mathcal{B}_1(X)$.Similarly,we have the super simplicial homology group follows from the $\mathbb{Z}_2$-graded chain complex [8.7] denoting as $\mathscr{H}_n(X)$,and superstrings are classified by $\mathscr{H}_1$.

Corollary1.7 A supergeometry is a geometry $\mathfrak{s}^n : \triangle^n = \mathrm{Spec}(A[\Delta^n]) \to \mathcal{X}$ for a super ring A,with $\mathbb{Z}_2$-grading

$$\mathfrak{s}^n : \mathrm{Spec}(A_0[\Delta^n]) \oplus \mathrm{Spec}(A_1[\Delta^n]) \to \mathcal{X} \qquad [8.9]$$

which gives a formal definition for superscheme below [7.28]

Definition1.8 A superscheme is a superspace $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ with topological space $\mathcal{X}$ a admitting structure $\mathfrak{s}$ [8.9] and structure sheaf dicussed below [7.28].

Proposition1.9 A family of open superstrings in X is a supergeometry $\mathfrak{s}^o = \mathfrak{s}^1$.And a family of closed superstrings in X is a supergeometry $\mathfrak{s}^c = \partial\mathfrak{s}^2$.

Corollary1.10 A generalized geometry is a geometry

$$\mathfrak{g}^n : \triangle^n = \mathrm{Spec}(R[\Delta^n]) \to X, \qquad R = R \oplus R^* \qquad [8.10]$$

Definition1.11 A D-string s_D is a D1-brane with T-dual space point above [6.21].In this case,a family of D-strings in X is a generalized geometry $\mathfrak{g}^1$.

Definition1.12 A super generalized geometry is following

$$(\mathfrak{g} \oplus \mathfrak{s})^n : \mathrm{Spec}(N_A[\Delta^n]) \to \mathcal{N}, \quad N_A = (A_0 \oplus A_1) \oplus (A_0 \oplus A_1)^* \qquad [8.11]$$

which gives us the definition of generalized superscheme from definition1.8.

Corrollary1.13 From the natural property of generalized geometry below [6.22],the **T**-fusion extends a pair of [8.11] to a M-brane $\mathscr{M}$ with specific limit

$$\begin{aligned}
\mathrm{Spec}(N_{\mathbb{C}}[\Delta^D]) &= \mathrm{Spec}(\mathbb{C}[\Delta^D])_R \oplus \mathrm{Spec}(\mathbb{C}^*[\Delta^D])_{R'} \\
&\cong \mathrm{Spec}(\mathbb{C}[x^1, .., x^D])_R \oplus \mathrm{Spec}(\mathbb{C}[x'^1, ..., x'^D])_{R'} \\
&= \mathrm{Spec}(\mathbb{C}[x^1, .., x^D])_{R\to 0} \oplus \mathrm{Spec}(\mathbb{C}[x'^1, ..., x'^D])_{R'\to\infty} \\
&= \cdot \oplus \mathrm{Spec}(\mathbb{C}[x'^1, ..., x'^D]) = \cdot \oplus \mathrm{Spec}(\mathbb{C}^{(0,D)}) \qquad [8.12]
\end{aligned}$$

$$\begin{aligned}
&(\mathrm{Spec}(N_{\mathbb{C}}[\Delta^D]), \mathrm{Spec}(N_{\mathbb{C}}[\Delta^D]))_{R\to 0} \\
&\cong (\mathrm{Spec}(\mathbb{C}^{(0;D,*)}), \cdot \oplus \mathrm{Spec}(\mathbb{C}^{(0;*,D)})) \\
&\cong \mathrm{Spec}(\mathbb{C}^{(0;D,*)}) \oplus_{(0,D+D)} \mathrm{Spec}(\mathbb{C}^{(0;*,D)}) \cong_{\mathrm{P}(\mathbf{T})} \mathscr{M}
\end{aligned}$$

where the subscripts are set for the radius in [6.10] and we will discuss in the definition1.15.Also,[8.12] gives formal definition to M-brane.

Definition1.14 A M-brane $\mathscr{M}$ with $D = 10$ is a projective symmetric generalized superscheme admitting the structure $\mathrm{P}(\mathbf{T}) \circ ((\mathfrak{g} \oplus \mathfrak{s})^D)^2$ that is

$$(\mathrm{Spec}(N_{\mathbb{C}}[\Delta^D]), \mathrm{Spec}(N_{\mathbb{C}}[\Delta^D])) \to (\mathcal{N}, \mathcal{N}) \to \mathscr{M} \quad [8.13]$$

And we see the supersymmetry naturally exists from $\mathbb{C}[x^1, ..., x^D] = \mathbb{C}^{D|D}$.And if we both have super and generalized settings,we ignore to denote symmetry because it naturally fit in.

Also,there is a subtile point,recall that the normalization in [8.4],actually the right hand side of the first line in [8.12] is normalized,in this case,we cannot operate the radius,this is answered by the following definition.

Definition1.15 A spontaneous regularity breaking is the normalized radius spontaneously breaks to those of double spaces in generalized geometry.

$$\Big(1 - \sum_i x^i\Big)_{N_{\mathbb{C}}[\Delta^D])} = \Big(R - \sum_i x^i\Big)_{\mathbb{C}[\Delta^D]} \times \Big(1/R - \sum_i x^i\Big)_{\mathbb{C}^*[\Delta^D]} \quad [8.14]$$

where we set $\alpha' = 1$ in [6.10].And $R \to 0$ actually means $R \to h - \epsilon$ to let it be unobservable and R' almost counts whole T-dual space now.

8.2 Algebraifold $\mathcal{A}$ and equivalence of categories

From an observation that a manifold is a continuous extension of a geometric object,we want to define an algebraifold for a continuous extension of an algebraic structure.

Definition2.1 An algebraifold $\mathcal{A}$ in topos T on the big étale site $\mathrm{Et}(X)$,is a locally ringed space locally isomorphic to a category AUU^ϵ with $h_{A_i} \in \mathrm{AS}/U$ and $Y/U^\epsilon \in \mathrm{AS}/U^\epsilon$ in category of algebraic spaces for an étale cover $U \in \mathrm{Et}(X)$

$$\mathcal{A}_A = \{h_{(A_i|U)} \to (Y/U^\epsilon_{/U}) \to h_{(A_j|U)}\}_{i,j \in I} \to U \quad [8.15]$$

where $A_i \in \mathrm{AS}/U \subset T$ on étale U-scheme U_i in $\mathrm{Cov}(X)$.And the evolution of relative properties is given by [9.130].If $U = \mathrm{Spec}(R)$,we must have an étale cover and it factors through an algebraic infinitesimal system

$$U = \mathrm{Spec}(R) \xleftarrow{\text{et}} \mathrm{Spec}(M/\epsilon^2) \xleftarrow{\text{et}} \mathrm{Spec}(M), \quad M/\epsilon^2 = (RM)/\epsilon^2 = R(M/\epsilon^2) \quad [8.16]$$

see [7.15] and we are in big étale site.In this case,$U^\epsilon = \mathrm{Spec}(R + \epsilon m)$

$$Y/U^\epsilon_{/U} = T|\mathrm{Et}(U) \to \mathrm{Hom}(T|\mathrm{Et}(U), B|(\mathrm{Spec}(R + \epsilon m))), \quad \epsilon m \in M/\epsilon^2 \quad [8.17]$$

with the following diagram commutes,with infinitesimal transition f^ϵ_U

$$\begin{array}{ccccc} \mathrm{AS}/U & \longrightarrow & \mathrm{AS}/U^\epsilon_{/U} \cong \mathrm{AUU}^\epsilon & \xrightarrow{(\sqrt{\Delta_{U^\epsilon/U}})^*} & \mathrm{AS}/U^\epsilon \\ \downarrow & & \downarrow & & \downarrow \\ U & \xrightarrow{f^\epsilon_U} & U^\epsilon_{/U} & \longrightarrow & U^\epsilon \end{array} \quad [8.18]$$

Proposition2.2 A function is a property $\{(x,y)|y=f(x)\}$.In this case,a sheaf is a category of properties.An algebraic space A over S-scheme U is equivalent to a fibered category of relative properties $\mathscr{C}_A$ and admits an étale presentation $X \to A$ over S.Which is equivalent to say for an étale equivalence relation which is $R \hookrightarrow X \times_S X$ and for étale projections $s,t : R \rightrightarrows X$ with $A = X/R \cong \mathscr{C}_A$ with $\#\mathscr{C}_A = \#(X/R)$.

Proposition2.3 If $A \in T$ satisfies [9.6] and extra condition for generalized geometry,we have a generalized superalgebraifold.

Theorem2.4 The category of properties is equivalent to the category of geometries $\mathscr{C}_{\mathcal{P}} \cong \mathscr{G}$ over a same base P.

Proof. If $\mathcal{P}$ are equivalent properties that are equivalent relations,the category of equivalent properties corresponds to an algebraifold $\mathcal{A}_A \in \mathscr{G}$ and a property corresponds to a $\triangle^n$ with $\partial\triangle^n = 0$.If $\mathcal{P}$ are properties that are not equivalent relations,a property corresponds to a $\triangle^n$ with $\partial\triangle^n \neq 0$. □

This proof is based on an observation that we can view a geometry with boundary corresponds to a relation that is not equivalent of boundaries,a geometry without boundary corresponds to a relation that is equivalent of boundaries.

Corollary2.5 by using the theorem2.4,a family of open string is a property that is not equivalent,a family of closed string is an equivalent property.And we know,the massless bosonic state of closed string gives the graviton which is the source of gravity.In this case,a family of gravitons actually is an equivalent property $\mathcal{P} \cong [s^c] : \partial\triangle^2 \to X$ in corollary1.6,with the first oscillation level reflected by the topology in X.

Theorem2.6 There is a natural existence of gravity in the structure constructed by algebraifolds $\mathcal{A}$.

An interesting thing is if we view scheme [8.16] as a functor [9.7] we have

$$\begin{tikzcd} U^\epsilon \arrow[r, mapsto] \arrow[d] & A^\epsilon \arrow[d] \\ h_{\mathrm{Spec}(R+\epsilon m)} \arrow[r] & \mathrm{Spec}(A^\epsilon(U^\epsilon)) \arrow[ul, "\mathrm{et}"'] \end{tikzcd} \qquad [8.19]$$

then if A is a sheaf of super structure,$R+\epsilon m \in S$ with S a symmetric super R-module,we must have $\epsilon m \in S_0/(\epsilon^2)$ or S_1 because $(x+\theta)^2 = 2x\theta \neq 0, \forall x,y \neq 0, x(\epsilon) \in S_0/(\epsilon^2), \theta \in S_1$.Then we can form a linear combination

$$\epsilon m \in M \oplus 1 \,\mathrm{or}\, 1 \oplus M^* \quad (\epsilon S_0/\epsilon^2) \oplus S_1 \cong M \oplus M^* \qquad [8.20]$$

If we let $\epsilon = \partial, m = Q, \epsilon m \in M \oplus 1$ with a conserved charge Q.This induces a split of dynamics on the generalized superalgebraifold $\underline{\mathcal{A}} = \mathcal{A} \oplus \mathcal{A}^*$

$$\begin{aligned} &\partial \oplus \partial^* = \partial \oplus 1 + 1 \oplus \partial^* = \hat{\partial} \in \underline{\mathcal{A}}, \quad \hat{\partial}(Q \oplus Q^*) \in \mathrm{Mod}_{\underline{\mathcal{A}}} \\ &\hat{\partial}(Q \oplus Q^*) = \partial Q \oplus Q^* + Q \oplus \partial^* Q^* \equiv \partial(Q \boxtimes Q^*) \end{aligned} \qquad [8.21]$$

where $\mathrm{Mod}_{\mathcal{A}}$ is the category of all $A(U)$-modules and from an observation that is a moving object can be separated to movement and object itself,the conserved quantity is reflected by the closure from the movement as an action of an algebraic structure on the object in the vector space.And we call $\otimes \to \boxtimes$ the **T**-fusion like,which makes generalized algebraifold on pre M-brane [6.22] where we loose the double setting because we **T**-fused the generalized geometry.

Theorem2.7 For a generalized superalgebraifold $\underline{\mathcal{A}}$,the dynamics of one part is open if and only if the dynamics of T-dual part is closed,with the conserved current $\partial(Q\boxtimes Q^*) \in \mathrm{Mod}_{\underline{\mathcal{A}}}$,also this split of dynamics induces a split of modules $\mathrm{Mod}_{\underline{\mathcal{A}}} = \mathrm{Mod}_{\mathcal{A}\oplus 1} \boxtimes \mathrm{Mod}_{1\oplus\mathcal{A}^*}$

8.3 Relative property and nonexistence

Definition3.1 A relative property is an étale equivalent property which is an étale equivalent relation.On the scheme level,that is a monomorphism $R \hookrightarrow X \times_s X$ which preserves for T-points $R(T) \subset X(T) \times X(T)$ and the projections are étale $R \hookrightarrow X \times_s X \rightrightarrows X$ where $S = \coprod \mathrm{Spec}(N_{\mathbb{C}}[\Delta^{10}])$.

Corollary3.2 By using the equivalence of categories in theorem2.4,an étale equivalent property is a class of open strings in the scheme with the endpoints gluing by étale morphism,from the notation we used in [7.32]

$$s(s^o) \xrightarrow{\text{et}} t(s^o) \quad \Leftrightarrow \quad s^c,\ (s(s^o), t(s^o)) = (s(s^o_D), t(s^o_D)) \qquad [8.22]$$

we used definition1.11.Which means an an étale equivalent property can be a class of closed strings or class of open strings with boundary attaching on a fixed plane.In this case,setting of étale is the source of open strings with D-branes.

Theorem3.3 The only type of strings in all string theories is the type of étale closed strings.The ordinary open string is just an ordinary closed string split by étale morphism.We will discuss this in section 12 and 13.

proof. From the difference of endpoints of open string [6.19] we have $X_i'^d = X_j'^d + \theta R'$ which means $X_i'^d = X_i'^d(X_j'^d)$,and $X_j'^d = X_j'^d(X_i'^d)$ then the relative differential $\Omega_{X_i'^d/X_j'^d} = 0$,which means the $X_i'^d \to X_j'^d$ is étale.

Definition3.4 A generalized super étale morphism is the étale morphism along each grading.For instance,$N_A \overset{\text{et}}{\to} N_B$ if and only if

$$A_0 \overset{\text{et}}{\to} B_0, A_1 \overset{\text{et}}{\to} B_1; A_0^* \overset{\text{et}}{\to} B_0^*, A_1^* \overset{\text{et}}{\to} B_1^* \qquad [8.23]$$

Definition3.5 A Dp-brane in super generalized geometry is a homotopy of super generalized étale morphisms in the section of the coherent sheaf

$$\begin{aligned} &\mathrm{D}p : (x' - (X_{ii}'^{p+1}, ...X_{ii}'^{p+1})) \in M^*, \times[0,1]^p \underset{\text{et}}{\overset{\text{et}}{\rightrightarrows}} \mathrm{Spec}(N_A[x^1, ..., x^D]) \\ &\mathrm{D}p \in \Gamma(\mathrm{Spec}(\mathbb{C}[x'^1, ..., x'^D]), \prod_d C^\infty[x'^1, ..., x'^d] \otimes_{\mathbb{C}[x'...]} M^{D-d}) \end{aligned} \qquad [8.24]$$

where we used [6.21],[8.11] and from discussion below [7.29],the sheaf is coherent $(C^\infty[x^{d+1}...x^D] \oplus C^\infty[x'^1, ..., x'^d])_{\mathbb{C}^{(D-d,d)}} \cong \prod_d C^\infty[x'^1, ..., x'^d] \otimes_{\mathbb{C}[x'...]} M^{D-d}$.

Another interesting thing is we combine [8.2] and [8.3] to discuss super generalized property,the supersymmetry gives a $\mathbb{Z}_2$ symmetry on the pair and the

T-duality also gives the symmetry on the it.This induces a spontaneous $\mathbb{Z}_2$ symmetry breaking for $N_A[x^1,...,x^D]$.

Definition3.6 A generalized super ring is the ring with structure

$$\begin{aligned}(A_0 \oplus A_1) \oplus (A_0 \oplus A_1)^* &\cong_{Bos} (A_0 \oplus A_0) \oplus (A_1 \oplus A_1)^* \\ &= (A \oplus A)_0 \oplus (A \oplus A)_1^*\end{aligned} \qquad [8.25]$$

where we used bosonisation [6.31].With super T-duality on it

$$\mathbf{T}^\delta : (A \oplus A)_0 \oplus (A \oplus A)_1^* \to (A \oplus A)_1^* \oplus (A \oplus A)_0 \qquad [8.26]$$

Corollary3.7 A generalized super ring has the following decomposition

$$\begin{aligned}(A \oplus A)_0 \oplus (A \oplus A)_1^* &= \frac{1}{2}((A \oplus A)_0 \oplus (A \oplus A)_1^*) + \frac{1}{2}((A \oplus A)_1^* \oplus (A \oplus A)_0) \\ {}_{\times\text{ordered}} &\to \frac{1}{2}((A \oplus A)_0 \times (A \oplus A)_1^*) + \frac{1}{2}((A \oplus A)_1^* \times (A \oplus A)_0)\end{aligned} \qquad [8.27]$$

Definition3.8 A spontaneous $\mathbb{Z}_2$ breaking on the generalized superscheme $\mathcal{X} \oplus \mathcal{X}^*$ level is the breaking of symmetric product scheme $\mathcal{X}_0 \times_s \mathcal{X}_1^*$ from [8.25] to two ordered products from [8.27] of the underlying generalized super ring.

$$(\mathcal{X}_0 \times_S \mathcal{X}_1^*)^{\text{unself } \mathbf{T}} =_{\times\text{ordered}} \mathcal{X}_0 \times_S \mathcal{X}_1^*/\mathbb{Z}_2 + \mathcal{X}_1^* \times_s \mathcal{X}_0/\mathbb{Z}_2 \qquad [8.28]$$

We will see a self T-dual case in [8.34].Following the breaking [8.28],we have a classification of generalized super étale equivalent properties

$$\mathcal{R} \hookrightarrow \mathcal{X}_0 \times_S \mathcal{X}_1^*, =_{\text{ordered}} \mathcal{R}^+ \hookrightarrow \mathcal{X}_0 \times_S \mathcal{X}_1^*/\mathbb{Z}_2, + \mathcal{R}^- \hookrightarrow \mathcal{X}_1^* \times_S \mathcal{X}_0/\mathbb{Z}_2 \quad [8.29]$$

Which gives a field,that is $\mathrm{F}^+ = \{\#\mathcal{R}^+\}$ and $\mathrm{F}^- = \{\#\mathcal{R}^-\}$ with additive identity $0_{\mathrm{P}(\mathbf{T})^{-1}Q\boxtimes Q^*}$ when $\mathcal{R}$ are properties on $\mathrm{Spec}(N_{\mathbb{C}}[x^1,...,x^D])$.And the field formed by étale equivalent properties on $\mathscr{M}$ we will discuss in [9.17].The subtile point is the former identity need to be **T**-fused to identity on the M-brane that we denote as $0_{Q\boxtimes Q^*}$ which cannot be ordinary zero we will see.

Definition3.9 The local nonexistence locally on $\mathrm{Spec}(N_{\mathbb{C}}[x^1,...,x^D])$ or $\mathscr{M}$ is total number of étale equivalent properties is equal to $0_{\mathrm{P}(\mathbf{T})^{-1}Q\boxtimes Q^*} = 0_{Q\boxtimes Q^*}$.

$$\sum_{\mathrm{P}(\mathbf{T})^{-1}\mathscr{M}} \{\#\mathcal{R}\}_{\mathcal{R}} = \sum_{\mathscr{M}} \{\#\mathcal{R}_{\mathscr{M}}\}_{\mathcal{R}} = 0_{\mathrm{P}(\mathbf{T})^{-1}Q\boxtimes Q^*} = 0_{Q\boxtimes Q^*} \neq 0 \qquad [8.30]$$

Definition3.10 The nonexistence is the global nonexistence with total number of étale equivalent properties is equal to 0,with no relative effect (independent of choices of reference frame).

8.4 Super generalized general relativity

Firstly,recall in double field theory,we have double copies of theories induced by T-duality,and where we live in depends on how to choose the dynamics (can be generated by winding number or angular momentum),the dynamics of one side must be closed when the dual side is open.And [8.21] is an algebraic expression of this and such freedom has relative effect.Now,we want to study relativity in super generalized geometry,which indeed makes sense based on the theorem2.7,which means we can put relative motion frame (open dynamics) in $\mathrm{Mod}_{\mathcal{A}\oplus 1}$ and corresponding rest frame (closed dynamics) to $\mathrm{Mod}_{1\oplus\mathcal{A}^*}$.

Definition4.1 A motion is an étale equivalent property that is a set of pairs $\{(x,y)|y = Lx, L \in SO(D-1,1)\}$.The general relativity is a fibered product $X \times_S X'$ where X' is a reference scheme compared to X.An interesting thing is

$$\sqrt{1-\frac{v^2}{c^2}} = R \Rightarrow R^2 = 1-\frac{v^2}{c^2} \Rightarrow 1 = \sqrt{R^2+\frac{v^2}{c^2}} \qquad [8.31]$$

which gives us a modified Lorentz correspondence

$$1 \times x \leftrightarrow_L \gamma(x-vt) \quad \Leftrightarrow \quad \sqrt{R^2+\frac{v^2}{c^2}}x \leftrightarrow \frac{x-vt}{R} \qquad [8.32]$$

Then,we compare [8.32] with [6.10] and we find Lorentz correspondence behaves like T-duality,with now the coordinates of dual space is $x' = x + vt$,and denote $M \oplus M'$ as generalized Lotentz module and grading extends to further algebraic structure.We can see the super generalized general relativity is used for explaining possibility in quantum theory [12.21].

Theorem4.2 There is a natural inclusion from a generalized Lorentz scheme to generalized superscheme with the classification of relative properties

$$\begin{tikzcd} X \oplus X' \arrow[r] \arrow[d, hook] & X \times_s X' \arrow[d, hook] & \mathcal{R}' \arrow[l, hook'] \arrow[d, hook] \\ \mathcal{X}_0 \oplus \mathcal{X}_1^* \arrow[r] & \mathcal{X}_0 \times_s \mathcal{X}_1^* & \mathcal{R} \arrow[l, hook'] \end{tikzcd} \qquad [8.33]$$

The physics meaning of [8.32] is the source of Lorentz transformation or special relativity is the compactness of underlying geometry.And the reference frame lives in the T-dual space with radius $1/R = \gamma$,when v changes the T-dual space radius changes and the ordinary space is always unchanged.This means the two parts in super generalized geometry in [8.11] are decoupled when $v \neq 0$ that is non-trivial case,which means they should be self T-dual,$R' \in \mathbb{C} \setminus \mathbb{R}_{\leq 1}$

$$\begin{aligned} &\mathrm{Spec}(N_{\mathbb{C}}[\Delta^D])^{\mathrm{dec}} \cong \mathrm{Spec}(\mathbb{C}[x^1,...,x^D])_{R=1} \coprod \mathrm{Spec}(\mathbb{C}[x'^1,...,x'^D])_{R'} \\ &\cong_{\mathrm{P}(\mathbf{T})} \mathrm{Spec}(\mathbb{C}[x^1,...,x^D,*]) \coprod \mathrm{Spec}(\mathbb{C}[*,x^1,...,x^D])_{R' \in \mathbb{R}^{\mathrm{norm}}_{>1}} \subset \mathscr{M}, \\ &\coprod_{\mathrm{P}(\mathbf{T})^{-1}} \mathrm{Proj}(\mathbb{R}[x^1,...,x^{D-1},*]) \oplus \mathrm{Proj}(\mathbb{R}[*,x'^1,...,x'^{D-1}]) \end{aligned} \qquad [8.34]$$

with trivial **T**-fusion.The explanation of [8.34] is given by combining physics meaning of automorphism with derived obstruction theory,we will see in [12.65].

And the black hole limit lets us construct $\mathbf{T}$-fusion dimension for dimension and combine [6.22],we have $\mathbf{T}$-fusion hierarchy

$$\mathrm{Proj}(N_{\mathbb{R}}[\Delta^{D-1}])^{\mathrm{dec}} \cong \mathrm{Proj}(\mathbb{R}[x^1,...,x^{D-1}])_{R=1} \coprod \mathrm{Proj}(\mathbb{R}[x^1,...,x^{D-1}])_{R'}$$
$$\cong_{\mathrm{P}(\mathbf{T})} \mathrm{Proj}(\mathbb{R}[x^1,...,x^{D-1},*,*]) \coprod \mathrm{Proj}(\mathbb{R}[*,*,x^1,...,x^{D-1}])_{R'\in\mathbb{R}^{\mathrm{norm}}_{>1}} \subset \mathscr{M}$$
$$\coprod_{\mathrm{P}(\mathbf{T})^{-1}} \mathrm{Spec}(\mathbb{C}[x^1,...,x^{D-2},*]) \oplus \mathrm{Spec}(\mathbb{C}[*,x'^1,...,x'^{D-2}])$$
[8.35]

the $\mathbb{R}$ with imaginary radius $\mathbf{T}$-fuse to $\mathbb{C}$ which is $(\mathbb{R}-)_{R'\in\mathbb{C}\setminus\mathbb{R}} \cong_{\mathrm{P}(\mathbf{T})} (\mathbb{C}-)$ and similarly for others.Also,we use $\mathbb{C}$ for super complex number for simplicity.

Definition4.3 A black hole on $D = 10$ decoupled super generalized geometry relative to self T-dual ordinary part is

$$\widetilde{\mathscr{B}\mathscr{H}} = \mathrm{Spec}(\mathbb{C}[x'^1,...,x'^D])^{\mathrm{self}\,\mathbf{T}}_{R'\in\mathbb{C}\setminus\mathbb{R}} \bigcap \cdot \oplus \mathrm{Proj}(\mathbb{R}[*,x'^1,...,x'^{D-1}])_{R'} \quad [8.36]$$

where we use the superscripts for self T-dual in decoupled case.

Remark4.4 We call the result theory that from viewing Lorentz correspondence as T-duality and include it into decoupled super generalized geometry,the super generalized general relativity.

Proposition4.5 We can understand above by putting [8.34] in [8.33],we get another decomposition from the self T-dual

$$(\mathcal{X}_0 \times_s \mathcal{X}_1^*)^{\mathrm{self}\ \mathbf{T}} =_{\mathrm{decoupled}} (\mathcal{X}_0 \times_s \mathcal{X}_0)_{\mathscr{M}} + (\mathcal{X}_1^* \times_s \mathcal{X}_1^*)_{\mathscr{M}} + \widetilde{\mathcal{X}}_1^* \times_s \widetilde{\mathcal{X}}_1^* \quad [8.37]$$

and the tilde is for unseen part (black hole,dark energy etc.) because we are in compact dimension with imaginary radius.And the decomposition [8.34] gives a self T-dual classification of generalized super étale equivalent properties

$$\mathscr{R} = \mathscr{R}^{++}_{\mathscr{M}} + \mathscr{R}^{--}_{\mathscr{M}} + \widetilde{\mathscr{R}} \quad [8.38]$$

where the subscript is for the relative properties on M-brane.Compared to discussion below [8.29],there is no a number counting for $\widetilde{\mathscr{R}}$ to construct a field but easily we can quantify it by [8.30] with definition3.10

$$0 = \sum\{\#\mathscr{R}\} = \sum\{\#\mathscr{R}^{++}_{\mathscr{M}} + \#\mathscr{R}^{--}_{\mathscr{M}}\} + \sum\{\#\widetilde{\mathscr{R}}\} \quad [8.39]$$

If we set it to cosmology constant Λ we have

$$\sum\{\#\widetilde{\mathscr{R}}\} = \Lambda = -0_{Q\boxtimes Q^*} \quad [8.40]$$

8.5 S-duality and U-duality on étale closed strings

Actually,[8.22] is the source of S-duality,an ordinary open string is dual to a D-string with agreement on endpoints.We know open strings live on Dd-brane which means they live in $D-d$ dimensional spacetime,from [6.21]

$$\mathbf{S} : C^\infty([x^{d+1},...,x^D] \oplus [x'^1,...,x'^d]) \to C^\infty([x'^1,...,x'^d] \oplus [x^{d+1},...,x^D]) \quad [8.41]$$

If we mod the **S**-duality,we get the generalized algebra from the coset

$$C^{\infty}([x^{d+1},...,x^{D}]\oplus[x'^{1},...,x'^{d}])+\mathbf{S}C^{\infty}([x^{d+1},...,x^{D}]\oplus[x'^{1},...,x'^{d}]) \quad [8.42]$$

Thus,we can regard S-duality as a fiber of T-duality on the generalized space with $\mathbf{S}C^{\infty}\neq C^{\infty}\mathbf{S}$.For generalized space $\mathbb{C}^{(D,D)}$,we have

$$\begin{tikzcd} \mathbf{S}^{(D,D)} \arrow[r, hook] \arrow[d] & \mathrm{End}(\Gamma(\mathrm{Spec}(\mathbb{C}^{(D,D)})) \arrow[d] \\ \mathbf{T}^{(D,D)} \arrow[r, hook] & \mathrm{End}(\mathrm{Spec}(\mathbb{C}^{(D,D)})) \end{tikzcd} \quad [8.43]$$

where if we set D to one direction and D^{*} to a prependicular direction,we find $\mathbf{S}^{(1,1)}{}_{D,D^{\perp}}\in SU(2,C^{\infty}\mathbb{C}^{(1,1)})\subset SL(2,\mathbb{C}^{(1,1)})$ where we set $f(x')\cong x'$.Now,we want to consider a fibered product and use the generalized super ring [8.11]

$$\begin{tikzcd} \mathrm{End}(\mathrm{Spec}(\mathbb{C}^{(D,D)}))\times_{\mathscr{M}}\mathrm{End}(\Gamma(\mathrm{Spec}(\mathbb{C}^{(D,D)})) \arrow[r, "\mathrm{P}(\mathbf{T})^{-1}"] \arrow[d, "\mathrm{P}(\mathbf{T})^{-1}"] & \mathrm{End}(\Gamma(\mathrm{Spec}(\mathbb{C}^{(D,D)})) \arrow[d] \\ \mathrm{End}(\mathrm{Spec}(\mathbb{C}^{(D,D)})) \arrow[r] & \mathscr{M} \end{tikzcd}$$

[8.44]

where we used **T**-fusion and recall the sheaf of properties on M-brane

$$\mathrm{End}(\mathrm{Spec}(\mathbb{C}^{(D,D)}))\times_{\mathscr{M}}\mathrm{End}(\Gamma(\mathrm{Spec}(\mathbb{C}^{(D,D)}))=_{\mathrm{P}(\mathbf{T})}\mathrm{End}(\mathscr{P}(\mathscr{M})) \quad [8.45]$$

Through the fiber product [8.44],we can define a new duality called U-duality

$$\mathbf{T}\times_{\mathscr{M}}\mathbf{S}=_{\mathrm{P}(\mathbf{T})}\mathbf{U}\in\mathrm{End}(\mathscr{P}(\mathscr{M})) \quad [8.46]$$

Also,we can apply S-duality on the coherent version of [8.41] in [8.24]

$$\mathbf{S}:C^{\infty}[x'^{1},...,x'^{d}]\otimes_{\mathbb{C}[x'...]}M^{D-d}\rightarrow C^{\infty}[x'^{d+1},...,x'^{D}]\otimes_{\mathbb{C}[x'...]}M^{*d} \quad [8.47]$$

Then we can let $C^{\infty}=\mathcal{T}:\mathrm{Spec}(N_{\mathbb{C}}[x'^{1},...,x'^{D}])_{\mathrm{Zar}}\rightarrow\mathrm{GSTen}$,to the generalized super tensor category based on [7.17] and super T-duality [8.26] with $\mathrm{Ob}(\mathrm{GSTen})=\mathrm{Ob}(\mathrm{GSTen}_{0})\oplus\mathrm{Ob}(\mathrm{GSTen}_{1}^{*})$

$$\mathrm{Ob}(\mathrm{GSTen}_{0})=\bigoplus_{0\leq n\leq D}M^{\otimes n},\quad \mathrm{Ob}(\mathrm{GSTen}_{1}^{*})=\bigoplus_{0\leq n\leq D}M^{*\otimes n} \quad [8.48]$$

morphisms are $M^{\otimes n}\xrightarrow{\otimes M^{*\otimes m}}M^{*\otimes(n+m)}$ and $M^{*\otimes n}\xrightarrow{\otimes M^{*\otimes m}}M^{\otimes(n+m)}$.And we apply [8.43],that means we should have a S-duality over the super T-duality

$$\mathbf{S}:M^{\otimes(D-d)}\rightarrow M^{*\otimes d},\qquad \mathbf{T}^{\delta}=\mathbf{S}\circ *:M^{\otimes d}\rightarrow M^{*\otimes d} \quad [8.49]$$

guided by [8.47] and the Hodge duality [5.25],we can express Hodge duality as

$$*=(\mathbf{T}^{\delta}\circ\mathbf{S})_{\mathrm{GSTen}} \quad [8.50]$$

which can be constructed by the dualities in superstring theory under super algebraic generalized geometry.

8.6 Stack generalized by dualities

For a stack $p : F \to C$ which is a category fibered in groupoids,for defining global descent theory,the presheaf $\underline{\mathrm{Isom}}(x, x')$ on $(C/X)^{\mathrm{op}}$ [9.48] is a sheaf.

Now,for a prestack $F \oplus F^* \to C$ generalized by a duality,we want to study the presheaf $\underline{\mathrm{Isom}}(x, x'), x \in F(X), x' \in F^*(X)$.The corresponding fibered category is $p : \mathscr{D} = (\underline{\mathrm{Isom}}(x, x')(C/X)^{\mathrm{op}}) \to (C/X)^{\mathrm{op}}$.For a X-scheme $Y \to X$ we have a X-morphism $(Y' \xrightarrow{f} Y)/X$.We can define a category $\mathscr{D}((Y' \xrightarrow{g} Y)/X)$ with an object (f, Θ),where we want to view f as a functor

$$f \in g^* \underline{\mathrm{Isom}}(x, x')(Y \to X) \hookrightarrow HOM_{(C/X)}(F(Y'), F^*(Y')) \qquad [8.51]$$

Then,the transition Θ is a natural transformation in $\mathscr{D}(Y' \times_Y Y'/X)$,that is $\Theta : \mathrm{pr}_1^* f \to \mathrm{pr}_2^* f$ such that the following diagram commutes

$$\begin{array}{ccccc} \mathrm{pr}_{12}^*\mathrm{pr}_1^* f & \xrightarrow{\mathrm{pr}_{12}^*\Theta} & \mathrm{pr}_{12}^*\mathrm{pr}_2^* f & = & \mathrm{pr}_{23}^*\mathrm{pr}_1^* f \\ \| & & & & \downarrow{\scriptstyle \mathrm{pr}_{23}^*\Theta} \\ \mathrm{pr}_{13}^*\mathrm{pr}_1^* f & \xrightarrow{\mathrm{pr}_{13}^*\Theta} & \mathrm{pr}_{13}^*\mathrm{pr}_2^* f & = & \mathrm{pr}_{23}^*\mathrm{pr}_2^* f \end{array} \qquad [8.52]$$

where Θ is called descent data for the functor f.A morphism is given by [9.53].

Definition6.1 A stack generalized by a duality is a category $F \oplus F^* \to C$ fibered in groupoids with ordinary stack conditions,with the extra condition

$$\mathscr{D}(\{(W_i \xrightarrow{g} Y)/X\}_{i \in I}) \cong \mathscr{D}(Y \to X) \qquad [8.53]$$

for all covering of X-scheme W,which is a global descent theory for gluing duality fusions in ordinary groupoid.So we have a global effective descent data in fibered 2-category which is a stack of dualities.

$$HOM_{(C/X)}(F, F^*) \to (C/X) \qquad [8.54]$$

Now,recall that the fibered category as a functor and scheme as a functor below [9.19],we need following fusion condition $C/X \cong X \cong F$,which means

$$\begin{aligned} HOM_{\bigcup_X (C/X)}(F, F^*) &\cong HOM_{\bigcup_X X}((C/X), F^*) \\ &= HOM_{\tilde{C}}((C/X), F^*) \cong F^*(X) \cong F^*(F) \end{aligned} \qquad [8.55]$$

where we used 2-Yoneda lemma [9.25],and we put it in [8.54]

$$p : (F^* \to F) \to \bigcup_X (C/X) = \tilde{C} \qquad [8.53]$$

which is a stack of fusions of dualities.

Definition6.2 A fusion of a duality is an object of $p(F^* \to F) \in \tilde{C}$.

For instance,if we set X to be a M-brane,we have $\tilde{C} = \bigcup_{\mathscr{M}}(C/\mathscr{M}) \cong (\mathscr{M})$.In this case,a $\mathbf{T}$-fusion is $F^*(\mathscr{M})(F(\mathscr{M})) \cong_{\mathrm{p}(\mathbf{T})} \mathscr{M}$.

8.7 Preview of M-theory

In this paper,we regard the modern algebraic geometry as the mathematical counterpart of M-theory,because we always follow a philosophy about everything is a reflection of properties.And in this case,every process should be a process of generating properties from the space or vacuum.Also,we have seen that every property is an algebraic structure corresponds to a geometric structure,in this case algebraic geometry should be a natural language.And we need a number counting for relative properties to achieve experiment-free.In the classical method (differential geometry),the first limit is we cannot correctly define and study the space or vacuum,the second limit is it can not provide a methodology to study a process of generating properties.

Definition7.1 A prespace (prevacuum) is an object for generating properties.In this case,we have a completely different way to understand the vacuum.For instance,a sheaf F is a prespace because it can generate a property $(X, F(X))$.Which enlarges the category of schemes to category of schemes with sheaves.Notice that we need relative properties for nonexistence of M-theory and we will see in [9.112] only a part of sheaves (algebraic spaces) are spaces.

Definition7.2 A preservation of universal property is a process admitting a relative 2-property below [9.110] along a covering.For instance,if conservation of positive energy is a relative property,positive energy is a relative 2-property.

Definition7.3 A space is a consistent prespace that guarantee preservations of universal properties.

Also,modern algebraic geometry started at viewing scheme as a functor,the physics meaning is the real vacuum meaning it is consistent,is not eventually nothing,it indeed has information for generating properties.Also,modern algebraic geometry supports to construct a theory of moduli.And the theory of moduli on the consistent site,unifying all superstring theories without verification of experiment,the M-theory.With further development of our theory we will give a full understanding of $D+1$-dim M-theory and D-dim superstring theories in [13.9].And the full process can be seen in diagrams above [13.1].We will give clear definition of M-theory at start and achieve the unification at end.

9 Modern super algebraic geometry I

9.1 The sheaf of properties $\mathscr{P}$

Supersymmetry is a background for describing geometry of our world adequately,and for the geometry,we want to apply modern AG based on the text [12] based on Grothendieck's philosophy that is points(closed sets) are not important,the importance is a collection of maps covering others(open sets),we want to ignore points completely,this shift from study points to maps onto open sets induce the generalization of standard AG.Which shifts the focus point from a category of schemes $(\mathrm{Sch}, \mathrm{Hom})$ to a site $(\mathrm{Sch}, \mathrm{Cov})$.

A Grothendieck topology on a category C is a set $\mathrm{Cov}(X) = \{\{X_i \to X\}_{i\in I}\}$

for all $X \in C$

(i)$\{V \cong X\} \in \mathrm{Cov}(X)$
(ii)$\{X_i \times_X Y \to Y\}_{i\in I} \in \mathrm{Cov}(X) \quad \forall \{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X), Y \to X \in C$
(iii)$\{V_{ij} \to X_i \to X\}_{i\in I, j\in J_i} \quad \forall \{X_i \to X\}_{i\in I}, \{V_{ij} \to X_i\}_{j\in J_i}$

[9.1]

A site is a category C with a Grothendieck topology.By the definition,a category $\mathrm{Op}(X)$ of open sets U of scheme X is a site with classical topology that is $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(X)$ with $\bigcup_i U_i = U$.A small étale site $\mathrm{et}(X)$,is a category of étale represented schemes X with objects are étale morphisms $U \to X$ we discussed in [7.30] and the Grothendieck topology $\mathrm{Cov}(U)$ is classical and the globalization $\coprod_i U_i \to U$ is surjective.A big étale site $\mathrm{Et}(X)$ is based on the strucuture of $\mathrm{et}(X)$ with more general topology that each $U_i \to U$ is étale with surjective globalization,it has enough coverings.

For a site C,a presheaf is a functor $F : C^{\mathrm{op}} \to \mathrm{Set}$.And it is a sheaf if it satisfies

$$\begin{aligned} &0 \to F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j) \to 0, \quad \forall U \in C \\ &F(U) = F(\coprod_i U_i) \cong \prod_{i,j\in I} F(U_i)/F(U_i \cap U_j) \end{aligned} \qquad [9.2]$$

which is an exact sequence for equalizer with a globalization $\coprod_i U_i \to U$ for a covering $\{U_i \to U\}_{i\in I}$.We can see an example [7.20] and the Gr operation is actually a sheafification.And the key observation is we ignore points (closed sets) and define sheaf only on open sets and their covering.

A topos T is an equivalent category of that of sheaves.Which is just a generalization of sheaf on a scheme to a category of sheaves on a site of schemes.And we denote topos on a small étale site X_{et},also we use S_{et} for the étale site of S [9.79],if not clear we use tilde for topos.In this case,we can do categorical algebra in topos equivalent to category of sheaves of sets to generate all sheaves of algebraic structures (group,ring etc.).For example,we want to generate a sheaf of generalized super ring.For $A \in T$ topos of sets with final object.For the additive Abelian group structure,the binary operation gives by composition of Homs

$$\begin{tikzcd} \mathrm{Hom}_C(A,A) \times \mathrm{Hom}_C(A,A) \arrow[r, "m"] \arrow[d] & \mathrm{Hom}_C(A,A) \arrow[d] \\ A \times A \arrow[r, "m"] & A \end{tikzcd} \qquad [9.3]$$

and identity from a final object e

$$\mathrm{id} : A \to \{*\} \times A \xrightarrow{e} A \times A \xrightarrow{m} A \qquad [9.4]$$

Associativity and commutativity are similar.Inverse is from the existence of limit

$$A \underset{\beta=\mathrm{id}}{\overset{\alpha=\mathrm{pro}(1\times A)}{\rightrightarrows}} A, \qquad A^{-1} = \mathrm{Eq}(\alpha,\beta) = \{a \in A | \alpha(a) = \beta(a)\} = A \qquad [9.5]$$

And the uniqueness of inverse is from the universal property of the equalizer.Similarly for multiplication,the only thing is multiplicative commutativity

$$\begin{array}{ccccc} A\times A & \longleftarrow & A(U)\times A(U) & \longleftarrow & ab \\ \downarrow{\scriptstyle c_{A,A}} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} \\ A\times A & \longleftarrow & A(U)\times A(U) & \longleftarrow & (-1)^{|a||b|}ba \end{array} \quad [9.6]$$

where c is the commutation in the superalgebra above [7.4].Around [7.22] and [7.31],we extract the commutation as a property from fields to the underlying geometry,but now,we further see this property comes from the underlying topos.This is a natural consequence,because the super ring is generated by the sheaf of super rings and the sheaf is generated by the topos on a site.The distributivity is from that of functions and we have generalized ring strucuture if it admits a spontaneous regularity breaking $A = A \oplus A^*$.A fact is different sites can induce equivalent topoi see [9.82],thus,we regard topos as the lowest level of logic of this generating process.

Now,we want to formalize the limit and colimit [7.23] in functorial approach.For a functor $F : I \to C$ the limit $\varprojlim F : C^{\mathrm{op}} \to \{h_X \to F\}$ where the representable functor $h_X = \mathrm{Hom}(\cdot, X)$ which is also called functor of points of X and it is completely determined by the underlying rings that is for a covering $\{X_i \to X\}_{i\in I}$,$h_X(X_i) = \mathrm{Hom}(\mathrm{Spec}(A_i), X)$.Also $X \cong Y \Leftrightarrow h_X \cong h_Y$ for schemes.Similarly,the colimit $\varinjlim F : C \to \{F \to h_X\}$.The Yoneda lemma is

$$(g : h_X \to F) \longleftrightarrow F(X) \quad [9.7]$$

we find $h_X \leftrightarrow \{\mathrm{id}_X : X \to X\}$ that is a point on scheme X corresponds a permutation of points of X,this is a trivial case that means the identity functor $\mathrm{id}_X \cong h_X$ and is represented by the scheme X,or the X is a moduli space of families of functions that keep identity globally.If a sheaf of curves $F \cong h_X$ is represented by scheme X,one point in X corresponds to a family of curves or equivalent class of curves where $F : (\mathrm{Sch}/X) \to (F(T \to X))$,this good property is from a point of scheme X is an ideal that is a subring I,then a natural T-point is $T = \mathrm{Spec}(I) \to X$ which corresponds to a correction to weirdness of Zariski topology.

A ringed topos is a ringed space (T, Λ) for topos,see below [7.23].Functor of sites $f : C' \to C$ is continuous if for every $X \in C'$ we have $\{f(X_i) \to f(X)\}_{i\in I}$ in $\mathrm{Cov}(f(X))$,if f commutates with fiber products then $f(X), f(X_i) \in C'$.And most of times the continuous map $f : C' \to C$ preserves functorial structure on their topoi $f_* : T \to f^*T'$ with $(f_*F)(X) = F(f(X)), X \in C'$,from

$$\prod_{i,j}(f_*)(X_i \times_X X_j) = \prod_{i,j} F(f(X_i \times_X X_j)) \xrightarrow{\simeq} \prod_{i,j} F(f(X_i) \times_{f(X)} f(X_j)) \quad [9.7]$$

And f_*, f^* are adjoint function that is $\mathrm{Hom}_T(f^*G, F) \cong \mathrm{Hom}_{T'}(G, f_*F)$.If f is not continuous,which means there will be mamy $U' \in C'$ such that $f(U') =$

U,which means we need to change f^* to $\hat{f}^*$ where

$$(\hat{f}^*G)(U) = \varinjlim_{i\in I} G(f^{-1}(U)) = G(\varinjlim_{i\in I} f^{-1}(U)), \quad \cong_I \equiv f(U'_{i\in I}) = U \qquad [9.8]$$

Then we combine with Yoneda lemma [9.7],we have

$$\mathrm{Hom}_T(f^*h_{X'}, F) \cong \mathrm{Hom}_{T'}(h_{X'}, f_*F) \longleftrightarrow f_*F(X') = F(f(X')) \qquad [9.9]$$

we find $f^*h_{X'}$ is represented by $f(X')$ that is $f^*h_{X'} \cong h_{f(X')}$.

Then,we want to talk about cohomology of sheaves.For a ring $\Lambda \in T$,we denote Mod_Λ as category of Λ-modules.Now,we want to assign each sheaf a point $\{x_i \to T\}_{i\in I}, \mathrm{pt}, F_i \in T$,which means $F_i \to x_i^*F$ is injective.If $F_i = \Lambda_i$,then x_i^*F is a Λ_i-module which is included to an injective module I_i,then we have $x_{i*}x_i^*F = F \hookrightarrow x_{i*}I_i$,the product of injective module is injecitve,we have $F \hookrightarrow x_{i*}I_i$ also,we have $F \to x_i^*F_i$ is injective with $F = \Lambda$ now we have $F_i \hookrightarrow x_{i*}I$.Above all ,we have $F_i \hookrightarrow I_i$ which means Mod_Λ has enough injectives.Then,from the duality of $\mathrm{Hom}_{\mathrm{Mod}_\Lambda}(\Lambda, F)$ and $\Lambda \otimes F$ in Abelian group,we have a left exact functor $\Gamma(T, -) : \mathrm{Mod}_\Lambda \to \mathrm{Ab}$,then we have the right derived functor $H^i(T, -)$ from injective resolution of every sheaf.Now,for a site C we have a trivial topos C/X,if $F \in \mathrm{Mod}_\Lambda$ is C-acyclic,we have $H^i((C/X), F) = 0$.For a covering $\mathscr{X} = \{X_i \to X\}_{i\in I}$,from [9.1] the fiber product is still in covering,we can form super fiber product $\mathcal{X}^{\underline{i}} = X_0^{\underline{i}} \oplus X_1^{\underline{i}}$ with $\underline{i} = (i_0, ..., i_r) \in I^{r+1}$

$$X_1^{\underline{i}} = (-1)^\sigma X_1^{\sigma(i_0)} \times_X X_1^{\sigma(i_1)} \cdots \times_X X_1^{\sigma(i_r)} \qquad [9.10]$$

for X a superscheme.Then,we have super Čech cohomology complex

$$\mathcal{C}^r(\mathscr{X}, F) = F(\coprod_{\underline{i}} X_0^{\underline{i}} \oplus X_1^{\underline{i}}) = \prod_{\underline{i}} F(X_0^{\underline{i}} \oplus X_1^{\underline{i}}) \qquad [9.11]$$

where F is a pre sheaf of Λ-modules.With the inverse boundary operator [8.6].

Now,we want to consider super version,recall that anticommutation corresponds to a change of orientation [7.31],thus we get super differential

$$d_r(X_0^{\underline{i}} \oplus X_1^{\underline{i}}) = \sum_{j=0}^{r+1} (-1)^j \oplus (-1)^{j+1} (X_0^{(\underline{i},*)})^2 = X_0^{(\underline{i},*)} \oplus X_1^{(\underline{i},*)} \qquad [9.12]$$

with generalized super Čech cohomology group

$$\underline{\check{\mathscr{H}}}^i(\mathscr{X}, F) = H^i(\mathcal{C}^\bullet(\mathscr{X}, F)) \oplus H^i(\mathcal{C}^{\bullet *}(\mathscr{X}, F)) \qquad [9.13]$$

the generalized super Čech (co)homology group $\underline{\check{\mathscr{H}}}^1(\mathscr{X}, \mathscr{P})$ classifies M-branes with generalized superalgebraifolds acting on.By Yoneda lemma we have

$$\mathrm{Hom}_{\mathrm{PMod}_\Lambda}(\bigoplus_{\underline{i}} \Lambda(h_{X_{\underline{i}}}), F) \cong \mathrm{Hom}_{\mathrm{PMod}_\Lambda}(\coprod_{\underline{i}} h_{X_{\underline{i}}}, F) \cong \prod_{\underline{i}} F(X_{\underline{i}}) = C^r(\mathscr{X}, F) \qquad [9.14]$$

We see that from [7.26],we form a sheaf of modules from $\mathscr{B}$-sheaf,now F is a presheaf of Λ-module but $\prod_{\underline{i}} F$ forms a $\mathscr{B}$-sheaf.Then,from [9.14],F is a sheaf of $\bigoplus_{\underline{i}} \Lambda(h_{X_{\underline{i}}})$-modules on scheme $\coprod_{\underline{i}} X_{\underline{i}}$.We can form a sheaf $\mathscr{P}$

$$\mathscr{P} : (C/\mathcal{X})/\mathscr{M} \to \Big(\bigoplus_{\underline{i}=(i_0,i_1),A} \underline{\mathcal{A}}\{- \subset \mathcal{X}^{\underline{i}} \oplus \mathcal{X}^{\underline{i}*}\}_{R\oplus R^*} \otimes (T \oplus T^*)_A \underline{\mathcal{A}}\mathscr{M} \Big) \quad [9.15]$$

we call it the sheaf of properties on the M-brane,which is a sheaf of $\mathcal{A}$-modules we discussed in [8.21],and the tangent bundle $\bigoplus_A (T \oplus T^*)_A = T \oplus T^*$.With the commutative diagram for a generalized super étale equivalence relation

$$\begin{tikzcd} \underline{\mathcal{A}}_A \arrow[r] \arrow[d] & \underline{\mathcal{A}} \arrow[d] \\ R \oplus R^* \arrow[r, "\subset"] & \mathcal{X}^{\underline{i}} \oplus \mathcal{X}^{\underline{i}*} \end{tikzcd} \quad [9.16]$$

where a collection of data $(\underline{\mathcal{A}}, \partial(Q \boxtimes Q^*)_{\mathscr{M}}) \in \mathscr{P}(\mathscr{M})$,where

$$(Q \boxtimes Q^*)_{\mathscr{M}} = \dim(\underline{\check{\mathscr{H}}}^1(\mathscr{M}, \mathscr{P})), \quad \sum_{\mathscr{M}} [(Q \boxtimes Q^*)_{\mathscr{M}}] = 0_{Q\boxtimes Q^*} \quad [9.17]$$

counting for the number of equivalent properties.And the constraint [9.17] ensures that the M-theory is a nonexistent theory which is experiment-free.In this case,we can get a field in the whole space from [9.17],denote as $\mathrm{F}^{++,--}_{Q\boxtimes Q^*}$ corresponding to the notation in [8.28] and [8.38] with decomposition

$$\mathrm{F}^{++,--}_{Q\boxtimes Q^*} \cong_{\mathrm{P}(\mathbf{T})^{-1}} [\mathbb{Z} \oplus \mathbb{C} \oplus \mathbb{Q} \oplus \mathbb{R} \oplus (\mathbb{Z}\mathbb{C}\mathbb{Q}\mathbb{R})]^{+,-}_{\mathrm{P}(\mathbf{T})^{-1}(Q\boxtimes Q^*)} \quad [9.18]$$

from inverse $\mathbf{T}$-fusion,corresponds to five $D = 10$ superstring theories.Notice that the sheaf on the M-brane [9.15] it is a presheaf on the localized étale site of M-brane,after discussion of descent theory,it become a sheaf globally [9.93].

9.2 Fibered category,2-Yoneda lemma and string-Space

Now,we want to discuss category as a moduli space.We know there is a unique morphism connecting fiber products

$$\begin{tikzcd} w \arrow[dr, "\lambda!"] \arrow[drr, bend left, "\psi"] \arrow[dd, dotted] \arrow[ddr, bend right, "y"'] & & \\ & u \arrow[r, "g"] \arrow[d] & v \arrow[d, "p"] \\ p(w) \arrow[r, dotted, "p(\lambda)"] & p(u) \arrow[r, "p(g)"] & p(v) \end{tikzcd} \quad [9.19]$$

We can see if we set $y = p(\lambda) \circ p$,we have another fiber product with $p \circ \lambda = p(\lambda) \circ p$,and a unique composition of two fiber products given by $\lambda!$ to that with $p \circ g \circ$

$\lambda = p(g) \circ p(\lambda) \circ p$.These form a category F over category C that is a pair (F, p) with $p : F \to C$ and morphisms are fiber products with universal property for composition with each $F(p(v)) = F/p(v)$ a category over $p(v)$.We call $(p : F \to C)/C$ a fibered category.Comapared to a scheme as a functor above[9.7],now we have a category as a functor corresponds to the 2-Yoneda lemma.Now we have a trivial fibered category $\mathrm{id} : C \to C$,for a functor $g : C/X \to C$,we have another fibered category $(C/X \to C)/C$,the morphisms are functors $C/X \to F$ over same C,if we collect them as objects and treat the natural transformations as morphisms we can construct a 2-category $HOM_C((C/X), F)$,and we have

$$\begin{tikzcd}
HOM_C(-, F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\xi}{\Longrightarrow}"] & F \arrow[d] \\
C/X \arrow[r] & X/X
\end{tikzcd} \qquad [9.20]$$

which gives $\xi : HOM_\xi((C/X), F) \to F(X)$.For seeing the structure

$$\begin{tikzcd}
(\phi : Y \to X) \in C/X \arrow[rr, "\eta_x"] \arrow[dr] & & \phi^* x \in F(Y) \subset F \arrow[dl] \\
& Y \in C &
\end{tikzcd} \qquad [9.21]$$

and η_x maps a trivial cartesian morphism of (C/X) to that of F over C

$$\begin{tikzcd}
Y'' \arrow[r, "!"] \arrow[dr, dotted, "\phi''!"'] & Y' \arrow[r, "\epsilon"] \arrow[d, "\phi'"] & Y \arrow[dl, "\phi"] \\
& X &
\end{tikzcd}
\xrightarrow{\eta_x(\epsilon)}
\begin{tikzcd}
\phi''^* ! x \arrow[r, dotted, "!"] \arrow[d, mapsto] & \phi'^* x \arrow[r] \arrow[d, mapsto] & \phi^* x \arrow[d, mapsto] \\
Y'' \arrow[r, "!"] & Y' \arrow[r, "\epsilon"] & Y
\end{tikzcd} \qquad [9.22]$$

so the functor η_x is a morphism of fibered categories and gives $\eta : x \in F(X), \mapsto \eta_x \in HOM_C((C/X), F)$.Also,for $f : x' \to x$ in $F(X)$ we have

$$\xrightarrow{\eta_f(\epsilon)}
\begin{tikzcd}
\phi''^* ! x' \arrow[r, dotted, "!"] \arrow[d] & \phi'^* x' \arrow[r, "\epsilon"] \arrow[d] & \phi^* x' \arrow[d] \\
\phi''^* ! x \arrow[r, dotted, "!"] & \phi'^* x \arrow[r, "\epsilon"] & \phi^* x
\end{tikzcd} \qquad [9.23]$$

so η_f is a morphism in the 2-category that gives $\eta : f \in F(x), \mapsto \eta_f$.Combing [9.22] and [9.23],we get a quasi-inverse η of functor ξ.Now from $\xi \circ \eta : x \mapsto \eta_x \mapsto \mathrm{id}_X^* x \cong x$,we get $\mathrm{id}_{F(X)} \cong \xi \circ \eta$.Also,we have $\xi : f \mapsto f(\mathrm{id}_X) \in F(X)$ and $\eta : f(\mathrm{id}_X) \mapsto \eta_{f(\mathrm{id}_X)} : C/X \to \phi^* f(\mathrm{id}_X)$,we are in category with pullbacks,thus $\eta_{f(\mathrm{id}_X)} : (C/X) \to f(\phi^* \circ \mathrm{id}_X) = f(\mathrm{id}_X \circ \phi)) \cong f(\phi)$

$$\begin{tikzcd}
f(\phi^* x) \cong \phi^* f(\mathrm{id}_X) \arrow[r] & f(\mathrm{id}_X) \in F(X) \\
\phi^* x \in Y \arrow[r, "\phi"] \arrow[u, "f"] & x \cong \mathrm{id}_X \in X \arrow[u, "f"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\phi^* \mathrm{id}_X \arrow[r] \arrow[d] & \mathrm{id}_X \arrow[d] \\
Y \arrow[r, "\phi"] & X
\end{tikzcd} \qquad [9.24]$$

we also have $\mathrm{id}_{HOM_C((C/X),F)} \cong \eta \circ \xi$.Therefore,we have 2-Yoneda lemma

$$HOM_C((C/X),F) \cong F(X) \tag{9.25}$$

Compared to Yoneda lemma,we can see (C/X) represented by scheme X,we also know scheme X is represented by h_X,and in this case,we can view the fibered category as a representable functor.If we let $F=(C/Y)$,we get a familiar connection $HOM_C((C/X),(C/Y)) \cong \mathrm{Hom}_C(X,Y)$.

A category fibered in sets over C is a fibered category $p : F \to C$ with identity morphisms as the only morphism in $F(U), U \in C$ and $F(U)$ is a set.Now in $HOM_C(F,G)$ for G is a category fibered in sets,we have two objects f,g and a morphism $\alpha : f \to g, \alpha_x : f(x) \mapsto g(x), x \in F$.But only morphism in G is identity morphism which means $f(x)=g(x), f=g$,thus,the 2-category $HOM_C(F,G)$ with G fibered in sets loose its categorical structure and becomes a set.And for $g : V \to U$ in category C,there is always a well-defined pullback map $g^* : F(U) \to F(V)$,it makes $\mathcal{F} : C \to (F \to C)$ behaves like a presheaf.Conversely,for a presheaf $F : C^{\mathrm{op}} \to \mathrm{Set}$,we know we just change the morphism g^{-1} to inclusion and g_* is the restriction on the level of sections,which gives g^* the pullback,then $p : F(C^{\mathrm{op}}) \to C^{\mathrm{op}}$ is a category fibered in sets.Then

$$\Gamma : (\text{presheaves on C}) \cong (\text{categories fibered in sets over C}) \tag{9.26}$$

where we discussed category of categories fibered in sets.

A splitting of a fibered category $p : F \to C$ is a subcategory $K \subset F$ with

$$\begin{aligned} &\text{(i) arrows in } K \text{ are cartesian} \\ &\text{(ii) } f : U \to V, v \in F(V) \text{ induces a unique } f : u \to v \in K(U) \to K(V) \\ &\text{(iii) } \mathrm{id}_u \in K \text{ for } u \in F(U), U \in C \end{aligned} \tag{9.27}$$

And we denote split fibered category as (F,K).Then from

$$\begin{tikzcd} C/V \arrow[rr, "g"] \arrow[dr, "v"'] \arrow[rr, phantom, shift right=3, "\overset{\alpha}{\Rightarrow}"] & & C/U \arrow[dl, "u"] \\ & F & \end{tikzcd} \tag{9.28}$$

The pairs in [9.28] (U,u) form a category $\widetilde{F}$,a morphism is a pair (g,α) with 3-isomorphism $\alpha : v \to u \circ g$.For a (W,w) and g',there is a unique 2-isomorphism $\alpha' : w \to v \circ g'$,let the following fit with axiom of fibered category

$$(C/W)/F \xrightarrow{(g',\alpha')} (C/V)/F \xrightarrow{(g,\alpha)} (C/U)/F \tag{9.29}$$

which means we have a fibered category of 3-catgories $HOM_C((C/U),F), U \in C$,that is $\tilde{F} = (HOM_C((C/U),F))$ with morphisms are 3-isomorphism,which is 3-category.From 2-Yoneda lemma,we have

$$\tilde{F} = (HOM_C((C/U),F)) \cong (F(U)) = F, \quad U \in C \tag{9.30}$$

And the pair $(\widetilde{F}, K)$ is a split fibered category with $K \subset \tilde{F}$ follows from counting all objects with the 3-isomorphisms choose to be 3-identities,actually $\widetilde{F}$ is a 3-category fibered in groupoid corresponding to an equivalent relation and if we mod this equivalent relation on the set $\mathrm{Ob}(\widetilde{F})$ we have $B\mathrm{Ob}(\widetilde{F}) = (* \to K)$ which is a classfying stack.

A groupoid is a category with objects forming a group see around [9.3],which is equivalent to say all morphisms are isomorphisms,it is transversal.A category fibered in groupoids over a category C is a fibered category $p : F \to C$ such that $F(U)$ is a groupoind for all $U \in C$.Similar to that in sets above [9.26],$HOM_C(F, F')$ is a groupoid for F, F' are categories fibered in groupoids.

One important construction of a groupoid in a category C with finite fiber products is a collection of data

$$(X_0, X_1, s, t, \epsilon, i, m) \tag{9.31}$$

with s, t, X_0, X_1 we have discussed above [7.32],and inverse and composition are

$$\epsilon : X_0 \to X_1 \quad i : X_1 \to X_1, \quad m : X_1 \times_{s,X_0,t} X_1 \to X_1 \tag{9.32}$$

with $s \circ m = s \circ \mathrm{pr}_2, t \circ m = t \circ \mathrm{pr}_1$.Associativity us given by

$$\begin{tikzcd} X_1 \times_{s,X_0,t} X_1 \times_{s,X_0,t} X_1 \arrow[r, "m\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times m"] & X_1 \times_{s,X_0,t} X_1 \arrow[d, "m"] \\ X_1 \times_{s,X_0,t} X_1 \arrow[r, "m"] & X_1 \end{tikzcd} \tag{9.33}$$

Identity factors through ϵ, m given by

$$\begin{tikzcd} & X_1 \times_{s,X_0} X_0 \arrow[dr, "\mathrm{id}\times\epsilon"] & & \\ X_1 \arrow[ur, equal] \arrow[dr, equal] & & X_1 \times_{s,X_0,t} X_1 \arrow[r, "m"] & X_1 \\ & X_0 \times_{X_0,t} X_1 \arrow[ur, "\epsilon\times\mathrm{id}"] & & \end{tikzcd} \tag{9.34}$$

Non-Abelianness factors through m, ϵ

$$\begin{tikzcd} X_1 \times_{s,X_0,t} X_1 \arrow[r, "\epsilon\times\epsilon"] \arrow[d, "m"] & X_1 \times_{t,X_0,s} X_1 \arrow[d, "m"] \\ X_1 \arrow[r, "\epsilon"] & X_1 \end{tikzcd} \tag{9.35}$$

which induces the following inverse diagrams

$$\begin{tikzcd} X_1 \arrow[r, "i\times\mathrm{id}"] \arrow[d, "t"] & X_1 \times_{t,X_0,s} X_1 \arrow[d, "m"] \\ X_0 \arrow[r, "m"] & X_1 \end{tikzcd} \qquad \begin{tikzcd} X_1 \arrow[r, "\mathrm{id}\times i"] \arrow[d, "s"] & X_1 \times_{t,X_0,s} X_1 \arrow[d, "m"] \\ X_0 \arrow[r, "m"] & X_1 \end{tikzcd} \tag{9.36}$$

Now,we consider a category $\{X_0(U)/X_1(U)\}$ whose objects are $u \in X_0(U)$,and for a morphism $\xi \in X_1(U), : u' \to u$,the composition $u'' \xrightarrow{\eta} u' \xrightarrow{\xi} u$ given by $(\eta, \xi) \in X_1(U) \times_{t,X_0,s} X_1(U)$,which is a groupoid over U.Following the diagram

$$\begin{tikzcd}
\{X_0(W)/X_1(W)\} \arrow[r, "g!"] \arrow[d] & \{X_0(V)/X_1(V)\} \arrow[r] \arrow[d] & \{X_0(U)/X_1(U)\} \arrow[d] \arrow[ll, bend right, "g^*!\circ f^*!=(fg)!^*"'] \\
W \arrow[r, shift left, "g"] & V \arrow[r, shift left, "f"] \arrow[l, shift left, "g^*!"] & U \arrow[l, shift left, "f^*!"]
\end{tikzcd} \qquad [9.37]$$

we get a category fibered in groupoid $p : \{X_0/X_1\} \to C$ with objects are pairs $(U, u), U \in C, u \in \{X_0(U)/X_1(U)\}$.A morphism is a pair $(f, \alpha) : (V, v) \to (U, u)$ with an isomorphism $\alpha \in \{X_0(V)/X_1(V)\}, : v \to f^*u$.Also,the composition

$$w \xrightarrow{\beta} g^*v \xrightarrow{g^*(\alpha)} (fg)^*u, \quad (W, w) \xrightarrow{(g,\beta)} (V, v) \xrightarrow{(f,\alpha)} (U, u) \qquad [9.38]$$

is $((f \circ g), g^*(\alpha) \circ \beta) : (W, w) \to (U, u)$.

Definition8.1 A string-Space $\mathcal{S}^{\text{pre}}$ is a $\mathbb{Z}_2$-graded category fibered in Lie groupoids over a big étale site,where

$$\mathcal{S}^{\text{pre}} : \Phi \oplus \Psi^* \to \text{ETSch}^{\text{SupGen}}(\mathscr{M}) \qquad [9.39]$$

of étale generalized superschemes over M-brane $\mathscr{M}$.We also have super T-duality $\mathbf{T}^\delta : \Phi \oplus \Psi^* \to \Psi^* \oplus \Phi$ from [8.26].Where Φ is a category fibered in Lie groupoids $\Phi : \{X_0/X_1\} \to \text{ETSch}_0^{\text{SupGen}}(\mathscr{M})$ and $\Psi^* : \{\psi_0/\psi_1\} \to \text{ETSch}_1^{\text{SupGen}*}(\mathscr{M})$.

For Φ,the $X_0(\mathcal{X}_0)$ collects all bosonic étale closed strings s living in $\mathcal{X}_0$ and a morphism $w : s_i \to s_f$ is an étale morphism along the time-evolution,which is a world-sheet.In this case,we collects all world-sheets made from the time-evolution of these strings to $X_1(\mathcal{X}_0)$ with $s(w)$ for initial states and $t(w)$ for final states.From discussion above [8.6],we see $X_0(\mathcal{X}_0) \to \mathcal{X}_0$,actually $\mathcal{X}_0$ is the moduli space of these strings,a point $U = \text{Spec}(R(\Delta^1)) \to \mathcal{X}_0$ corresponds to a class of étale closed strings,if we let $\mathscr{G}_0$ be the moduli space of world-sheets

$$\begin{array}{ccc} X_0(\mathcal{X}_0) & \longrightarrow & \mathcal{X}_0 \\ \downarrow & \overset{\text{rep}}{\Longrightarrow} & \downarrow \\ X_1(\mathcal{X}_0) & \longrightarrow & \mathscr{G}_0 \end{array} \ \to_{\mathbf{T}^\delta} \ \begin{array}{ccc} \psi_0(\mathcal{X}_1^*) & \longrightarrow & \mathcal{X}_1^* \\ \downarrow & \overset{\text{rep}}{\Longrightarrow} & \downarrow \\ \psi_1(\mathcal{X}_1^*) & \longrightarrow & \mathscr{G}_1^* \end{array} \qquad [9.40]$$

where we combined super T-duality.These give them smoothness where we assumed that the moduli spaces are differentiable.To verify the properness

$$\begin{array}{ccccc} X_0(\mathcal{X}_0) \times_{X_0} X_0(\mathcal{X}_0) & \xrightarrow{\epsilon \times \epsilon} & X_1 \times_{t,X_0,s} X_1 & \xrightarrow{\quad m \quad} & X_1 \\ \uparrow & & \uparrow & & \uparrow \\ (s_1, s_2) & \longrightarrow & (w_1, w_2) & \longrightarrow & w_1 \times_{\text{glue}} w_2 \end{array} \qquad [9.41]$$

It just ends with an interactive world-sheet,and must be proper.Above all,the definition7.1 is indeed consistent with Lie groupoid structure with properness naturally comes from the physics.

The super T-dual part Ψ,we have already found in [7.32].Also,we need to combine the facts we have discussed,we have R and NS sector for fermions [3.51].From the classfication of string theories,we can group the vertex of étale closed strings to ori = {(NS,NS),(R,R)} and orb = {(R,NS),(NS,R)},with

$$\text{ori}: \quad \psi\tilde{\psi} = -\tilde{\psi}\psi, \qquad \text{orb}: \quad \psi\tilde{\psi} = \tilde{\psi}\psi \tag{9.42}$$

on the reflection point.These corresponds to two types of orbifolds,the first is ordinary orientifold [6.24] and the second should correspond to orbifold [7.31] if they are defined over unordered and ordered set

$$\begin{tikzcd} \text{ori} \arrow[r, leftrightarrow, "\Omega"] \arrow[d] & \text{orb} \arrow[d] \\ (\psi,\tilde{\psi}) =_{\text{unordered}} -(\psi,\tilde{\psi}) \arrow[r, leftrightarrow, "\Omega"] & (\psi,\tilde{\psi}) =_{\text{ordered}} -(\psi,\tilde{\psi}) \end{tikzcd} \tag{9.43}$$

where,the ordered set is for orbifold and unordered set is for orientifold that is unoriented on world-sheet.Also [9.43] induces the decomposition of super T-dual part $\Psi = \Psi \oplus \tilde{\Psi}$ where the tilde part is for orientifolds.Which induces a further decomposition of string-Space

$$(\Phi \oplus \tilde{\Phi}) \oplus (\Psi \oplus \tilde{\Psi}) \cong_{Bos} (\Phi \oplus \Phi) \oplus (\tilde{\Psi} \oplus \tilde{\Psi})^* \tag{9.44}$$

with Ω-super T-duality on the string-Space

$$\mathbf{T}^{\delta}_{\Omega} : (\Phi \oplus \Phi) \oplus (\tilde{\Psi} \oplus \tilde{\Psi})^* \to (\tilde{\Psi} \oplus \tilde{\Psi})^* \oplus (\Phi \oplus \Phi) \tag{9.45}$$

For a category fibered in groupoids $p : F \to C$,we can made another category $p_{/X} : F_{/X} \to (C/X)$ which is also fibered in groupoids.And an object of $F_{/X}$ is a pair $(y, f : p(y) \to X), y \in F$ and a morphism is $(g, p(g)), g : y \to y'$

$$p(g) : p(y') \to p(y), \quad f' = f \circ p(g) \tag{9.46}$$

And the functor $p_{/X}$ sends $(y, f : p(y) \to X)$ to $f \in C/X$.For any $f : Y \to X$

$$\begin{tikzcd} F(Y) \arrow[r, "\simeq"] \arrow[d] & F_{/X}(f : Y \to X) \arrow[d] \\ Y \arrow[r, "\simeq"] & f : Y \to X \end{tikzcd} \tag{9.47}$$

the $F_{/X}$ is a category fibered in groupoids.Recall that the relation between fibered category and presheaf [9.26],for a fiber $F(X) \cong F_{/X}(\mathrm{id}_X)$,we have

$$\underline{\text{Isom}}(x, x') : (C/X)^{\text{op}} \to \text{Set} \tag{9.48}$$

for $x, x' \in F(X)$.For any morphism $f : Y \to X$,with pullbacks f^*x, f^*x'

$$\underline{\mathrm{Isom}}(x, x')(f : Y \to X) = \mathrm{Isom}_{F(Y)}(f^*x, f^*x') \qquad [9.49]$$

where the set of isomorphisms is an object and a morphism is (g, g^*)

$$g^* : \underline{\mathrm{Isom}}(x, x')(f : Y \to X) \to \underline{\mathrm{Isom}}(x, x')(fg : Z \to X) \qquad [9.50]$$

where $Z \xrightarrow{g} Y \xrightarrow{f} X$.If $x = x'$,the section is an automorphism group

$$\underline{\mathrm{Isom}}(\mathrm{id}_x)(f : Y \to X) = \mathrm{Aut}_{F(Y)}(\mathrm{id}_{f^*x}) \qquad [9.51]$$

Which means the presheaf becomes $\underline{\mathrm{Aut}}(\mathrm{id}_x) : (C/X)^{\mathrm{op}} \to \mathrm{Groups}$.

9.3 Descent theory and a pre M-theory $\mathcal{M}^{\mathrm{pre}}$

Descent theory of fibered category gives us a method to glue schemes (topology and section) by different morphisms,which is a high dimensional representation of gluing axiom of sheaves.For a fibered category $p : F \to C$,we can define $F(X \xrightarrow{f} Y)$ for each f,the object is a pair (E, σ) with $E \in F(X)$ and the isomorphism $\sigma : \mathrm{pr}_1^* E \to \mathrm{pr}_2^* E$ in $F(X \times_Y X)$ as a data of gluing

$$\begin{tikzcd}
X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
X \arrow[r] & X \cap X \in Y
\end{tikzcd} \qquad [9.52]$$

which is a high dimensional representation descending data to transition map.

$$\sigma|_{X \cap X} = \mathrm{pr}_1^* E|_{X \cap X} \to \mathrm{pr}_2^* E|_{X \cap X} = f(E)|X \xrightarrow[\simeq]{r_{X,X}} f(E)|X \qquad [9.53]$$

where we want to abuse f with f_* for simplicity.And following the diagram in $F(X \times_Y X \times_Y X)$ for the composition axiom of category

$$\begin{tikzcd}
 & \mathrm{pr}_{23}^*\sigma \circ \mathrm{pr}_{12}^*\sigma = \mathrm{pr}_{13}^*\sigma \arrow[rr] \arrow[dl, dash] \arrow[dd] & & \sigma \arrow[dl] \arrow[dd] \\
\sigma' \arrow[rr] \arrow[dd] & & \sigma \arrow[dd] & \\
 & F(X \times_Y X \times_Y X) \arrow[rr] \arrow[dl, "\mathrm{pr}_{12}"'] \arrow[dr, "\mathrm{pr}_{13}"] & & F(X \times_Y X) \arrow[dl] \\
F(X \times_Y X) \arrow[rr, "\mathrm{pr}_{23}"] & & F(X \times_Y X) &
\end{tikzcd}$$

[9.54]

descending data to composition of isomorphisms on gluing area.A morphism in $F(X \xrightarrow{f} Y)$ is the morphism $g : (E', \sigma') \to (E, \sigma)$ following the diagram

$$\begin{tikzcd} \mathrm{pr}_1^* E \arrow[r, "\sigma"] & \mathrm{pr}_2^* E \\ \mathrm{pr}_1^* E' \arrow[r, "\sigma'"] \arrow[u, "\mathrm{pr}_1^* g"] & \mathrm{pr}_2^* E' \arrow[u, "\mathrm{Pr}_2^* g'"'] \end{tikzcd} \qquad [9.55]$$

And for a pair (E, σ) in the category $F(X \xrightarrow{f} Y)$,the isomorphism σ is the descent data for the object E.For a functor $\epsilon : F(Y) \to F(X \xrightarrow{f} Y)$,we have $E_0 \in F(Y)$.And we can pullback it to the section on the fiber product $X \times_Y X \to Y$,we have $\sigma_{\text{can}} : \mathrm{pr}_1^* f^* E_0 \to \mathrm{pr}_2^* f^* E_0$ which is a canonical isomorphism.Then,ϵ sends E_0 to $(f^* E_0, \sigma_{\text{can}})$.In general,we want to study fibered category over a site with covering which means we have a covering $\{X_i \to X\}_{i \in I}$,we need $F(\{X_i \to Y\})$ with objects as collections of data $(\{E_i\}_{i \in I}, \{\sigma_{ij}\}_{i,j \in I}), E_i \in F(X_i)$ and the isomorphism is $\sigma_{ij} : \mathrm{pr}_1^* E_i \to \mathrm{pr}_2^* E_j$,based on gluing axiom of sheaf

$$\begin{tikzcd} f(E_i)|f(X_i) \arrow[r, "\sigma_{ij}|_{f(X_i)\cap f(X_j)}"] \arrow[rr, bend left, "\sigma_{ik}|_{f(X_i)\cap f(X_k)}"] & f(E_j)|f(X_j) \arrow[r, "\sigma_{jk}|_{f(X_j)\cap f(X_k)}"] & f(E_k)|f(X_k) \end{tikzcd} \qquad [9.56]$$

translated to descent theory,we have

$$\begin{tikzcd} \mathrm{pr}_{12}^* \mathrm{pr}_1^* E_i \arrow[r, "\mathrm{pr}_{12}^* \sigma_{ij}"] \arrow[d, equal] & \mathrm{pr}_{12}^* \mathrm{pr}_2^* E_j \arrow[r, equal] & \mathrm{pr}_{23}^* \mathrm{pr}_1^* E_j \arrow[d, "\mathrm{pr}_{23}^* \sigma_{jk}"] \\ \mathrm{pr}_{13}^* \mathrm{pr}_1^* E_i \arrow[r, "\mathrm{pr}_{13}^* \sigma_{ik}"] & \mathrm{pr}_{13}^* \mathrm{pr}_2^* E_k \arrow[r, equal] & \mathrm{pr}_{23}^* \mathrm{pr}_2^* E_k \end{tikzcd} \qquad [9.57]$$

and the isomorphims σ_{ij} are descent data of $\{E_i\}_{i \in I}$.Also,a natural functor is

$$\begin{tikzcd} F(Y) \arrow[r, "\epsilon_i"] \arrow[rr, bend right, "\epsilon"'] & F(\{X_i \to Y\}) \arrow[r, hook, "i"] & F(X \xrightarrow{f} Y) \end{tikzcd} \qquad [9.58]$$

The collection of morphisms $\{X_i \to Y\}$ is of effective descent for F if $\epsilon = i \circ \epsilon_i$ in [9.58] induces an equivalence of categories.If $(\{E_i\}, \{\sigma_{ij}\}) \in i^{-1}(F(X \xrightarrow{f} Y))$,we call $\{\sigma_{ij}\}$ is effective.

For instance,if we have $F(Q \to Y)$ with $Q = \coprod_i X_i$,this scheme Q is trivially coverd by these subschemes.If $F(Y) \cong F(\{X_i \to Y\})$,it means a $(\{E_i, E_j\}, \sigma_{ij})$ maps to a $(\{f(E_i), f(E_j), r_{i,j}\}) \in F(Y)$ such that

$$f(E_i)|_{f(X_i)\cap f(X_j)} \cong_{r_{ij}} f(E_j)|_{f(X_i)\cap f(X_j)} \Rightarrow \coprod_i f(E_i)!|\coprod_i f(X_i) \in Y \qquad [9.59]$$

Then the pullback along $f : X \to Y$ in [9.58]

$$\{\prod_i E_i, \mathrm{pr}^*\mathrm{id}_X\} = \prod_i F(X_i \to Y) = F(\coprod_i X_i \to Y) = F(Q \to Y) \qquad [9.60]$$

Reversely,if we have $F(Y) \cong F(Q \to Y) = \prod_i F(X_i \to Y)$ which means we have a global section $E \in \prod_i F(X_i)$,such that

$$\begin{aligned} &(f(E)|_{f(X_i)})|_{f(X_i)\cap f(X_j)} = (f(E)|_{f(X_j)})|_{f(X_i)\cap f(X_j)} \\ &\Rightarrow f(E_i)|_{f(X_i)\cap f(X_j)} \cong_{r_{ij}} f(E_j)|_{f(X_i)\cap f(X_j)} \end{aligned} \qquad [9.61]$$

along the pullback throgh $\{f : X_i \to Y\}_{i\in I}$,we

$$\{(\{E_i\}, \{\mathrm{pr}^* f^* r_{ij}\})\} = F(\{X_i \to Y\}) \qquad [9.62]$$

which gives a lemma with $Q = \coprod_i X_i$

$$F(Y) \cong F(\{X_i \to Y\}) \Leftrightarrow F(Y) \cong F(Q \to Y) \qquad [9.63]$$

giving an understanding of descend theory,the first equivalence of categories for effective descent gives a glued scheme Q.Also,if $f : X \to Y$ is a morphism such that ϵ induces an equivalence of categories,we call f is an effective descent morphism for F.Next,we want to give several examples of such morphisms.

Descent for sheaves in a site. For a site C associated with topos $(C/X)\breve{}$,a morphism $f : X \to Y$ induces $f : (C/X)\breve{} \to (C/Y)\breve{}$ with pullback

$$f^*(F)(W \to X) = F(W \to X \to Y), \quad F \in (C/Y)\breve{} \qquad [9.64]$$

Define category Sh with object $(X, E), X \in C, E \in (C/X)\breve{}$.A morphism is a pair $(f, \epsilon) : (X, E) \to (Y, F)$ where $f : X \to Y, \epsilon : E \to f^*E$ and composition is similar to [9.38].A point is sheaf is defined on a scheme,this sheaf is a presheaf on a site see above [9.15],this induces the ideal about gluing sheaf

$$\{\text{sheaf } E \text{ on } X | X \in C\} \Leftrightarrow \text{A presheaf of sheaves on } C, \cong p : \mathrm{Sh} \to C \qquad [9.65]$$

where we used [9.26].This gives us a fibered category with $\mathrm{Sh}(X) = (C/X)\breve{}$.We first have $\mathrm{Sh}(X \to Y)$ with objects are pairs $(E, \sigma), E \in \mathrm{Sh}(X), \sigma \in \mathrm{Sh}(X \times_Y X)$ satsifying the cocycle condition.Along $g : X \times_{f,Y,f} X \to Y$,we have

$$f_*\mathrm{pr}_{1*}\sigma, f_*\mathrm{pr}_{2*}\sigma^{-1} : f_*E \to g_*\mathrm{pr}_2^*E, \quad f_*\mathrm{pr}_{i*} \cong g_* \qquad [9.66]$$

By the fiber product,we have a inverse functor of ϵ above [9.56]

$$\eta : \mathrm{Sh}(X \to Y) \to \mathrm{Sh}(Y), \qquad (E, \sigma) \mapsto \mathrm{Eq}(f_*\mathrm{pr}_{1*}\sigma, f_*\mathrm{pr}_{2*}\sigma^{-1}) \qquad [9.67]$$

Composite functor $\eta \circ \epsilon$ follows from $E_0 \in \mathrm{Sh}(Y), f_*f^*E_0 \to g_*\mathrm{pr}_2^*f^*E_0$

$$\mathrm{id}_{\mathrm{Sh}(Y)} \cong \eta \circ \epsilon \quad \Leftrightarrow \quad \eta \circ \epsilon : E_0|_{Y \leftarrow X\times_Y X} \overset{\simeq}{\mapsto} \mathrm{Eq}(E_0 \rightrightarrows g_*\mathrm{pr}_2^*f^*E_0) \qquad [9.67]$$

And for the functor $\epsilon \circ \eta : (E,\sigma) \mapsto (f^*F, \sigma_{\text{can}}), F = \eta(E,\sigma)$.We have

$$\begin{tikzcd} \mathrm{Sh}(X) \arrow[r, "\text{restriction}"] \arrow[d, "f"] & \mathrm{Sh}(X \times_Y Y') \arrow[d, "f"] \\ \mathrm{Sh}(Y) \arrow[r, "\text{restriction}"] & \mathrm{Sh}(Y') \end{tikzcd} \tag{9.68}$$

for an inclusion $Y' \xrightarrow{\subseteq} Y \in \{Y'_i \to Y\}_{i\in I}$.Which means it is equivalent to show $(E \cong f^*F)|_{X\times_Y Y'}$,if $Y' = X$,we see it is the pullback of an identity morphism

$$(E \cong f^*F)|_{X\times_Y X} = g^*\mathrm{id}_{\mathrm{Sh}(Y)} =\cong_{F(X\times_Y X)} \tag{9.69}$$

Above all,we have the equivalence of categories and each $f : X \to Y$ in $\mathrm{Cov}(Y)$ of C is an effective descent morphism for Sh

$$\mathrm{Sh}(X \to Y) \cong \mathrm{Sh}(Y) \tag{9.70}$$

Next,we want to discuss sheaves of modules.For a scheme X,We need the category (Sch/X) which is a fppf site with $\mathrm{Cov}(U) = \{\{U_i \to U\}_{i\in I}\}$ with each $U_i \to U$ is flat and locally of finite presentation and the map $\coprod_{i\in I} U_i \to U$ is surjective.We have adequate properties of sheaves of modules on this site.

Descent for quasi-coherent sheaves. For a fppf site $C = (\mathrm{Sch}/S)$ with scheme S.We have a presheaf of rings $\mathscr{O} : C \to (\Gamma(T, \mathcal{O}_T)), T \in C$,and this fibered category $(p : \Gamma(T, \mathcal{O}_T) \to C) \cong (h_T) \cong (\mathrm{Sch}/X)$ because $\mathcal{O}_T \cong h_T$ is represented by T.Theorem 4.1.2 in [12] tolds us for any morphism $X \to Y$ of category of Y-schemes with fppf or étale topology,h_X is a sheaf.In this case,we find our familiar structure sheaf $\mathcal{O}_T$ below [7.23].Now,for a category of quasi-coherent sheaves on S,denote as $\mathrm{Qcoh}(S)$,we have a presheaf F_{big} of $\mathscr{O}$-modules,$F_{\text{big}} : (T \to S) \mapsto \Gamma(T, f^*F)$ where f^*F is a quasi-coherent sheaf on T by pullback.Then,we need to know a big Zariski site of a scheme S is a category of S-schemes with $\mathrm{Cov}(U)$ for $(U \to S)$ be $\{\{U_i \to U\}_{i\in I}\}$ for each $U_i \to U$ is an open embedding and $U = \bigcup_{i\in I} U_i$.In this case,F_{big} is a sheaf on big Zariski site because $\mathscr{O}$ is a sheaf.And we want to extend to fppf site,starting at a big Zariski cover $\mathrm{Spec}(B) \to T = \mathrm{Spec}(A)$,we have the short exact sequence

$$0 \to \mathcal{O}_T(T) \to (B \leftarrow A) \rightrightarrows B \otimes_A B \to 0 \tag{9.71}$$

from sheaf axiom [9.2] of sheaf of rings.If $A \to B$ is faithfully flat,we have

$$0 \to \mathcal{O}_T(T) \otimes_A M \to B \otimes_A M \rightrightarrows (B \otimes_A B) \otimes_A M \to 0 \tag{9.72}$$

from discussion above [7.29] for a flat A-module M.But [9.72] gives sheaf axiom of F_{big} with $f^*F(T) = \mathcal{O}_T(T) \otimes_A M$.Also,$\mathrm{Spec}(B) \to T$ is faithfully flat from that on ring level,which means it is also a fppf cover.Therefore,$(F_{\text{big}})_T$ is a sheaf for any quasi-coherent sheaf F on S and F_{big} is a sheaf on fppf site.

And by restriction of sheaf,we have $(F_{\text{big}})_S \xrightarrow{\simeq} F \Leftrightarrow F|_T \xrightarrow{\simeq} F_{\text{big}}$,which induces the definition of big quasi-coherent sheaf on S that is a sheaf F of $\mathscr{O}$-modules on C,$F_T = (F_{\text{big}})_T$ and $g^*F_T \xrightarrow{\simeq} F_{T'}$ for every morphism $g : T \to T'$.In this case,we have a fibered category in general

$$p : \text{QCOH} \to (\text{schemes}), \qquad \text{QCOH}((\text{schemes})|_{S_{\text{fppf}}}) = \text{Qcoh}(S_{\text{fppf}}) \qquad [9.73]$$

And we claim that for each $f : X \to Y$ in $\text{Cov}(Y)$ of fppf site

$$\text{QCOH}(X \to Y) \cong \text{QCOH}(Y) \qquad [9.74]$$

Compared to [9.70],the extra thing is the preserving of quasi-coherence for [9.67].We see above,fppf site can be viewed as an extension of Zariski site,in this case,we have a Zariski covering $Y = \bigcup_i Y_i, f^{-1}(Y_i) \in X, = \bigcup_j X_{ij}$ with Y_i affine and X_{ij} is quasi-compact through f.The morphism $X_{ij} \to Y_i$ and this quasi-seperated cover gives us the diagram

$$\begin{tikzcd}
\text{QCOH}(X_{ij} \coprod X_{ij'}) \arrow[r] \arrow[d, shift right] & \text{QCOH}(X_{ij'}) \arrow[d] \\
\text{QCOH}(X_{ij}) \arrow[r] \arrow[u, bend right, shift right] & \text{QCOH}(Y_i) \arrow[l, bend right, "f^*"']
\end{tikzcd} \qquad [9.75]$$

Follows from the fiber product we have an equivalence of categories

$$\{F_i \in \text{QCOH}(Y_i)\} \leftrightarrow \{\{(f^*F_i)_j\}, \{\sigma_{jj'} : (f^*F_i)_j \xrightarrow{\simeq} (f^*F_i)_{j'}\}\} \qquad [9.76]$$

$\{(f^*F_i)_j\}$ is an isomorphic class from $\sigma_{jj'}$ along f^{-1}.Then,we get

$$\text{QCOH}(Y_i) \cong \text{QCOH}(f^{-1}(Y_i) \to Y_i) \qquad [9.77]$$

where we used the fact that the quasi-coherence preserves along the affine cover and the isomorphism let us glue X_{ij} through j,after gluing through i

$$\begin{aligned} &\text{QCOH}(Y) \cong \text{QCOH}(X \to Y) \\ &\{\prod_i F_i\} \leftrightarrow \{\{(\prod_j (f^*F_i)_j\}, \{\alpha_{ii'} : (f^*F_i)_j \xrightarrow{\simeq} (f^*F_{i'})_j\}\} \end{aligned} \qquad [9.78]$$

where $\{\{(f^*F_i)_j\}, \{\sigma_{jj'}\}\} = \{\prod_j (f^*F_i)_j\}$ with cocycle condition and we repeated [9.75] for index i and we see the descent structure [9.54] becomes an usual process to glue Zariski sheaf in [9.75] by using this quasi-separated cover,and the extra quasi-coherence preserves because of fppf cover is Zariski.

Also,from the definition above [9.71],we find equivalence of categories

$$\begin{aligned} &(S, \mathcal{O}_S) \xrightarrow{\simeq} (S_{\text{Zar}}, \mathscr{O}_{S_{\text{Zar}}}) \xrightarrow{\simeq} (S_{\text{fppf}}, \mathscr{O}_{S_{\text{fppf}}}), \qquad f_{\text{fppf}} : U \to S \\ &(S \mapsto \prod_i C^\infty(U_i)) \to (U_i \subset S, \mapsto C^\infty(U_i)) \to (f_{\text{fppf}} \mapsto \Gamma(U_{\text{Zar}}, f^*\mathscr{O}_{\text{Zar}})) \end{aligned} \qquad [9.79]$$

Similarly for étale morphism f,we get ringed topoi on small étale site.And based on this ringed structure,we have equivalence of categories of quasi-coherent sheaves.For $\eta : (S_{\mathrm{et}}, \mathscr{O}_{S_{\mathrm{et}}}) \to (S_{\mathrm{Zar}}, \mathscr{O}_{S_{\mathrm{Zar}}})$ induced by $f_{\text{étale}}$ and a quasi-coherent sheaf $F \in \mathrm{Qcoh}(S_{\mathrm{Zar}})$,we have $\eta^* F$ is a quasi-coherent sheaf on S_{et},sending $f_{\text{étale}}$ to $\Gamma(U_{\mathrm{Zar}}, g^* F)$.And we can define a sheaf $\mathscr{O}_{S_{\mathrm{et}}}$-module E is quasi-coherent if $E \cong \eta^* F$.Because product of A-modules is an A-module,we naturally have

$$\begin{tikzcd}
\mathrm{Qcoh}(U_{\mathrm{Zar}}) \times_{\mathrm{Qcoh}(U'_{\mathrm{et}})} \mathrm{Qcoh}(S_{\mathrm{Zar}}) \cong \mathrm{Qcoh}((U \coprod U')_{\mathrm{Zar}}) \arrow[r] \arrow[d] & \mathrm{Qcoh}(U'_{\mathrm{Zar}}) \arrow[d, shift left] \\
\mathrm{Qcoh}(U_{\mathrm{Zar}}) \arrow[r, "\eta^*"] & \mathrm{Qcoh}(U_{\mathrm{et}}) \arrow[u, shift left, "\eta_*"]
\end{tikzcd} \tag{9.80}$$

then,for $F' \in \mathrm{Qcoh}(U_{\mathrm{Zar}}), F \in \mathrm{Qcoh}(U'_{\mathrm{Zar}})$ we have $F' \cong \eta_* \eta^* F'$,by using this

$$\mathrm{Hom}_{S_{\mathrm{Zar}}}(F, F') \cong \mathrm{Hom}_{U'_{\mathrm{Zar}}}(F, \eta_* \eta^* F') \cong \mathrm{Hom}_{f^*_{\mathrm{et}}(U'_{\mathrm{Zar}})}(\eta^* F, \eta^* F') \tag{9.81}$$

which means the functor η^* is fully faithful,then we have

$$\eta^* : \mathrm{Qcoh}(S_{\mathrm{Zar}}) \cong \mathrm{Qcoh}(S_{\mathrm{fppf}}) \xrightarrow{\simeq} \mathrm{Qcoh}(S_{\mathrm{et}}) \tag{9.82}$$

which is an equivalence of subcategories of topoi $T_{\mathrm{Zar}}, T_{\mathrm{fppf}}$ and T_{et},which means these sites can induce a same subtopos.With the diagram of sites in [9.82]

$$\begin{tikzcd}
S_{\mathrm{et}} \arrow[r] \arrow[rr, bend right, "\mathrm{et}"'] & S_{\mathrm{fppf}} \arrow[r, "\mathrm{fppf}"] & S_{\mathrm{Zar}}
\end{tikzcd} \tag{9.83}$$

If put [9.70],[9.74] and [9.82] together,we further have

$$\mathrm{QCOH}(S_{\mathrm{et}}) \xrightarrow{\simeq} \mathrm{QCOH}(S_{\mathrm{fppf}}) \xrightarrow{\simeq} \mathrm{QCOH}(S_{\mathrm{Zar}}) \cong \mathrm{QCOH}(S) \tag{9.84}$$

Then,we get a further definition above [9.80],that is E on S_{et} is quasi-coherent if and only if E restricts to each $S_{i,\mathrm{et}}$ is quasi-coherent for an étale covering $\{S_i \to S\}_{i \in I} \in S_{\mathrm{et}}$.

Torsors and an example. A $\boldsymbol{\mu}$-torsor on a site C with $\boldsymbol{\mu}$ a sheaf of groups,is a sheaf $\mathscr{T}$ such that for every $X \in C$ has a covering $\{X_i \to X\}_{i \in I}$ the section $\mathscr{T}(X_i) \neq \varnothing$ for all i and the action $\boldsymbol{\mu}(X)\mathscr{T}(X)$ is simply transitive.Which means for all $t \in \mathscr{T}$ we can find a $g \in \boldsymbol{\mu}$ to let $t = gt'$.We claim that this is equivalent to say

$$(\boldsymbol{\mu} \times \mathscr{T}, \cdot \times \cup) \xrightarrow{\simeq} (\mathscr{T} \times \mathscr{T}, \cup \times \cup), \quad (g, t) \mapsto (t, gt) \tag{9.85}$$

Indeed,it is a homomorphism because

$$(g_1, t_1)(g_2, t_2) = (g_1 g_2, t_1 \cup t_2) \to (t_1 \cup t_2, g_1 t_1 \cup g_2 t_2) = (t_1, g_1 t_1)(t_2, g_2 t_2) \tag{9.86}$$

And $e_{\mathscr{T} \times \mathscr{T}} = (\varnothing, \varnothing)$ so $\ker = (e, \varnothing) \to (\varnothing, e\varnothing), = e_{\boldsymbol{\mu} \times \mathscr{T}}$ which means it is injective.And surjection follows from simply transitive action.A torsor $(\mathscr{T}, \rho)$ with a left action ρ is trivial if it has a global section which means if s is a global section $\rho(s) = s$,the uniqueness gives us

$$\boldsymbol{\mu} = \mathrm{stab}_{\boldsymbol{\mu}}(s) \xrightarrow{\varphi, \simeq} \mathscr{T} = \mathrm{fix}_{\varphi^{-1}(s)}(\mathscr{T}) \tag{9.87}$$

which gives a global section $\varphi^{-1}(s)$ of $\boldsymbol{\mu}$ and let $\boldsymbol{\mu}$ identifies $\mathscr{T}$.Now,we want to consider category of $\boldsymbol{\mu}$-torsors with a morphism satisfying the diagram

$$\begin{tikzcd}
\boldsymbol{\mu} \times \mathscr{T} \arrow[r, "\mathrm{id}_{\boldsymbol{\mu}} \times f"] \arrow[d, "\rho"] & \boldsymbol{\mu} \times \mathscr{T}' \arrow[d, "\rho'"] \\
\mathscr{T} \arrow[r] & \mathscr{T}'
\end{tikzcd} \qquad [9.88]$$

We have seen in [9.3] that we can generate group structure by topos,also different sites can induce a same topos we should put torsors in topos but not on sites.And we want to consider an example when $\boldsymbol{\mu}_n(X) = \{f \in \mathcal{O}_X^* | f^n = 1\}$ with $X \in C$,similarly to [9.83],we can consider category $\mathrm{Tor}(\boldsymbol{\mu}_n)$ on X_{et}.Let Σ_n be the category of pairs (L, σ),where L is a graded $\mathcal{O}_X$-module of degree 1,also an invertible sheaf with trivialization $\sigma : L^{\otimes n} \to \mathcal{O}_X$.A morphism is an isomorphism on the level of line bundles and satisfies the diagram

$$\begin{tikzcd}
 & \mathcal{O}_X & \\
L^{\otimes n} \arrow[ur, "\sigma"] \arrow[rr, "\rho^{\otimes n}"] & & L'^{\otimes n} \arrow[ul, "\sigma"']
\end{tikzcd} \qquad [9.89]$$

where $\rho : L \xrightarrow{\simeq} L'$.For a pair (L, σ),we have a sheaf on X_{et} that is $\mathscr{T}_{(L,\sigma)}$ sending $U \xrightarrow{\mathrm{et}} X$ to $\sigma|_U : L|_U \to \mathscr{O}_U$ satisfying $\mathrm{id}_{\mathscr{O}_U} = \sigma|_U{}^{\otimes n} \circ \sigma|_U{}^{\otimes(-n)}$.Putting a constraint $f^n = 1$ where $f \in \mathcal{O}_U^*$ makes $\mathscr{T}_{(L,\sigma)}$ a $\boldsymbol{\mu}_n$-torsor on étale site.And we use étale here gives us a reason why we want to study étale morphism because étale and Zariski sites induce equivalent of categories [9.82] and sometimes we cannot find Zariski cover,but we have enough étale cover because $\mathbb{C}$ is algebraically closed.Now a subscheme of $X = \underline{\mathrm{Spec}}(R)$ is $U = \underline{\mathrm{Spec}}(R[T])$ from ring extension $R \subset R[t]$ and a Zariski cover of this scheme is $\underline{\mathrm{Spec}}(R[T]/f)$ where $f \in R[T]^*$,which is not for a structure $\boldsymbol{\mu}_n$-torsor living on because we are lack of the above constraint,in this case we can use

$$U_{T^n - f} = \underline{\mathrm{Spec}}(R[T]/(T^n - f)) \to U \subset X, \quad \mathscr{T}_{(L,\sigma)}|_{U_{T^n-f}} = \mathscr{O}_{U_{T^n-f}} \qquad [9.90]$$

which naturally gives $T^n = 1 \in \mathscr{O}_U(U_{T^n-f})$ which is in the section of graded $\mathscr{O}_X$-module and this cover is étale because we need $R = \mathbb{C}$ for every f it can be expressed as T^n and property of étale follows from

$$\begin{tikzcd}
\Omega_R \arrow[r] & \Omega_A \\
R = \mathbb{C}[T] \arrow[u, "D"] \arrow[r, "f"] & A = \mathbb{C}[T]/(T^n - f) \cong \Omega_R \arrow[u, "D"']
\end{tikzcd} \qquad [9.91]$$

with a derivation $D : T^n \mapsto nT^{(n-1)}$ inducing isomorphism and $\Omega_{A/R} = 0$ follows from $\Omega_A \cong \Omega_R$.Also,by descent of quasi-coherent sheaves,We find the fibered category $p : \mathrm{Tor}(\boldsymbol{\mu}_n) \to X_{\mathrm{et}}$ is equivalent to a sheaf $\mathrm{Tor}(\boldsymbol{\mu}_n)$ by

[9.26].And by [9.87],it is identified by $\boldsymbol{\mu}_n$.Then,we want to apply these things we have developed to generalized superalgebraifold in [9.15] and string-Space.

Definition8.2 A consistent string-Space $\mathcal{S}$ is a string-space which is isomorphic to a stack $\mathrm{Tor}_{\underline{\boldsymbol{\mathcal{A}}}}$ of $\underline{\boldsymbol{\mathcal{A}}}$-torsors over the $\mathrm{ETSch}^{\mathrm{SupGen}}(\mathscr{M})$

$$(\mathcal{S} : \Phi \oplus \tilde{\Psi}^* \to \mathrm{ETSch}^{\mathrm{SupGen}}(\mathscr{M})) \cong \mathrm{Tor}_{\underline{\boldsymbol{\mathcal{A}}}} \qquad [9.92]$$

with an action $(\Gamma\underline{\mathcal{A}}) \times (\Phi \oplus \Psi^*) \to (\Phi \boxtimes \tilde{\Psi}^*)$ making it to an $\Gamma\underline{\mathcal{A}}$-module.Descent of quasi-coherent sheaves make it becomes a stack generalized by Ω-super T-duality with corresponding fusion of duality.Which also means previous category of sheaves is a category of sheaves of properties in [9.15] corresponding to the string-Space can be descended to a sheaf of properties on M-brane

$$\mathscr{P}(\mathrm{ETSch}^{\mathrm{SupGen}}(\mathscr{M})) \xrightarrow{\simeq} \mathscr{P}(\mathscr{M}) \qquad [9.93]$$

with the number counting [9.18] that becomes a consistent isomorphism

$$\mathrm{F}^{++,--}_{Q\boxtimes Q^*} \cong \boxtimes[\mathbb{Z} \oplus \mathbb{C} \oplus \mathbb{Q} \oplus \mathbb{R} \oplus (\mathbb{ZCQR})]^{+,-\mathrm{self\ T}}_{\mathrm{P}(\mathbf{T})^{-1}(Q\boxtimes Q^*)} \qquad [9.94]$$

A subtile corollary is we have a relative property which is étale equivalent

$$\mathrm{F}^{\tilde{\mathscr{R}}} = \mathrm{F}^{++,--}_{Q\boxtimes Q^*} \times_{0_{Q\boxtimes Q^*}} \mathrm{F}^{+,-\mathrm{self\ T}}_{\mathrm{P}(\mathbf{T})^{-1}Q\boxtimes Q^*} \to_{\mathrm{induce}} \tilde{\mathscr{R}} \qquad [9.95]$$

corresponding to the notation in [8.38] with additive identity is the cosmological constant we have seen in [8.40],with the further fiber product

$$\begin{tikzcd}
\mathrm{F}^{++,--}_{Q\boxtimes Q^*} \coprod \mathrm{F}^{\tilde{\mathscr{R}}} \arrow[r] \arrow[d] & \mathrm{F}^{\tilde{\mathscr{R}}} \arrow[d] \\
\mathrm{F}^{++,--}_{Q\boxtimes Q^*} \arrow[r] & 0
\end{tikzcd} \qquad [9.96]$$

Definition8.3 An universe evolution picture $\boldsymbol{\Pi}$ is the following diagram which is the original description of evolution of our universe

$$\begin{tikzcd}
0 & \mathrm{F}^{++,--}_{Q\boxtimes Q^*} \coprod \mathrm{F}^{\tilde{\mathscr{R}}} \arrow[l, shift left] \arrow[l, shift right, "\tilde{\mathscr{R}}"'] \arrow[r, shift left] \arrow[r, shift right] & \mathrm{F}^{+,-\mathrm{self\ T}}_{\mathrm{P}(\mathbf{T})^{-1}Q\boxtimes Q^*} \arrow[ll, bend left, "\tilde{\mathscr{R}}"]
\end{tikzcd} \qquad [9.97]$$

with arrows denoting the subtile directions of evolution.

Definition8.4 A pre M-theory $\boldsymbol{\mathcal{M}}^{\mathrm{pre}}$ is a geometric (based on simplicial settings [8.11]) stack for which we express it in a collection of data

$$\boldsymbol{\mathcal{M}}^{\mathrm{pre}} = (\mathrm{ETSch}^{\mathrm{SupGen}}_{\mathrm{eff}}(\mathscr{M}), \mathrm{P}(\mathbf{T}), \mathscr{P}, \boldsymbol{\Pi}) \qquad [9.98]$$

The subscript means coverings are effective descent morphisms [9.93] for $\mathscr{P}$.

9.4 Stacks (2-preschemes) and Yoneda duality

For a site C,a category fibered in groupoids $p : F \to C$ is a stack (we call 2-prescheme see [9.104]) if and only if the following conditions hold

$$\begin{array}{l} \text{(i) For any presheaf } \underline{\mathrm{Isom}}(x, y) \text{ on } C/X \text{ in [9.49] is a sheaf} \\ \text{(ii) For any covering } \{X_i \to X\}\text{,any data } \sigma \text{ on [9.52] is effective} \end{array} \qquad [9.99]$$

The (i) is for global descent of groupoid structures and (ii) is for that of fibers.To interpret it,we first study two sections with overlap in topological space and how can we topologically retract them will not affect the relative property above [8.22] that is equivalent to a class of closed strings,the answer is

$$F(X_i \times_X X_j) \to_{\mathrm{retract}} R \subset F(X_i) \times_{F(X_i \cap X_j)} F(X_j) \qquad [9.100]$$

where R is an equivalence relation.If only (i) satisfies it is a prestack.

Definition9.1 A spontaneously breaking of equivalence relation is

$$R \subset F(X_i) \times_{F(X_i \cap X_j)} F(X_j) \to_{\mathrm{breaking}} F(X_i) \coprod F(X_j) \qquad [9.101]$$

The information is contained in the equivalence relation (overlap) and breaks to two degenerate states.But to achieve such retract in [9.100],we need F to be a sheaf because $F(X \times_Y X) \cong F(X) \times_{F(Y)} F(X)$.Thus,we have a natural retract

$$(\text{Fibered categories with global descent}) \to_{\mathrm{retract,over}\, C} (\text{Prestacks}) \qquad [9.102]$$

Notice that C does not neccesary preserves C_{eff}.We can apply [9.98] in,that is

$$(\text{Superstring theory}^{\mathrm{type}}) \to_{\mathrm{retract,over}\, \mathrm{ETSch}^{\mathrm{SupGen}}_{\mathrm{eff}}(\mathscr{M})} (\mathcal{S}^{\mathrm{type}}) \in \boldsymbol{\mathcal{M}}^{\mathrm{pre}} \qquad [9.103]$$

where a type of theory is a fibered category and category of categories induced by dualities.And global descent [9.93] glues these string-Spaces to a pre M-theory which gives us the unification of superstring theories.Intuitively

$$\mathrm{retract} : \mathrm{Fibcat.} \xrightarrow{\simeq} \mathrm{presheaf} \xrightarrow{\mathrm{Isom}} \mathrm{sheaf} \to \mathrm{fibers}^{\mathrm{group.}} \to \mathrm{prestack} \qquad [9.104]$$

Then to stack which is a method to make the global section of a scheme be scheme-like,so we call a stack a 2-prescheme,we may regard the latter algebraic stacks as 2-schemes.Secondly,stackification (retract) [9.104] let us focus on the relative properties above [8.22] in the world.

Before we discuss algebraic space,we need to recap the Yoneda lemma because something we have not captured about representable functor.First,we have a fibered category $p : \mathrm{Sch}^{\mathrm{rep}} \to \mathrm{Sch}$ with étale topology where $\mathrm{Sch}^{\mathrm{rep}}(X) = h_X$ and we claim that it is a stack,because h_X is a sheaf we have global descent [9.70],the only thing is to verify it is a category fibered in groupoid

$$T \overset{\simeq}{\nrightarrow} X \to_{\mathrm{replace}} T \xrightarrow{P^2} X \qquad [9.105]$$

where $(T \to X) \in \mathrm{Sch}/X \cong h_X$.Follows from [9.105],we have the theorem.

Theorem8.5 Sch/X is a quasi-groupoid if and only if $P^2 \simeq\cong$ in RHS of [12.19] and $\to\cong\simeq_{\mathrm{weak}}$ see double-weak diagram [12.85],which means if and only if Sch/X admits a relative 2-property see below [9.113].

Theorem8.6 Sch/X admit a relative 2-property if and only if $\mathbf{Ret}_*\mathrm{Sch}/X$ is a category and objects have relative properties.But these are just natural settings for [12.4],so Sch/X is a quasi-groupoid in LHS with $\simeq_{\mathrm{weak}}$ and we can perform a weak version of stack (quasi-stack) on it.

Also,we have $h_X \cong X$ by Yoneda lemma,which makes us put them into a quasi-stack generalized by duality,and we call this as a Yoneda duality $\mathbf{Y}$

$$p : \mathrm{Sch}^{\mathrm{rep}} \oplus \mathrm{Sch} \to \tilde{C} \qquad [9.106]$$

with $\mathbf{Y}$-fusion $\mathrm{P}(\mathbf{Y}) : \mathrm{Sch}^{\mathrm{rep}}(\mathrm{Sch}) \in \tilde{C}$.An interesting thing is generalized super version has LEE and High energy representation,that is

$$\begin{tikzcd}
\mathbf{U} \arrow[r] & (\mathscr{M})^{\mathrm{rep}} \oplus (\mathscr{M}) \\
\mathbf{Y} \arrow[u] \arrow[r] & (\mathrm{ETSch}_{\mathrm{eff}}^{\mathrm{SupGen}}(\mathscr{M}))^{\mathrm{rep}} \oplus \mathrm{ETSch}_{\mathrm{eff}}^{\mathrm{SupGen}}(\mathscr{M}) \arrow[u, "\mathrm{P}(\mathbf{T})"] \\
\mathrm{W} \leftrightarrow \mathrm{P} \arrow[u] \arrow[r] & (\mathcal{X}_0/\mathscr{M})^{\mathrm{rep}} \oplus \mathcal{X}_0/\mathscr{M} \arrow[u]
\end{tikzcd} \qquad [9.107]$$

where $\mathrm{W} \leftrightarrow \mathrm{P}$ is wave-particle duality.

9.5 Relative 2-properties and algebraic spaces

Now,we are able to discuss algebraic spaces.A class of objects in a site C is a subcategory $S \subseteq C$ which is stable if for every $U \in S$,every covering $\{U_i \to U\} \in S$.If a stable class of objects with a global property P,we call it a stable property P of objects.For instance,in big Zariski site,we can collect all coverings of an affine schemes $U \in C$ to form a stable class,the locally noetherian is global that is stable,because for every cover $\mathrm{Spec}(R) \to \mathrm{Spec}(A)$ we have

$$\begin{tikzcd}
\cdots \arrow[r, hook] & (br) \arrow[r, hook] & (ar) \arrow[r, hook] & (ar)_{\max} \\
\cdots \arrow[r, hook] & (b) \arrow[u] \arrow[r, hook] & (a) \arrow[u] \arrow[r, hook] & (a)_{\max} \arrow[u]
\end{tikzcd} \qquad [9.108]$$

for every ascending chain and A is a noetherian ring and R becomes an A-module.For a site C,if a subcategory $D \subseteq C$ contains all isomorphisms in C and a morphism $f \in D$ if and only if pullback of it is in D we call it a closed subcategory of S.And if $C = (C, \mathrm{Cov})$ is a site for each morphism $f : X \to Y$

in D if and only if

$$\begin{tikzcd} f^*\mathrm{Cov}(Y) \arrow[r] \arrow[d] & \mathrm{Cov}(Y) \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \in D \tag{9.109}$$

which is a fibered product,we call the closed subcategory stable.And a stable closed subcategory $D \subseteq C$ is local on domain,if for every $f \in D$ only if

$$\begin{tikzcd} & \mathrm{Cov}(X) \arrow[r] \arrow[d] & \mathrm{Cov}(Y) \arrow[d] \\ X_i \arrow[r, "x_i"] \arrow[ur, "i\in I"] & X \arrow[r, "f"] & Y \end{tikzcd} \in D \tag{9.110}$$

Let D_P denote the subcategory of C with the same objects and Hom changes to Hom_P which is a hom set preserving the property P.Now we have

$$\begin{aligned} &\text{(i) } \Leftrightarrow f^*\mathrm{Cov}(Y) \in \mathrm{Hom}_P(-,X), \mathrm{Cov}(Y) \in \mathrm{Hom}_P(-,X) \\ &\text{(ii) } \Leftrightarrow \text{Based on (i)}, \mathrm{Cov}(X) \in \mathrm{Hom}_P(-,X) \end{aligned} \tag{9.111}$$

for every $f \in D_P, : X \to Y$.The property is stable if it satisfies (i),and is local on domain if it satisfies (ii).In summary,by using the philosophy above [9.1] we want to study a property of morphisms (morphisms preserving the property) which shifts the focus point from C to Cov and we get Cov_P inherited from hom set.Recall the definition above [8.1],now we shift to morphisms of properties

$$\begin{tikzcd} P \arrow[r, hook] \arrow[d, "\simeq"] & X \times_S X \arrow[d, "f\times f"] \\ P' \arrow[r, hook] & Y \times_S Y \end{tikzcd} \tag{9.112}$$

And if we apply [9.110],we get P is local on domain $\Leftrightarrow$ for every $f \in C$

$$\begin{tikzcd} P \arrow[r, hook] \arrow[d, "\simeq"] & \mathrm{Cov}(X) \times_S \mathrm{Cov}(X) \arrow[d, "f\times f"] \\ P' \arrow[r, hook] & \mathrm{Cov}(Y) \times_S \mathrm{Cov}(Y) \end{tikzcd} \tag{9.113}$$

Definition9.2 A 2-property of relative properties P is a relation of stable relative properties P based on the site $(C, \mathrm{Cov}_{\mathscr{C}_P})$.A relative 2-property of relative properties is a pair (P, P') with the strucuture in [9.113] at least on the level of stable properties.For instance,the we have a 2-property from [8.50],$(*, (\mathbf{T}^\delta \circ \mathbf{S}))$ on the site GSTen and each is a stable relative property along coverings.

Corollary9.3 For find a general site for properties being stable,$\mathrm{Cov}_{\mathscr{C}_P} = \mathrm{Et}$.

Let $f : F \to G$ be a morphism of sheaves on Sch/S with the étale topology.f is represented by schemes if for every S-scheme T and morphism $T \to G$ the fiber product $F \times_G T$ is a scheme.If f is representable by schemes,we say f

has (preserves) property if for every S-schemes T,$\mathrm{pr}_2 : F \times_G T \to T$ preserves property P.Which means a relative 2-property on f by the structure [9.113] and inherited from schemes level to represented sheaves level.For instance,if $f : h_X \to h_Y$ is a morphism with a relative 2-property P^2 on the $(\mathrm{Sch}/S, \mathrm{Et})$

$$\begin{tikzcd} h_X \times_{h_Y} T \arrow[r, "P^2"] \arrow[d] & T \arrow[d] \\ h_X \arrow[r, "P^2"] & h_Y \end{tikzcd} \cong \begin{tikzcd} X \times_Y T \arrow[r, "P^2"] \arrow[d, "\mathrm{stab}"] & T \arrow[d, "\mathrm{stab}"] \\ X \arrow[r, "P^2"] & Y \end{tikzcd} \qquad [9.114]$$

Because f is representable by schemes,$h_X \times_{h_Y} T \cong X \times_Y T$.Which means every $X \xrightarrow{P^2} Y \Leftrightarrow h_X \xrightarrow{P^2} h_Y$ on a stable site $(\mathrm{Sch}/S, \mathrm{Cov}_{\mathscr{C}_P})$.Exactly

$$\begin{tikzcd} h_X \arrow[r] \arrow[d] & h_Y \arrow[d, "\text{representable}^{\text{pres.}}"] \\ X \arrow[r, "\text{stable}^{\text{pres.}}"] & Y \end{tikzcd} \qquad [9.115]$$

And we also call a relative 2-property universal property see below [9.130].

Definition9.4 A general $\mathrm{Cov}^{\text{cons.}}$ is a collection of preservations of universal properties,which means $\mathrm{Cov}_{\mathscr{C}_P} \coprod \mathrm{Rep}$ is general,$(h_Y \to Y) \in \mathrm{Rep}$.And we call a site generalized by the covering which containing sheaves admitting preservation of universal properties as extra objects,a general site denoting by cons..

We claim that for a sheaf F on Sch/S with étale topology,the diagonal morphism is representable by schemes,then for any scheme T, T',the morphism $f : T \to F$ is representable by schemes.Indeed,we have

$$\begin{tikzcd} T \times_F T' \arrow[r] \arrow[d, "f \coprod g"] & T \times_S T' \arrow[d, "f \times g"] \\ F \arrow[r, "\Delta"] & F \times F \end{tikzcd} \qquad [9.116]$$

where $\Delta^*(f \times g) = f \coprod g$ and $(f \coprod g)^* F = T \times_F T'$ and because diagonal morphism is representable by schemes,$T \times_F T'$ is a scheme which means any morphism $f : T \to F$ with T a scheme is representable by schemes.

An algebraic space over S is a sheaf $X : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$ on big étale site with a diagonal morphism $\Delta : X \to X \times_S X$ represented by schemes and there exists an étale presentation that is a surjective étale morphism $U \to X$ from a S-schemes giving a covering $X \to U \in \mathrm{Rep}$ in a general $\mathrm{Cov}^{\text{cons.}}$.And we see schemes over S are algebraic spaces over S.Let AS/S is a category of algebraic spaces over S,we see from [9.115]

$$\mathrm{AS}/S = ((\mathrm{Sch}/S)^{\text{cons.}}, \mathrm{ET} \coprod \mathrm{Rep}) \qquad [9.117]$$

where we use ET for coverings of ordinary big étale site.For a morphism of schemes $g : S' \to S$,we have a category ASS' with objects are pairs $(X, f_{/S})$ with

$X\in \mathrm{AS}/S$ and $f:X\to S'$,a morphism is $(t,\circ t^{-1}):(X',f'_{/S})\to(X,f_{/S})$.For an algebraic space $Y\in \mathrm{AS}/S'$,we have a functors

$$Y_S:(\mathrm{Sch}/S)^{\mathrm{op}}\to((\epsilon,Y(\epsilon:T\to S'))),\quad f_{Y/S}:(Y_S\to S')/S \tag{9.118}$$

with $\epsilon:T\to S'$ where T is a S-scheme making ϵ a S-morphism.First,T is a scheme which is a sheaf over S' and Y is a sheaf over S' so Y_S is an étale sheaf.Then,by $\Delta^*_{S'/S}(f_{Y/S}\times f_{Y/S})=(f_Y\coprod f_Y)$,we have

$$\begin{tikzcd}
{} \arrow[r] \arrow[d] & \color{red}{Y\times_{Y\times_{S'}Y}(S'\times_S S')\times_{S'\times_S S'}S'} \arrow[r] \arrow[d] \arrow[dl, dotted, red] \arrow[dr, dotted, red] & {} \arrow[d] \\
\color{blue}{Y\times_{Y\times_{S'}Y}(S'\times_S S')} \arrow[r] \arrow[d] \arrow[drr, dotted, red] \arrow[ddrr, dotted, blue] & {} \arrow[r] \arrow[d] & Y_S \arrow[d, "\Delta_{Y_S}"] \\
Y \arrow[r, "\Delta_Y"] & Y\times_{S'}Y \arrow[r] \arrow[d] \arrow[dr, dotted, blue, leftrightarrow] & Y_S\times_S Y_S \arrow[d, "f_{Y/S}\times f_{Y/S}"] \\
& S' \arrow[r, "\Delta_{S'/S}"] & S'\times_S S'
\end{tikzcd} \tag{9.119}$$

where Y is algebraic space,S,S' are schemes and colored fiber products are scheme because of representable diagonal morphisms.And the red product is isomorphic to $Y\times_{Y\times_{S'}Y}(S'\times_S S')\times_{Y_S\times_S Y_S}Y_S$.Thus,we find Δ_{Y_S} is representable by schemes.Next,we have an étale presentation $U/S'\to Y$ corresponding to a global section $u\in Y(U/S')$,which gives a global section $(\epsilon,u|(\epsilon:U/S'))$ in $Y_S(U/S)$,so we have an surjective morphism $U/S\to Y_S$.From [9.119]

$$\begin{tikzcd}
U \arrow[r, "\mathrm{et}"] \arrow[dr] & Y \arrow[d, "\sqrt{\Delta_{S'/S}}"] \\
& Y_S
\end{tikzcd} \tag{9.120}$$

which means $U/S\to Y_S$ is étale.Therefore,$Y_S\in \mathrm{AS}/S$.Then,we claim that

$$\mathrm{AS}/S'\cong \mathrm{ASS}',\quad Y\mapsto(Y_S,f_{Y/S}) \tag{9.121}$$

Indeed,for a $Y_S\in \mathrm{ASS}'$ with $f_{Y/S}:Y_S\to S$,we can recover $Y\in \mathrm{AS}/S'$ from $(\sqrt{\Delta_{S'/S}})^*f_{Y/S}$.Also,from $Y\in \mathrm{AS}/S'$,we can recover $Y_S\in \mathrm{ASS}'$ along $(\sqrt{\Delta_{S'/S}})_*\circ\sqrt{\Delta_Y}_*$ with $(\sqrt{\Delta_{S'/S}})_*\circ\sqrt{\Delta_Y}_*(f_Y)=f_{Y/S}$.And the uniqueness of pullback and pushforward gives the correspondence.

Now,for a relative property R and $T,X\in(\mathrm{Sch}/S,\mathrm{Et})$ we have

$$\begin{tikzcd}
R\times_{X\times_S X}X \arrow[r] \arrow[d] & X \arrow[d, "\Delta_X"] \\
R \arrow[r, hookrightarrow] & X\times_S X
\end{tikzcd} \tag{9.122}$$

which makes X to be a principle R-bundle,now we want to define a $\mathscr{B}$-sheaf that is an assignment $T\mapsto X(T)$ because $(T\to S)\in \mathrm{Et}(S)$ this extents to an étale sheaf $X \cong h_X : (\mathrm{Sch}/S)^{\mathrm{op}} \to (\mathrm{Sch}/X_{/S})$,then we have a h_R-torsor h_X

$$h_X/h_R = h_{X/R} \cong X/R \in \mathrm{AS}/S \tag{9.123}$$

An interesting thing is $X, X/R \in (\mathrm{Sch}/S)^{\mathrm{cons.}}$ gives a double counting of relative property R,so we have reason to regard this as a problem similar to gauge fixing.In this case,actually the site should be general in a pre M-theory [9.98]

$$\mathrm{ETSch}_{\mathrm{eff}}^{\mathrm{SupGen,cons.}}(\mathscr{M}) = (\mathrm{ETSch}_{\mathrm{eff}}^{\mathrm{SupGen}}(\mathscr{M}))^{\mathrm{rep}} \oplus \mathrm{ETSch}_{\mathrm{eff}}^{\mathrm{SupGen}}(\mathscr{M}) \tag{9.124}$$

Because we cannot distinguish representable sheaves and schemes by Y-duality [9.106] the problem of a pre M-theory cannot be consistent is because we want to do number counting of relative properties to achieve nonexistence in [9.97],but we know algebraic spaces can also be in site with relative properties,so we cannot have a consistent number counting on the site which can be general.

Theorem9.5 A number counting of relative properties is not unique and depends on the underlying site.

Remark9.6 Combing with definition3.10,the nonexistence for pre M-theory is local because it changes with the number countings of relative properties.

Definition9.7 The M-theory is consistent if and only if there is an unique number counting of properties to achieve nonexistence.Which is equivalent to say a pre M-theory is M-theory if and only if it is experiment-free.

Theorem9.8 A number counting of generalized super relative 2-properties is unique on the site $\mathrm{ETSch}_{\mathrm{eff}}^{\mathrm{SupGen,cons.}}(\mathscr{M})$.

Proof. The general site is from Y-duality as a part of U-duality,we get these properties after **U**-fusion [9.148].The number counting of each generalized super relative 2-property is 0,thus there is no double counting problem,which means this number counting is unique,$(F_{Q\boxtimes Q^*})^{\mathrm{rep}} \oplus (F_{Q\boxtimes Q^*}) \cong_{\mathrm{P}(\mathbf{Y})} 0$!. $\square$

Back to algebraic spaces as sheaf quotients [9.123].To verify the axioms below [9.113],we have seen it is an étale sheaf.And let $Y = X/R$,based on [9.119]

$$\begin{tikzcd} R_U \arrow[r, hook] \arrow[d, shift left] & U \times_S U \arrow[d, "j\times j"] & R_U \times_{U\times_S U} U \arrow[r] \arrow[d] & U \arrow[d, hook, "j"'] \\ R \arrow[r, hook] \arrow[u, shift left, "(j\times j)^*"] & X \times_S X & R \times_{X\times_S X} X \arrow[r] & X \end{tikzcd} \tag{9.125}$$

where a Zariski morphism $j : U \hookrightarrow X$ and induces $\bar{j} : U/R_U \to Y$.Also,we have $R_U \hookrightarrow R$ induced by the cover,which gives the diagram

$$\begin{tikzcd} R_U \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] \arrow[d, hook, "j"'] & s(t^{-1}(j^{-1}\circ f)(T)) \arrow[r] \arrow[d, "\mathrm{et}"'] \arrow[dr, phantom, "\square"] & U/R_U \arrow[d, "\bar{j}"'] \\ R \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & f(T) \arrow[r] & Y \end{tikzcd} \tag{9.126}$$

where we work étale locally $f : T \xrightarrow{\mathrm{et}} X, j^{-1} \circ f : (T \to U)/X$.And we can see $\bar{j}$ is representable by open embedding through étale T-points.For seeing Δ_Y is representable by schemes,we need to gather information in [9.116] and [9.122]

$$\begin{tikzcd}
R \times_{X\times_S X} X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "\Delta"] & \textcolor{red}{Y} \\
R \arrow[r, hook] \arrow[ur, dotted, red] \arrow[rr, dotted, red, bend right] & X \times_S X & X \arrow[l] \arrow[u]
\end{tikzcd} \tag{9.127}$$

for a general étale covering $f : X \to Y$,and we get $R \cong X \times_Y X$.Then,for W, S affine schemes $W \to Y \times_S Y$ where we work Zariski locally

$$\begin{tikzcd}
R \arrow[r, "\Delta^*(f_{/S}\times f_{/S})"] \arrow[d] & Y \arrow[d, "\Delta"] & Y \times_{Y\times_S Y} W \arrow[l] \arrow[d] & F' \arrow[l] \arrow[d] \\
X \times_S X \arrow[r, "f_{/S}\times f_{/S}"] & Y \times_S Y & W \arrow[l, "\mathrm{et}"'] & W' \arrow[l, "\mathrm{et}"'] \arrow[lll, dotted, bend left, "\mathrm{et}"]
\end{tikzcd} \tag{9.128}$$

where $\Delta^*(f_{/S} \times f_{/S}) = f \coprod f$,it follows that

$$F' = Y \times_{Y\times_S Y} W \times_W W' \cong Y \times_{Y\times_S Y} W' \cong R \times_{X\times_S X} W' \tag{9.129}$$

Now W is affine and also need to be quasi-compace which means W' is affine,the diagonal morphism from F' makes W' be separated.Also diagonal morphism is a monomorphism and F' need to be an separated scheme.For a $((\mathrm{Sch}/S)^{\text{cons.}}, \mathrm{Et})$ the coverings $\mathrm{Et}(S)$ induces a category of local relative properties,for example,an étale morphism $X \to S$ induces a relative property $X \times_S X$ localized by this S-scheme.Based on this local relative property,a S-morphism induces a sub relative property R by [9.127],if we add a further étale morphism and change the notation

$$\begin{tikzcd}
 & & \cdots & X''/\ldots \arrow[r] \arrow[d, equal] & (X''/\ldots)/\ldots \arrow[d, equal] \\
{} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & X'' \arrow[r] & \textcolor{red}{X'} \arrow[r] & \textcolor{blue}{X} \\
X'' \times_X X'' \arrow[r, hook] \arrow[urr, dotted, blue] \arrow[rrr, dotted, blue, bend right] & X'' \times_{X'} X'' \arrow[r, hook] \arrow[ur, dotted, red] \arrow[rr, dotted, red, bend right] & X'' \times_S X'' \arrow[r] \arrow[u] \arrow[dr] & X'' \arrow[r] \arrow[u] & X \arrow[u] \\
 & & \cdots & Y \times_S Y & \cdots
\end{tikzcd} \tag{9.130}$$

where we find Et induces an evolution of relative properties,relative properties was transferred along morphisms in coverings and may be same and may be changed and back to classfy X' and X by sheaf quotient of X'' by corresponding relative property.This also gives us an understanding of why we regard relative 2-property as universal property.Formally,we use [9.104]

$$\text{Fibcat. of stacks of rel.properties} \to_{\text{retract}} \text{stack of rel.2-properties} \tag{9.131}$$

And we will see it in the generalized super case,this retract is **U**-fusion.And this chain as ∞-category is truncated by physics.

Now,back to math,apply [9.130] in [9.128] we classify $F = F'/(F' \times_F F')$ where $F = Y \times_{Y\times_S Y} W$.If we denote $g : F' \to W'$,we have

$$\begin{array}{ccc} F' & \underset{\Delta^*_{F'/F}}{\overset{\Delta_{F'/F}}{\rightleftarrows}} & F' \times_F F' \\ \downarrow g & & \downarrow \\ W' & \underset{\Delta^*_{W'/W}}{\overset{\Delta_{W'/W}}{\rightleftarrows}} & W' \times_W W' \end{array} \qquad [9.132]$$

which is $F \to W$ is monomorphism g^* is well defined and makes [9.132] cartesian.Because scheme W' is an algebraic space and F' is a scheme,$F' \times_F F'$ is a scheme with étale topology.Now,we repeate [9.126] but focus on F'

$$\begin{array}{ccc} s(t^{-1}(U')) & \longrightarrow & U'/R'_{U'} \\ \big\downarrow & & \big\downarrow \bar{j} \\ F' & \longrightarrow & F'/R' \end{array} \qquad [9.133]$$

where we denoted $R' = F' \times_F F'$ and a quasi-compact open subscheme $U' \subset F'$,which is cartesian and $s(t^{-1}(U'))$ is an quasi-compact open subset of F'.In this case,we put [9.133] back to [9.128],we get

$$s(t^{-1}(U')) \cong U'/R'_{U'} \times_W W' \qquad [9.134]$$

Then,if $s(t^{-1}(U'))$ is a scheme,$U'/R'_{U'}$ is a scheme then F is scheme.Now,the set $s(t^{-1}(U'))$ is a set of quasi-affine schemes and W' is quasi-compact and quasi-seperated,which means by Zariski's main theorem

$$s(t^{-1}(U')) \xhookrightarrow{j} F' \xrightarrow{\Delta} W' \qquad (\Delta \circ j)^{-1}(W'', \subset W') \subset s(t^{-1}(U')) \quad [9.135]$$

By definition, $(\Delta \circ j)$ is quasi-affine.With a fibered catgegory

$$\begin{array}{ccc} ((\Delta \circ j)^{-1}W'' \to W'') & \xrightarrow{p} & (W'') \\ \big\downarrow & & \big\downarrow \\ \mathrm{Aff} & \xrightarrow{p} & \mathrm{Sch} \end{array} \qquad [9.135]$$

By global descent of quasi-affine morphisms over fppf coverings in 4.4.17 in [12] and also for étale coverings,or just by knowing that gluing affine schemes follows from gluing structure sheaves on them by using [9.70].Thus,we have a glued affine scheme $s(t^{-1}(U'))$.Back to [9.28] the diagonal morphism of $Y = X/R$ is representable by schemes.The last thing is étale surjection,we claim that a natural étale presentation is $X \to X/R$,the quotient let it be already epimorphism of étale sheaves.For a morphism $T \to Y$ and it factor through X

we have the cartesian square with $\Delta^*_{X/Y}(f\times_Y \mathrm{id}) = f_{/Y}$

$$\begin{tikzcd}
T\times_Y X \arrow[r, "f\times_Y \mathrm{id}"] \arrow[d] & X\times_Y X \arrow[r] \arrow[d, shift left, "\mathrm{et}"] & X \arrow[d] \\
T \arrow[r, shift left] & X \arrow[l, shift left, "\Delta^*_{X/Y}(f\times_Y \mathrm{id})"] \arrow[u, shift left, "\Delta_{X/Y}"] \arrow[r] & Y
\end{tikzcd} \qquad [9.136]$$

because diagonal morphism of Y is representable,$X\times_Y X$ is scheme by [9.116] so $\Delta_{X/Y}$ is étale and $X\to Y$ is étale.By definition below [9.116],X/R is an algebraic space and comes into $((\mathrm{Sch}/S)^{\mathrm{cons.}}, \mathrm{Et}\coprod \mathrm{Rep})$.Now,an algebraic space X can have stable relative property from étale presentation in general site with coverings adimitting preservations of relative 2-properties,$U\xrightarrow{P^2} X$.Also,by [9.115] morphisms of algebraic spaces admit preservations of relative 2-properties.

Algebraic spaces are fppf sheaves. For an algebraic space $X|S$ which is an algebraic space X over a scheme S,that is an étale sheaf and at least a fppf presheaf.If we put fppf topology in,we claim that

$$q : X|S_{\mathrm{fppf}} \xrightarrow{\simeq} \overline{X}|S_{\mathrm{fppf}} \qquad [9.137]$$

where $\overline{X}$ is a fppf sheaf.Definition 5.4.7 in [12] tells us a morphism $X\to S$ over S is quasi-separated if the diagonal $\Delta_{X/S}$ is quasi-compact.So,we want to assume $\Delta_{X/S}$ is quasi-compact,in this case,for a fppf morphism $U\to S$

$$\begin{tikzcd}
U\times_X U \arrow[r] \arrow[d, "\Delta^*_{X/S}(f\times_S f)"] & U\times_S U \arrow[d, "f\times_S f"] \\
X \arrow[r, "\Delta_{X/S}"] & X\times_S X
\end{tikzcd} \qquad [9.138]$$

where $\Delta^*_{X/S}(f\times_S f) = f\coprod f$ and define f by $X(U)$.Because,X is an algebraic space,$U\times_X U$ is a fppf sheaf recall that below [9.70] and X is a separated presheaf.Also,the diagonal morphism is monomorphism,so q is injective with $\overline{X} = U\times_X U$.To see surjectivity of q is to verify if $s\in \overline{X}(U)$ is in the image of q and it suffices to consider U is quasi-compact,in this case,we can decompose the separated presheaf $X = \bigcup_i X_i$ such that

$$\lim_i X_i|S_{\mathrm{fppf}}, \quad X_i(U) = (\lim_i X_i)(S_{\mathrm{fppf}})|_{U_i} \qquad [9.139]$$

they are locally matching,because U is quasi-compact we may therefore also assume that X is quasi-compact.For an étale presentation $X_0\to X$

$$\begin{tikzcd}
U\times_{\overline{X}} X_0 \arrow[r] \arrow[d] & X_0 \arrow[r, "\mathrm{et}"] \arrow[d] & X_i \arrow[dl, "q"] \\
U_i \arrow[r, "s_i"] \arrow[urr, dotted, "\tilde{s}_i"] & \overline{X} \arrow[ur, dotted, bend right] &
\end{tikzcd} \qquad [9.140]$$

where we used a fppf covering $\{U_i \to \overline{X}\}_{i\in I}$,we find q adimits a section $\tilde{q} : \tilde{s}_i \circ s_i^{-1}$,where we want to apply the descent of morphisms admits a section in 4.2.9 in [12],which gives us for $\overline{X} = U \times_X U$ is a quasi-compact fppf sheaf

$$(U \times_{\overline{X}} X_0)(\{X_i \to \overline{X}\}) \cong (U \times_{\overline{X}} X_0)(\overline{X}) \qquad [9.141]$$

where we used property of quasi-compact,separated,scheme is fppf sheaf and fppf presheaf is locally a fppf sheaf.And surjectivity is given by [9.141].Thus,we get a result [9.137] that is consistent to [9.79].

9.6 Generalized super relative 2-properties with U-fusion

Now we have one duality that hasn't been applied but living in M-theory that we discussed in section 8.5,which is U-duality.From [8.28],we have

$$(\mathcal{X}_0 \times_S \mathcal{X}_1^*/\mathbb{Z}) \boxtimes_{\mathbf{T}} (\mathcal{X}_1^* \times_S \mathcal{X}_0/\mathbb{Z}) \subset (\mathcal{X}_0 \times_S \mathcal{X}_1^*)^{\text{self } \mathbf{T}} \qquad [9.142]$$

where we T-fused them see below [8.21] and [8.37].After **T**-fusion,the theory becomes self T-dual and self S-dual that is meaningless to perform further fusion of these dualities.Fortunately,we have U-duality on the level of M-brane.Guided by the evolution picture [9.97],A **U**-fusion is the following process

$$\mathrm{P}(\mathbf{U}) : (\mathcal{X}_0 \times_S \mathcal{X}_0)_{\mathscr{M}} \mapsto (\mathcal{X}_0 \times_S \mathcal{X}_0)_{\mathscr{M}} \boxtimes_{\mathbf{U}} (\widetilde{\mathcal{X}}_1^* \times_S \widetilde{\mathcal{X}}_1^*) \qquad [9.143]$$

we need to perform it on two copies first,and an **U**-fusion is

$$\begin{aligned}
&\mathrm{P}(\mathbf{U})(\mathcal{X}_0 \times_S \mathcal{X}_1^*)^{\text{self } \mathbf{T}} \\
&= (\mathcal{X}_0 \times_S \mathcal{X}_0)_{\mathscr{M}} \boxtimes_{\mathbf{U}} (\widetilde{\mathcal{X}}_1^* \times_S \widetilde{\mathcal{X}}_1^*) \oplus (\mathcal{X}_1^* \times_S \mathcal{X}_1^*)_{\mathscr{M}} \boxtimes_{\mathbf{U}} (\widetilde{\mathcal{X}}_1^* \times_S \widetilde{\mathcal{X}}_1^*) \\
&= [(\mathcal{X}_0 \times_S \mathcal{X}_0)_{\mathscr{M}} \oplus (\mathcal{X}_1^* \times_S \mathcal{X}_1^*)_{\mathscr{M}}] \boxtimes_{\mathbf{U}} (\widetilde{\mathcal{X}}_1^* \times_S \widetilde{\mathcal{X}}_1^*) \qquad [9.144] \\
&= (\mathcal{X}_0 \boxtimes \mathcal{X}_1^* \times \mathcal{X}_0 \boxtimes \mathcal{X}_1^*)_{\mathscr{M}} \boxtimes_{\mathbf{U}} (\widetilde{\mathcal{X}}_1^* \times_S \widetilde{\mathcal{X}}_1^*) \\
&\cong_{\mathrm{P}(\mathbf{U})} (\mathcal{X}_1 \times_{\mathscr{M}} \mathcal{X}_1) \boxtimes_{\mathbf{U}} (\widetilde{\mathcal{X}}_1^* \times_{\widetilde{\mathscr{M}}} \widetilde{\mathcal{X}}_1^*)
\end{aligned}$$

where we preserved the parity.An observation is our universe is evolving and decaying at same time,and the U-duality is a duality of flipping them

$$\mathbf{U} : (\mathcal{X}_1 \oplus \mathcal{X}_1) \vee (\widetilde{\mathcal{X}}_1^* \oplus \widetilde{\mathcal{X}}_1^*) \to (\widetilde{\mathcal{X}}_1^* \oplus \widetilde{\mathcal{X}}_1^*) \vee (\mathcal{X}_1 \oplus \mathcal{X}_1) \qquad [9.145]$$

Guided by [9.145],the general generalized super site [9.124] should be

$$\mathrm{ETSch}_{\text{eff}}^{\text{SupGen}}(\mathscr{M})_{\Lambda}^{\text{rep}} \vee \mathrm{ETSch}_{\text{eff}}^{\text{SupGen}}(\mathscr{M})_{-\Lambda} \qquad [9.146]$$

with changing to vee meaning that these two copies are overlapping with each other.Also,they are characterized by cosmological constant see [8.40] because the identity is unique.In this case, $\widetilde{\mathscr{M}} = \mathscr{M}^{\text{rep}}$.Then the **U**-fusion of [8.38] is

$$(\mathrm{P}(\mathbf{U})\mathscr{R})^{\text{unself}\,\mathbf{U}} = (\mathcal{X}_1 \times_{\mathscr{M}} \mathcal{X}_1) \boxtimes_{\mathbf{U}} (\widetilde{\mathcal{X}}_1^* \times_{\widetilde{\mathscr{M}}} \widetilde{\mathcal{X}}_1^*) = \mathscr{R}_{\mathscr{M}} \boxtimes_{\mathbf{U}} \widetilde{\mathscr{R}}_{\mathscr{M}^{\text{rep}}} \qquad [9.147]$$

and end with generalized super raltive 2-properties.The number counting is subtile that is counting for zero

$$\#(\mathscr{R}_{\mathscr{M}} \boxtimes_{\mathbf{U}} \widetilde{\mathscr{R}}_{\mathscr{M}^{\mathrm{rep}}}) = \sum\{\#\mathscr{R}^{++}_{\mathscr{M}} + \#\mathscr{R}^{--}_{\mathscr{M}}\} + \sum\{\#\widetilde{\mathscr{R}}\} = \Lambda - \Lambda = 0 \quad [9.148]$$

each generalized super relative 2-property counting zero which means it is independent of choice of $\mathscr{M}$ in different sites.which gives the uniqueness of number counting.Based on $\mathbf{U}$-fusion,we can consider a further site which is general

$$\mathrm{ETSch}^{\mathrm{SupGen,cons.}}_{\mathrm{eff}}(\mathscr{M}^{\mathrm{rep}}_{\Lambda} \vee \mathscr{M}_{-\Lambda}) \quad [9.149]$$

The 0 in [9.148] can be seen from the global zero section of sheaf of (co)homology on $\boldsymbol{\mathcal{M}}$ in [13.10] fused from [9.17].

10 Modern super algebraic geometry II

10.1 Invariants and quasi-coherent sheaves on $(\mathbf{Sch}/X)^{\mathbf{cons.}}$

For a groupoid in schemes $s, t : G \rightrightarrows X_0 \xrightarrow{f} T$ with f is a morphism of algebraic spaces is called invariant if $f \circ s = f \circ t$.A theorem 6.2.2 in [12] is

$$\begin{tikzcd}
 & & s(t^{-1}(x)) \in U \arrow[dl] & \\
X_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[urr, bend left, dotted] & X_0 \arrow[r, "f"] \arrow[rr, bend right, "f'"'] \arrow[l, bend left, dotted] & Y & Z \arrow[l, "!"]
\end{tikzcd} \quad [10.1]$$

where f, f' are invariant morphisms to affine schemes with s, t are finite and flat and U is an affine open subset.Now,we want to study topological properties of algebraic spaces.For a quasi-separated algebraic space X over S,we have an étale presentation $U \to X$ with U a scheme.See around [9.90] if we set g : $\mathrm{Spec}(K) \to X$,it factors through U,because we can choose K to be a finite separable field extension over the underlying ring of étale cover,denote as g_U.If g is epimorphism then $\mathrm{im}(g_U) \to X$ is also epimorphism.We can assume U is quasi-compact because $\mathrm{im}(g_U) \subset V$ which is a quasi-compact connected open subset of U.Repeat [9.116] we have

$$\begin{tikzcd}
U \times_X U \arrow[r] \arrow[d, "\Delta^*_{X/S}(g \times_S g)"] & U \times_S U \arrow[d, "g \times_S g"] \\
X \arrow[r, "\Delta_{X/S}"] & X \times_S X
\end{tikzcd} \quad [10.2]$$

where $\Delta^*_{X/S}(g \times_S g) = g \coprod g$.So $U \times_X U$ is quasi-compact because the diagonal morphism is quasi-compact see below [9.137].Also,we can pullback

$$\begin{array}{ccc} \mathrm{Spec}(K) \times_X U & \longrightarrow & U \times_X U \\ \big\downarrow{\scriptstyle f^*\mathrm{id} \times_X \mathrm{id}} & & \big\downarrow{\scriptstyle \mathrm{id} \times_X \mathrm{id}} \\ \mathrm{Spec}(K) & \xrightarrow{\quad f \quad} & U \end{array} \tag{10.3}$$

where $f^*\mathrm{id} \times_X \mathrm{id} = f \times_X \mathrm{id}$.Now,$\mathrm{Spec}(K) \times_X U$ is quasi-compact and étale over $\mathrm{Spec}(K)$,thus,it can be a finite disjoint union of spectra of field extensions of K.The epimorphism $\mathrm{pr}_2 : \mathrm{Spec}(K) \times_X U \to U$ gives U also a similar structure of disjoint union.But connected means glued affine schemes which means glued structure sheaves,so U is a spectrum of a field.Thus,we can replace U with $\mathrm{Spec}(K)$ being the étale presentation of X.And the $\mathrm{Spec}(K) \times_X \mathrm{Spec}(K)$ is quasi-compact and étale over $\mathrm{Spec}(K)$ which is also a disjoint union.Corollary 6.2.14 in [12] tells us if there exists a finite flat surjection $Y \to X$ of constant rank with Y is an affine schemes,then X is also an affine scheme.Fitting in this case,X is affine and by $\mathrm{Spec}(K) \times_X \mathrm{Spec}(K) \rightrightarrows \mathrm{Spec}(K) \to X \cong \mathrm{Spec}(L)$

$$\mathrm{Spec}(K) \xrightarrow{\mathrm{et}} X \cong \mathrm{Spec}(\mathrm{Eq}(K \rightrightarrows \Gamma(R, \mathscr{O}_R))) = \mathrm{Spec}(K)/R \tag{10.4}$$

where $R = \mathrm{Spec}(K) \times_X \mathrm{Spec}(K)$ and we used [9.130]

$$\begin{array}{ccccc} & & \mathrm{Spec}(K) & \longrightarrow & X = \mathrm{Spec}(K)/R \\ & & \big\downarrow & & \big\downarrow \\ R & \hookrightarrow & \mathrm{Spec}(K) \times_S \mathrm{Spec}(K) & \longleftarrow & \mathrm{Spec}(K) \end{array} \tag{10.5}$$

Notice that X is quasi-separated for $R = \coprod \mathrm{Spec}(K(x)) \cong \mathrm{Spec}(\bigotimes_L K(x))$,as we have global descent over fppf coverings that is finitely presented morphisms [9.135].By definition below [9.137] $\Delta_{X/S}$ is quasi-compact and we have

$$\begin{array}{ccc} R & \longrightarrow & \mathrm{Spec}(K) \times_S \mathrm{Spec}(K) \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{\Delta_{X/S}} & X \times_S X \end{array} \tag{10.6}$$

which means R is quasi-compact,so we can have effective decent to glue the disjoint union to an affine scheme

$$\mathrm{Aff}(\{\mathrm{Spec}(K(x_i)) \to R\}_{x_i \in R}) \cong \mathrm{Aff}(R) \tag{10.7}$$

with trivial projection here $p = \mathrm{id}$.Also [10.4] gives us k-points of algebraic space X similarly to schemes,then we can define topological space

$$|X| = \{l_i : \mathrm{Spec}(k_i) \to X\}_{i \in I}/\sim, \quad l_1 \sim l_2 : \mathrm{Spec}(k_1) \cong_{/X} \mathrm{Spec}(k_2) \tag{10.8}$$

The closed set is $|Y| \subset |X|$ from a closed subspace $Y \subset X$.And $\coprod_i \mathrm{Spec}(k_i)$ is a scheme by [10.7] if X is quasi-separated over S.For such an algebraic space X/S,let $Y \to X$ be an étale presentation with Y a scheme,so $\mathrm{Spec}(k) \times_{l,X} Y \to \mathrm{Spec}(k)$ is étale so $Z = \mathrm{Spec}(k) \times_{l,X} Y = \coprod_{z \in Z} \mathrm{Spec}(k(z))$ which is a glued scheme with each a spectrum of a residue field of a point $z \in Z$.Let $h = \mathrm{pr}_2 : \mathrm{Spec}(k(z)) \mapsto \mathrm{Spec}(k(h(z))) \subset Y$ and $T = \coprod_{h(z) \in Y} \mathrm{Spec}(k(h(z)))$.Now we have

$$\begin{tikzcd} & T \subset Y \arrow[r] \arrow[d] & T/\overline{R} \\ \overline{R} \arrow[r, hook] & T \times_X T & T \subset Y \arrow[l] \arrow[u] \end{tikzcd} \qquad [10.9]$$

If we let $Z \times_{\mathrm{Spec}(k(x))} Z \to \overline{R} \subset h(Z \times_X Z) \subset T \times_X T \subset Y \times_X Y$ with $\overline{R}$ being the scheme-theoretic image and by [10.9] $\overline{R} = T \times_{T/\overline{R}} T$,we have

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r, "g"] \arrow[rr, bend left, "l"] & T/\overline{R} \arrow[r, "l'"] & X \end{tikzcd} \qquad [10.10]$$

Also,$\overline{R}$ is pullback of $Y \times_X Y$ along $T/\overline{R} \to Y$,so $T/\overline{R} \to X$ is monomorphism see below [9.138].Also g in [10.10] is epimorphism,by [10.4],$T/\overline{R}$ is a spectrum of a field.Then a point l factors through a point l' with $T/\overline{R} = \mathrm{Spec}(k')$ for X is quasi-separated.Actually,[10.10] gives a categorical structure of $|\mathrm{AS}/S|$

$$|f| : |X| \to |Y|, \quad (\mathrm{Spec}(k) \to X) \mapsto (\mathrm{Spec}(k) \to Y) \qquad [10.11]$$

which also gives functorial strucutre of $|\,| : \mathrm{AS}/S \to |\mathrm{AS}/S|$ and we have

$$\begin{tikzcd} {|f^{-1}(Z)|} & {|Z|} \arrow[l, "|f|^{-1}"'] \\ f^{-1}Z \arrow[u, "|\,|"] & Z \arrow[l, "f^{-1}"] \arrow[u, "|\,|"] \end{tikzcd} \qquad [10.12]$$

for a closed subspace $Z \subset Y$,making $|f|$ become a continuous map of topological spaces.Conversely,a morphism f has a property if morphism of underlying topological space $|f|$ has a property.We have seen that for quasi-separated algebraic spaces,the source $s(|X|)$ is a glued scheme below [10.8] and so it is consistent with discussion below [9.114] that is the a property of morphism of algebraic spaces is from that of projective morphism of schemes.

For an quasi-separated algebraic space X/S with S an affine scheme,for an étale cover $U \to X$,it induces a relative property $U \times_X U \hookrightarrow U \times_S U$ see [9.130].

$$\begin{tikzcd} R_W = R \times_{U \times_S U} (W \times_S W) \arrow[r, hook] \arrow[d] & W \times_S W \arrow[d] \\ R = U \times_X U \arrow[r, hook] & U \times_S U \end{tikzcd} \qquad [10.13]$$

By a base change along $W \subset U$ and we get an open embedding $W/R_W \to X$ follows from $X = U/R$.Also,X is quasi-separated,we can let U be quasi-compact.Now,for a groupoid in scheme $s,t : R \rightrightarrows U$ with $W^{(n)} \subset U$ is the largest open subset over t with rank n.We have

$$\begin{array}{ccc} R \times_{s,U,t} R & \xrightarrow{m} & R \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle t} \\ R & \xrightarrow{t} & U \end{array} \qquad \begin{array}{ccc} R \times_{s,U,t} R & \xrightarrow{\mathrm{pr}_2} & R \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle t} \\ R & \xrightarrow{s} & U \end{array} \tag{10.14}$$

with $m : R \times_{s,U,t} R \to R$ see [9.32],so we can see $t^{-1}(W^{(n)}) = s^{-1}(W^{(n)}) = R_W^{(n)}$ which is the largest open subset over pr_1,that reduces the groupoid to a subgroupoid $R_W^{(n)} \rightrightarrows W^{(n)}$,an invariant map $W^{(n)} \to X_W^{(n)} \subset X$ follows from that for each n and along these invariant maps,the subgroupoids descent to $X^{(n)} \subset X$ that gives us an invariant map f

$$R \underset{t}{\overset{s}{\rightrightarrows}} U \xrightarrow{f} X^{(n)} = \bigcup_W X_W^{(n)} \xhookrightarrow{i} X \tag{10.15}$$

and union of $X_W^{(n)}$'s with each the largest is a dense open subset in X.By theorem [10.1],$X^{(n)}$ is an affine scheme with i a dense open embedding.Globally,for S is a scheme,$X^{(n)}$ is a scheme.

Definition10.1 A Et $\coprod$ Rep is a general Cov with Rep a collections of coverings of representable sheaves admitting preservations of relative 2-properties from Et which is a big étale topology of schemes.

$$\{Y_i \to Y\}_{i\in I} \in \mathrm{Et} \coprod \mathrm{Rep} \Leftrightarrow Y_i, Y \in \mathrm{AS}/X, \coprod_{i\in I} Y_i \to_{\mathrm{surj.}} Y \tag{10.16}$$

where we put the topology on Sch/X and becomes $(\mathrm{Sch}/X)^{\mathrm{cons.}}$.A fact of [9.79] is for a full subcategory $\mathrm{Et}'(X) \hookrightarrow \mathrm{Et}(X)$,like what we do for a sheaf,that induces a morphism of topoi $X_{\mathrm{et}'} \overset{\simeq}{\longleftarrow} X_{\mathrm{et}}$,sometimes we are easier to define sheaves on $\mathrm{Et}'(X)$,in that case it is X_{Zar} and is easy to define structure sheaves.Now,we want to let $\mathrm{Et}'(X) = \mathrm{Et}(X)|_{Y\to X} \cong \mathrm{Et}(Y) \hookrightarrow \mathrm{Et}(X)$ which gives us equivalence of topoi $X_{\mathrm{et}} \cong Y_{\mathrm{et}}$.Now for an algebraic space X we have $U \times_X U \rightrightarrows U \to X$ with an étale presentation and associated relative property,also we have $\mathrm{pr}_{12}, \mathrm{pr}_{23}, \mathrm{pr}_{13} : U \times_X \times_X U \to U \times_X U$.Recall the descent theory [9.52] and [9.54],we can define category $(R \rightrightarrows U)_{\mathrm{et}}$ with objects are pairs (F_U, ϵ) with F_U an étale sheaf on U and $\epsilon : s^*F_U \overset{\simeq}{\longrightarrow} t^*F_U$ and $\mathrm{pr}_{23}^*\epsilon \circ \mathrm{pr}_{12}^*\epsilon = \mathrm{pr}_{13}^*\epsilon$ given by

$$\begin{array}{ccc} (U \times_X U) \times_{t,U,s} (U \times_X U) \times_{s,U.t} (U \times_X U) & \overset{m\times\mathrm{id}}{\longrightarrow} & (U \times_X U) \times_{s,U,t} (U \times_X U) \\ \downarrow{\scriptstyle \mathrm{id}\times m} & & \downarrow{\scriptstyle m} \\ (U \times_X U) \times_{s,U,t} (U \times_X U) & \xrightarrow{m} & U \times_X U \end{array} \tag{10.17}$$

which is the associativity [9.33] of groupoid axioms.By [9.10],we have

$$(R \rightrightarrows U)_{\mathrm{et}} \cong U_{\mathrm{et}} \cong X_{\mathrm{et}} \tag{10.18}$$

Now,if these are trivial topoi,[10.18] give us a global descent theory over étale coverings to glue the relative properties generated by [9.130].Now,we can discuss,category of $\mathscr{O}_X$-modules in X_{et},that is equivalent to category of pairs $(M_U, \epsilon) \in (R \rightrightarrows U)_{\text{et}}$,with M_U is a sheaf $\mathscr{O}_U$-module on U_{et}.Also,a sheaf of modules $\mathscr{O}_X$-module M is quasi-coherent if there M_U is quasi-coherent sheaf on U for an étale presentation $U \to X$ of an algebraic space X.And if X is locally noetherian from scheme level [9.108],then M is called coherent sheaf if M_U is coherent.And if X is a scheme by [9.79],we get the usual notions.If M_U is a quasi-coherent sheaf we get get another one by

$$\begin{tikzcd} V \times_X U \arrow[r, "p_2"] \arrow[d, "p_1"] & U \arrow[d] \\ V \arrow[r] & X \end{tikzcd} \qquad M_V \cong p_{1*}p_2^* M_U \in \mathrm{Qcoh}(V_{\text{et}}) \qquad [10.19]$$

where we used [9.74] and [9.83].And this gives us M is quasi-coherent (coherent) on X (locally noetherian) if and only if for every étale morphism $V \to X$,M_V is a quasi-coherent (coherent) sheaf on V.

10.2 Algebraic stacks and the M-theory $\boldsymbol{\mathcal{M}}$

A morphism of stacks $f : \mathscr{X} \to \mathscr{Y}$ is representable by algebraic spaces if for every scheme U and morphism $y : U \to \mathscr{Y}$,the fiber product

$$\begin{tikzcd} \mathscr{X} \times_{\mathscr{Y},y} U \arrow[r] \arrow[d] & U \arrow[d] \\ \mathscr{X} \arrow[r] & \mathscr{Y} \end{tikzcd} \qquad [10.20]$$

is an algebraic space.A lemma 8.1.3 in [12] tells us if f above is already representable,then every algebraic space V,the fiber product $V \times_{\mathscr{Y}} \mathscr{X}$ is an algebraic space.

A stack $\mathscr{X}/S$ is an algebraic stack if it satisfies

(i) The diagonal is representable, $\quad \Delta_{\mathscr{X}/S} : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$

(ii) A smooth presentation that is a smooth surjection [10.21]
with an algebraic space X, $\quad X \to \mathscr{X}$

Definition10.2 The M-theory $\boldsymbol{\mathcal{M}}$ is a pre M-theory satisfies

(i) It admits a reverse **U**-fusion called **U**-breaking

$$\mathrm{P}(\mathbf{U})^{-1} : \boldsymbol{\mathcal{M}} \to \boldsymbol{\mathcal{M}}^{\text{pre.rep.}}_{\text{cons.}} \vee \boldsymbol{\mathcal{M}}^{\text{pre.}}_{\text{cons.}}$$

which is representable by generalized super relative 2-properties [10.22]

(ii) A smooth presentation from general M-brane in [9.149]

$$\mathscr{M}^{\text{rep}}_{\Lambda} \vee \mathscr{M}_{-\Lambda} \to \boldsymbol{\mathcal{M}}$$

where $\mathscr{M}_{\Lambda}^{\text{rep}}, \mathscr{M}_{-\Lambda}$ are M-branes which are $\mathbf{U}$-dual to each other,that are also generalized super algebraic spaces in the generalized super site [9.149].

Theorem10.3 The M-theory $\boldsymbol{\mathcal{M}}$ is an algebraic stack in super algebraic generalized geometry,called generalized super algebraic geometric stack.

Proof. Indeed,combining the definition

$$\mathscr{M}_{\Lambda}^{\text{rep}} \vee \mathscr{M}_{-\Lambda} \to \boldsymbol{\mathcal{M}} \to \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}} \vee \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.}} \tag{10.23}$$

which gives us grading by U-duality,with each pre M-theory

$$(\mathscr{M}_{\Lambda}^{\text{rep}} \to \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}}) \vee (\mathscr{M}_{-\Lambda} \to \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.}}) \tag{10.24}$$

and each pre M-theory has a smooth presentation.Also from (i) in [10.22]

$$\begin{tikzcd}
\boldsymbol{\mathcal{M}} \times_{\dots} (\widetilde{\mathcal{X}} \vee \mathcal{X}) \arrow[r] \arrow[d] & \widetilde{\mathcal{X}} \vee \mathcal{X} \arrow[r] \arrow[d] & (\widetilde{\mathcal{X}}_0 \oplus \widetilde{\mathcal{X}}_1^*) \vee (\mathcal{X}_0 \oplus \mathcal{X}_1^*) \arrow[d] \\
\boldsymbol{\mathcal{M}} \arrow[r, "\mathrm{P}(\mathbf{U})^{-1}"] & \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}} \vee \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.}} \arrow[r, "\mathrm{P}(\mathbf{T}^{\delta})^{-1}"] & \mathcal{S}_{\text{cons.}}^{\text{rep}} \vee \mathcal{S}_{\text{cons.}}
\end{tikzcd} \tag{10.25}$$

where $\widetilde{\mathcal{X}} \vee \mathcal{X}$ denoted as generalized super relative 2-properties [9.144],and $\mathcal{S}_{\text{cons.}}$ is the consistent string-Space [9.92] over the general site [9.149] and $\mathrm{P}(\mathbf{T}^{\delta})^{-1}$ means at least the super $\mathbf{T}$-breaking [9.45].Also,the fiber product has grading by $\mathbf{U}$-duality,with each an generalized super algebraic space

$$\widetilde{\mathcal{X}} \cong \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}} \times_{\Delta_{\mathbf{T}}^{\text{rep}}, \mathcal{S}_{\text{cons.}}^{\text{rep}}} (\widetilde{\mathcal{X}}_0 \oplus \widetilde{\mathcal{X}}_1^*) \tag{10.26}$$

where $\widetilde{\mathcal{X}}$ is an algebraic space in generalized super site and $\widetilde{\mathcal{X}}_0 \oplus \widetilde{\mathcal{X}}_1^*$ is a generalized super scheme above [8.28].Similarly for the U-dual part,we have

$$\mathcal{X} \cong \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.}} \times_{\Delta_{\mathbf{T}}, \Delta \mathcal{S}_{\text{cons.}}} (\mathcal{X}_0 \oplus \mathcal{X}_1^*) \tag{10.27}$$

So the settings of the M-theory make each pre M-theory an generalized super algebraic stack.With the $\mathbf{U}$-breaking of diagonal morphism of the M-theory

$$\Delta_{\mathbf{U}} = \mathrm{P}(\mathbf{U})^{-1} = \Delta_{\mathbf{T}}^{\text{rep}} \boxtimes_{\mathbf{U}} \Delta_{\mathbf{T}} = \mathrm{P}(\mathbf{T})_{\text{rep}}^{-1} \vee \mathrm{P}(\mathbf{T})^{-1} \tag{10.28}$$

Thus,we find each of the two copies in [10.24] is an algebraic stack.Then

$$\boldsymbol{\mathcal{M}} \times_{\boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}} \vee \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.}}} (\widetilde{\mathcal{X}} \vee \mathcal{X}) \cong (\boldsymbol{\mathcal{M}} \times_{\boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}}} \widetilde{\mathcal{X}}) \vee (\boldsymbol{\mathcal{M}} \times_{\boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.}}} \mathcal{X}) \tag{10.29}$$

where we used [10.25] and the fiber product is equivalent to a generalized super algebraic space follows from [10.30].Thus,$\mathrm{P}(\mathbf{U})^{-1}$ is representable by generalized super algebraic spaces which satisfying the (i) in [10.21]. □

Similarly to algebraic space [9.116],for any scheme $U \to \mathscr{X}$ to an algebraic stack is representable by algebraic spaces

$$\begin{tikzcd}
U \times_{\mathscr{X}} T \arrow[r] \arrow[d, "\Delta^*_{\mathscr{X}/S}(u \times t)"] & U \times_S T \arrow[d, "u \times t"] \\
\mathscr{X} \arrow[r, "\Delta_{\mathscr{X}/S}"] & \mathscr{X} \times_S \mathscr{X}
\end{tikzcd} \tag{10.30}$$

where T is a scheme,which makes the smooth presentation make sense.Further than stack in [9.99],now the sheaf is an algebraic space by

$$\begin{array}{ccc} \underline{\mathrm{Isom}}(x_1,x_2) & \longrightarrow & X\times_S X \\ \Big\downarrow{\scriptstyle \Delta^*_{\mathscr{X}/S}(x_1\times_S x_2)} & & \Big\downarrow{\scriptstyle x_1\times_S x_2} \\ \mathscr{X} & \xrightarrow{\Delta_{\mathscr{X}/S}} & \mathscr{X}\times_S\mathscr{X} \end{array} \qquad [10.31]$$

where the $X\to\mathscr{X}$ is an smooth presentation and the Isom sheaf [9.49] is on $(\mathrm{Sch}/X)^{\mathrm{op}}$ now.Similarly to [10.19],we can get another smooth presentation by an étale covering $X'\to X$ and an exercise 5.G in [12] tells us if $\underline{\mathrm{Isom}}(x_1,x_2)$ is an alegbraic space over X if and only if for an étale cover the pullback of the sheaf is an algebraic space,which means

$$\begin{array}{ccccc} f^*\underline{\mathrm{Isom}}(x_1,x_2) & \longrightarrow & \underline{\mathrm{Isom}}(x_1,x_2) & & \\ \Big\downarrow & & \Big\downarrow & & \\ X' & \xrightarrow[\mathrm{et}]{f,\mathrm{et}} & X & \xrightarrow{\mathrm{et}} & \mathscr{X} \end{array} \qquad [10.32]$$

where $f^*\underline{\mathrm{Isom}}(x_1,x_2)=\underline{\mathrm{Isom}}(f^*x_1,f^*x_2)$,which also implies the global descent of sheaves [9.10] but now these sheaves are algebraic space.Thus,the diagonal $\Delta_{\mathscr{X}/S}$ is representable if and only if for every smooth presentation $X\to\mathscr{X}$ with X an algebraic space,the corresponding Isom sheaf is an algebraic space.Also,if we use algebraic spaces in [10.30],the fiber product is algebraic space.

Now,we can define a stack $[X/G]$ with X an algebraic space and G a smooth group scheme,which has objects that are triples $(T,\mathscr{T},\pi)$ with T a scheme and $\mathscr{T}$ is a G_T-torsor above [9.85] which is an sheaf on the big étale site.We have

$$\begin{array}{ccccc} G\times_S T & \longrightarrow & T & \longrightarrow & X\times_S T \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ G & \longrightarrow & S & \longleftarrow & X \end{array} \qquad [10.33]$$

over this base,we have an action by $G_T=G\times_S T$

$$\begin{array}{ccc} G_T\times_T(X\times_S T) & \longrightarrow & X\times_S T \\ \Big\downarrow & & \Big\downarrow \\ G_T & \longrightarrow & T \end{array} \qquad [10.34]$$

which makes $X\times_S T$ a G_T-torsor.Also notice that we work on a general site which means schemes can be representable sheaves.And this define a morphism $\pi:\mathscr{T}\to X\times_S T$ a G_T-equivaraint morphism of sheaves on $(\mathrm{Sch}/T)^{\mathrm{cons.}}$.A morphism of triples is a pair $(f_{/S},f^b_{/S}):(T',\mathscr{T}',\pi')\to(T,\mathscr{T},\pi)$ with S-moprhism

$f_{/S} : T' \to T$ and $f^b_{/S}$ is an isomorphism of $G_{T'}$-torsors,such that

$$\begin{tikzcd} \mathscr{T}' \arrow[rr, "f^b_{/S}", "\simeq"'] \arrow[dr, "\pi'"] & & f^*_{/S}\mathscr{T} \arrow[dl, "f^*_{/S}\pi"] \\ & X \times_S T' & \end{tikzcd} \qquad [10.35]$$

the diagram commutes.The [9.70] gives us a global descent theory and it is a groupoid because of the sense of [9.100],for a global glued sheaf $\mathcal{T}$,we have

$$(\mathcal{T}|_{X\times_S T'})|_{X\times_S T'\cap f^*_{/S}(X\times_S T)} \cong f^*_{/S}((\mathcal{T}|_{X\times_S T})|_{f_{/S*}(X\times_S T')\cap X\times_S T}) \qquad [10.36]$$

Thus,it is a stack from [9.104].Now,we want to define a Isom sheaf

$$I = \underline{\mathrm{Isom}}((\mathscr{T}_1, \pi_1), (\mathscr{T}_2, \pi_2)), \qquad (T' \to T) \to (\mathscr{T}_1|_{T'} \to \mathscr{T}_2|_{T'}) \qquad [10.37]$$

on $(\mathrm{Sch}/T)^{\mathrm{cons.}}$,which is compatible with π.Now,we assume $\mathscr{T}_i$ is globally defined (trivial),and we can perform gauge-fixing $\sigma_i : \mathscr{T}_i \cong G_T$,the equivariant morphism is $\pi_i : G_T \to X\times_S T$,Then,[10.37] becomes $(T' \to T) \to (G_{T'} \to G_{T'})$ induced by right multipilcation m_g by $g \in G(T')$,with $G_{T'} = G_T|_{T'}$,and satisfies

$$\begin{tikzcd} G_{T'} \arrow[rr, "m_g", "\simeq"'] \arrow[dr, "\pi_1"] & & G_{T'} \arrow[dl, "\pi_2"] \\ & X \times_S T & \end{tikzcd} \qquad [10.38]$$

Similarly to [10.31],we have a cartesian diagram

$$\begin{tikzcd} I \cong \underline{\mathrm{Isom}}(\pi_1(e), \pi_2) \arrow[r] \arrow[d, "\Delta^*_{X_T/T}(\pi_1(e)\times_T \pi_2)"] & G_T \times_T G_T \arrow[d, "\pi_1(e)\times_T \pi_2"] \\ X_T \arrow[r, "\Delta_{X_T/T}"] & X_T \times_T X_T \end{tikzcd} \qquad [10.39]$$

where $X_T = X\times_S T$,$\pi_1(e) = \pi_2(\cdot g)$ and $\Delta^*_{X_T/T}(\pi_1(e)\times_T \pi_2) = \pi_1(e)\coprod \pi_2$,also notice that we used the trivialness of the torsors.Because X is an algebraic space $\Delta_{X/T}$ is representable by schemes,$\Delta_{X_T/T}$ is also representable by schemes,which means I is a scheme.Back to [10.33],if we change T to X and [10.34] becomes

$$\begin{tikzcd} G_X \times_X (X\times_S X) \arrow[r] \arrow[d] & X \times_S X \arrow[d] \\ G_X \arrow[r] & X \end{tikzcd} \qquad [10.40]$$

this let us define a map $(\mathscr{T}_X, \rho) : X \to [X/G]$ with G_X-equivariant morphism $\rho : \mathscr{T}_X \to X\times_S X$,which gives us

$$\begin{tikzcd} \mathscr{T} \arrow[r] \arrow[d] & X \arrow[d, "(\mathscr{T}_X,\rho)"] \\ T \arrow[r, "(\mathscr{T},\pi)"] & {[X/G]} \end{tikzcd} \qquad [10.41]$$

$\mathscr{T}$ is the fiber product follows from combining [10.34] and [10.40]

$$\begin{tikzcd} G_T \times_T (X \times_S X) \arrow[r] \arrow[d] & X \times_S X \arrow[d] \\ G_T \times_T (X \times_S T) \arrow[r] \arrow[d] & X \times_S T \arrow[d] \\ G_T \arrow[r] & T \end{tikzcd} \quad [10.42]$$

which makes G_X-torsor become a G_T-torsor,and denote a $G_T \times_S G_X$-torsor as $\mathscr{T}$.Recall,that G is a smooth group scheme over S so G_T is a smooth over T,and $\mathscr{T}$ is trivial and étale over T,which means $X \to [X/G]$ is a smooth surjection.Thus,[X/G] is an algebraic stack.A proposition 4.5.6 in [12] tells us

$$(\text{Principle } G\text{-bundles on } X) \cong (\boldsymbol{\mu}\text{-torsors on X}) \quad [10.43]$$

when G is affine and we can apply it to [10.41],we get

$$\left(\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d] & X \\ T & \end{tikzcd} \right) \to_{\text{retract}} [X/G](T) \quad [10.44]$$

where P is a principle G-bundle on T.Also,recall that we mentioned a stack below [9.30],now we regard a relative property as G over S,then the corresponding classifying stack of G is $BG = [S/G]$,where we put a point $T \to S$ into a point $[S/G](T)$,and by using [10.44] and [9.87],an element corresponds to an equivalence class of groups in the sections of h_G.Then,we want to discuss

$$\begin{tikzcd} \mathscr{W} = \mathscr{Y} \times_{\mathscr{Z}} \mathscr{X} \arrow[r] \arrow[d] & \mathscr{X} \arrow[d, "c"] \\ \mathscr{Y} \arrow[r, "d"] & \mathscr{Z} \end{tikzcd} \quad [10.45]$$

which is the fibered product of algebraic stacks,it is a stack because this is fibered product of sheaves see above [9.102].An element of a point $\mathscr{W}$ can be expressed as $A = (x, y, \sigma)$ with $x \in \mathscr{X}, y \in \mathscr{Y}$ and $\sigma : c(x) \xrightarrow{\simeq} d(y)$,we can have

$$\begin{tikzcd} \underline{\text{Isom}}(A, A') \arrow[r] \arrow[d, "\Delta^*_{\mathscr{X}/S}(A \times_S A')"] & T \times_S T \arrow[d, "A \times_S A'"] \\ \mathscr{W} \arrow[r, "\Delta_{\mathscr{X}/S}"] & \mathscr{W} \times_S \mathscr{W} \end{tikzcd} \quad [10.46]$$

Because the triples and $A \cong A'$,we can decompose it to

$$\begin{tikzcd} \underline{\text{Isom}}(x, x') \arrow[r] \arrow[d] & T \times_S T \arrow[d, "x \times_S x'"] \\ \mathscr{X} \arrow[r, "\Delta_{\mathscr{X}/S}"] & \mathscr{X} \times_S \mathscr{X} \end{tikzcd} \quad \begin{tikzcd} \underline{\text{Isom}}(y, y') \arrow[r] \arrow[d] & T \times_S T \arrow[d, "y \times_S y'"] \\ \mathscr{Y} \arrow[r, "\Delta_{\mathscr{Y}/S}"] & \mathscr{Y} \times_S \mathscr{Y} \end{tikzcd} \quad [10.47]$$

and a fiber product connecting them

$$\begin{tikzcd} c(x) \arrow[r, "\simeq"] \arrow[d, "\simeq"] \arrow[dr, Rightarrow] & c(x') \arrow[d, "\simeq"] \\ d(y) \arrow[r, "\simeq"] & d(y') \end{tikzcd} \qquad \begin{tikzcd} \underline{\mathrm{Isom}}(c(x), d(y')) \arrow[r] \arrow[d] & T \times_S T \arrow[d, "c(x) \times_S d(y')"] \\ \mathscr{X} \arrow[r, "\Delta_{\mathscr{X}/S}"] & \mathscr{X} \times_S \mathscr{X} \end{tikzcd} \tag{10.48}$$

Each Isom sheaf is an algebraic space,in [10.47] and [10.45].Equate with [10.46]

$$\begin{tikzcd} \underline{\mathrm{Isom}}((x, y, \sigma), (x', y', \sigma')) \arrow[r] \arrow[d] & \underline{\mathrm{Isom}}(x, x') \times \underline{\mathrm{Isom}}(y, y') \arrow[d, "c \times d"] \\ \underline{\mathrm{Isom}}(c(x), d(y')) \arrow[r, "\Delta"] & \underline{\mathrm{Isom}}(c(x), d(y')) \times \underline{\mathrm{Isom}}(c(x), d(y')) \end{tikzcd} \tag{10.48}$$

where we used the gluing axiom of sheaves

$$\begin{aligned} &\underline{\mathrm{Isom}}(c(x), d(y')) \times \underline{\mathrm{Isom}}(c(x), d(y')) \\ &\quad = \underline{\mathrm{Isom}}(c(x) \cong d(y'), c(x) \cong d(y')) \\ &\quad\quad = \underline{\mathrm{Isom}}(c(x), c(x')) \times \underline{\mathrm{Isom}}(d(y), d(y')) \end{aligned} \tag{10.49}$$

And fiber product in [10.48] is an algebraic space,so diagonal morphism in [10.46] is representable see below [10.32].Now,we want to find a smooth presentation for [10.45].If c is representable,we can form a diagram

$$\begin{tikzcd} Y'' \times_{Y'} (Y' \times_{\mathscr{Y} \times_{\mathscr{Z}} \mathscr{X}} T) \arrow[r, "x"] \arrow[d] & Y' \times_{\mathscr{Y} \times_{\mathscr{Z}} \mathscr{X}} T \arrow[r, "\mathrm{pr}_2"] \arrow[d] & T \arrow[d] & \\ Y'' \arrow[r, "\mathrm{et}"] \arrow[rr, bend right, dotted, red, "f"'] & Y' \arrow[r] \arrow[d] & \mathscr{Y} \times_{\mathscr{Z}} \mathscr{X} \arrow[r] \arrow[d] & \mathscr{X} \arrow[d, "c"] \\ & Y \arrow[r, "y"] & \mathscr{Y} \arrow[r, "d"] & \mathscr{Z} \end{tikzcd} \tag{10.50}$$

with étale surjective $Y'' \to Y'$ and y a smooth presentation of $\mathscr{Y}$.Also

$$Y' \times_{\mathscr{Y} \times_{\mathscr{Z}} \mathscr{X}} T \cong Y \times_{\mathscr{Y}} T \tag{10.51}$$

so pr_2 is smooth.Similarly,f is smooth follows from

$$Y'' \times_{Y'} (Y' \times_{\mathscr{Y} \times_{\mathscr{Z}} \mathscr{X}} T) \cong Y'' \times_{\mathscr{Y} \times_{\mathscr{Z}} \mathscr{X}} T \tag{10.52}$$

because composition of a smooth morphism and a étale morphism is smooth.In this case,f gives it a smooth presentation.Thus,the fiber product of algebraic stacks is still an algebraic stack.For general case,we let z becomes a smooth presentation,by above [10.30],$Z \times_S \mathscr{X}$ is an algebraic space

$$\begin{tikzcd} Z \times_{\mathscr{Z}} \mathscr{X} \arrow[r] \arrow[d, "\Delta^*_{\mathscr{Z}/S}(z \times_S c)"] & Z \times_S \mathscr{X} \arrow[d, "z \times_S c"] \\ \mathscr{Z} \arrow[r, "\Delta_{\mathscr{Z}/S}"] & \mathscr{Z} \times_S \mathscr{Z} \end{tikzcd} \tag{10.53}$$

so the fiber product is an algebraic space which means we have an étale presentation $z' : Z' \to Z \times_{\mathscr{Z}} \mathscr{X}$,similarly to [10.50] we have

$$\begin{tikzcd} \mathscr{W} = \mathscr{Y} \times_{\mathscr{Z}} \mathscr{X} \arrow[r] \arrow[d] & \mathscr{X} \arrow[d, "c"] & Z \times_{\mathscr{Z}} \mathscr{X} \arrow[l] \arrow[d] \\ \mathscr{Y} \arrow[r, "d"] & \mathscr{Z} & Z \arrow[l, "z"'] \end{tikzcd} \qquad [10.54]$$

let the $z'' : Z'' \to Z$ is a smooth surjection and repeat the method in [10.50],we get a smooth presentation $d^* z_*(z' \circ z'') : W \to \mathscr{W}$.

Now,we back to the M-theory [10.22],it **U**-breaks to two algebraic pre M-theories which are U-dual to each other,with smooth presentation [10.24] from M-branes.By using $[X/G]$ above [10.33] we have

$$\boldsymbol{\mathcal{M}}^{\text{pre.rep.}}_{\text{cons.}} = [\mathscr{M}^{\text{rep}}_{\Lambda}/\check{G}] \vee [\mathscr{M}_{-\Lambda}/G] = \boldsymbol{\mathcal{M}}^{\text{pre.}}_{\text{cons.}} \qquad [10.55]$$

with U-duality flipping them

$$\mathbf{U}: \quad [\mathscr{M}^{\text{rep}}_{\Lambda}/\check{G}] \vee [\mathscr{M}_{-\Lambda}/G] \quad \leftrightarrow \quad [\mathscr{M}_{-\Lambda}/G] \vee [\mathscr{M}^{\text{rep}}_{\Lambda}/\check{G}] \qquad [10.56]$$

which naturally gives us Langlands dual group $\check{G}$ for a group scheme G,which makes Langlands duality become a natural result of the M-theory.

10.3 Quasi-coherent sheaves on algebraic stacks

For an algebraic stack $\mathscr{X}/S$,an $\mathscr{X}$-space is a pair (T,t) with T an algebraic space and a morphism $t : T \to \mathscr{X}$.A morphism is $(f, f^b) : (T', t) \to (T, t)$ with $f : T \to T'$ and an isomorphism $f^b : t' \to t \circ f$ and the composition is given by

$$\begin{tikzcd} T'' \arrow[r, "g"] \arrow[dr, "t''"'] & T' \arrow[r, "f"] \arrow[d, "t'"] & T \arrow[dl, "t"] \\ & \mathscr{X} & \end{tikzcd} \qquad [10.57]$$

$(g, g^b) \circ (f, f^b) = (g \circ f, g(f^b))$ with $g(f^b) : t'' \to t \circ f \circ g$.And we denote the category of $\mathscr{X}$-spaces AS/$\mathscr{X}$,also the category of $\mathscr{X}$-schemes $(T,t) \in \text{Sch}/\mathscr{X}$ is a full subcategory.By 2-Yoneda lemma [9.25]

$$\mathscr{X}(T') \cong HOM_{\text{AS}}((\text{AS}/T'), \mathscr{X}) \qquad [10.58]$$

thus,f^b is a 2-isomorphism in the 2-category $\mathscr{X}(T')$.For two morphisms of algebraic stacks $f : \mathscr{Y} \to \mathscr{X}$ and $f' : \mathscr{Y}' \to \mathscr{X}$,an $\mathscr{X}$-morphism is (g, σ) with $g : \mathscr{Y} \to \mathscr{Y}'$ and $\sigma : f \to f' \circ g$ which is a 2-isomorphism.Collection of such morphisms forms a 2-category $HOM_{\mathscr{X}}(\mathscr{Y}, \mathscr{Y}')$,a morphism is is an 2-isomorphism $\lambda : g \to g'$ such that $\lambda \circ \sigma = \sigma'$

$$\begin{tikzcd} \mathscr{Y} \arrow[rr, bend left, "g'"{name=U}] \arrow[rr, "g"'{name=D}] \arrow[dr, "f"] & & \mathscr{Y}' \arrow[dl, "f'"] \\ & \mathscr{X} & \arrow[from=D, to=U, phantom, "\lambda\Uparrow"] \end{tikzcd} \qquad [10.59]$$

If f, f' are representable morphism of algebraic stacks [10.20],we have

$$\begin{tikzcd} \mathscr{Y} \times_{\mathscr{X}} U \arrow[r, "!"] \arrow[d] \arrow[drr, dotted] & \mathscr{Y}' \times_{\mathscr{X}} U \arrow[r] \arrow[d, "t"] \arrow[dr, dotted, "t'"] & U \arrow[d, "h"] \\ \mathscr{Y} \arrow[r] & \mathscr{Y}' \arrow[r, "f'"] & \mathscr{X} \end{tikzcd} \qquad [10.60]$$

follows from [9.19],where U is an algebraic space.And back to [10.57],we have

$$(f, f^b)! : (\mathscr{Y} \times_{\mathscr{X}} U) \to \mathscr{Y}' \times_{\mathscr{X}} U, \qquad \lambda = f'^* h_* f^b! : g = f'^* h_* t \to g' = h_* f \qquad [10.61]$$

which is a unique morphism in $\mathscr{X}$-space.And in this case,we have an unique 2-ismorphism which makes $HOM_{\mathscr{X}}(\mathscr{Y}, \mathscr{Y}')$ be a set.We can therefore define relative space over $\mathscr{X}$,RS/$\mathscr{X}$ with objects are representable morphism of algebraic stacks and a morphism is an isomorphism class of (g, σ).Also,by the representibility we have AS/$\mathscr{X} \hookrightarrow$ RS/$\mathscr{X}$.

In this case,we can define a localized site which is the lisse-étale site

$$(T, t) \in \text{Lis-Ét}(\mathscr{X}) \cong (\text{Ét}(T) \to \mathscr{X}) \subset \text{AS}/\mathscr{X} \qquad [10.62]$$

for an algebraic stack $\mathscr{X}$ and a smooth morphism $t : T \to \mathscr{X}$.A covering is $\{(f_i, f^b) : (T_i, t_i) \to (T, t)\}$ over an étale covering (for preserving smoothness) $\{f_i : T_i \to T\}$.And we denote $\mathscr{X}_{\text{lis-ét}}$ as the topos on the site.Also,we can view this as a smooth presentation $\text{Ét}(T) \to \mathscr{X}$.Similarly to [10.19],a presheaf $F \in \mathscr{X}_{\text{lis-ét}}$ is a sheaf if and only if for every $(T, t), F|_T$ is a sheaf.In this case,we define $\mathscr{O}_{\mathscr{X}}$ sending (T, t) to $\Gamma(T, \mathscr{O}_T)$.Then,we can see clearly about $\mathscr{X}_{\text{lis-ét}}$ as

$$((\{F_{(T,t)}\}, \{\rho_{(f,f^b)}\})), \quad \rho_{(f,f^b)} : f^{-1}F_{(T,t)} \to F_{(T',t')} \qquad [10.63]$$

see [10.57],with f is étale,$\rho_{(f,f^b)}$ is an isomorphism.Also,

$$\mathscr{X}_{\text{lis-ét}} \cong (HOM_{\text{Sh}}(\text{Sh}/F_{(T,t)}), \mathscr{X})) \cong (\mathscr{X}(F_{(T,t)})) \qquad [10.64]$$

with Sh is the category of sheaves [9.65] over a general site.With

$$\text{Sh}/F_{(T,t)} \cong \text{Sh}(T/\mathscr{X}) \cong (\text{AS}/(T/\mathscr{X}))^{\sim} \qquad [10.65]$$

For understanding [10.64],first for any composition below [10.57],we have the following diagram by global descent theory over étale covering for Sh [9.70]

$$\begin{tikzcd} g^{-1}f^{-1}F_{(T,t)} \arrow[r, "g^{-1}\rho_{(f,f^b)}"] \arrow[d, "\simeq"] & g^{-1}F_{(T'.t')} \arrow[d, "\rho_{(g,g^b)}"] \\ (fg)^{-1}F_{(T,t)} \arrow[r, "\rho_{(g\circ f, g(f^b))}"] & F_{(T'',t'')} \end{tikzcd} \qquad [10.66]$$

which makes $\mathscr{X}_{\text{lis-ét}}$ be a fibered category,and similarly to [10.58]

$$\begin{tikzcd} & F_{(T,t)}|T \arrow[dr] \arrow[d, phantom, "\simeq\Uparrow"] & \\ f^{-1}(F_{(T,t)}) \cong F_{(T',t')}|T' \arrow[ur] \arrow[rr] & & \mathscr{X} \end{tikzcd} \qquad [10.67]$$

which make $\mathscr{X}_{\text{lis-ét}}$ be category fibered in groupoid.And the global descent theory of $\mathscr{X}_{\text{lis-ét}}$ is given from $\mathscr{X} \to \text{AS}$,that is

$$\begin{array}{ccc} \mathscr{X}(\{F_{(T',t')} \to F_{(T,t)}\}) & \xrightarrow{\simeq} & \mathscr{X}(F_{(T,t)}) \\ \downarrow & & \downarrow \\ \mathscr{X}(\{T' \to T\}) & \xrightarrow{\simeq} & \mathscr{X}(T) \end{array} \qquad [10.68]$$

We then find that the topos on Lis-Ét$(\mathscr{X})$ (on an algebraic stack) is a stack

$$\mathscr{X}_{\text{lis-ét}} \cong (p : \mathscr{X} \to ((T/\mathscr{X})\tilde{)})) \qquad [10.69]$$

For a sheaf of rings Λ on Lis-Ét$(\mathscr{X})$ (globalize on the $\mathscr{X}$) in $\mathscr{X}_{\text{lis-ét}}$ (localized on every (T,t)),a sheaf Λ-module is cartesian if for every $(f, f^b) : (T', t') \to (T, t)$

$$f^* F_{(T,t)} = f^{-1} F_{(T,t)} \otimes_{f^{-1}\Lambda} \Lambda'_{(T',t')} \cong F_{(T',t')} \qquad [10.70]$$

where for the fibered category (especially $\mathscr{X}$),we use cartesian product because scheme is representable sheaf and we can regard an algebraic stack as a representable sheaf and for concrete elements we use the pushforward as the left evolution.Similarly to [10.19],a sheaf $\mathscr{O}_{\mathscr{X}}$-module is quasi-coherent if F is cartesian and for every (T,t),$F_{(T,t)}$ is quasi-coherent on T.An algebraic stack $\mathscr{X}$ is locally noetherian if an only if it localizes to every (T,t) along the site Lis-Ét$(\mathscr{X})$ with T is locally noetherian.Then,a quasi-coherent sheaf F on locally noetherian 2-scheme $\mathscr{X}$ (globalized along Lis-Ét$(\mathscr{X})$) is coherent if each $F_{(T,t)}$ is coherent.The F on the algebraic stack is understood by descent from a glued $F_{(T,t)}$ by global descent theory of stack.And we denote the category on an algebraic stack (2-scheme) QCoh$(\mathscr{X})$.Then,we want to treat [10.62] in detail.

For a smooth presentation $X \to \mathscr{X}$ with a $(Z, z) \in$ Lis-Ét$(\mathscr{X})$,we have

$$\begin{tikzcd} Z' \arrow[r, "s"] \arrow[dr, "f"'] & Z \times_{\mathscr{X}} X \arrow[r] \arrow[d, "p"] & X \arrow[d, "x"] \\ & Z \arrow[r, "z"] & \mathscr{X} \end{tikzcd} \qquad [10.71]$$

where p is smooth from smooth x,and a étale morphism $f : Z' \to Z$ factors through an étale morphism s.In this case,we have

$$x_* \circ x^*(z) \circ s^* \circ f^* : Z \to X \to \mathscr{X}, \qquad F_{(Z,z)} \cong (x^*(z) \circ s^* \circ f^*)^* F_{(X,x)} \quad [10.72]$$

For a quasi-coherent sheaf F on $\mathscr{X}$,and for every Z we give a index i

$$X/\mathscr{X} = \lim_i Z^i/\mathscr{X}, \quad F_{(X,x)} = F_{\operatorname{colim}_i (Z^i, z^i)} \cong \operatorname{colim}_i F_{(Z^i, z^i)} \qquad [10.73]$$

the $F_{(T',t')}$ is quasi-coherent,then $F_{(X,x)}$ is quasi-coherent,also if $F_{(X,x)}$ is quasi-coherent,then every $F_{(T',t')}$ is quasi-coherent,then we get for a smooth presentation from a scheme X,a cartesian sheaf $\mathscr{O}_{\mathscr{X}}$-module is quasi-coherent (coherent) if and only if $F_{(X,x)}$ is quasi-coherent (coherent) on X.If $\mathscr{X}/S$ is Deligne-Mumford stack which is an algebraic stack with étale presentation,we can consider Ét$(\mathscr{X}) \subset$ Lis-Ét$(\mathscr{X})$ with objects (T,t),where t is étale morphism from algebraic spaces and we write the associate topos $\mathscr{X}_{\text{ét}}$.Similarly to the case of algebraic spaces below [10.16],the inclusion induces a restriction of ringed topoi

$$\text{Ét}(\mathscr{X}) \hookrightarrow \text{Lis-Ét}(\mathscr{X}), \quad (\mathscr{X}_{\text{lis-ét}}, \mathscr{O}_{\mathscr{X}}) \to (\mathscr{X}_{\text{ét}}, \mathscr{O}_{\mathscr{X}_{\text{ét}}}) \qquad [10.74]$$

Similarly to [9.82],for a Delign-Mumford stack $\mathscr{X}$

$$r_* : \text{QCoh}(\mathscr{X}_{\text{lis-ét}}) \xrightarrow{\simeq} \text{QCoh}(\mathscr{X}_{\text{ét}}) \qquad [10.75]$$

For an étale morphism $f : W \to \mathscr{X}$ with W a scheme from [10.71]

$$\begin{tikzcd} T \arrow[d, "\text{smo}"'] \arrow[dr, "\text{smo}", "t"'] & \\ W \arrow[r, "\text{et}"] & \mathscr{X} \end{tikzcd} \qquad [10.76]$$

A restriction $\text{Lis-Ét}(\mathscr{X})|_W \cong (\text{id}, f^*)(\text{Lis-Ét}(\mathscr{X})) = \text{Lis-Ét}(W)$,which means $W_{\text{lis-ét}} \cong \mathscr{X}_{\text{lis-ét}}|_W$,then we have $W_{\text{Zar}} \hookrightarrow \text{Lis-Ét}(W)$,which induces equivalence on topoi see below [10.16].Thus,we can let W be an affine scheme.In this case,a sheaf M in $\mathscr{X}_{\text{ét}}$ is quasi-coherent which means $M|_W$ is quasi-coherent.Then for every smooth $r : T \to W$,$r^*(M|_W) \simeq M|_{r^*W}$ is quasi-coherent sheaf which means M is also a quasi-coherent sheaf in $\mathscr{X}_{\text{lis-ét}}$,so we have $M \cong r_* r^* M$ for M in $\mathscr{X}_{\text{lis-ét}}$,and for quasi-coherent N in $\text{X}_{\text{lis-ét}}$,we know étale morphism is smooth,we have $r^* r_* N_T \cong r^* N_W \cong N_T$ because N is cartesian [10.70].

For instance,let G a finite group scheme over S,we want to describe étale topos of BG.The site for the classifying stack is G-$\text{Ét}(S)$ with objects étale morphisms $T \to S$ with a G-action,from $G \times_S T' \to G \times_S T$,a morphism is a pair $(f, g) : T' \to T, g \in G$ with $\text{pr}_{1*}\text{pr}_2^*(f) \cong g$,which is g-equivariant

$$\begin{tikzcd} G \times_S T' \arrow[rr, "\text{pr}_2^*(f)"] \arrow[dr] & & G \times_S T \arrow[dl] \\ & G & \end{tikzcd} \quad g^{-1}f(gt) = g^{-1}\text{pr}_{2*}\text{pr}_1^* ggt = f(t) \qquad [10.77]$$

A composition is $(f \circ f', gg')$.A collection of morphisms $\{(f_i, g_i) : T' \to T\}_{i \in I}$ is a covering in G-equivariant étale site G-$\text{Ét}(S)$ if $\{T' \to T\}_{i \in I}$ is an étale covering.There is a functor $Y : G\text{-Ét}(S) \xrightarrow{\simeq} \text{Ét}(BG), T/S \to T/BG$ which defined by a trivial G-torsor.Define $\mathscr{O}_{G-S}$ sending every T/S to $\Gamma(T, \mathscr{O}_T)$ and we can discuss $\mathscr{O}_{G-S}$-module.We have a functor $\text{Ét}(S) \to G\text{-Ét}(S), \text{id}_T \mapsto (\text{id}_T, \text{id}_G)$,we know from [10.77] an G-equivariant morphism is equivalent to a group action,thus for a $\mathscr{O}_{G-S}$-module,the inverse functor gives us an $\mathscr{O}_S$-module with morphisms $f : T' \to T$ behaves like a left group action.In this case

$$\text{QCoh}(BG) \cong \text{Qcoh}(S) \,\text{with left}\, G\text{-action} \qquad [10.78]$$

10.4 Ind-coherent sheaves on AlStk

This starts at that F representing the fibered category $p : F \to C$ [9.19] is a category of categories $F(X), X \in C$,which means adding with DG setting F is a

$(\infty,1)$-category.And we base on [17],[18],[19],[20].Another view of quasi-coherent sheaves on algebraic stack is following

$$\begin{tikzcd} & \mathscr{Y} \in \mathrm{AlStk} \arrow[dr] & \\ S \in \mathrm{AS} \arrow[ur, "\mathrm{Kan}", "\mathrm{smooth}"'] \arrow[rr] & & \mathrm{DGCat} \end{tikzcd} \tag{10.79}$$

which is a left Kan extension for AlStk $\to$ DGCat with AlStk category of algebraic stacks.Which gives us

$$\mathrm{QCoh}_{\mathrm{co}}(\mathscr{Y}) = \mathrm{ProQCoh}(\mathscr{Y}) \simeq_{\mathrm{Kan}} \operatorname*{colim}_{S\to\mathscr{Y}} \mathrm{QCoh}(S) \tag{10.80}$$

And a DG category is a category $(C, \bigoplus_n \mathrm{Hom}_n(A_l, B_{l+n}))$ where C is a category with DG objects.The graded hom set is additive with translation functor $d : X \mapsto X[1]$,for $f : A_l \to B_{l+n}$ a moprhism,df_l given by the diagram

$$\begin{tikzcd} A_l \arrow[r, "f_l"] \arrow[d, "d"] & f_l A_l \arrow[d, "d"] \\ A_{l+1} \arrow[r, "f_{l+1}"] & B_{l+n+1} \end{tikzcd} \tag{10.81}$$

for $n \in \mathbb{Z}$,this is a ∞-category with df_l a 2-morphism.In this case

$$\begin{tikzcd} \mathrm{H.}_1(A_l, A_{l+1}) \arrow[rr] & & \mathrm{H.}_1(B_{l+n}, B_{l+n+1}) \\ & \mathrm{Hom}_n(A_l, B_{l+n}) \simeq \mathrm{Hom}_n(A_l, B_{l+n})[1] \arrow[ul, "f"] \arrow[ur] & \end{tikzcd} \tag{10.82}$$

where $\mathrm{Hom}_n(A_l, B_{l+n})[1] = \mathrm{Hom}_n(A_l[1], B_{l+n}[1])$,for every f,we have such triangular.Notice that $df : \mathrm{Hom}_n(-,-) \to \mathrm{Hom}_{n+1}(-,-), \in \mathrm{Hom}_1(-,-)$ and this DG category is triangulated [17].Now,we can view $0 \in \mathbb{Z}$ as an zero object of C,for every $(g : X_n \to Y_m) \in C$ with $f : m/n \to X_n$,we have $f^*g : m/n \to 0$ which means every morphism admits a kernal and cokernal and it is exact if and only if it is coexact,which means DG category is a stable $(\infty,1)$-category which is triangulated by [10.81].Then,we can discuss t-structure on DG category.

Now,let $\mathscr{Y}$ be an algebraic stack and a morphism $\tilde{\mathscr{Y}} \to \mathscr{Y}$ with $(n+1)$-folded algebraic stack $\tilde{\mathscr{Y}}$.Its Čech nerve $\tilde{\mathscr{Y}}^\bullet$ makes $\mathrm{QCoh}_{\mathrm{co}}(\tilde{\mathscr{Y}}^\bullet)$ become a cosimplicial category.we can discuss descent theory of $\mathrm{QCoh}_{\mathrm{co}}$,for an étale covering (at least fppf [9.74])$\tilde{\mathscr{Y}}^\bullet \to \mathscr{Y}$

$$\mathrm{QCoh}_{\mathrm{co}}(\{\tilde{\mathscr{Y}}^\bullet \to \mathscr{Y}\})^{\geq -\infty} \simeq \mathrm{QCoh}_{\mathrm{co}}(\mathscr{Y})^{\geq -\infty} \tag{10.83}$$

Because the colimit commutes with the tensor product let us have a functor $\Omega_{\mathscr{Y}}$

$$\begin{tikzcd} \mathrm{QCoh}_{\mathrm{co}}(\mathscr{Y}_1) \arrow[r, "f_*"] \arrow[d, "\Omega_{\mathscr{Y}}"] & \mathrm{QCoh}_{\mathrm{co}}(\mathscr{Y}_2) \arrow[d, "\Omega_{\mathscr{Y}}"] \\ \mathrm{QCoh}(\mathscr{Y}_1) \arrow[r, "f_*"] & \mathrm{QCoh}(\mathscr{Y}_2) \end{tikzcd} \tag{10.84}$$

which is well defined on the level of fibered category.Let $\mathscr{Y}$ is a quasi-compact algebraic stack with an affine diagonal,for a smooth presentation $X \to \mathscr{Y}$ with X an algebraic space,we have $\mathscr{Y} = [X/G]$ with G is a finite group scheme.If it is eventually connective and almost of finite type,the functor $\Omega_{\mathscr{Y}}$ induces an equivalence of category.By using this theorem,we have

$$\mathrm{QCoh}_{\mathrm{co}}(\mathscr{Y}) \simeq \mathrm{QCoh}(\operatorname*{colim}_{[X/G](S_i)}[X/G](S_i)) = \operatorname*{colim}_{i}\mathrm{QCoh}(\mathscr{Y}_i) \qquad [10.85]$$

Now let $\mathscr{Y}$ is the case in [10.85] the quasi-coherent sheaves are almost finitely presented above [7.30].To see this,the extra condition is preserve of t-structure.In the setting above [10.93],it suffices to show that for ≥ 0 case.For $\mathscr{Y} \simeq \operatorname*{colim}_{i\in I} Y_i$ which let us see Zariski locally,and the transition map is $f_{ij} : Y_i \to Y_j$.Pullback along $Y_i \to \mathscr{Y}$,we have $\tilde{Y}_i^{\bullet} = \tilde{\mathscr{Y}}^{\bullet} \times_{\mathscr{Y}} Y_i$.In this case,$\tilde{\mathscr{Y}}^m \simeq \operatorname*{colim}_{i\in I}\tilde{Y}_i^m$,and $f_{ij}^m : \tilde{Y}_i^m \to \tilde{Y}_j^m$.Attaching with quasi-coherent sheaves,we have from [12.14]

$$\mathrm{QCoh}_{\mathrm{co}}(\mathscr{Y}) \simeq \lim_{i\in I^{\mathrm{op}}}\mathrm{QCoh}(Y_i), \quad \mathrm{QCoh}_{\mathrm{co}}(\tilde{\mathscr{Y}}^m) \simeq \lim_{i\in I^{\mathrm{op}}}\mathrm{QCoh}(\tilde{Y}_i^m) \qquad [10.86]$$

where we rewrite the colimit to limit with a flipping of order.With right adjoint $(f_{ij})_*^R, (f_{ij})_*^R$.For $\phi : [m] \to [n], \in \Delta$,and we have

$$g^{\phi} : \tilde{\mathscr{Y}}^n \to \tilde{\mathscr{Y}}^m, \quad g_i^{\phi} : \tilde{Y}_i^n \to \tilde{Y}_i^m \qquad [10.87]$$

Also,cosideration of transition f_{ij},we have a commutative square,based on this

$$\begin{array}{ccc} \mathrm{QCoh}(\tilde{Y}_i^m) & \xrightarrow{(f_{ij}^m)_*} & \mathrm{QCoh}(\tilde{Y}_j^m) \\ \big\downarrow{\scriptstyle (g_i^{\phi})^*} & & \big\downarrow{\scriptstyle (g_j^{\phi})^*} \\ \mathrm{QCoh}(\tilde{Y}_i^n) & \xrightarrow{(f_{ij}^n)_*} & \mathrm{QCoh}(\tilde{Y}_j^n) \end{array} \qquad [10.88]$$

which is cartesian,which gives us a 2-isomorphism

$$(g_i^{\phi})^* \circ (f_{ij}^m)_*^R \to (f_{ij}^n)_*^R \circ (g_j^{\phi})^* \qquad [10.89]$$

because of the uniqueness of global section from the global descent theory in [9.74],as we work Zariski locally and the fppf cover preserves sheaves property and its quasi-coherence.Thus,we focus on ≥ 0 and get a well-defined functor

$$\Delta \times I \to (\mathrm{QCoh}_{\mathrm{co}}^{\geq 0} \to (\tilde{Y}_{\bullet}^{\bullet})), \quad (m,i) \mapsto \mathrm{QCoh}_{\mathrm{co}}^{\geq 0}(\tilde{Y}_i^m)^{\geq 0} \qquad [10.90]$$

Also,we can rewrite the fibered category

$$\mathrm{QCoh}_{\mathrm{co}}^{\geq 0} \to (\tilde{Y}_{\bullet}^{\bullet}) = \lim_{m\in\Delta}\lim_{i\in I^{\mathrm{op}}}\mathrm{QCoh}(\tilde{Y}_i^m)^{\geq 0} \qquad [10.91]$$

Then,we have a diagram as $\tilde{Y}_i^{\bullet}$ over Y_i above [12.40]

$$\begin{array}{ccc} \mathrm{QCoh}_{\mathrm{co}}^{\geq 0} \to (\tilde{Y}_{\bullet}^{\bullet}) & \xrightarrow{\simeq} & \lim_{m\in\Delta}\lim_{i\in I^{\mathrm{op}}}\mathrm{QCoh}(\tilde{Y}_i^m)^{\geq 0} \\ \big\downarrow & & \big\downarrow \\ \mathrm{QCoh}_{\mathrm{co}}(\mathscr{Y})^{\geq 0} & \xrightarrow{\simeq} & \lim_{i\in I^{\mathrm{op}}}\mathrm{QCoh}(Y_i)^{\geq 0} \end{array} \qquad [10.92]$$

by using the theorem above [10.93],the ins_i is t-exact the vertical arrow preserves t-structure, and combine with [10.68] then we get the descent [10.83]. For $C_i \in$ DGCat,let $C = \operatorname*{colimit}_i C_i$ with $\mathrm{ins}_i : C_i \to C$.Suppose each C_i has a t-structure which gives an orientation and along $C_i^{\geq 0}[-n]$,C_i is closed under filtered colimits.Also,we assume $F_{i,j} : C_i \to C_j$ is t-exact.We let ins_i be right t-exact which means $\mathrm{ins}_i(C_i^{\leq 0}) \subset C^{\leq 0}$.For $i \in I$,the index set I is filtered,then ins_i is t-exact.To verifies it,first a ∞-category Cat is presentable if and only if

$$\begin{aligned} &\text{(i) Cat} \simeq P(C, R), C \text{ is a small } \infty\text{-category,} R \in \mathrm{Map}(\mathrm{PreStk(C)}) \\ &\text{(ii) } P(C,R) \text{ is full subcategory of PreStk(C),generated by } F \\ &\qquad \text{under colimit if } C \text{ with } R \text{ is a set of isomorphisms of } \infty\text{-gpds} \end{aligned} \qquad [10.93]$$

where $F \in$ PreStk and the freely generated under colimit see [7.20] that is a filtered colimit.Now,let the $\mathrm{Groth}^{\mathrm{lex}}$ be the category of presentable stable ∞-categories with right complete t-structure and t-exact colimit preserving functors.For category of presentable stable ∞-categories pr^{L}

$$(C, C^{\leq 0}) \mapsto C^{\leq 0} \quad \hookrightarrow \quad \infty\text{-Cat}^{\mathrm{lax}} \to \mathrm{pr}^{\mathrm{L}} \qquad [10.94]$$

with the (co)limit-preserving functors as the morphisms and the lax means $D(X) \times D(Y) = D(X \times Y)$.The pr^{L} has all small limits,forgetful functors $\infty\mathrm{Cat} \to \mathrm{pr}^{\mathrm{L}}$ and all small colimits $\mathrm{pr}^{\mathrm{L}} \to \infty\text{-Cat}^{\mathrm{op}}$,there exist a H-T duality on it see [14.68],$C \otimes D \simeq \underline{HOM}_{\mathrm{pr}^{\mathrm{L}}}(C, D)$.Over this,we can have $\mathrm{ComAlg}(\mathrm{pr}^{\mathrm{L}}, \otimes)$ and $\mathrm{Mod}_C(\mathrm{pr}^{\mathrm{L}})$ with objects are C-linear presentable ∞-categories.A spectrum is an infinite sequence $\{X_i\}_{i \geq 0}$ of pointed topological spaces with homotopy equivalence $X_i \simeq \Omega X_{i+1}$.A spectrum is a spectrum object of ∞-category S of pointed spaces and we denote the ∞-category of spectra as $\mathrm{Sp}(S) = \mathrm{Stab}(S)$ which is the stabilization.Now,back to [12.18],the functor of presentable ∞-categories admits preserving of filtered colimits.Then we have $C \simeq \mathrm{Stab}(C^{\leq 0})$,and Stab preserves colimits.So $\mathrm{ins}_i : C_i \to C^{\leq 0} \to C$ is t-exact from $\mathrm{ins}_i(C_i^{\geq 0}) \subset C^{\geq 0}$.And for $c \in C$,we have an adjunction

$$c \simeq \operatorname*{colimit}_{i \in I} \mathrm{ins}_i \circ \mathrm{ins}_i^R(c) \qquad [10.95]$$

ins_i^R is the right adjoint of a right t-exact functor,which is left t-exact,which means $\mathrm{ins}_i^R(c) \in C_i^{\geq 0}$ for $c \in C^{\geq 0}$,which means $C^{\geq 0}$ is generated by colimits of essential images of $C_i^{\geq 0}$ along ins_i.A t-structure should be viewed as an orientation structure of an algebraic sequence.

Now,we can apply above to $\mathrm{QCoh}(S) \in$ DGCat has a t-structure.And for affine $\mathscr{Y}$-schemes S we have $\mathrm{QCoh}_{\mathrm{co}}(\mathscr{Y})^{\leq 0}$ is generated by colimits of essential images of $\mathrm{QCoh}(S)^{\leq 0}$,we then get a corollary A.2.7 in [20]

$$\begin{aligned} &\text{(a) The t-strucuture on } \mathrm{QCoh}_{\mathrm{co}}(\mathscr{Y}) \text{ commutes with filtered colimits.} \\ &\text{(b) } \forall S/\mathscr{Y}, \mathrm{ins}_i : \mathrm{QCoh}(S) \to \mathrm{QCoh}_{\mathrm{co}}(\mathscr{Y}) \text{ is t-exact.} \end{aligned} \qquad [10.96]$$

For Vect $=$ QCoh(Spec(k)),we can let ${}^{\leq n}$DGAff denote for the category of $\leq n$-folded affine schemes

$$^{\leq n}\mathrm{DGAff} = (\mathrm{ComAlg}(\mathrm{Vect}^{\geq -n, \leq 0}))^{\mathrm{op}} \tag{10.97}$$

We can see clearly from an affine scheme dgaff over k-module

$$\begin{aligned} \mathrm{dgaff}^{\leq n} &= \mathrm{Spec}(k\text{-mod}) = (\mathrm{Spec}(k\text{-mod}/(x^{\leq n}))) \\ &= \mathrm{Spec}(\mathcal{O}_{k\text{-mod}}(\mathrm{Spec}(k\text{-mod}/(x^{\leq n}))))^{\mathrm{op}} \\ &= \mathrm{Spec}(\mathcal{O}_{k\text{-mod}}(\mathrm{Spec}(k\text{-mod}^{\geq -n, \leq 0})))^{\mathrm{op}} \end{aligned} \tag{10.98}$$

Now,for the category of DGCat,we can glue by Stab below [12.18]

$$\mathrm{DGAff} \simeq \mathrm{Stab}((^{\leq n}\mathrm{DGAff})) = \lim_n (^{\leq n}\mathrm{DGAff}) \tag{10.99}$$

Let ${}^{\leq n}\mathrm{DGAff}_{\mathrm{ft}} \subset {}^{\leq n}\mathrm{DGAff}$ be the full subcategory of n-coconnective affine schemes almost of finite type (quasi-compact and locally of finite type).By filtered colimits see [7.20],we have ${}^{\leq n}\mathrm{DGAff} \simeq \mathrm{Pro}({}^{\leq n}\mathrm{DGAff}_{\mathrm{ft}})$.We also denote

$$\mathrm{DGAff}_{\mathrm{aft}} = \lim_n (^{\leq n}\mathrm{DGAff}_{\mathrm{ft}}) \subset \mathrm{AffSch} \tag{10.100}$$

to be the full subcategory of affine schemes almost of finite type (of finite type after each connective truncation).In this case,we can define

$$\mathrm{AlStk}_{\mathrm{aft}} \simeq \lim_n (^{\leq n}\mathrm{AlStk}_{\mathrm{ft}}),\ {}^{\leq n}\mathrm{PreStk}_{\mathrm{ft}} \simeq \mathrm{Funct}((^{\leq n}\mathrm{DGAff})^{\mathrm{op}}, \infty\text{-Grpd})) \tag{10.101}$$

Back to [10.79],we can have the following left Kan extension

$$\begin{tikzcd} & S \in (^{\leq n}\mathrm{DGAff})^{\mathrm{op}} \arrow[dr, "^{\leq n}\mathrm{IndCoh}^!"] & \\ S_0 \in (^{\leq n}\mathrm{DGAff}_{\mathrm{ft}})^{\mathrm{op}} \arrow[ur, hook, "\mathrm{kan}"] \arrow[rr, "^{\leq n}\mathrm{IndCoh}"] & & \mathrm{DGCat} \end{tikzcd} \tag{10.102}$$

Similarly to [12.2],we have after taking limit in [12.49]

$$\mathrm{IndCoh}^!(S) = \operatorname*{colim}_{S_0 \to S} \mathrm{IndCoh}(S_0) \tag{10.103}$$

Now,from [12.48] we find the underlying ring k-mod$/(x^{\leq n}) \simeq k$-mod$/(x^{\leq n+1})$.In this case,we have a 2-isomorphism ${}^{\leq n}\mathrm{IndCoh}^! \simeq^{\leq n+1} \mathrm{IndCoh}^!$,then by 2-Yoneda lemma [9.25] we have a (∞,2)-groupoid

$$\begin{aligned} \infty\text{-Grpd}(^{\leq n+1}\mathrm{DGAff}) &\simeq HOM(^{\leq n}\mathrm{DGAff}/^{\leq n+1}\mathrm{DGAff}, \infty\text{-Grpd}) \\ \infty\text{-Grpd}((^{<\infty}\mathrm{DGAff})^{\mathrm{op}}) &\simeq HOM(\operatorname*{colim}_n {}^{\leq n}\mathrm{DGAff}, \infty\text{-Grpd}) \end{aligned} \tag{10.104}$$

for $\operatorname*{colim}_n {}^{\leq n}\mathrm{DGAff} = \lim_n (^{\leq n}\mathrm{DGAff})^{\mathrm{op}}$ with [12.51],we have an embedding

$$\mathrm{Kan} : (^{<\infty}\mathrm{DGAff})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk})^{\mathrm{op}} \tag{10.105}$$

which can be used for right Kan extension Kan^* with $\mathrm{colim}^{\mathrm{op}}$

$$\lim_{S\to\mathscr{Y},S\in{}^{<\infty}\mathrm{DGAff}}\mathrm{IndCoh}^!(S)=\mathrm{IndCoh}^!(\mathscr{Y}) \qquad [10.106]$$

The good thing is we shift the $\mathscr{Y}$ to category of $\mathscr{Y}$-spaces AS/$\mathscr{Y}$ see [10.57].

$$p:\mathrm{IndCOH}_{(\infty,1)}\to\mathrm{DGAff}_{\mathrm{ft}}/(\mathrm{AlStk}_{\mathrm{aft}})^{\mathrm{op}},\qquad \mathrm{IndCOH}(\mathscr{Y})\mapsto\mathrm{DGAff}_{\mathrm{ft}}/\mathscr{Y} \qquad [10.107]$$

where we combined with [12.52],which is a fibered category fibered ind-ly in categories of coherent sheaves,quasi-coherent to coherent is from the settings of finite type and ind-completion see above [7.30] and [9.108].Now,let $\mathscr{Y}$ be an algebraic stack with a smooth presentation $S\to\mathscr{Y}$ we have,for an étale morphism $S'\to S$.From the global descent theory [12.39]

$$\mathrm{IndCoh}(S'\times_S\mathscr{Y})\simeq\mathrm{Coh}(S')\times_{\mathrm{Coh}(S)}\mathrm{IndCoh}(\mathscr{Y}) \qquad [10.108]$$

The ind means filtered colimits on $\mathrm{dgAlg}^{\geq 0}$ and filtered limits on $\mathrm{DGAff}^{\geq 0}$,the reason why we use filtration is because the free collection [7.21] and [11.32] to get good space [11.33].In this case,we can define monoidal structure

$$\mathrm{IndCoh}^!(S)\times\mathrm{IndCoh}^!(S)\to\mathrm{IndCoh}^!(S\times S)\to\mathrm{IndCoh}^!(S) \qquad [10.109]$$

which lifting the [12.1] to

$$\begin{tikzcd} & \mathrm{ComAlg}(\mathrm{DGCat})=\mathrm{DGCat}^{\mathrm{SymMon}} \arrow[d] \\ (\mathrm{PreStk})^{\mathrm{op}} \arrow[ur] \arrow[r, "\mathrm{IndCoh}^!"] & \mathrm{DGCat} \end{tikzcd} \qquad [10.110]$$

10.5 Representation of affine Lie algebra over Ran

Sec 4.1 in [18]. For a prestack $\mathscr{Y}$,we have a new prestack $\mathscr{Y}_{\mathrm{dr}}$ by [10.57] for $C=\mathrm{DGAff}_{\mathrm{cl,red.}}$ through étale coverings which is a ∞-groupoid,by [9.25]

$$\mathscr{Y}_{\mathrm{dR}}=\mathscr{Y}(S)\simeq HOM_C(C/S,\mathscr{Y})=\mathrm{DGAff}_{\mathrm{cl,red.}}/\mathscr{Y} \qquad [10.111]$$

where cl and red denote for closed reduces schemes.We have

$$\text{D-mod}(\mathscr{Y})=\mathrm{QCoh}(\mathscr{Y}_{\mathrm{dr}})\simeq\mathrm{QCoh}(\mathrm{DGAff}_{\mathrm{cl,red.}}/\mathscr{Y}(S)) \qquad [10.112]$$

gives us way to define functor of points of the prestack.Also,

$$\mathrm{Ran}(X)=\mathrm{Hom}(\mathrm{DGAff}_{\mathrm{cl,red.}},S)\to X\simeq\mathrm{DGAff}_{\mathrm{cl,red.}}/X \qquad [10.113]$$

where we used [9.7] and it is a prestack.Also,follows from [10.111]

$$\mathrm{Ran}(X)_{\mathrm{dR}}=\mathrm{DGAff}_{\mathrm{cl,red.}}/\mathrm{Ran}(X)\simeq\mathrm{Ran}(X) \qquad [10.114]$$

By the descent of affine morphisms [9.135],we are able to discuss global section $S \times X = \coprod \underline{x}$,with a étale point $\underline{x} : ({}^{\mathrm{cl}}S)_{\mathrm{red}} \to X$,let $\underline{x} \in \Gamma_{\underline{x}}$ be a set of points

$$S \times X \setminus \Gamma_{\underline{x}} \xrightarrow{\mathrm{et}}_{\mathrm{open}} S \times X \qquad [10.115]$$

this is a concrete way to shift Zariski glued schemes to étale glued schemes and gives us a way to see around $\underline{x}$ by formal completion to formal scheme along different topology.Similarly to [9.3],the Ran can be viewed as a sheaf $\mathrm{DGAff}_{\mathrm{cl,red.}}/X \in \widetilde{X_{\mathrm{et}}}$.And by using [9.3],we can easily generate a semi group

$$\coprod : \mathrm{Ran}(X) \times \mathrm{Ran}(X) \to \mathrm{Ran}(X), \quad \underline{x}_1 \times \underline{x}_2 \mapsto \underline{x}_1 \coprod \underline{x}_2 \qquad [10.116]$$

From [9.82] and below [10.16],we have a weak contractibility of Ran space

$$\mathrm{QCoh}(\underline{x}) \xrightarrow[p^*]{\simeq} \mathrm{QCoh}(\mathrm{Ran}(X)) \qquad [10.117]$$

where $p : \mathrm{Ran} \to \underline{x}$ is a projection.Now,for a quotient stack $\mathscr{Y} = [\underline{x}/\check{G}]$,attaching with $\mathscr{Y}_{\mathrm{Ran}(X)} = [\mathrm{Ran}(X)/\check{G}]$,a S-point of it is a pair

$$\mathscr{Y}_{\mathrm{Ran}(X)}(S) = (\underline{x}, y) \quad \underline{x} : S \to \mathrm{Ran}(X), y : (X_{\underline{x}})_{\mathrm{dr}} \times_{S_{\mathrm{dr}}} S \to \mathscr{Y} \qquad [10.118]$$

where $X_{\underline{x}}$ is the formal scheme.Let $\underline{x} = \{x_i\}_{i \in I}$

$$\begin{tikzcd} \mathscr{Y}_{\mathrm{Ran}} \times_{\mathrm{Ran}(X)} x_i \arrow[r] \arrow[d] & x_i \arrow[d, "p^*"] \\ \mathscr{Y}_{\mathrm{Ran}(X)} \arrow[r] & \mathrm{Ran}(X) \end{tikzcd} \qquad [10.119]$$

decompose $\mathscr{Y}_{\mathrm{Ran}(X)}$ and $(X_{\underline{x}})_{\mathrm{dr}}$.For smooth presentation $f : \mathrm{Ran}(X) \to \mathscr{Y}_{\mathrm{Ran}(X)}$

$$\begin{tikzcd} \mathscr{Y}_{\mathrm{Ran}(X)} \times_{\mathscr{Y}_{\mathrm{Ran}(X)}} \mathscr{Y}_{\mathrm{Ran}(X)} \arrow[r] \arrow[d, "f_* \Delta^*_{\mathrm{Ran}/X}"] & \mathrm{Ran}(x) \times_X \mathrm{Ran}(x) \arrow[d, "\Delta^*_{\mathrm{Ran}/X}"] \\ \mathscr{Y}_{\mathrm{Ran}(X)} \arrow[r, "f^*"] & \mathrm{Ran}(X) \end{tikzcd} \qquad [10.120]$$

the fiber product is equivalent to

$$\mathscr{Y}_{\mathrm{Ran}(X)} \times_{\mathscr{Y}_{\mathrm{Ran}(X)}} \mathscr{Y}_{\mathrm{Ran}(X)} \xrightarrow{\simeq} \mathscr{Y}_{\mathrm{Ran}(X)} \times_{\mathrm{Ran}(X)} \mathrm{Ran}(X) \times_X \mathrm{Ran}(X) \quad [10.121]$$

the reverse gives us a restriction for $X_{\underline{x}} \in \mathrm{Ran}(X)$,we have $p^* y_*(X_{\underline{x}}) \in \mathscr{Y}_{\mathrm{Ran}(X)}$

$$p^* y_* \Delta^*_{\mathrm{Ran}/X} : \mathscr{Y}_{\mathrm{Ran}(X)} \times_{\mathscr{Y}_{\mathrm{Ran}(X)}} \mathscr{Y}_{\mathrm{Ran}(X)} \to \mathscr{Y}_{\mathrm{Ran}(X)} \qquad [10.122]$$

where $f_* = p^* y_*$.Based on this we have a natural tensor categorical structure

$$p^* y_* \Delta^*_{\mathrm{Ran}/X} : \mathrm{QCoh}(\mathscr{Y}_{\mathrm{Ran}(X)}) \otimes \mathrm{QCoh}(\mathscr{Y}_{\mathrm{Ran}(X)}) \to \mathrm{QCoh}(\mathscr{Y}_{\mathrm{Ran}(X)}) \quad [10.123]$$

In this case,we can define the category of representation of $\check{G}$ over Ran space.

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} = \mathrm{QCoh}([\mathrm{Ran}(X)/\check{G}]) \tag{10.124}$$

which gives a good space to let representations living on.Guided by [10.44]

$$\begin{tikzcd} \mathrm{LocSys}_{\check{G}} \arrow[r] \arrow[d] & \mathrm{Ran}(X) \arrow[d] \\ {[\mathrm{Ran}(X)/\check{G}]} \arrow[r] & \mathrm{Ran}(X) \end{tikzcd} \tag{10.125}$$

where $\check{G}$ is a group scheme over X.Where the stack of $\check{G}$-local systems is the stack of principle $\check{G}$-bundle on $\mathrm{Ran}(X)$ in the classfying stack of $\check{G}$.Thus

$$\begin{tikzcd} \mathrm{LocSys}_{\check{G}} \times_X \mathrm{Ran}(X) \arrow[r, "\mathrm{ev}"] \arrow[d, "p^* y_* \Delta^*_{\mathrm{Ran}/X}"] & {[\mathrm{Ran}(X)/\check{G}]} \\ \mathrm{LocSys}_{\check{G}} & \end{tikzcd} \tag{10.126}$$

where we used [10.122] and [10.123].Then,we have

$$\mathrm{Loc}^{\mathrm{Spec}}_{\check{G}} : \mathrm{QCoh}([\mathrm{Ran}(X)/\check{G}]) \simeq \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} \xrightarrow{\simeq} \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \tag{10.127}$$

see [9.82],they generated same topos.

For a group scheme G^α over $S \in \mathrm{Ran}(X)$ with a index set A^{op},we have

$$\mathfrak{L}^+(G) = \lim_{\alpha \in \mathrm{A}^{\mathrm{op}}} G^\alpha \simeq \mathrm{AS}/\mathfrak{L}^+(G) \tag{10.128}$$

The reason we do this because we work over étale site (at least fppf),we want to glue group schemes and [12.73] by étale morphsms.For every G^α,we can discuss quasi-coherent sheaves on the algebraic stack

$$\mathrm{Rep}(G^\alpha) = \mathrm{QCoh}([\mathrm{pt}/G^\alpha]), \quad \mathrm{Rep}(\mathfrak{L}^+(G))^{\mathrm{pre}} = \lim_{\alpha \in A^{\mathrm{op}}} \mathrm{Rep}(G^\alpha) \tag{10.129}$$

see [10.69] and $\mathrm{pt} = T/S$.By [12.28] we let $f : S \to X^I \in \mathrm{Ran}(X)$ with a group scheme G/X,we have a the following diagram and let $S \times_X G = G^{\alpha_0}$

$$\begin{tikzcd} S \times_X G \arrow[r] \arrow[d] & G \times_X X^I \arrow[r] \arrow[d] & G \arrow[d] \\ S \arrow[r, "f"] & X^I \arrow[r] & X \end{tikzcd} \tag{10.130}$$

Which gives us $S \times_X G \simeq S \times_{X^I} (G \times_X X^I)$.Also in the topos

$$\mathrm{Rep}(G) \to \mathrm{Rep}(G \times_X X^I) \to \mathrm{Rep}(S \times_X G) \tag{10.131}$$

Letting α_0 be an initial object is equivalent to let X^I be an initial object in the localized site Ran(X),and the closed and reduced setting [12.24] makes $\mathrm{Rep}(G \times_X X^I)$ be compactly generated.Then

$$\mathrm{Rep}(\mathfrak{L}^+(G))^{\mathrm{pre}} = \lim_I \mathrm{Rep}(S \times_{X^I} (G \times_X X^I)) \tag{10.132}$$

which is compactly generated.We see the topos on the stack is a stack [10.69] which means [12.62] is a unretracted version of the stack of representations.

$$\mathrm{Rep}(\mathfrak{L}^+(G))^{\mathrm{pre}}_{\mathrm{retracted}} \simeq \mathrm{QCoh}([\mathrm{pt}/\mathfrak{L}^+(G)]) = \mathrm{Rep}(\mathfrak{L}^+(G)) \tag{10.133}$$

11 Dynamics (Stackified) of the M-theory

Introduction

A complete physics theory not only constructs fields but also describes the dynamics of these fields.Now,we have complete the first part for the M-theory based on schemes (from regular functions),now for the dynamics,we need to based on DG schemes (from differential equations) see the former sections about Lagrangian,energy-momentum tensor etc..Similarly for schemes,we also need generalized super settings for correctly describe our world.We will see that solving the dynamics of M-theory is equivalent to solving the Geometric Langlands conjecture.We can find the basics in the former section and the section 12.1 about DG settings.

11.1 2-nonexistence and $\mathcal{M}$-flow

Definition11.1 A DG generalized superscheme ${}^{DG}\mathcal{X}$ is a generalized superscheme $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_1^*$ with a $\mathbb{Z}_2$-graded DG $\mathcal{O}_{\mathcal{S}}$-algebra given by [4.53],which is a ring denoting as $\mathcal{O}^\bullet_{\mathcal{X}} = \mathcal{O}^\bullet_{\mathcal{X}_0} \oplus \mathcal{O}^\bullet_{\mathcal{X}_1^*}$,such that $\mathcal{O}_{\mathcal{S}} \to \mathscr{H}_0(\mathcal{O}^\bullet_{\mathcal{S}})$ is surjective we used the notation for super simplicial cohomology group see below [8.8].Put it back to [8.12],we have a DG-M-brane denoting as ${}^{DG}\mathscr{M}$ after super $\mathbf{T}$-fusion.

Compared to notation in [9.73],for DG schemes,we use $\mathrm{QCoh}({}^{DG}S)$ for derived category $\mathrm{Qcoh}({}^{DG}S)$ with unbounded cohomologies,which is a DG category.Now,for a category $\mathrm{Coh}({}^{DG}S)$,we have a DG category $\mathrm{IndCoh}({}^{DG}S)$ with objects are functors $F : I \to \mathrm{Coh}({}^{DG}S)$ see above [9.7] which is an ind-completion.Because t-structure is compatible with F,which means we have

$$\Upsilon_S : \mathrm{IndCoh}({}^{DG}S) \to \mathrm{QCoh}({}^{DG}S), \quad \Upsilon_S \in \mathrm{DGCat}_{\mathrm{cont}} \tag{11.1}$$

which is a t-exact functor.Now,we can back to generalized super algebraifold $\underline{\mathcal{A}}$

$$\underline{\mathcal{A}}\text{-Mod} \cong \mathrm{IndCoh}^{\mathrm{rep}}({}^{DG}\mathcal{X}) \to \mathrm{QCoh}^{\mathrm{rep}}({}^{DG}\mathcal{X}) \to \mathrm{ProCoh}^{\mathrm{rep}}({}^{DG}\mathcal{X}) \tag{11.2}$$

see below [8.20] and Pro means it is from pro-completion.First,we perform a Y-duality which makes representable sheaves to schemes in a big étale site

$$\mathbf{Y} : \mathrm{ProCoh}^{\mathrm{rep}}({}^{DG}\mathcal{X}) \to \mathrm{EtProCoh}({}^{DG}\mathcal{X}^{\mathrm{rep}}) \subset \mathrm{ETSch}^{\mathrm{SupGen}}_{\mathrm{eff}}(\mathscr{M}^{\mathrm{rep}}_{\Lambda}) \tag{11.3}$$

Performing Y-duality on two sides and $\underline{\mathcal{A}}$-Mod becomes a fibered category

$$\begin{aligned} p : \mathrm{IndCoh}^{\mathrm{rep}}({}^{DG}\mathcal{X}) \vee \mathrm{ProCoh}^{\mathrm{rep}}({}^{DG}\mathcal{X}) \\ \to \mathrm{EtProCoh}({}^{DG}\mathcal{X}^{\mathrm{rep}}) \vee \mathrm{EtIndCoh}({}^{DG}\mathcal{X}^{\mathrm{rep}}) \end{aligned} \qquad [11.4]$$

Because U-duality is a generalized super relative 2-property

$$\begin{tikzcd} \mathrm{IndCoh}^{\mathrm{rep}}({}^{DG}\mathscr{M}) \arrow[r, "\vee"] \arrow[d, "\vee"] \arrow[dr, phantom, "\Rrightarrow \mathbf{U}"] & \mathrm{ProCoh}^{\mathrm{rep}}({}^{DG}\mathscr{M}) \arrow[d, "\vee"] \\ \mathrm{EtProCoh}({}^{DG}\mathscr{M}^{\mathrm{rep}}) \arrow[r, "\vee"] & \mathrm{EtIndCoh}({}^{DG}\mathscr{M}^{\mathrm{rep}}) \end{tikzcd} \qquad [11.5]$$

where we performed super $\mathbf{T}$-fusion and gives the meaning of overlap we mentioned below [9.146].Now,we are facing a subtile thing that is the U-dual parts originally behave like [10.55] that only connection should be the U-duality,but now [11.5] means they are twisted with each other,the explanation is following.

Definition11.2 The 2-nonexistence $\bigcirc$ is the absolute nonexistence based on existing of global nonexistence,with total number of generalized super relative 2-properties vanishes.Which is from vanishing of existence of nonexistence

$$(F_{Q\boxtimes Q^*})^{\mathrm{rep}} \boxtimes (F_{Q\boxtimes Q^*}) + \hat{F} = 0 - 0 = |_{\mathrm{Max}}\bigcirc \qquad [11.6]$$

and it indicates the maximal length of chain of vanishing relative properties.If

$$(F_{Q\boxtimes Q^*})^{\mathrm{rep}} \boxtimes (F_{Q\boxtimes Q^*}) = (F_{Q^{\mathrm{rep}}\boxtimes Q^{*\mathrm{rep}}}) \boxtimes (F_{Q\boxtimes Q^*}) \qquad [11.7]$$

In this case,we get an overlapping counting field

$$\hat{F} = (F_{Q^{\mathrm{rep}}\boxtimes Q^*}) \boxtimes (F_{Q\boxtimes Q^{*\mathrm{rep}}}) \qquad [11.8]$$

Theorem11.3 The M-theory indeed have dynamics but not canceling by **U**-fusion,the existence of the dynamics of the M-theory is to cancel the existence of the global nonexistence to give a nonexistence on the level of generalized super relative 2-property, which is the 2-nonexistence $\bigcirc$.

In this case,studying the dynamics of the M-theory is equivalent to study the overlapping of two U-dual pre M-theories.

Definition11.4 An $\underline{\mathcal{A}}$-gerbe over DG-ETSch$^{\mathrm{SupGen}}_{\mathrm{eff}}(\mathscr{M}_{-\Lambda})$ is

$$p : \mathrm{Tor}^{\mathrm{twist}}_{\mathrm{grop.}}(\underline{\mathcal{A}}) \to C = \text{DG-ETSch}^{\mathrm{SupGen}}_{\mathrm{eff}}(\mathscr{M}_{-\Lambda}) \qquad [11.9]$$

in the stack $\mathrm{Tor}_{\underline{\mathcal{A}}}$ following from [9.92] with grop. denoting for groupoid and now $\underline{\mathcal{A}}$ is a sheaf of groups,which is a substack with an isomorphism see [9.51].

$$l_x : \underline{\mathcal{A}}|_{C/p(x)} \to \underline{\mathrm{Aut}}_x, \quad \forall x \in \mathrm{Tor}^{\mathrm{twist}}_{\mathrm{grop.}}(\underline{\mathcal{A}}) \qquad [11.10]$$

such that it satisfies the $\boldsymbol{\mu}$-gerbe axioms 12.2.2 in [12] and extra constraints

(i) $\underline{\mathcal{A}}$ is self $\mathbf{U}$-dual $\quad\Leftrightarrow\quad$ $\underline{\mathcal{A}} \cong G^{\mathrm{rep}} \vee G$ (ii) $\underline{\mathcal{A}}$-gerbe $\cong G^{\mathrm{rep}} \vee G$-gerbe

(iii) $\underline{\mathcal{A}}$-gerbe is a subalgebraic stack, $\subset \boldsymbol{\mathcal{M}}^{\mathrm{pre.}}_{\mathrm{cons.}}$ in the M-theory

[11.11]

the pre M-theory is algebraic by [10.24] with diagonal morphism given by [10.25].Along this diagonal morphism,we have spontaneous $\mathbb{Z}_2$ breaking [8.28]

$$\underline{\mathcal{A}} \cong \mathscr{M}^{\text{rep}} \vee_{\bigcirc} \mathscr{M}/G \boxtimes \check{G} \qquad G \equiv G^{\text{rep}} \vee G/\mathbb{Z}_2, \quad \check{G} \equiv G \vee G^{\text{rep}}/\mathbb{Z}_2 \quad [11.12]$$

where these are Langlands dual to each other in [10.56].Which means a $\underline{\mathcal{A}}$-gerbe is equivalent to a $G \boxtimes \check{G}$-gerbe.By [10.43],we have

$$G \boxtimes \check{G}\text{-gerbe} \cong G\text{-gerbe} \boxtimes \check{G}\text{-gerbe} \cong \text{Bun}_G^{\text{twisted}} \boxtimes \check{G}\text{-gerbe} \qquad [11.13]$$

where $\text{Bun}_G^{\text{twisted}}$ is stack of twisted principle G-bundles [19] guided by the sheaf of properties on the M-brane [9.15],we have an isomorphism

$$\text{Bun}_G^{\text{twisted}} \boxtimes \check{G}\text{-gerbe} \cong \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \qquad [11.14]$$

where the right hand side is a DG category of D-modules with D formed by super $\mathbf{T}$-fusing all generalized super derivations.

Definition11.5 The dynamics of the M-theory is called $\boldsymbol{\mathcal{M}}$-flow

$$\boldsymbol{\mathcal{M}}\text{-flow} \equiv \text{D}^{\leq 11}\text{-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{DGETSch}_{\text{eff}}^{\text{SupGen,cons.}}(\mathscr{M}_{\Lambda}^{\text{rep}} \vee \mathscr{M}_{-\Lambda}) \quad [11.15]$$

which is a well-defined stack retracted from the non-perturbative (not well-defined) flow approaching D0-brane [6.22] and it describes the evolution of generalized super relative 2-properties compared to [9.15],that is

$$\begin{tikzcd}
\cdots & \mathscr{M} \arrow[r] & {[\mathscr{M}/G']} \arrow[r] & {[\mathscr{M}/(G'G'')]\cdots} \\
\mathscr{M}^{\text{rep}} \vee_{[\mathscr{M}/G']} \mathscr{M} \arrow[r, hook] & \mathscr{M}^{\text{rep}} \vee_{\bigcirc} \mathscr{M} \arrow[r] \arrow[u] & \mathscr{M}^{\text{rep}} \arrow[r] \arrow[u] & \mathscr{M}'^{\text{rep}}\cdots \arrow[u] \\
\mathscr{M}^{\text{rep}} \vee_{[\mathscr{M}/G'G'']} \mathscr{M} \arrow[u, hook] & \cdots \arrow[l, hook']
\end{tikzcd} \qquad [11.16]$$

with the global group scheme G in [11.15] given by

$$G \cong \prod_i \mathscr{M}^{\text{rep}} \vee_{[\mathscr{M}/G_i]} \mathscr{M} \qquad [11.17]$$

Recall the Lie groupoid for string-Space [9.39],the half-twist in [11.15] is because the stack Bun_G is retracted along normalized determinant line bundle

$$\det_{\text{Bun}_G} : (P_G)^{\text{op}} \to (\det(\Gamma(\mathscr{M}, g_{P_G})) \otimes \det(\Gamma(X, g_{P_G^0}))^{\otimes -1}) \qquad [11.18]$$

where P_G^0 is a trivial bundle and g_{P_G} is sheaf of Lie algebras see [7.4].From the stack generalized by U-duality see [8.54],we have from [11.8]

$$\underline{\text{Aut}}_x \xrightarrow{\simeq} \underline{\text{Isom}}(x, \mathbf{U}x) \to_{\#} \hat{F}, \qquad x \in \text{Tor}_{\underline{\mathcal{A}}} \qquad [11.19]$$

over the overlap counting field.For $\hat{G}$,we have a functor following the diagram

$$\begin{tikzcd} \mathrm{Rep}(\check{G}) \arrow[rr, "\mathrm{Loc}^{\mathrm{Spec}}_{\check{G}}"] & & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \\ & {}^{DG}\mathrm{Gr}(\mathfrak{U}(\Gamma g_{\mathrm{P}(\mathbf{T}^{\delta})^{-1}\check{G}})_{\mathrm{GSTen}} \arrow[ul] \arrow[ur] & \end{tikzcd} \quad [11.20]$$

where we used [7.20] and [8.48],the bottom is a fibered category with fibers are the corresponding tensor categories,over the general site [9.149].For an element $\mathrm{Gr}(\mathfrak{U})$ in it,we have $\mathrm{Hom}(\mathrm{Gr}(\mathfrak{U}), -) \cong \bigoplus_n \mathrm{Hom}_{\mathrm{filtered}_n}(\mathfrak{U}, -)$ which is compatible with filtered colimits,which means $\mathrm{Gr}(\mathfrak{U})$ is a compact object.For seeing further and combing with physics,some information we have not captured.

11.2 DG Lie (pre)scheme and Quantum collapse

Recall theorem3.3,we know the only type of strings is that of étale closed strings corresponding to the relative properties,also vibration of closed strings gives us a theory of gravity.For the purpose to study the dynamics of the M-theory,we need clearly discuss the matter and quantum effect under the view of algebraic geometry.And our start point is to define an adequate space which let the behaviors of matters living on them.

Definition12.1 A DG Lie generalized super(pre)scheme on 2-d world-sheet under $D = 10$ dimensional spacetime is a spectrum of affine Lie algebra

$$\mathrm{Spec}(\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[\mathbf{t}, \mathbf{t}^{-1}])^{\leq 10}, \quad D^{n \leq 10} : \mathcal{O}^{\Delta}_{\tilde{\mathfrak{g}}} \to \mathcal{O}^{\Delta}_{\tilde{\mathfrak{g}}} \qquad [11.21]$$

with generalized super grading and bounded cohomologies $n \leq 10$,which makes the structure sheaf become $D^{\leq 10}$-module $\mathcal{O}^{\bullet}_{\tilde{\mathfrak{g}}}$.So called pre because the algebra of quantum fields is not commutative,by [11.32] it loses representability because of the automorphism,we can define it as [11.21] because we will use free collection [7.21] in [11.40] and we will define it more correctly by using adic space below [13.39].And the bold t is for representing the coordinates we need (not imperative).A field is a function $f \in \mathcal{O}^{\bullet}_{\tilde{\mathfrak{g}}}$.The nontrivial behaviors are given by gluing axiom

$$f(\mathcal{O}^{\bullet}_{\tilde{\mathfrak{g}}}(U_i))|_{f(U_i)\cap f(U_j)} \cong f(\mathcal{O}^{\bullet}_{\tilde{\mathfrak{g}}}(U_j))|_{f(U_i)\cap f(U_j)} \qquad [11.21]$$

where we used étale covering $\{f : U_i \to U\}_{i \in I}$.By uncertainty principle

$$f(\mathcal{O}^{\bullet}_{\tilde{\mathfrak{g}}}(U_f)) \times_{=} f(\mathcal{O}^{\bullet}_{\tilde{\mathfrak{g}}}(U_g)) \cong \prod_j f(\mathcal{O}^{\bullet}_{\tilde{\mathfrak{g}}}(U_j))/z_j \cong \prod_{z_j} f(\mathcal{O}^{\bullet}_{\tilde{\mathfrak{g}}})(U_{z_j}) \qquad [11.22]$$

where $z \in \mathfrak{p} \subset V(1/fg)$,which is an algebraic formulation of OPE (11.5.1) of current algebra in [3].Now,we have two problems in [11.22],the first is this fields are not clearly described as quantum fields,the second is the gluing property of sheaves breaks or is modified.

Definition12.2 An étale sheaf of wavefunctions of string theory is

$$\mathcal{O}^{\bullet\mathrm{qut}}_{\tilde{\mathfrak{g}}} : ({}^{\mathrm{DG,Lie}}\mathrm{Sch}/\mathrm{Spec}(\tilde{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}])^{\leq D})^{\mathrm{op}} \to (\mathscr{A}[\phi_b]) \qquad [11.23]$$

where we used the state-operator mapping in CFT by path integral 2.74 in [4].

$$(f_w : T_{-\infty} \to^{\mathrm{DG,Lie}} X) \mapsto \int_{\phi'_b=0}^{\phi_b=1} [\mathcal{D}\phi'_b][\mathcal{D}\phi_i]_{f_z:\phi'_b\to\phi_b} e^{-S[\phi_i]} r^{L_0+\bar{L}_0} A(0) \quad [11.24]$$

where we performed a conformal transformation $z = e^{-iw}$,$A(0)$ is a functional on $T_{-\infty}$.Geometrically,summing over all possible field configurations is

$$\mathcal{O}^{\bullet\mathrm{qut}}_{\tilde{\mathfrak{g}}}(f_w) \cong h^{A(0)} \to h_{\mathscr{A}[\phi_b]} \qquad [11.25]$$

Suprisingly,we find it is consistent with the definition of algebraifold [8.15].

$$h^{A(0)} \to h_{\mathscr{A}[\phi_b]} \in h_{A(0)} \to F \to h_{\mathscr{A}[\phi_b]} \qquad [11.26]$$

which means the setting of algebraifold makes it naturally of quantum.

Definition12.3 An étale quantum sheaf of wavefunctions is

$${}^{\mathrm{Betti}}\mathcal{O}^{\bullet\mathrm{qut}}_{\tilde{\mathfrak{g}}} : (T \to \mathscr{X}) \mapsto \mathscr{A}[\phi_b]_0 \oplus \mathscr{A}[\phi_b]_1^* \mapsto \dim(\underline{\check{\mathscr{H}}}^{D-1}(T/\mathscr{X}, \mathscr{P})) \quad [11.27]$$

see [9.17] and $\mathscr{X} =^{\mathrm{DG,Lie}} \mathcal{X}$,which is a Betti sheaf of $D^{\leq 10}$-modules.We want to let the cohomology to detect the number of $D-1$ dimensional holes as the topological information.Now,we have a tool to study the gluing problem.The singular point is given by $=\in\cong$,in this contacting point the variation of momentum should be enough larger to form a generalized super black hole

$$\begin{tikzcd} \mathcal{O}^{\bullet\mathrm{qut}}_{\tilde{\mathfrak{g}}}(U_i) \times_{\widetilde{\mathscr{B}\mathscr{H}}} \mathcal{O}^{\bullet\mathrm{qut}}_{\tilde{\mathfrak{g}}}(U_j) \arrow[r] \arrow[d] & \mathcal{O}^{\bullet\mathrm{qut}}_{\tilde{\mathfrak{g}}}(U_j) \arrow[d] \\ \mathcal{O}^{\bullet\mathrm{qut}}_{\tilde{\mathfrak{g}}}(U_i) \arrow[r] & \widetilde{\mathscr{B}\mathscr{H}} \end{tikzcd} \qquad [11.28]$$

for an étale covering $\{U_i \to^{\mathrm{DG,Lie}} \mathcal{X}\}$ with the descent data identity morphism $\sigma : \mathrm{pr}_1^* E_i = \mathrm{pr}_2^* E_j$.Also we have inequality of topological information

$${}^{\mathrm{Betti}}\mathcal{O}^{\bullet\mathrm{qut}}_{\tilde{\mathfrak{g}}}(U_i \times_{f(U_i)\cap f(U_j)} U_j) =^{\mathrm{Betti}} \mathcal{O}^{\bullet\mathrm{qut}}_{\tilde{\mathfrak{g}}}(U_i) +^{\mathrm{Betti}} \mathcal{O}^{\bullet\mathrm{qut}}_{\tilde{\mathfrak{g}}}(U_j) + 1 \quad [11.29]$$

which means we gain topological information during the contaction or a superposition state.Also,[11.28] follows from that we are in region of relative properties and this relative property is given by $=$ is heavily strong than $\cong$,which means the first type of étale closed strings in [8.22] dominates the behaviors,that is ordinary closed strings (theory of gravity is dominant).Now,we need back to descent theory [9.70],global descent should gives

$$\mathcal{O}^{\bullet\mathrm{qut}}_{\tilde{\mathfrak{g}}}(\{U_i \to^{\mathrm{DG,Lie}} \mathcal{X}\}) \xrightarrow{\simeq} \mathcal{O}^{\bullet\mathrm{qut}}_{\tilde{\mathfrak{g}}}({}^{\mathrm{DG,Lie}}\mathcal{X}), \quad (E_i, \sigma_{ij}) \mapsto f_* E_i \qquad [11.30]$$

which is information-preserving,the phyics behavour breaks the global descent as the descent data σ_{ij} is not effect before collapse to a certain state.

Definition12.4 A quantum collapse is a process that the local descent theory becomes to a global descent theory along an étale coverings

$$\begin{aligned} &\mathcal{O}^{\bullet\mathrm{qut}}_{\tilde{\mathfrak{g}}}(\{U_i \to^{\mathrm{DG,Lie}} \mathcal{X}\}) \\ &\quad \to \mathcal{O}^{\bullet\mathrm{qut}}_{\tilde{\mathfrak{g}}}(\{U_i \to^{\mathrm{DG,Lie}} \mathcal{X}\}) \setminus \{(\{E_i\},\{=\})\} \cong \mathcal{O}^{\bullet\mathrm{qut}}_{\tilde{\mathfrak{g}}}({}^{\mathrm{DG,Lie}}\mathcal{X}) \end{aligned} \qquad [11.31]$$

for the quantum sheaf.An interesting thing is for math,the objects we exclude are that of automorphisms,and the isomorphisms give good property for descent theory but these are trivial behaviors,which gives us a motivation to focus on automorphism [11.10] meaning there are contactions.See the problem for constructing representable functor of the second paragraph in P69 in [12],physics gives a really concrete explanation that why automorphisms prevent us to achieve representable (unapproachable to global descent).

Corollary12.5 A covering of a scheme is an effective descent morphism for F of algebra if and only if the free collection [7.21] of every ideal in the algebra is the ideal itself.Combing [11.19],we have a connection that we need to know

$$\begin{tikzcd}
\text{Contaction} \arrow[rr, leftrightarrow] \arrow[dr, "\widetilde{\mathscr{BH}}", "\text{gr.dominance}"'] & & \text{Not effect descent} \arrow[dl, "\underline{\text{Aut}}"] \\
& \text{Self U-dual} &
\end{tikzcd} \qquad [11.32]$$

also we have a diagram for our familiar copy in [11.6]

$$\begin{tikzcd}
\text{Free} \arrow[rr, leftrightarrow] \arrow[dr, "\text{regular}", "\text{qut.dominance}"'] & & \text{Effective descent} \arrow[dl, "\underline{\text{Isom}}"] \\
& \text{Unself U-dual} &
\end{tikzcd} \qquad [11.33]$$

[11.33] is for getting good space (i.e.algebraic space,stack etc.) and [11.32] should correlate with the objects (non-trivial behaviors) living on the good space.

11.3 F-duality and the Geometric Langlands

In this subsection,we try to combine the math(GLC) with the physics (M-theory).From [11.32] and [11.33],we find a duality between self U-dual and unself U-dual copies gives us a duality of gravity theory and quantum theory,which gives us motivation to define an unified field.

Definition12.5 The F-duality over the 2-nonexistence $\bigcirc$ is a duality between existence and nonexistence,denoting as $\mathbf{F}$ see [11.7] and [11.8]

$$\mathbf{F} : (F_{Q^{\mathrm{rep}}\boxtimes Q^{*\mathrm{rep}}}) \boxtimes (F_{Q\boxtimes Q^*}) \leftrightarrow (F_{Q^{\mathrm{rep}}\boxtimes Q^*}) \boxtimes (F_{Q\boxtimes Q^{*\mathrm{rep}}}) \qquad [11.34]$$

Which is a duality on the $\boldsymbol{\mathcal{M}}$-flow over $\bigcirc$.In summary,we have

$$\bigcirc \Leftarrow \mathbf{F} \Leftarrow \mathbf{U} \Leftarrow \mathbf{T} \times_{\mathscr{M}} \mathbf{S} \qquad [11.35]$$

where we used [8.46].Notice the direction of the arrow similarly to that of [9.97],which is for canceling the property of right-evolution,we want to call this direction the left-evolution.

For the understanding of [11.33],we know a consistent string-Space [9.92],a pre M-theory [9.98] and the M-theory [10.22] are just a good living space for nontrivial behaviors which exists by F-duality.In this case,we have

$$\begin{tikzcd} \boldsymbol{\mathcal{M}}\text{-flow} \arrow[r] \arrow[d, "\mathbf{F}"] & (\boldsymbol{\mathcal{M}}^{\text{pre.rep.}}_{\text{cons.}} \vee \boldsymbol{\mathcal{M}}^{\text{pre.}}_{\text{cons.}})\text{-flow} \arrow[d] \\ \boldsymbol{\mathcal{M}} \arrow[r, "\mathrm{P}(\mathbf{U})^{-1}"] & \boldsymbol{\mathcal{M}}^{\text{pre.rep.}}_{\text{cons.}} \vee \boldsymbol{\mathcal{M}}^{\text{pre.}}_{\text{cons.}} \end{tikzcd} \quad [11.36]$$

And this should be $\mathbf{T}^{\delta}$-breaking to an $(10^{\mathfrak{P}}, 1)$-stack $\mathcal{S}$-flow

$$(\mathcal{S}^{\text{rep}}_{\text{cons.}} \vee \mathcal{S}_{\text{cons.}})\text{-flow}_{10^{\mathfrak{P}}} \to^{\text{DG,Lie}} (\mathcal{S}^{\text{rep}}_{\text{cons.}} \vee \mathcal{S}_{\text{cons.}})_{10^{\mathfrak{P}}} \quad [11.37]$$

over the DG Lie algebraic stack based on $\mathcal{S}^{\text{rep}}_{\text{cons.}} \vee \mathcal{S}_{\text{cons.}}$ with $D^{n\leq 10}$ with $10^{\mathfrak{P}}$-morphisms defined by morphisms of relative properties [9.130],and it is non-perturbative because we are on level of relative properties in theorem9.5 below [9.124].Naturally,it is presentable stable and we focus on the seen part

$$\mathcal{S}^{\text{rep}}_{\text{cons.}} \vee \mathcal{S}_{\text{cons.}} \simeq (\text{pts}, \mathcal{S}_{\text{cons.}}) \quad [11.38]$$

follows from that the unseen part of universe is a point (unseen) observed in seen part.Which makes [11.37] to a stack of pointed spaces.By [9.92] and [11.21]

$$^{\text{DG,Lie}}\mathcal{S}_{\text{cons.}} :^{\text{DG,Lie}} (\Phi \oplus \tilde{\Psi}^*) \to^{\text{Lie}} \text{DGETSch}^{\text{SupGen}}(\mathscr{M}) \quad [11.39]$$

And we use [7.20] and guided by [11.31] that we need to do free collection

$$^{\text{DG,Lie}}(\Phi \oplus \tilde{\Psi}^*) \simeq^{\text{DG,Lie}} \text{Gr}(\mathfrak{U}(\Gamma\mathcal{O}^{\bullet\text{qut}}_{\tilde{\mathfrak{g}}}))^{\text{retracted}}_{\text{GSTen}} \quad [11.40]$$

retracting [9.104] through admitting of preservation of relative property.Follows from the diagram above [4.48],that means the affine Lie algebra descents to a group scheme with $R \to 0$ is for simulating $(0, \xi, \overline{\xi})$,G here is for [4.46]

$$g_{\mathscr{D}} : (\tilde{\mathfrak{g}}, (1/i\epsilon)\delta_Q \mathscr{A}) \to G(x^1, ..., x^{10})_{R\to 0} \oplus G(\theta^1, ..., \theta^{10})_{R\to\infty} \to G_{\tilde{\mathfrak{g}}_0} \oplus G_{\tilde{\mathfrak{g}}_1^*} \quad [11.41]$$

where we used [3.42],[4.47] and [7.4] for $Q, \mathscr{A} \in \tilde{\mathfrak{g}}$.We also used the the existence of Majorana-Weyl condition in superstring theory below [5.15].

Then,we get a generalized super Lie group scheme from all such pushforward

$$\mathcal{G}_{\tilde{\mathfrak{g}}_0} \oplus \mathcal{G}_{\tilde{\mathfrak{g}}_1^*} \simeq (g_{\mathscr{D}*}(\tilde{\mathfrak{g}}, (1/i\epsilon)\delta_Q \mathscr{A})) \quad [11.42]$$

Then,we can back to [11.20],and use [11.12] we get

$$^{DG}\text{Gr}(\mathfrak{U}(\Gamma g_{\mathrm{P}(\mathbf{T}^{\delta})^{-1}\check{G}})^{\text{retracted}}_{\text{GSTen}} \simeq^{\text{DG,Lie}} (\Phi \oplus \tilde{\Psi}^*)^{\text{rep}} \vee^{\text{DG,Lie}} (\Phi \oplus \tilde{\Psi}^*) \quad [11.43]$$

with the U-duality twists to F-duality $\mathbf{F} : G \vee \check{G} \leftrightarrow \check{G} \vee G$ in [11.12]

$$\check{G} \vee G \simeq \mathrm{P}(\mathbf{T}^{\delta}) \left[(g_{\mathscr{D}*}(\tilde{\mathfrak{g}}, (1/i\epsilon)\delta_Q \mathscr{A}))^{\mathrm{rep}} \vee (g_{\mathscr{D}*}(\tilde{\mathfrak{g}}, (1/i\epsilon)\delta_Q \mathscr{A})) \right] \quad [11.44]$$

Also,we have by using [11.42] from [11.20]

$$\mathrm{Rep}(\check{G}) \vee \mathrm{Rep}(G) \simeq \mathrm{P}(\mathbf{T}^{\delta})^{DG} \mathrm{Gr}(\mathfrak{U}(\Gamma g_{[(\mathcal{G}_{\tilde{\mathfrak{g}}_0} \oplus \mathcal{G}_{\tilde{\mathfrak{g}}_1^*})^{\mathrm{rep}} \vee (\mathcal{G}_{\tilde{\mathfrak{g}}_0} \oplus \mathcal{G}_{\tilde{\mathfrak{g}}_1^*})]})^{\mathrm{retracted}}_{\mathrm{GSTen}} \quad [11.45]$$

In this case,we can combine [11.39],[11.43] and [11.45] together

$$\begin{aligned} {}^{\mathrm{DG,Lie}}(\mathcal{S}^{\mathrm{rep}}_{\mathrm{cons.}} \vee \mathcal{S}_{\mathrm{cons.}}) :& \mathrm{P}(\mathbf{T})^{-1}(\mathrm{Rep}(\check{G}) \vee \mathrm{Rep}(G)) \\ & \to^{\mathrm{Lie}} \mathrm{DGETSch}^{\mathrm{SupGen}}_{\mathrm{eff.cons.}}(\mathscr{M}^{\mathrm{rep}} \vee \mathscr{M}) \end{aligned} \quad [11.46]$$

which lets non-perturbative theory living on as the double counting of the relative properties below [9.124].An interesting thing is if we let 1.2.2 in [19].

$$(\mathrm{P}(\mathbf{U})\mathrm{Rep}(G)) \vee \simeq \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}(-), \quad \mathrm{Rep}(\check{G}) = \mathrm{QCoh}([\mathscr{M}^{\mathrm{rep}}/\check{G}]) \quad [11.47]$$

recall the self duality gives a fusion-like when we calculate in [8.34]

$$\mathrm{P}(\mathbf{U})(\mathrm{Rep}(\check{G}) \vee \mathrm{Rep}(G)) \simeq (\mathrm{P}(\mathbf{U})\mathrm{Rep}(G) \vee)\mathrm{Rep}(\check{G}) \quad [11.48]$$

see [11.66] and [12.38].Pullback along [10.25],we get

$$\begin{aligned} \boldsymbol{\mathcal{M}} &\simeq \mathrm{Loc}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}} \mathrm{P}(\mathbf{T}^{\delta})^{\mathrm{DG,Lie}}(\mathcal{S}^{\mathrm{rep}}_{\mathrm{cons.}} \vee \mathcal{S}_{\mathrm{cons.}})_{10^{\mathfrak{P}}} \\ &\simeq \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})^{\leq 11} \to^{\mathrm{Lie}} \mathrm{DGETSch}^{\mathrm{SupGen,cons.}}_{\mathrm{eff}}(\mathscr{M}^{\mathrm{rep}} \vee \mathscr{M}) \end{aligned} \quad [11.49]$$

Combining with [11.15] and the F-duality [11.36],we get

$$\begin{tikzcd} \mathrm{D}^{\leq 11}\text{-}\mathrm{mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \arrow[rr, leftrightarrow, "\mathbf{F}"] \arrow[dr] & & \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})^{\leq 11} \arrow[dl] \\ & {}^{\mathrm{Lie}}\mathrm{DGSch}^{\mathrm{SupGen}}_{\mathrm{eff,cons.}}(\mathscr{M}^{\mathrm{rep}} \vee \mathscr{M}) & \end{tikzcd}$$

[11.50]

where compared to that below [11.46],the generalized super algebraic stack $\boldsymbol{\mathcal{M}}$ which is a $(11^{\mathfrak{P}^2}, 1)$-stack with well defined (perturbative) theory [11.15] living on as the generalized relative 2-properties [9.148],from theorem9.8.

In this case,we see the geometric langlands correspondence is from F-duality which is a twisted form of U-duality between M-theory and its dynamics

$$\mathrm{GLC}(\boldsymbol{\mathcal{M}} \leftrightarrow \boldsymbol{\mathcal{M}}\text{-flow}) \hookrightarrow \mathbf{F} \quad [11.51]$$

Combining [11.32] and [11.33] with [11.50],the LHS (geometric n-stack) which is algebraic with (non)representability of localities which are nontrivial behaviors in QFT (embedding gravity effect i.e.black hole) and the RHS (geometric n-stack) which is algebraic with representability of spaces themselves which are in GR (spreading quantum effects),we see the QFT includes gravity effect and GR include quantum effect and this is explained by [11.85].Thus,the

F-duality is a duality between non-representability and representability,recall [9.107],it should be understood as a twisted form of U-duality.Now,back to GLC [11.50],the LHS collects all information of automorphisms from self U-duality and the RHS collects all information of pure isomorphisms ($\cong \setminus =$).Also,we need to notice that the truncation ≤ 11 is for the degree of geometric n-stack see [12.81] and [12.101],that is for the spacetime dimensions.

11.4 Ran space and the Unified field$^!$ theory

This subsection is for compensation of details in the above discussion.For a D-dimensional generalized superscheme $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_1^*$,we have

$$\mathrm{Ran}(\mathcal{X}) = \mathrm{Hom}(\mathrm{Clabi}_{\mathrm{mir}}^{\mathrm{retracted}}, S) \to \mathcal{X} \simeq \mathrm{Clabi}_{\mathrm{mir}}^{\mathrm{retracted}}/\mathcal{X} \qquad [11.52]$$

from [10.113],localized from mirror pairs of retracted Calabi-Yau manifolds that is $\mathrm{Clabi}_{\mathrm{mir}}^{\mathrm{retracted}} \simeq \mathbf{Ret}_*\mathrm{Clabi}_{\mathrm{mir}}$ see [12.19].A point $\underline{x} \in \mathrm{Ran}(\mathcal{X})$ is a closed reduced étale $\mathcal{X}$-scheme $M \in \mathrm{Clabi}_{\mathrm{mir}}^{\mathrm{retracted}}$.By formal completion

$$\mathcal{X}_{\underline{x}} \simeq \mathcal{X}_{f_*M}, \quad (f : M \to \mathcal{X}) \in \mathrm{Ran}(\mathcal{X}) \qquad [11.53]$$

is a formal glued scheme with étale topology given by [10.115].By [10.117]

$$p : \mathrm{Ran}(\mathcal{X}) \to \underline{x} \qquad [11.54]$$

which is a projection because they generate same topos.In this case,we can have

$$\mathrm{Ran}(\mathcal{X}) \simeq \coprod \mathcal{X}_{\underline{x}} \times \underline{x} \simeq \coprod_{\underline{x}} \mathcal{X}_{\underline{x}} \times \mathrm{Cla} \simeq \mathrm{EtSch}_{\mathrm{eff,Cla}}^{\mathrm{SupGen}}/\mathcal{X} \qquad [11.55]$$

where Cla is the global mirror pair by the global descent theory.Which makes the Ran space of $\mathcal{X}$ a stack over the localized site.The groupoid is from

$$\mathrm{Ran}(\mathcal{X})^{\mathrm{retracted}} \simeq (\text{1-properties}\, P)_{\mathrm{Ran}(\mathcal{X})} \qquad [11.56]$$

by theorem2.4 such that for $P \in (\text{1-properties})_{\mathrm{Ran}(\mathcal{X})}$ it follows the diagram

$$\begin{tikzcd} \underline{x}_1 \arrow[r] \arrow[d] & \underline{x}_2 \arrow[d, "f"] \\ P_1 \arrow[r, "P_1 \times P_2"] & P_2 \end{tikzcd} \qquad f^*(P_1 \times P_2) \simeq M_1 \times_s M_2 \hookrightarrow \mathcal{X}_0 \times_s \mathcal{X}_1^* \qquad [11.57]$$

with isomorphism replaced by admit of preservation of relative property.A point on $\mathrm{Ran}(\mathcal{X})$ is a 4-dimensional spacetime.Also,if we view $\check{G}$ as a $\check{G}$-torsor

$$\begin{tikzcd} \check{G} \arrow[r] \arrow[d, "\Delta_{B\check{G}}^*(x \times x)"] & \mathrm{Spec}((F_{Q^{\mathrm{rep}} \boxtimes Q^*}) \boxtimes (F_{Q \boxtimes Q^{*\mathrm{rep}}})) \arrow[d, "x \times x"] \\ {[\mathscr{M}^{\mathrm{rep}}/\check{G}]} \arrow[r, "\Delta_{B\check{G}}"] & B\check{G} \times B\check{G} \end{tikzcd} \qquad [11.58]$$

where we used [11.8],by the overlap counting field of generalized super relative 2-properties,which should be equivalent to [11.44].And for [11.39]

$$\begin{tikzcd} \mathrm{Rep}(\mathcal{G}_{\tilde{\mathfrak{g}}_0}\oplus\mathcal{G}_{\tilde{\mathfrak{g}}_1^*}) \arrow[r] \arrow[d, "\mathrm{Lie}^*"] & {}^{\mathrm{Lie}}\mathrm{DGETSch}^{\mathrm{SupGen}}_{\mathrm{eff}}(\mathscr{M}) \arrow[d, "\mathrm{Lie}^*"] \\ \Phi\oplus\tilde{\Psi}^* \arrow[r] & \mathrm{ETSch}^{\mathrm{SupGen}}_{\mathrm{eff}}(\mathscr{M}) \end{tikzcd} \quad [11.59]$$

that follows this diagram,and the stack of representation is isomorphic to

$$^{\mathrm{DG,Lie}}(\Phi\oplus\tilde{\Psi}^*)=(\Phi\oplus\tilde{\Psi}^*)\times_{\mathrm{ETSch}^{\mathrm{SupGen}}_{\mathrm{eff}}(\mathscr{M}),\mathrm{Lie}^*}{}^{\mathrm{Lie}}\mathrm{DGETSch}^{\mathrm{SupGen}}_{\mathrm{eff}}(\mathscr{M}) \quad [11.60]$$

And the Lie group-Lie algebra correspondence,it should be trivial to distinguish representation of Lie group and that of Lie algebra on tensor level.

$$\mathrm{Rep}(\mathcal{G}_{\tilde{\mathfrak{g}}_0}\oplus\mathcal{G}_{\tilde{\mathfrak{g}}_1^*})\simeq\mathrm{Rep}(\Gamma\mathcal{O}^{\bullet\mathrm{qut}}_{\tilde{\mathfrak{g}}}) \quad [11.61]$$

Now,we back to Ran space and from [11.49],we have

$$\mathrm{Loc}^{\mathrm{Spec}}_{\check{G},\mathrm{Ran}}\mathrm{Rep}(\check{G})=\mathrm{Loc}^{\mathrm{Spec}}_{\check{G}}\mathrm{Rep}(\check{G})_{\mathrm{Ran}}=\mathrm{Loc}^{\mathrm{Spec}}_{\check{G}}\mathrm{QCoh}([\mathrm{Ran}(\mathscr{M}^{\mathrm{rep}})/\check{G}]) \quad [11.62]$$

From [11.38],like what we use Spec for schemes,we have by colimits above [10.95]

$$\mathrm{Stab}(\mathcal{S}^{\mathrm{rep}}_{\mathrm{cons.}}\vee\mathcal{S}_{\mathrm{cons.}})\simeq\mathcal{S}^{\heartsuit}_{\mathrm{cons.}} \quad [11.63]$$

for presentable stable ∞-category.Also by $\mathbf{T}^{\delta}$-breaking,

$$\mathrm{QCoh}([\mathrm{Ran}(\mathscr{M}^{\mathrm{rep}})/\check{G}])\simeq\mathrm{QCoh}([\mathrm{EtSch}^{\mathrm{SupGen}}_{\mathrm{eff,Cla}}\setminus\mathscr{M}^{\mathrm{rep}}/\check{G}]) \quad [11.64]$$

which is a double quotient.[11.62] means the heart of t-structure.To see this clearly,we can view U-duality gives us two DG sequences with rep part ≤ 0 and U-dual copy for ≥ 0,and the existence of F-duality makes them has intersections.This based on the way we define the geometry [8.11] and above [8.22],which makes our universe triangulated.And we can naturally glue the self U-dual part into heart of t-structure,which follows that

$$\begin{tikzcd} (\mathscr{M}^{\mathrm{rep}}_{\Lambda}\vee\mathscr{M}_{-\Lambda})^{\mathrm{unself}\,\mathbf{U}}\simeq\mathscr{M}^{\leq 0}\oplus\mathscr{M}^{\geq 0} \arrow[r, leftrightarrow, "\mathbf{F}"] \arrow[d] & \mathscr{M}^{\heartsuit} \arrow[d] \\ \mathrm{Spec}((F_{Q^{\mathrm{rep}}\boxtimes Q^{*\mathrm{rep}}})\boxtimes(F_{Q\boxtimes Q^*})) \arrow[r, leftrightarrow, "\mathbf{F}"] & \mathrm{Spec}((F_{Q^{\mathrm{rep}}\boxtimes Q^*})\boxtimes(F_{Q\boxtimes Q^{*\mathrm{rep}}})) \end{tikzcd} \quad [11.65]$$

which gives us an understanding of the spontaneously $\mathbb{Z}_2$ breaking [11.12]

$$\begin{aligned} \boldsymbol{\mathcal{M}}^{\mathrm{pre.rep.}}_{\mathrm{cons.}}\vee\boldsymbol{\mathcal{M}}^{\mathrm{pre.}}_{\mathrm{cons.}} &= [\mathscr{M}^{\mathrm{rep}}\vee\mathscr{M}/G^{\mathrm{rep}}\vee G] \\ &\simeq [(\mathscr{M}^{\leq 0}\oplus\mathscr{M}^{\geq 0})\vee\mathscr{M}^{\heartsuit}/G^{\mathrm{rep}}\vee G] \\ &= [\mathscr{M}^{\heartsuit}/\check{G}]\vee_{\bigcirc}[\mathscr{M}^{\leq 0}\oplus\mathscr{M}^{\geq 0}/G] \end{aligned} \quad [11.66]$$

where we used [10.55] with $\mathscr{M}^{\text{rep}} \vee_{\bigcirc} \mathscr{M} \simeq \mathscr{M}^{\heartsuit} \vee_{\bigcirc} (\mathscr{M}^{\leq 0} \oplus \mathscr{M}^{\geq 0})$.Which is consistent with [11.17] if and only if

$$\prod_i [\mathscr{M}/G_i] = \boldsymbol{\mathcal{M}}^{\text{pre.}}_{\text{cons.}} \simeq \text{Spec}(F), \quad \vee_{\text{Spec}(F)} = \times_{\text{Spec}(F)} \tag{11.66}$$

where we used F for unself U-dual field in [11.64].Which indeed gives us the experiment-free M-theory from [10.22]

$$\boldsymbol{\mathcal{M}} \to \boldsymbol{\mathcal{M}}^{\text{pre.rep.}}_{\text{cons.}} \vee \boldsymbol{\mathcal{M}}^{\text{pre.}}_{\text{cons.}} \simeq \text{Spec}(\hat{F}) \vee_{\bigcirc} \text{Spec}(F), \quad \boldsymbol{\mathcal{M}} \simeq \bigcirc \tag{11.67}$$

Notice that the $\bigcirc$ is the only ideal and the left evolution is guaranteed

$$\text{Spec}(\bigcirc) = \bigcirc, \qquad \bigcirc \Leftarrow \hat{F} \vee_{\bigcirc} F \tag{11.68}$$

where We left with F-duality for verifying their are information in this absolute nonexistence which make it able to generate our universe by left-evolution [11.35]

$$\boldsymbol{\mathcal{M}}\text{-flow} \simeq \bigcirc\text{-flow} = \bigcirc \Leftarrow \mathbf{F} \tag{11.69}$$

Now,back to [11.63],the double quotient is equivalent to

$$\begin{aligned} &[\text{EtSch}^{\text{SupGen}}_{\text{eff,Cla}} \setminus \mathscr{M}^{\text{rep}}/\check{G}] \\ &\quad \simeq [\text{EtSch}^{\text{SupGen}}_{\text{eff,Cla}}(\mathcal{X}_0 \oplus \mathcal{X}_1^*)^{\heartsuit}/\text{P}(\mathbf{T})^{-1}\check{G})/(\mathscr{M}^{\heartsuit}/\check{G})] \\ &\quad \simeq \text{Ran}(\text{Stab}(\mathcal{S}^{\text{rep}}_{\text{cons.}} \vee \mathcal{S}_{\text{cons.}}))/\boldsymbol{\mathcal{M}}^{\text{pre.rep.}}_{\text{cons.}} \simeq \text{Ran}(\mathcal{S}^{\heartsuit}_{\text{cons.}})/\boldsymbol{\mathcal{M}}^{\text{pre.rep.}}_{\text{cons.}} \end{aligned} \tag{11.70}$$

we can get further information by [10.33],we have

$$\begin{tikzcd} \check{G} \times_{\text{Spec}(\hat{F})} \text{Ran}(\mathcal{X}_0 \oplus \mathcal{X}_1^*)^{\heartsuit} \arrow[r] \arrow[d, "\text{P}(\mathbf{T})"] & \text{Ran}(\mathcal{X}_0 \oplus \mathcal{X}_1^*)^{\heartsuit} \arrow[d] & \text{Ran}(\mathcal{S}^{\heartsuit}_{\text{cons.}}) \arrow[l] \arrow[d, "\text{P}(\mathbf{T})"] \\ \check{G} \arrow[r] & \text{Spec}(\hat{F}) & \boldsymbol{\mathcal{M}}^{\text{pre.rep.}}_{\text{cons.}} \arrow[l] \end{tikzcd} \tag{11.71}$$

the right cartesian diagram is from [10.25],where $(\tilde{\mathcal{X}} \vee \mathcal{X})^{\heartsuit} \in \text{Spec}(\hat{F})$ and the $\text{P}(\mathbf{T})^{-1}\check{G}$ in [11.70] is the fiber product of the left cartesian diagram in [11.71].Also from [11.70] we get the diagram

$$\begin{tikzcd} \check{G} \times_{\text{Spec}(\hat{F})} \text{Ran}(\mathcal{X}_0 \oplus \mathcal{X}_1^*)^{\heartsuit} \times_{\text{Ran}(\mathcal{X}_0 \oplus \mathcal{X}_1^*)^{\heartsuit}} \text{Ran}(\mathcal{S}^{\heartsuit}_{\text{cons.}}) \arrow[r] \arrow[d] & \text{Ran}(\mathcal{S}^{\heartsuit}_{\text{cons.}}) \arrow[d] \\ \check{G} \times_{\text{Spec}(\hat{F})} \text{Ran}(\mathcal{X}_0 \oplus \mathcal{X}_1^*)^{\heartsuit} \arrow[r] \arrow[d] & \text{Ran}(\mathcal{X}_0 \oplus \mathcal{X}_1^*)^{\heartsuit} \arrow[d] \\ \check{G} \times_{\text{Spec}(\hat{F})} \boldsymbol{\mathcal{M}}^{\text{pre.rep.}}_{\text{cons.}} \arrow[r] & \boldsymbol{\mathcal{M}}^{\text{pre.rep.}}_{\text{cons.}} \end{tikzcd} \tag{11.72}$$

the bottom line is from [11.71] with

$$\mathrm{Ran}(\mathcal{X}_0 \oplus \mathcal{X}_1^*)^{\heartsuit} \simeq \mathrm{Spec}(\hat{F}) \times_{\boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}}} \mathrm{Ran}(\mathcal{S}_{\text{cons.}}^{\heartsuit}) \qquad [11.73]$$

Also,[11.65] shrinks the previous $\mathbf{U}$-breaking diagram [11.36]

$$\begin{tikzcd}
\boldsymbol{\mathcal{M}}\text{-flow} \arrow[r] \arrow[d, "\mathbf{F}"] & \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.}}\text{-flow} \simeq \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}} \arrow[r] \arrow[d, "\mathrm{QG}"] & \mathrm{Ran}(\mathcal{S}_{\text{cons.}}^{\heartsuit}) \arrow[d] \\
\boldsymbol{\mathcal{M}} \arrow[r, "\mathrm{P}(\mathbf{U})^{-1}"] & \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.}} \arrow[r, "\mathrm{P}(\mathbf{T})^{-1}"] & \mathrm{Ran}(\mathcal{S}_{\text{cons.}}^{\text{rep}} \times_{\mathrm{Spec}(F)} \mathcal{S}_{\text{cons.}})
\end{tikzcd} \qquad [11.74]$$

Now,we need to open the settings of DG Lie by [11.46],[11.47] and [11.58]

$$\begin{tikzcd}
\boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.}}\text{-flow} \simeq \boldsymbol{\mathcal{M}}_{\text{cons.Ran.}}^{\text{pre.rep.}} \arrow[r, "\mathrm{P}(\mathbf{T})^{-1}"] \arrow[d, "\mathrm{QG}"] & \mathrm{Ran}(\mathcal{S}_{\text{cons.}}^{\heartsuit}) \arrow[d] \\
\mathrm{QCoh}(\boldsymbol{\mathcal{M}}_{\text{cons.Ran.}}^{\text{pre.rep.}}) \arrow[r, "\mathrm{P}(\mathbf{T})^{-1}"] & {}^{\mathrm{DG,Lie}}\mathrm{Ran}(\mathcal{S}_{\text{cons.}}^{\text{rep}} \times_{\mathrm{Spec}(F)} \mathcal{S}_{\text{cons.}})
\end{tikzcd} \qquad [11.75]$$

From [11.72],we get a relative property of $\mathrm{Ran}(\mathcal{S}_{\text{cons.}}^{\heartsuit})$

$$(\check{G} \times_{\mathrm{Spec}(\hat{F})} \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}}) \times_{\boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}}} \mathrm{Ran}(\mathcal{S}_{\text{cons.}}^{\heartsuit}) \to \mathrm{Ran}(\mathcal{S}_{\text{cons.}}^{\heartsuit}) \qquad [11.76]$$

which inspires us to define an algebraic space see below [9.116],[9.123]

$$\mathrm{Ran}(\mathcal{S}_{\text{cons.}}^{\heartsuit})/(\check{G} \times_{\mathrm{Spec}(\hat{F})} \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}}) : \mathrm{Ran}(\mathcal{S}_{\text{cons.}}^{\heartsuit})^{\Leftarrow} \to \mathrm{QCoh}(\boldsymbol{\mathcal{M}}_{\text{cons.Ran.}}^{\text{pre.rep.}}) \qquad [11.76]$$

where we changed op to a left evolution.To see this we want use a trick

$$\mathrm{Rep}(G) \vee \Big([\mathscr{M}^{\heartsuit}/(-)] \times_{\mathrm{Spec}(F)} \mathrm{Rep}(-) \Big) \circ \Big(\check{G} \times_{\mathrm{Spec}(\hat{F})} \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}} \Big) \qquad [11.77]$$

where we used [11.47],and equipped with $\mathbf{U}$-fusion,we get

$$(\mathrm{P}(\mathbf{U})\mathrm{Rep}(G) \vee \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.}}) \vee_{\bigcirc} \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}} \simeq \boldsymbol{\mathcal{M}} \vee_{\bigcirc} \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}} \qquad [11.78]$$

where we used [11.67],then we use the GLC correspondence

$$\Big(\mathrm{GLC} \vee_{\bigcirc} \mathrm{id}\Big) \circ \Big(\boldsymbol{\mathcal{M}} \vee_{\bigcirc} \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}}\Big) \simeq \boldsymbol{\mathcal{M}}\text{-flow} \vee_{\bigcirc} \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}} \qquad [11.79]$$

Then,we want to use $\mathbf{U}$-breaking

$$\Big(\mathrm{P}(\mathbf{U}) \vee_{\mathrm{Spec}(\hat{F})} \mathrm{id}\Big) \circ \Big(\boldsymbol{\mathcal{M}}\text{-flow} \vee_{\bigcirc} \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}}\Big) \simeq \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}} \vee_{\mathrm{Spec}(\hat{F})} \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}} \qquad [11.80]$$

And by the global descent theory,this is equivalent to $\boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}}$.Then

$$\mathrm{Ran}(\mathcal{S}_{\text{cons.}}^{\heartsuit})/(\check{G} \times_{\mathrm{Spec}(\hat{F})} \boldsymbol{\mathcal{M}}_{\text{cons.}}^{\text{pre.rep.}}) \xrightarrow{\mathrm{QG}} [\mathrm{EtSch}_{\text{eff,Cla}}^{\text{SupGen}} \setminus \mathscr{M}^{\text{rep}}/\check{G}] \qquad [11.81]$$

where we used [11.70] from combining [11.77] to [11.80]

$$\begin{gathered}\mathrm{QG} = \Big(\mathrm{P}(\mathbf{U}) \vee_{\mathrm{Spec}(\hat{F})} \mathrm{id}\Big)\circ \\ \Big(\mathrm{GLC} \vee_{\bigcirc} \mathrm{id}\Big) \circ \Big(\mathrm{P}(\mathbf{U})\mathrm{Rep}(G) \vee \Big([\mathscr{M}^{\heartsuit}/(-)] \times_{\mathrm{Spec}(F)} \mathrm{Rep}(-)\Big)\Big)\end{gathered} \qquad [11.82]$$

By this we have a stack generalized by F-duality [8.53]

$$\Big(\boldsymbol{\mathcal{M}}^{\text{pre.rep.}}_{\text{cons.Ran.}} \vee \boldsymbol{\mathcal{M}}^{\text{pre.}}_{\text{cons.Ran.}} \to \boldsymbol{\mathcal{M}}\Big) \simeq \mathrm{QG} \qquad [11.83]$$

By using the 2-Yoneda lemma [9.25],it is equivalent to a $(11^P, 2)$-stack

$$HOM(\mathrm{EtSch}^{\mathrm{SupGen}}_{\mathrm{eff,Cla}}/\boldsymbol{\mathcal{M}}^{\text{pre.rep.}}_{\text{cons.Ran.}}, \boldsymbol{\mathcal{M}}^{\text{pre.}}_{\text{cons.Ran.}}) \simeq \boldsymbol{\mathcal{M}}^{\text{pre.}}_{\text{cons.Ran.}}(\boldsymbol{\mathcal{M}}^{\text{pre.rep.}}_{\text{cons.Ran.}}) \qquad [11.84]$$

which is an unification of quantum and gravity on the level of relative property see [11.37],that is non-perturbative.To see clearly,from [11.32] and [11.33]

$$\begin{tikzcd} \mathbf{F} \simeq & \boldsymbol{\mathcal{M}}\text{-flow} \arrow[rr, bend right, "\text{qut.dominance}"'] & G \vee \hat{G} \arrow[l, "\text{gerbe}"'] \arrow[r, "\text{Rep}"] & \boldsymbol{\mathcal{M}} \arrow[ll, bend right, "\text{gr.dominance}"'] & \simeq \mathrm{P}(\mathbf{U})\boldsymbol{\mathcal{M}}^{\text{pre.}}_{\text{cons.Ran.}}(\boldsymbol{\mathcal{M}}^{\text{pre.rep.}}_{\text{cons.Ran.}}) \end{tikzcd} \qquad [11.85]$$

where,we combined with [11.12] and [11.44].

Definition12.6 The **Unified field$^!$ theory UFT** is the diagram [11.85] representing the generalized super algebraic $(11^{\mathfrak{P}^2}, 2)$-stack.The ! denotes for the underlying number counting field [11.67].

Definition12.7 We stackified the M-theory and its dynamics and regard them as algebraic objects.In this case,we want a overall terminology to denote this theory studding the properties and connections of these objects as

$$\textbf{the } \boldsymbol{\mathcal{M}}^!\textbf{-theory} \qquad [11.86]$$

which is a field$^!$ theory based on super algebraic generalized geometry.

12 Modern super algebraic geometry III

12.1 Weak homotopy equivalence,D-brane and Possibility

Now,we want to give an detailed expatiation of the retract [9.104] and the replacement [9.105].An observation is that,h is a homotopy between F_U and F_V

$$\begin{tikzcd} U \cup V \arrow[r, bend left, "F_U"] \arrow[r, bend right, "F_V"'] \arrow[r, phantom, "\Downarrow h"] & F(U) \times F(V) \end{tikzcd} \qquad [12.1]$$

if and only if there exist $F|_{U\cap V} = F_U|_{U\cap V} \cong F_V|_{U\cap V}$

$$\begin{tikzcd} F(U)|_{U\cap V} \arrow[r, "\simeq"] & F(V)|_{U\cap V} \\ F_U \arrow[u] \arrow[r, "F|_{U\cap V}"] & F_V \arrow[u] \end{tikzcd} \qquad [12.2]$$

with $(F_U \to F_V)|_{U\cap V} = F|_{U\cap V}(U \to V) = \cong$ which means the overlap $|_{U\cap V}$ gives us a weak equivalence relation.The relative properties living in ordinary fibered category and behave like (unstable) weak equivalence relations,$P = U \times_{U\cap V} V$ here,over étale site,from descent theory [9.53] relative properties (descent data) can be localized to be represented by an overlap but it is unstable and the retract [9.104] push things to high energy level and things are highly unified relatively and relative properties are stable.

Definition13.1 A retract is a forgetful functor let us focus on structure of relative properties and forget the remaining structures,denote as **Ret**

$$\begin{tikzcd} X \arrow[d] & P \arrow[l, Leftarrow] \arrow[d, "PR"] & P^2 \cdots \arrow[l, Leftarrow] \arrow[d] \\ Y & R \arrow[l, Leftarrow] & R^2 \cdots \arrow[l, Leftarrow] \end{tikzcd} \quad , \quad \mathbf{Ret} : C \to \infty\text{-Grpd} \qquad [12.3]$$

with PR a relative 2-properties,sending relative n-properties to n-isomophism

$$\mathbf{Ret}(X \xrightarrow[\simeq_{\text{weak}}]{P} Y) = P : \mathbf{Ret}(X) \to \mathbf{Ret}(Y) \simeq\cong_{\text{strong}} \in \infty\text{-Grpd} \qquad [12.4]$$

We also need to find a space that the relative properties truly lives in (behaves as $\cong$),if we donot do super **T**-fusion,there is no cut off of relative 2-property [11.6],in the LEE of M-theory,relative properties lives in an 10-Grpd,denote as

$$^{\mathscr{P}}\mathbf{Ret}(\to)^{\leq 10}, \bigoplus_{n+1}^{\mathscr{P}} \mathrm{Hom}_n(P^n, R^n)), n \leq 10, \in \mathbb{Z}^+ \qquad [12.5]$$

As the descent of string-Spaces to a pre M-theory [9.93],we can use [7.25] that $D = 10$ dimensional theory is shrinked to a point (be a solution in D=11) in the extra dimension thus,in M-theory the highest degree of relative property is 2.And we can regard a scheme or algebraic space as a relative 0-property.In the ∞-groupoid every n-isomorphism is a relative n-property.And for a topos T on an étale site,we have for P a relative property in $\mathbf{Ret}(\rightarrow)^{\leq 10}_{\mathscr{P}}$

$$F \in HOM(T, {}^{\mathscr{P}}\mathbf{Ret}(\rightarrow)^{\leq 10}), F|_P \text{ is a weak homotopy equivalence} \qquad [12.6]$$

And combing the [8.24],we can further define D-branes in this derived category.

Definition13.2 A Dn-brane is a weak homotopy equivalence which is the global section of an étale sheaf F_{P^n} localized by a generalized super relative n-property $P^n \in {}^{\mathscr{P}}\mathbf{Ret}(\rightarrow)^{\leq 10}_{\text{GenSup}}$ and coherent sheaf $F \in T$ on the site [9.149].Thus we consider the derived algebraic geometry for studying LEE of the $\boldsymbol{\mathcal{M}}^!$-theory,we based on [21].Loosely,a D1-brane is an étale morphism and D2-brane is a continuously extension of an étale morphism etc..

A DG (differential graded) $\mathbb{Z}_2$-graded (generalized super) algebra is

$$A_{0\bullet} \oplus A^*_{1\bullet} = (\{A_{0i} \oplus A^*_{1i}\}_{i\in\mathbb{Z}}, \partial), \quad \partial : A_i \rightarrow A_{i-1}, \partial^2 = 0 \qquad [12.7]$$

which is a family of simplicial $N_{\mathbb{C}}$-modules with the conditions in [7.4].And we denote $\text{dgAlg}^{\geq 0}_k$ the category of commutative DG k-algebras for $n \in \mathbb{Z}^+$ and denote the $\text{DGAff}^{\geq 0}$ as that of affine DG schemes.

$$\text{Spec}(A_{0\bullet} \oplus A^*_{1\bullet}) \simeq \text{Spec}(A_{0\bullet}) \oplus \text{Spec}(A^*_{1\bullet}) \qquad [12.8]$$

with similicial super $\mathbf{T}$-duality $\mathbf{T}^{\delta}_{\bullet} : A_{0\bullet} \oplus A^*_{1\bullet} \rightarrow A^*_{1\bullet} \oplus A_{0\bullet}$.Notive that the $A_{00} \oplus A_{10}$ is the final object of the chain complex and by left evolution,we get cochain complex on the scheme.And a generalized derived superscheme

$$(\pi^0(\mathcal{X}_{0\bullet} \oplus \mathcal{X}^*_{1\bullet}) \subseteq \underline{\mathcal{X}}_0, \mathscr{H}_{\bullet}(\mathscr{O}_{\underline{\mathcal{X}}_{\bullet}})), \quad \pi^0\underline{\mathcal{X}}_{\bullet} = \underline{\text{Spec}}(\mathscr{H}_0(\mathscr{O}_{\underline{\mathcal{X}}_{\bullet}})) \qquad [12.9]$$

where we denote the generalized DG superscheme as $\underline{\mathcal{X}}_{\bullet} = \mathcal{X}_{0\bullet} \oplus \mathcal{X}^*_{1\bullet}$,and we used the notation below [8.8].the structure sheaf $\mathscr{H}_i(\mathscr{O}_{\underline{\mathcal{X}}_{\bullet}})$ is a $\mathscr{H}_0(\mathscr{O}_{\underline{\mathcal{X}}_{\bullet}})$-module which is quasi-coherent by the setting.Which means we can perform [10.83]

$$H_0(\mathscr{O}_{X_\bullet}(U)) \otimes_{H_0(\mathscr{O}_{X_\bullet}(V))} H_i(\mathscr{O}_{X_\bullet}(V)) \cong H_i(\mathscr{O}_{X_\bullet}(U)) \qquad [12.10]$$

for an étale cover $U \rightarrow V \in \pi^0 X_\bullet$,by setting ≤ 10 $\underline{\mathcal{X}}^{\leq 10}_{\bullet}$ gives us the full 10 dimensional space time constructing by $\mathscr{H}_i(\mathscr{O}_{\underline{\mathcal{X}}_{\bullet}})$ with left evolution

$$\text{D0-brane} \in \mathscr{H}_0(\mathscr{O}_{\underline{\mathcal{X}}_{\bullet}}\text{-mod}) \overset{\partial}{\leftarrow} \mathfrak{s}^c, \mathfrak{s}^c_{\text{weak}} \in \mathscr{H}_1(\mathscr{O}_{\underline{\mathcal{X}}_{\bullet}}\text{-mod}) \cdots \qquad [12.11]$$

where we used notation below [8.9] and theorem3.3 below [8.22] before the $\mathbf{T}^{\delta}_1$-fusion,$\mathfrak{s}^c_{\text{weak}}$ behaves like open superstrings induced by a weak homotopy equivalence.Which means in ths string-Space we have a closed-like superstring induced by a ordinary openstring with endpoints attaching with a D-string.After the fusion,we only have one type of closed string corresponding to the étale equivalence relations called étale closed string recall the $\mathbf{F}$-duality is about self-duality

[11.34],we can see this by

$$\begin{tikzcd}
\mathbf{S}^{\text{self}} : [\mathfrak{s}^c] \arrow[rr, leftrightarrow, "\mathrm{P}(\mathbf{T})^{-1}\mathrm{P}(\mathbf{U})^{-1}\mathbf{F}"] \arrow[dr, "\mathrm{P}(\mathbf{T}^\delta)"'] & & (\mathbf{S}^{\text{unself}} : [\mathfrak{s}^c_{\text{weak}}](\mathfrak{s}^o \leftrightarrow \text{D-string}) \arrow[dl, "\mathrm{P}(\mathbf{T}^\delta)"] \\
& \mathfrak{s}^c_{\text{et}} &
\end{tikzcd} \qquad [12.12]$$

which induces one duality fused version of [12.11],we should view this chain with the étale closed string generates the unified field$^!$ theory [11.85].

Now,we want to discuss the quasi-coherent complex,An $A_\bullet$-module $M_\bullet$ is a chain complex $M_\bullet$ with $A_\bullet \otimes_k M_\bullet \to M_\bullet$.For a derived scheme $(\pi^0 X_\bullet, H_\bullet(\mathscr{O}_{X_\bullet}))$ with structure sheaf $H_0(\mathscr{O}_{X_\bullet}) = \mathscr{O}_{\pi^0 X_\bullet}$ and a complex of presheaves for $i > 0$ by definition [12.9],a simplicial sheaf $\mathscr{O}_{X_\bullet}$-module behaving like [12.10] is a quasi-coherent^1 complex^2 if for every degree $\mathscr{O}_{X_\bullet}\text{-mod}(U) \cong \mathscr{O}_{X_\bullet}(U)\text{-mod}^1$ and for every homology presheaves $H_i(\mathscr{O}_{X_\bullet}\text{-mod}) \cong H_i(\mathscr{O}_{X_\bullet})\text{-mod}^2$ which means $H_i(\mathscr{O}_{X_\bullet})\text{-mod}(U) \cong H_i(\mathscr{O}_{X_\bullet}(U))\text{-mod}$ by [12.10],it satisfies

$$H_0(\mathscr{O}_{X_\bullet}(U)) \otimes_{H_0(\mathscr{O}_{X_\bullet}(V))} H_i(\mathscr{O}_{X_\bullet}\text{-mod})(V) \cong H_i(\mathscr{O}_{X_\bullet}\text{-mod})(U) \qquad [12.13]$$

Now,we want to study ∞-category.A topological category C is a category enriched in topological spaces with $\mathrm{Hom}_C(X,Y)$.A homotopy category $\mathrm{Ho}(C)$ over a topological category C is C with $HOM_C(C,C) \cong (\pi^0 \mathrm{Hom}_C(X,Y))$,which means it is homotopy enriched.A functor $F : C = \mathrm{Top}(U) \to D = \mathrm{Top}(F(U))$ is a quasi-equivalence equivalent if for all $U, V \in C, \pi_i(U) \cong \pi_i(F(U))$ with commutative square which means the weak equivalent diagram [12.1] and $\mathrm{Ho}(F)$ is an equivalence.Below [9.30],that is a case of relative category which means a category with equivalence which is relative to a subcategory with weaker equivalence in that case $(\cong, =)$ we also discuss such things around [11.31].Also see [12.4] and [12.5] the morphism with preservation of relative property is an isomorphism (relatively strong) in the groupoid,this should be understood by the existence of the strong equivalence (relative properties [9.130]) which is descent data induces weak equivalence by homotopy [9.100]

Theorem13.3 A a breaking of duality is a relatively weak projection to relative properties of degree n from relative property of degree $n+1$.Reverse of the realization gives an explanation of the **T**-fusion [6.22].

Recall generalized super algebraifold that is **T**-fusion like below [8.21],which is a manifold in the string-Space in M-theory which means the unified field$^!$ theory with étale closed string living on it see [11.12] and [11.85].

Next,we want to study derived functor,let $(C, \mathrm{Ho}(C))$ and $(D, \mathrm{Ho}(D))$ are relative category with the second is a subcategory of the first,with stronger equivalence.For the fibered category $p : \mathrm{Ho} \to \mathrm{Top}$ from category of homotopy categories to category of topological categories,the right derived functor is the 2-category $\mathbf{R} = HOM_{\mathrm{Top}}(\mathrm{Top}, \mathrm{Ho})$,a derived functor of F is the following deformation of diagram with $\mathbf{R}_{C/D} \cong HOM_{\mathrm{Top}}(C/D, \mathrm{Ho}(D))$

$$F: C \xrightarrow{F} D \;\to\; \begin{tikzcd} \mathrm{Ho}(C) \arrow[r, "\mathrm{Ho}(F)"] & \mathrm{Ho}(D) \\ C \arrow[u, "\lambda_C"] \arrow[r, "F"'] \arrow[ur, phantom, "\Nwarrow \eta"] & D \arrow[u, "\lambda_D"'] \end{tikzcd} \;\cong\; \begin{tikzcd} \mathrm{Ho}(C) \arrow[r, bend left, "\mathrm{Ho}(F)'"] \arrow[r, "\mathrm{Ho}(F)"'] & \mathrm{Ho}(D) \\ & D \arrow[u] \arrow[ul] \end{tikzcd} \qquad [12.14]$$

the η is unique because [12.1] up to weak equivalence in the homotopy enriched category $\mathrm{Ho}(F) \xrightarrow{\simeq} \mathrm{Ho}(F)'$.The dual notion (reverse arrow in [12.14])gives us the left derived functor,and we let $\mathbf{R}_{C/D} \cong \mathbf{R}F$ for a left exact functor we get the long exact sequence along $\lambda_D \circ F$ with naturally injective resolution

$$\begin{aligned} &0 \to A \to B \to C = 0 \xrightarrow{F} 0 \to F(A) \to F(B) \to F(C) \to \\ &0 \to F(A) \to F(B) \to F(C) \xrightarrow{\mathrm{d}} \lambda_D(F(I^1)) \cdots \end{aligned} \qquad [12.15]$$

where C has enough injectives similarly above [9.10],for each A with an injective resolution,we can see for a projective morphism $P(A) \to A \to 0$,canonically

$$0 \to A \xrightarrow{\varepsilon} P(A) \to A/\mathrm{im}(\varepsilon) \cong \ker(\varepsilon) \xrightarrow{\varepsilon_2} P(\ker(\varepsilon)) \to \ker(\varepsilon)/\mathrm{im}(\varepsilon_2) \cdots \quad [12.16]$$

In such case,we extract $0 \to A \to H_1(A_\bullet) \to \cdots$ as the injective resolution and $\mathrm{Ho}(D) \subseteq D$,so $\lambda_D(F(A)) = F(A)$.Then,we back to [12,15],left exact functor let rows of short exact be left exact,and λ_D sends them to homotopy enriched category giving us δ and make them into long exact sequence.Then first order is $F(A)$,the second order is the homology group $\lambda_D \circ F(H_1(A_\bullet)) \cong H_1(\lambda_D \circ F(A_\bullet))$.

Then,a model category is a relative category (C, W) with fibrations and cofibrations.First is a retract from [12.1] by a weak homotopy equivalence [12.2]

$$\begin{tikzcd} F(U) \arrow[r, "F(r_{UV})"'] \arrow[d] \arrow[rr, bend left] & F(V) \arrow[r, "F(r_{VU})"'] \arrow[d] & F(U) \arrow[d] \\ U \arrow[r, "r_{UV}"] \arrow[rr, bend right] & V \arrow[r, "r_{VU}"] & U \end{tikzcd} \qquad \begin{tikzcd} F(U) \arrow[r] \arrow[d] & F(V) \arrow[d] \\ U \arrow[r] \arrow[ur, phantom, "\xRightarrow{F|_{U\cap V}}"] & V \end{tikzcd} \qquad [12.17]$$

induced by the LEE of relative properties [12.4],which make them into same week homotopy equivalence class which gives a categorical explaination of the retract [9.104] that is mod this weak equivalence to stablized the relative property in ordinary fibered category see below [12.2].

Corollary13.4 Combining the theorem13.3 above [12.13],the n-groupoid ${}^{\mathscr{P}}\mathbf{Ret}(\to)$ is a space of dualities.A tract is a stablization of relative property from the homotopy type of ordinary properties induced by the relative property.

$$\begin{tikzcd} \text{homotopy type of ordinary properties} \arrow[r, "\text{stabilize}"] \arrow[d, hook] & \text{stable relative } P \arrow[d, hook] \\ \text{ordinary fibered category} \arrow[r, "\mathbf{Ret}"] & \text{prestack} \end{tikzcd} \qquad [12.18]$$

Based on the unified field$^!$ theory [11.85],we want to do a summarization

$$\begin{tikzcd}
 & \text{unknown} \arrow[rr, two heads] & & \text{known}\,\mathrm{P}(\mathbf{U})^{-1}\mathbf{F} & \\
 & \text{non-solvable theory} \arrow[u, "\mathbf{Ret}^*\mathrm{P}(\mathbf{T}_0^\delta)^{-1}"] \arrow[rr, two heads, "\mathbf{Ret}"] & & \text{solvable}\,\mathcal{S}^{\text{rep}}(\mathcal{S}) \arrow[u, "\mathrm{P}(\mathbf{T}_\bullet^\delta)"'] & \\
\text{(Closed)} \arrow[ur] \arrow[r] \arrow[ur, phantom, "\Leftarrow \mathrm{Sh}_{\text{D-branes}}"] & \text{(Open)} \arrow[u] \arrow[rr, two heads] \arrow[rr, bend right, "\mathbf{Ret}"'] & & \mathcal{S}_{\text{cons.}} \arrow[u] \arrow[r, phantom, "\overset{\boldsymbol{\mathcal{A}}}{\Rightarrow}"] & \mathcal{S}^{\text{pre}}_{\text{cons.}} \arrow[l] \arrow[ul]
\end{tikzcd} \qquad [12.19]$$

where (open) means superstring theory of open strings,blue means it is not a full theory it is a space for nontrivial behaviors (red) living on.Recall definition13.2 above [12.7],a D-brane is a continuous extension of étale morphisms (étale relations),thus after retract,we get the generalized superalgebraifold is a contineous extension of étale equivalence relation (generalized super relative properties).The strings in the last row of [12.19] are

$$\text{closed str} \qquad \text{open str attaching on D-brane} \twoheadrightarrow [\mathfrak{s}^c_{\text{weak}}] \qquad [\mathfrak{s}^c] \qquad [12.20]$$

Back to model category,the second axiom is composition of 2-morphisms.In fibered category $p : F \to C$ which is called Grothendieck's fibration which is given by the 2-categorical structure [9.25].And in model category,we need to realize that simplicial structure give us a way to construct higher morphism

$$\begin{tikzcd}
A_1 \arrow[r, dotted] \arrow[d, dotted] & A_2 \arrow[r, "f"] \arrow[d, "p"] & A_3 \arrow[d, dotted] \\
B_1 \arrow[ur, "h_1"] \arrow[r, "g"] & B_2 \arrow[ur, "h_2"] \arrow[r, dotted] & A_3
\end{tikzcd} \qquad [12.21]$$

such 1-morphism p in model 1-category is (co)fibration lifting by $(h_1)h_2$.In this case,we have a homotopy between 1-morphisms $f \circ h_1 \to h_2 \circ g$.But for physics,recall we regard the corepresentable representable pair [11.25] as the time evolution (sum over all possible paths) and the existence of 2-morphism gives us a way to continuously extent the path (1-morphism).Also evolution is a motion,we need to consider super generalized general relativity [8.33],actually the motion happens locally in self T-dual (self rest-motion) generalized Lorentz module in $D = 10$ over $\mathbb{C}$,for an evolution of bosonic field

$$^{\text{DG,Lie}}(\mathcal{X}_0 \oplus \mathcal{X}_1^*) \rightsquigarrow_{\text{local,self}} {}^{\text{DG,Lie}}(X_0 \boxtimes \bar{X}_0) \hookrightarrow {}^{\text{DG,Lie}}\mathcal{X}_0 \qquad [12.22]$$

where the [5.13] tells us we can use complex conjugation as a self-dual structure.And by using the quantum sheaf we used in [11.23],we have

$$\mathcal{O}^{\bullet\text{qut}}_{\tilde{\mathfrak{g}}}({}^{\text{DG,Lie}}(X_0 \boxtimes \bar{X}_0)) \cong \mathcal{O}^{\bullet\text{qut}}_{\tilde{\mathfrak{g}}}({}^{\text{DG,Lie}}X_0) \times \bar{\mathcal{O}}^{\bullet\text{qut}}_{\tilde{\mathfrak{g}}}({}^{\text{DG,Lie}}\bar{X}_0) = 1 \qquad [12.23]$$

Recall self T-duality [8.14] which has the regularity or renormalization.For further discussion,some information we have not acquired.

12.2 The natural of QFT and GR and *Nonexpressibility*

we need to combine the [8.28],[8.37] and [12.12],we have

$$\begin{array}{ccc} \mathcal{R}^+,\mathcal{R}^- & \xleftrightarrow{\mathrm{P}(\mathbf{T})^{-1}\mathrm{P}(\mathbf{U})^{-1}\mathbf{F}} & \mathscr{R}^{++}_{\mathscr{M}},\mathscr{R}^{--}_{\mathscr{M}},\widetilde{\mathscr{R}} \\ \downarrow & & \downarrow \\ \mathfrak{s}^c_{\text{weak}} \text{ in } D & \longleftrightarrow & \mathfrak{s}^c \text{ in } D+1 \end{array} \quad [12.24]$$

we see the LEE of the F-duality,along the [12.19] we have

$$\mathrm{P}(\mathbf{T}^\delta)^{-1}\mathrm{P}(\mathbf{U})^{-1}\mathbf{F} \cong \mathbf{Ret}_*\mathbf{Ads/Cft} \quad [12.25]$$

actually is a retracted holographic duality,which is a duality between self in the bulk and weak form of closed string on the boundary over étale site.We know the **F**-fusion gives us the 2-nonexistence [11.6],so the question is we have not calculated for the $\hat{F}$.We know see this by

$$\mathrm{P}(\mathbf{T}^\delta)^{-1}\mathrm{P}(\mathbf{U})^{-1} = \mathrm{P}(\mathbf{T}^\delta \times_{\mathscr{M}^\heartsuit} \mathbf{U})^{-1} \quad [12.26]$$

from observation of [12.12] and we use this to fuse [8.28] and [8.37]

$$\begin{aligned} &(\mathcal{X}_0 \times_S \mathcal{X}_1^*)^{\text{unself}\,\mathbf{T}} \boxtimes_{\mathbf{T}^\delta \times_{\mathscr{M}^\heartsuit}\mathbf{U}} (\mathcal{X}_0 \times_S \mathcal{X}_1^*)^{\text{self}\,\mathbf{T}} \\ &= ((\mathcal{X}_0 \times_S \mathcal{X}_1^*) \oplus (\mathcal{X}_1^* \times_S \mathcal{X}_0)) \boxtimes_{\mathbf{T}^\delta \times_{\mathscr{M}^\heartsuit}\mathbf{U}} (\widetilde{\mathcal{X}}_1^* \oplus \widetilde{\mathcal{X}}_1^*) \cdots \\ &= (\mathcal{X}_0 \times_S \mathcal{X}_1^*) \boxtimes_{\mathbf{T}^\delta \times_{\mathscr{M}^\heartsuit}\mathbf{U}} \widetilde{\mathcal{X}}_1^* \oplus (\mathcal{X}_1^* \times_S \mathcal{X}_0) \boxtimes_{\mathbf{T}^\delta \times_{\mathscr{M}^\heartsuit}\mathbf{U}} \widetilde{\mathcal{X}}_1^* \cdots \\ &= ((\mathcal{X}_0 \times_{\mathscr{M}^{\text{rep}}} \widetilde{\mathcal{X}}_0) \oplus (\mathcal{X}_1^* \times_{\mathscr{M}^{\text{rep}}} \widetilde{\mathcal{X}}_1^*)) \boxtimes_{\mathbf{T}^\delta \times_{\mathscr{M}^\heartsuit}\mathbf{U}} ((\mathcal{X}_0 \times_{\mathscr{M}} \mathcal{X}_0) \\ &\qquad \oplus (\mathcal{X}_1^* \times_{\mathscr{M}} \mathcal{X}_1^*)) \\ &\cong_{\mathrm{P}(\mathbf{T}^\delta \times_{\mathscr{M}^\heartsuit}\mathbf{U})} ((\mathcal{X}_0 \times_{\mathscr{M}^{\text{rep}}} \widetilde{\mathcal{X}}_0) \oplus (\mathcal{X}_0 \times_{\mathscr{M}^{\text{rep}}} \widetilde{\mathcal{X}}_0))/\mathbb{Z}_2 \\ &\qquad + ((\mathcal{X}_1^* \times_{\mathscr{M}} \widetilde{\mathcal{X}}_1^*) \oplus (\mathcal{X}_1^* \times_{\mathscr{M}} \widetilde{\mathcal{X}}_1^*))/\mathbb{Z}_2 \\ &\cong_{\mathrm{P}(\mathbf{U})} (\mathcal{X}_0 \times_{\mathscr{M}^{\text{rep}}} \widetilde{\mathcal{X}}_0) \boxtimes_{\mathbf{U}} (\mathcal{X}_1^* \times_{\mathscr{M}} \widetilde{\mathcal{X}}_1^*) \end{aligned} \quad [12.27]$$

And compared with [9.144] and [11.8],it indeed corresponds to generalized super relative 2-properties counting for the field for self U-duality $\hat{F}$ [11.8]

$$(\mathrm{P}(\mathbf{U})\mathscr{R})^{\text{self}\,\mathbf{U}} = (\mathcal{X}_0 \times_{\mathscr{M}^{\text{rep}}} \widetilde{\mathcal{X}}_0) \boxtimes_{\mathbf{U}} (\mathcal{X}_1^* \times_{\mathscr{M}} \widetilde{\mathcal{X}}_1^*) = \hat{\mathscr{R}}_{\mathscr{M}^{\text{rep}}} \boxtimes_{\mathbf{U}} \hat{\mathscr{R}}_{\mathscr{M}} \quad [12.28]$$

Based on this and [9.144],we want to form a combination

$$\begin{aligned} &((\mathcal{X}_1 \times_{\mathscr{M}} \mathcal{X}_1) \boxtimes_{\mathbf{U}} (\widetilde{\mathcal{X}}_1^* \times_{\widetilde{\mathscr{M}}} \widetilde{\mathcal{X}}_1^*)) \oplus ((\mathcal{X}_0 \times_{\mathscr{M}^{\text{rep}}} \widetilde{\mathcal{X}}_0) \boxtimes_{\mathbf{U}} (\mathcal{X}_1^* \times_{\mathscr{M}} \widetilde{\mathcal{X}}_1^*)) \\ &= ((\mathcal{X}_1 \times_{\mathscr{M}} \mathcal{X}_1) \vee_{\bigcirc} (\mathcal{X}_0 \times_{\mathscr{M}^{\text{rep}}} \widetilde{\mathcal{X}}_0)) \oplus ((\widetilde{\mathcal{X}}_1^* \times_{\mathscr{M}^{\text{rep}}} \widetilde{\mathcal{X}}_1^*) \\ &\qquad \vee_{\bigcirc} (\mathcal{X}_1^* \times_{\mathscr{M}} \widetilde{\mathcal{X}}_1^*)) = \mathscr{R}_{\mathscr{M}} \vee_{\bigcirc} \hat{\mathscr{R}}_{\mathscr{M}^{\text{rep}}} \oplus \widetilde{\mathscr{R}}_{\mathscr{M}^{\text{rep}}} \vee_{\bigcirc} \hat{\mathscr{R}}_{\mathscr{M}} \end{aligned} \quad [12.29]$$

See the definition [10.22],which is a smooth presentation of $\boldsymbol{\mathcal{M}}$

$$\text{Ét}(\mathscr{R}_{\mathscr{M}} \vee_{\bigcirc} \hat{\mathscr{R}}_{\mathscr{M}^{\text{rep}}} \oplus \widetilde{\mathscr{R}}_{\mathscr{M}^{\text{rep}}} \vee_{\bigcirc} \hat{\mathscr{R}}_{\mathscr{M}})/\boldsymbol{\mathcal{M}} \equiv \text{Lis-Ét}^{\text{GenSup}}_{\text{cons.}}(\boldsymbol{\mathcal{M}}) \quad [12.30]$$

we can discuss the relative space of the generalized super algebraic stack $\boldsymbol{\mathcal{M}}$

$$\textbf{the } \boldsymbol{\mathcal{M}}^{!}\textbf{-theory} \Rightarrow (\mathrm{RS}^{\mathrm{GenSup}}_{\mathrm{cons.}}/\boldsymbol{\mathcal{M}}) \supset \text{Lis-Ét}^{\mathrm{GenSup}}_{\mathrm{cons.}}(\boldsymbol{\mathcal{M}}) \qquad [12.31]$$

which gives as a fibered category over the relative space.And a fiber is a local unified field$^{!}$ theory.Which means we can apply it locally like [12.22].And to see the morphism of stacks is representable [10.20]

$$\begin{tikzcd} \boldsymbol{\mathcal{M}}^{\mathrm{pre.}}_{\mathrm{cons.Ran.}} \times_{\boldsymbol{\mathcal{M}}} (\cdots) \arrow[r] \arrow[d, "\mathrm{P}(\mathbf{U})^{-1}\mathrm{P}(\mathbf{F})\cong\Delta_{\mathbf{U}*}(\mathrm{P}(\mathbf{F}))"] & \mathscr{R}_{\mathscr{M}} \vee_{\bigcirc} \hat{\mathscr{R}}_{\mathscr{M}^{\mathrm{rep}}} \oplus \widetilde{\mathscr{R}}_{\mathscr{M}^{\mathrm{rep}}} \vee_{\bigcirc} \hat{\mathscr{R}}_{\mathscr{M}} \arrow[d, "\mathrm{P}(\mathbf{F})"] \\ \boldsymbol{\mathcal{M}}^{\mathrm{pre.}}_{\mathrm{cons.Ran.}} \arrow[r, "\mathrm{P}(\mathbf{U})"] & \boldsymbol{\mathcal{M}} \end{tikzcd} \qquad [12.32]$$

where by the [10.28] and [12.25],we have

$$\mathrm{P}(\mathbf{T}^{\delta})\mathrm{P}(\mathbf{Ret}_{*}\mathbf{Ads/Cft}) \cong \mathrm{P}(\mathbf{U})^{-1}\mathrm{P}(\mathbf{F}) \cong \Delta_{\mathbf{U}*}(\mathrm{P}(\mathbf{F})) \qquad [12.33]$$

Theorem13.6 A relative effect is generated by a property P evolves in a direction away from its own property P,or towards property that does not exist relative to its own property,which is equivalent to say relative effect is a long range effect.

To understand the theorem,the Lorentz transformation gives an example that a relative effect is about $z' = z + vt$ (motion frame) relative to z (rest-frame).But the quantum field theory is a theory about locality,which mean the nontrivial behavior happens in $z \to z'$ [3.12].In this case,the reason why we cannot combine theory of quantum and that of gravity is just because the ordinary space is not good enough to have a local relative effect,also the ordinary superstring theory is just to contain quantum (open string) and gravity (closed string),but it is not about combing quantum and gravity,it cannot because localization on the D-brane (D0-brane) which is equivalent to a localization of openstring [8.22],we know every string theories need to have closed string,so the theory of D0-brane must contain the localization of closed string,and the space is not good enough,thus the theory becomes non-solvable [12.19].Conversely,quantum effect is a local effect,we do not have a long range quantum effect in the ordinary space (cannot explain the quantum entanglement).

Corollary13.7 The general relativity is a theory describing continuous extension of long range relative effects (global properties).The quantum field theory is a theory describing discontinuous extension of local quantum effects (local properties).

Theorem13.8 A theory quantifying gravity must describe local properties and describe the global properties **at the same time**.

It is completely meaningless for an ordinary space which likes the equivalence [7.25] is meaningless analytically.Now,we are in the right hand side of [12.19],the fiber product of [12.31] is isomorphic to

$$\boldsymbol{\mathcal{M}}^{\mathrm{pre.}}_{\mathrm{cons.Ran.}} \times_{\boldsymbol{\mathcal{M}}} (\cdots) \cong (\mathscr{R}_{\mathscr{M}} \vee_{\boldsymbol{\mathcal{M}}^{\mathrm{pre.}}_{\mathrm{cons.Ran.}}} \hat{\mathscr{R}}_{\mathscr{M}^{\mathrm{rep}}} \oplus \widetilde{\mathscr{R}}_{\mathscr{M}^{\mathrm{rep}}} \vee_{\boldsymbol{\mathcal{M}}^{\mathrm{pre.}}_{\mathrm{cons.Ran.}}} \hat{\mathscr{R}}_{\mathscr{M}}) \quad [12.34]$$

which is an algebraic space by [10.30] as each pre M-theory is algebraic below [10.27].Recall [11.6],the number counting of [12.29] and [12.34] gives 2-nonexistence.Which means over the site [12.30] and [12.31],the M-theory globally is $\bigcirc$ and locally is $\bigcirc$.In this case,we localized the global relative effect to local relative effect.Which means when making a certain property exist,the property also tends to not exist **at the same time** to cancel the existence of the property,which is guaranteed by the local 2-nonexistence $\bigcirc$.And we call this as the **super generalized relative principle**.

Now,we can back to derived algebraic geometry at [12.21].The category with simplicial structure is homotopy enriched for retraction and lifting.Thus we have a naturally model structure on $\mathrm{dgAlg}_k^{\geq 0}$ with the weak equivalence as the quasi-isomorphism that is with the projective resolution (reverse of injective [12.16])

$$\begin{tikzcd} & A_i \arrow[d, "\text{surj}"] \\ B_{i+1} \arrow[ur, "\simeq_{\text{weak}}"] \arrow[r, "\text{inj}"'] & B_i \end{tikzcd} \quad 0 \to H_{i+1}(B_{i+1}) \xrightarrow{\simeq} H_i(A_i) \to 0 \qquad [12.35]$$

where we used $B_{i+1} \to A_i \cong B_{i+1} \to B_i \to A_i$,and such surjection $A_i \to B_i$ is fibration with left lifting property and cofibrations are $A_\bullet \to B_\bullet$ with right lifiting properties with respect to the fibrations.Recall the left evolution above [12.11],for $\mathrm{DGAff}^{\geq 0}$ it should be a injective resolution,and the fibrations in the former corresponds to the cofibrations in the latter.An object in model category is fibrant if the map to final object is a fibration and it is cofibrant if the map to initial object is cofibration.For a weak equivalent $A \to A'$ attaching with fibration $A' \to B$ where B is a final object we call A' a fibrant replacement.And in $\mathrm{dgAlg}_k^{\geq 0}$,the map from every object to final object is automatically surjective so every object is fibrant.For a scheme X over S,we have

$$\begin{tikzcd} P \times_X X \arrow[r] \arrow[d, "\Delta^*_{X/S}(p \times_S x)"] & P \times_S X \arrow[d, "p \times_S x"] \\ X \arrow[r, "\Delta_{X/S}"'] \arrow[ur, "h"] & X \times_S X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, bend left=60] \arrow[r, bend left=20] \arrow[r] \arrow[r, bend right=20] \arrow[r, bend right=60] \arrow[r, phantom, "\Uparrow h"] & Z \end{tikzcd} \xrightarrow[\text{local}\ \bigcirc]{\subset} {}^{\mathrm{DG,Lie}}(X_0 \boxtimes \bar{X}_0)$$

[12.36]

we denote $P \times_S X$ is the path object of X,with the weak equivalence h,we call $P \times_S X$ a path of X.We can see the weak equivalence induces paths of X,which explain the measure of path integral in the topological space and the paths are confined in a local region by the super generalized general relativity [12.23] with local relative effect (in the unified field$^!$ theory).Loosely speaking,when it loses local property it gains local property to cancel it,and it is confined.And such elegant model will be homotopy weakly projected to our world by $\mathbf{Ret}^*$

Definition13.9 *Nonexpressibility* means when any operation happens (observation,discussion imagination etc.),it induces generating properties **at the same time**,which affect the system with local 2-nonexistence.Which should be regarded as the most abstract and difficult thing with $\bigcirc$ of **the $\mathcal{M}^!$-theory**.This means the truth (in non-perturbation) is in confinement.

A functor $F : C \to D$ of model categories is (left)right Quillen if has a (right)left adjoint G and F preserves (co)fibrations and trivial (co)fibrations.If $G \dashv F$ an adjunction which means $\mathrm{Hom}_C(G(d), c) \cong \mathrm{Hom}_D(d, F(c))$ for all objects and satisfy the above conditions we call this is a Quillen adjunction.From [12.14],for above F which is right Quillen,we have

$$\begin{tikzcd} & A' \arrow[d, "\text{fibrant}"] \\ A \arrow[ur, "\simeq_{\text{weak}}"] \arrow[r] & B \end{tikzcd} \xrightarrow{F} \begin{tikzcd} & F(A') \arrow[d, "\text{fibrant}"] \\ F(A) \arrow[ur] \arrow[r] & F(B) \end{tikzcd} \xrightarrow{\lambda_D} \begin{tikzcd} & F(A') \arrow[d] \\ F(A) \arrow[ur, "\simeq"] \arrow[r] & \lambda_D \circ F(B) \end{tikzcd}$$

[12.37]

the Quillen functor preserves the fibration,thus gives us a right derived functor $\mathbf{R}F$ sending A to the $F(A')$ with A' the fibrant replacement of A.If we have equivalence of category $\mathbf{R}F : \mathrm{Ho}(C) \xrightarrow{\simeq} \mathrm{Ho}(D)$ with quasi-inverse $\mathbf{L}G$,then $G \dashv F$ is said to be a Quillen equivalence.Which means following [12.14],for a unit $'F(A)$ which is a cofibrant in $\mathrm{Ho}(D)$,$G('F(A)) \xrightarrow{\simeq}_{\text{weak}} A$ in C and for a co-unit $G(B)'$ which is a fibrant in $\mathrm{Ho}(C)$ we have $B \xrightarrow{\simeq}_{\text{weak}} F(G(B)')$ in D.We know that $\lim_I : C^I \to C$ is a functor,so we have homotopy limit which is right derived functor of the limit functor $\mathbf{R}\lim_I$ for objects C^I quotient the weak homotopy equivalence see difference with [7.23].And we denote the homotopy fiber product by $X \times^h_Y Z$.If Y' is a fibrant,the homotopy fiber product is $X \times^h_{Y'} Z$ with $X \xrightarrow{\simeq}_{\text{weak}} Y'$ and $Z \xrightarrow{\simeq}_{\text{weak}} Y'$ are fibrant replacement.A way to construct is for [12.36],we have the diagram where we denote $PY = P \times_S Y$

$$\begin{tikzcd} X \times^h_Y Z \arrow[r] \arrow[d] \arrow[rr, bend left] & Z \arrow[d] & Z \times_{Y,\mathrm{ev}_1} PY \arrow[l] \arrow[d] \arrow[dl, "h_1"'] \\ X \arrow[r] & Y & PY \arrow[l, "\mathrm{ev}_1"] \end{tikzcd}$$

[12.38]

with $\mathrm{ev}_0, \mathrm{ev}_1$ as the first and second projections so we have

$$X \times^h_Y Z \cong X \times_{Y,h_1} (Z \times_{Y,\mathrm{ev}_1} PY) \cong X \times_{Y,\mathrm{ev}_0} PY \times_{\mathrm{ev}_1,Y} Z \qquad [12.39]$$

as we have $h_1 = \mathrm{ev}_0$ a weak homotopy equivalence.Now,based on the evolution of relative property [9.127] and [9.130],if we combine [12.36] we can see further evolution of property (around the relative property) in the homotopy type induced by the homotopy weak projection $\mathbf{Ret}^*$ of the stable relative property

$$\begin{tikzcd} & X \arrow[dl, bend right] \arrow[d] & X/R \arrow[l] \arrow[d] & X/(X \times^{h^*}_S X) \arrow[l, twoheadrightarrow, "\mathbf{Ret}"'] \arrow[ll, bend right] \arrow[dl, bend left] \\ X \times^{h^*}_S X \arrow[r, twoheadrightarrow, "\mathbf{Ret}"] & X \times_S X & X \arrow[l] \arrow[ll, bend left] & \end{tikzcd}$$

[12.40]

where the diagram is a relative system and we observe in the retracted frame so we used h^*,which means in retracted frame we get stable relative property $X \times_S X$ but in retracted* frame it behaves like $X \times_S^h X$ (no star).Combing the diagram [12.19] and put the generalized super setting in,we get $X \times_S^h X$ is a D-brane where we want to discuss in the next section.

Definition13.10 A D-brane (based on definition13.2) is a section of sheaf quotient of a weak homotopy equivalence satsifying the diagram [12.40].Also,see the algebraic space as a sheaf quotient of a relative property [9.123].

For a trivial resolution $A \mapsto (A \leftarrow 0 \leftarrow 0 \cdots)$ of [12.16],we have an embedding,if we view it as a trivial functor $\mathrm{id} : \mathrm{Alg}_k \subseteq \mathrm{dgAlg}^{\geq 0}$,then we can send it to Ho along [12.14],in this case we get a right derived functor

$$\mathbf{R}\mathrm{id} : \mathrm{Alg}_k \to \mathrm{Ho}(\mathrm{dgAlg}^{\geq 0}), \quad \mathrm{Hom}_{\mathrm{dgAlg}^{\geq 0}}(A_\bullet, B) = \mathrm{Hom}_{\mathrm{Alg}_k}(H_0(A_\bullet), B) \tag{12.41}$$

also recall the setting for the derived functor below [12.16] with $A_\bullet \in \mathrm{dgAlg}^{\geq 0}$ and $B \in \mathrm{Alg}_k$.For fibrant replacements $A_\bullet \xrightarrow{\simeq}_{\text{weak}} A_\bullet$ and $B \xrightarrow{\simeq}_{\text{weak}} B$

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dgAlg}^{\geq 0})}(A_\bullet, B) &\cong A_\bullet \times^h_{A_\bullet \times_B B} B \cong A_\bullet \times_B B \in \mathrm{dgAlg}^{\geq 0} \\ &\cong \mathrm{Hom}_{\mathrm{dgAlg}^{\geq 0}}(A_\bullet, B) = \mathrm{Hom}_{\mathrm{Alg}_k}(H_0(A_\bullet), B) \end{aligned} \tag{12.42}$$

where we used every object is fibrant above [12.36] and homotopy fiber product around [12.38],so the functor [12.41] is fully faithful and by [12.9],we get

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{DGAff}^{\geq 0})}(X, Y_\bullet) \cong \mathrm{Hom}_{\mathrm{Aff}}(X, \pi^0 Y_\bullet) \tag{12.43}$$

for a scheme X and derived scheme $Y_\bullet$.

For $A_{0\bullet} \oplus A^*_{1\bullet}, B_{0\bullet} \oplus B^*_{1\bullet} \in \mathrm{dgAlg}^{\geq 0}_{\mathrm{GenSup}}$,we have the graded tensor product

$$\begin{aligned} (A_{0\bullet} \oplus A^*_{1\bullet}) \otimes (B_{0\bullet} \oplus B^*_{1\bullet}) &= [A_{0\bullet} \otimes B_{0\bullet} \oplus A^*_{1\bullet} \otimes B^*_{1\bullet}] \\ &\oplus [A_{0\bullet} \otimes B^*_{1\bullet} \oplus A^*_{1\bullet} \otimes B_{0\bullet}] = \bigoplus_{i+j=\bullet} (A_i \otimes_k B_j)_0 \oplus (A_i \otimes_k B_j)^*_1 \\ &= (A \otimes_k B)_{0\bullet} \oplus (A \otimes_k B)^*_{1\bullet} \end{aligned} \tag{12.44}$$

for the differential we first consider a bilinear map $A_i \times M_j \to M_{i+j}$

$$\begin{aligned} 0 = \partial\partial(a_i m_j) &= \partial(\partial a_i m_j + (-1)^{\deg(a_i)} a_i \partial m_j) \\ &= (-1)^{\deg(a_i)-1} \partial a_i \partial m_j + (-1)^{\deg(a_i)} \partial a_i \partial m_j \end{aligned} \tag{12.45}$$

So,we can let $\partial = \partial_0 \oplus \partial_1^*$ and $\partial_0(a_{0i} \otimes b_{0j}) = \partial_0 a_{0i} \otimes b_{0j} + (-1)^{\deg(a_{0i})} a_{0i} \otimes \partial_0 b_{0j}$ for $a_{0i} \otimes b_{0j} \in (A \otimes_k B)_{0(i+j)}$ and ∂_1^* for $a_{0i} \otimes b_{1j} \in (A \otimes_k B)^*_{1(i+j)}$.And the multiplication is $(a \otimes b) \cdot (a' \otimes b') = (-1)^{\deg(a')\deg(b)+F}(aa' \otimes bb')$ where F is the world-sheet spinor number.In [12.44],$\otimes_k : \mathrm{dgAlg}^{\geq 0}_k \times \mathrm{dgAlg}^{\geq 0}_k \to \mathrm{dgAlg}^{\geq 0}_k$ and we have an adjunction giving us right adjoint $C_\bullet \mapsto C_\bullet \otimes_k C_\bullet$

$$\mathrm{Hom}_{\mathrm{dgAlg}^{\geq 0}_k}((A \otimes_k B)_\bullet, C_\bullet) \cong \mathrm{Hom}_{\mathrm{dgAlg}^{\geq 0}_k \times \mathrm{dgAlg}^{\geq 0}_k}(A_\bullet \otimes_k B_\bullet, C_\bullet \otimes_k C_\bullet) \tag{12.46}$$

the right adjoint is right Quillen above [12.37],and by a lemma 2.29 in [21],for $G \dashv F$ of model categories,G is right Quillen if and only if F is left Quillen,so the tensor product of model categories $\otimes_k$ is left Quillen.In this case,along [12.14] we can define the derived tensor graded tensor product

$$\otimes_k^{\mathbf{L}} = \mathbf{L}\otimes_k : \mathrm{Ho}(\mathrm{dgAlg}_k^{\geq 0}) \times \mathrm{Ho}(\mathrm{dgAlg}_k^{\geq 0}) \to \mathrm{Ho}(\mathrm{dgAlg}_k^{\geq 0}) \qquad [12.47]$$

Now,for $A_\bullet \in \mathrm{dgAlg}_k^{\geq 0}$ the $A_\bullet$-module is quasi-free if the underlying graded module is flat over underlying graded algebra see [12.7].For $B_\bullet \in \mathrm{Ho}(\mathrm{dgAlg}_k^{\geq 0})$,if $A_\bullet \xrightarrow{\simeq}_{\mathrm{weak}} B_\bullet$ is a quasi-isomorphism,we say $A_\bullet$ is a model of $B_\bullet$ with $A_\bullet$ defined up to isomorphism and $B_\bullet$ defined up to quasi-isomorphism.See above [7.29],we let $B_\bullet$ be (quasi-)flat then the $A_\bullet \otimes_k (-)$ is a right exact functor with left derived functor $\mathrm{Tor}_i^k(-, B_\bullet) = H_i((-) \otimes_k B_\bullet)$.By below [12.37] and [12.46],the we can give a cofibration replacement of $A_\bullet, B_\bullet$ we have

$$\begin{aligned} \mathrm{Tor}_i^k(A_\bullet, B_\bullet) &= H_i(A_\bullet \otimes_k B_\bullet) = H_i('A_\bullet \cong A_\bullet \otimes_k {}'B_\bullet \cong B_\bullet) \\ &= H_i(('A \otimes_k {}'B)_\bullet) = H_i((A \otimes_k^{\mathbf{L}} B)_\bullet) \end{aligned} \qquad [12.48]$$

where we used the notation [12.47],back to ordinary category,we get quasi-isomorphism $(A \otimes_k B)_\bullet \to (A \otimes_k^{\mathbf{L}} B) \in \mathrm{Ho}$,so the former is a model of the left derived graded tensor product.In $\mathrm{DGAff}^{\geq 0}$,we have derived scheme

$$X \times_Z^h Y = \mathrm{Spec}(A \otimes_C^{\mathbf{L}} B),\ X = \mathrm{Spec}(A_\bullet), Y = \mathrm{Spec}(B_\bullet), Z = \mathrm{Spec}(C_\bullet) \quad [12.49]$$

Now for an example,we want to calculate $k \otimes_{k[t]}^{\mathbf{L}} k$,by below [12.37] and [12.48],we should start with $k[t] \cdot s \otimes_{k[t]} k$ with a weak equivalence $k[t] \xrightarrow{\simeq}_{\mathrm{weak}} k$ induced by $k[t] \to k[t] \cdot s$ over k and which give us a short exact sequence

$$\to k[t] \cdot s \otimes_{k[t]} k \xrightarrow{\partial} k[t] \otimes_{k[t]} k \xrightarrow{\partial} 0 \qquad [12.50]$$

We should set $\deg(t) = 0, \deg(s) = 1, \partial s = 1$ and we have

$$\partial^2(ts) = \partial(\partial ts + t) = \partial^2 ts + \partial t \partial s + \partial t \qquad [12.51]$$

which means $\partial t = 0$ and by the definition of derived scheme [12.9]

$$\pi^0 \mathrm{Spec}(k \otimes_{k[t]}^{\mathbf{L}} k) = \mathrm{Spec}(H_0(k[t] \otimes_{k[t]} k)) \cong \mathrm{Spec}(k[t]), \quad \partial t = 0 \qquad [12.52]$$

In this case,we can see the derived intersection scheme $\{0\} \times_{\mathbb{A}^1}^h \{0\}$ gives loop (no boundary) paths,which can be viewed as generating closed strings from vacuum which is a right evolution compared to [12.11].Compared to the stable relative property $\mathrm{Spec}(k \otimes_{k[t]} k) = \mathrm{Spec}(k) = \{0\}$.

For dual numbers we used in [7.15] and [8.20]

$$R[\epsilon]/(\epsilon^2) = R \oplus R\epsilon, \quad C^\infty(\mathbb{R})[\sigma]/(\sigma^2), \quad \mathscr{O}^{\mathrm{hol}}(\mathbb{C})[z]/(z^2) \qquad [12.53]$$

For a smooth scheme X,the $k[\epsilon]$-points $\mathrm{Spec}(k[\epsilon]) \to X$ form a tangent space

$$X(k \oplus k\epsilon) \cong \underline{\mathrm{Spec}}(\mathscr{O}_k[\epsilon]/(\epsilon^2)) \cong \{(x, v) | (x) \in \mathcal{O}_k, v \in \mathcal{O}_k[\epsilon]/(\epsilon^2)\} \qquad [12.54]$$

For a nilpotent surjection of rings $f : A \to B$,we have for $b^2 = 0$ we have $f(a)^2 = b^2 = f(a^2) = 0$,which means such surjection can be written as square zero extensions with $xy = 0, \forall x, y \in \ker(f)$,such f gives us an isomorphism

$$A \otimes_B A \cong A \otimes_B (B \oplus \ker(f)), \quad (a, a') \mapsto (a, (f(a), a - a')) \qquad [12.55]$$

Taking for schemes,we have $X(A) \times_{X(B)} X(A) \cong X(A) \times_{XB} X(B \oplus \ker(f))$.Then by [12.54],we see the tangent space $X(B \oplus (\ker(f)))$ acts on the fibers $X(A)$ over $X(B)$.Similarly to problem of representability see [11.32] and above,if we use ${}^{\mathrm{DG,Lie}}X$ the nontrivial behaviors on the DG Lie scheme give obstructions to lift B-points to A-points.For a square zero extension with $\ker(f) = I$,like [12.16] we have $\tilde{B}_\bullet = (A \hookleftarrow I \leftarrow 0 \cdots)$ with $\tilde{B}_\bullet \xrightarrow{\simeq}_{\mathrm{weak}} B$,for a surjection $u : \tilde{B}_\bullet \to (B \overset{0}{\leftarrow} I \leftarrow) \cdots) = \tilde{B}_\bullet \oplus I[1]$ to the sequence killing the image of I,we have $A = \tilde{B}_\bullet \otimes^{\mathbf{L}}_{\tilde{B}_\bullet \oplus I[1]} B \in \mathrm{dgAlg}_k^{\geq 0}$.A functor $F : \mathrm{Ho}(\mathrm{dgAlg}_k^{\geq 0}) \to \mathrm{Set}$ is half-exact if $F(0) \cong *$,factorization $F((A \otimes B)_\bullet) \cong F(A_\bullet) \times F(B_\bullet) \forall A_\bullet, B_\bullet \in \mathrm{dgAlg}_k^{\geq 0}$ and surjective factorization of pushout $A_\bullet \otimes^{\mathbf{L}}_k B_\bullet$ for which any representable functor $h^{B_\bullet}$ is half-exact.Now,for a representable functor F on $\mathrm{Ho}(\mathrm{dgAlg}_k^{\geq 0})$

$$F(A) \to_{\mathrm{surj}} F(B) \times_{u, F(\tilde{B}_\bullet \oplus I[1]), 0} F(B), \quad u(x) = (x, 0) \Leftrightarrow x \in \mathrm{Im}(F(f)) \qquad [12.56]$$

In this case,we get a tangent space which is an obstruction space $(F(\tilde{B}_\bullet \oplus I[1]), u)$ a element $y \in I[1]$ gives $B \setminus f(y)$ which means a nontrivial element in the obstruction space gives an obstruction to lift elements from $F(B)$ to $F(A)$.

Next,we study cotangent complex,which gives a clarification of [7.27],[9.91].A morphism $R_\bullet \to A_\bullet$ gives us complex of Kähler differential

$$a \otimes b - b \otimes a + I^2 \in \Omega^1_{A/R_\bullet} = (I = \ker((A \otimes_R A)_\bullet) \to A_\bullet)/I^2 \qquad [12.57]$$

with derivation $d : A_\bullet \to \Omega^1_{A/R_\bullet}, a \cdot 1 \mapsto da \otimes 1 - 1 \otimes da + I^2$.In general for a cofibrant replacement $\tilde{A}_\bullet \to A_\bullet$,along the pushout gives us

$$\mathbb{L}^{A/R} = (\Omega^1_{\tilde{A}/R} \otimes_{\tilde{A}} A)_\bullet \in \mathrm{dgMod}_{A_\bullet} \qquad [12.58]$$

Recall the Quillen adjunction above [12.37],we have such adjunction here

$$G : B_\bullet/A_\bullet \mapsto (\Omega^1_{B/R} \otimes_B A)_\bullet, \quad M_\bullet/(\epsilon^2) \mapsto A_\bullet \oplus M_\bullet/(\epsilon^2) \qquad [12.59]$$

where we used a dual number [12.53].The first is left Qullen preserving cofibration with final object $A_\bullet$ from $A_\bullet$-augmented $R_\bullet$-algebra to $\mathrm{dgMod}_{A_\bullet}$ and [12.58] is the left derived functor taking by cofibrant replacement

$$\begin{tikzcd} & \tilde{A} \arrow[dl, "\text{weak}"', "\simeq"] \arrow[d, "\text{cofibrant}"] \\ B_\bullet \arrow[r] & A_\bullet \end{tikzcd} \qquad \mathbf{L}G \cong \mathbb{L}^{A/R} \qquad [12.60]$$

Then by the definition [12.57],$\mathbb{L}^{A/R} = J/J^2, J = \ker((\tilde{A} \otimes A)_\bullet \to A_\bullet)$.Based on this,we have André-Quillen cohomology

$$D^i_{R_\bullet}(S_\bullet, f_* M_\bullet) = \mathbb{E}\mathrm{xt}^i_{A_\bullet}(\mathbb{L}^{S/R}, f_* M_\bullet) = H_i(\mathrm{Map}_{\mathrm{dgAlg}^{\geq 0}_{R_\bullet}}(S_\bullet, B_\bullet \oplus M_\bullet)) \quad [12.61]$$

where $f : S_\bullet \to M_\bullet$ with $f_* M_\bullet = f_* S_\bullet \oplus M_\bullet$ and (homotopy) fibered category fibered in mapping groupoids with $S_\bullet \xrightarrow{\simeq}_{\mathrm{weak}} B_\bullet \oplus M_\bullet$

$$p : \mathrm{Map}_{\mathrm{dgMod}_{A_\bullet}}(S_\bullet, B_\bullet \oplus M_\bullet) \to \mathrm{Map}_{\mathrm{dgMod}_{A_\bullet}}(S_\bullet, B_\bullet) \quad [12.62]$$

In this case,$\mathbb{L}^{S/R} = (\Omega^1_{(B\oplus M)/R} \otimes_{B \oplus M} S)_\bullet$.Now,see [12.56],we have a representable functor on homotopy category $\mathrm{Map}(B_\bullet, -)$,then we have

$$\mathrm{Map}_{\mathrm{dgAlg}^{\geq 0}_k}(B_\bullet, B_\bullet \oplus I[1]) = D^0_{A_\bullet}(B_\bullet, I[1]) = D^0_{A_\bullet}(B_\bullet, I)[1] \quad [12.63]$$

which is the corresponding obstruction space.Also by definition of $\mathbb{E}\mathrm{xt}^i$

$$\begin{aligned} D^0_{A_\bullet}(B_\bullet, I[1]) &= \mathrm{Hom}_{\mathrm{dgAlg}^{\geq 0}_k}(\mathbb{L}^{B/A}, I[1]) \\ &= \mathrm{Hom}_{\mathrm{dgAlg}^{\geq 0}_k}((\Omega^1_{\tilde{B}/A} \otimes_{\tilde{B}} B)_0, I[1]) \\ &\cong \mathrm{Hom}_{\mathrm{dgAlg}^{\geq 0}_k}(J = \ker(B \otimes_A B \to B)_0 / J^2), \ker(f)[1]) \\ &= \mathrm{Hom}_{\mathrm{dgAlg}^{\geq 0}_k}(\{da \otimes db - db \otimes da + J^2\}, \ker(f)[1]) \end{aligned} \quad [12.64]$$

The interesting thing is,see [7.18],[7.21] and [7.27],to put physics in we can have

$$da \otimes db - db \otimes da + J^2 \leftrightarrow X \otimes Y - Y \otimes X - [X, Y], X, Y \in \mathfrak{g}_0 \quad [12.65]$$

where we used generalized super affine Lie algebra in [11.21].

Theorem13.11 An obstruction corresponding to a nontrivial element in obstruction space $D^0_{\mathfrak{g}_0 \boxtimes_{\mathbf{T}} \mathfrak{g}_1^*}(\tilde{\mathfrak{g}}, I[1])$ corresponds to a nontrivial behaviors on the generalized super DG Lie scheme induced by nontrivial contaction.

$$D^0_{\mathfrak{g}_0 \boxtimes_{\mathbf{T}} \mathfrak{g}_1^*}(\tilde{\mathfrak{g}}, I[1]) \cong \mathrm{Hom}_{\mathrm{dgAlg}^{\geq 0}_{\mathrm{GenSup}}}(J(X, Y, [X, Y] \neq 0), I[1]) \quad [12.67]$$

$$\begin{tikzcd} {}^{\mathrm{DG,Lie}}(\mathcal{X}_0 \boxtimes \mathcal{X}_0)_{R' \in \mathbb{R}^{\mathrm{norm}}_{>1}} \arrow[rr, bend left, "\mathbf{F}([X,Y]\neq 0)"] & \bigcirc \arrow[l, "\mathrm{qut.dominance}"', "{[12.23]}"] \arrow[r, "{[11.8]}", "\mathrm{gr.dominance}"'] & (\mathcal{X}_1^* \oplus \tilde{\mathcal{X}}_1^*)^{\Leftarrow \tilde{\mathscr{R}} \simeq \tilde{\mathscr{R}}^*_{\lim}} \end{tikzcd}$$

Corollary13.12 Because an obstruction corresponds to an obstruction of lifting,thus a nontrivial behavior on the field corresponds to an obstruction of lifting.If we set the lifting to be **T**-fusion this explain the calculation [8.34] for the generalized super black hole also see [12.94].

$$\begin{tikzcd} \widetilde{\mathscr{B}\mathscr{H}} \arrow[r, "\underline{\mathrm{Aut}}", "{[11.32]}"'] \arrow[d, "\mathcal{S}^{\leq 0} \oplus \mathcal{S}^{\geq 0}"', "\underline{\mathrm{Isom}}"] & {[X, Y] \neq 0} \arrow[r, "\mathrm{Thm13.12}"'] & \text{An obstruction of } \mathbf{T}\text{-fusion in } \mathcal{X}_1^* \arrow[d, "{[12.94]}"', "\tilde{\mathscr{R}}^*_{\lim}"] \\ {[8.35]} & & \text{Reduce one dimension by (dark energy)}^* \arrow[ll, "{[11.66]}", "\mathrm{P}(\mathbf{T}^\delta_{D-1})^{-1}{,}\mathcal{S}^{\heartsuit}"'] \end{tikzcd}$$

[12.68]

Because we are in étale site,and if $R \to S$ is a smooth morphism of k-algebras,we have $\mathbb{L}^{S/R} \simeq \Omega^1_{S/R}$.We have exact sequence by proposition 8.3A in [6]

$$\mathbb{L}^{C/B}[-1] \to (\mathbb{L}^{B/A} \otimes_B C)_\bullet \to \mathbb{L}^{C/A} \to \mathbb{L}^{C/B} \tag{12.69}$$

as for exact sequence of homomorphisms $A_\bullet \to B_\bullet \xrightarrow{\psi} C_\bullet$,we have

$$\Omega^1_{B/A} \otimes_B C \xrightarrow{u} \Omega^1_{C/A} \xrightarrow{v} \Omega^1_{C/B} \to 0 \tag{12.70}$$

with $d_{B/A}(b) \otimes c \mapsto c \cdot d_{C/A}\psi(b) \mapsto c \cdot d_{C/B}\psi(b)$,$v$ is surjective,because $\Omega^1_{C/B}$ is a B-module,we have $d_{C/B}\psi(b) = \psi(d_{C/B}b) = 0$,so we get the zero in the right hand side $v \circ u$=0.By above [12.46],any representable functor is half-exact,so the it suffices to show the sequence is exact after acting a representable functor

$$\mathrm{Hom}_C(\Omega^1_{B/A} \otimes_B C, T) \leftarrow \mathrm{Hom}_C(\Omega^1_{C/A}, T) \leftarrow \mathrm{Hom}_C(\Omega^1_{C/B}, T) \tag{12.71}$$

for $\mathrm{Hom}(\Omega^1_{B/A} \otimes_B C, T) \simeq \mathrm{Hom}(\Omega^1_{B/A}, T) \simeq \mathrm{Der}_k(B, T)$,it becomes

$$\mathrm{Der}_k(B, T) \leftarrow \mathrm{Der}_k(C, T) \leftarrow \mathrm{Der}_k(C, T) \tag{12.72}$$

this is exact by setting and [12.70] is exact.In this case,we have the exact triangle [12.69].To show the equivalence,we work étale locally with U affine

$$\begin{tikzcd} U \arrow[r, "f", "\mathrm{et}"'] \arrow[d, "g"', "\mathrm{et}"] & \mathrm{Spec}(S) \arrow[d] \\ \mathbb{A}^n_R \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}, \quad \mathbb{L}^{U/R} \simeq_{\mathrm{weak}} (\mathbb{L}^{S/R} \otimes_S f^*S) \simeq f^*\mathbb{L}^{S/R} \tag{12.73}$$

based on $\Omega^1_{U/R} \simeq \Omega^1_{\mathrm{Spec}(S)/R} \otimes_S f^*S \simeq f^*\Omega^1_{\mathrm{Spec}(S)/R}$ and $\mathbb{L}^{U/S} \simeq_{\mathrm{weak}} \Omega^1_{U/S} = 0$ by [9.91],which gives us the weak version on cotangent complex.We étale $U \to \mathbb{A}^n_R = \mathrm{Spec}(R[x_1, ..., x_n])$ with the affine space is cofibrant over R,by discussion below [12.37] and combine [12.73] we get a weak equivalence version

$$\mathbb{L}^{U/R} \simeq_{\mathrm{weak}} g^*\mathbb{L}^{\mathbb{A}^n_R/R} \simeq g^*\Omega^1_{\mathbb{A}^n_R/R} \simeq \Omega^1_{U/R}, \mathbb{L}^{S/R} \simeq_{\mathrm{weak}} f_*\Omega^1_{U/R} \simeq \Omega^1_{S/R} \tag{12.74}$$

Then,we go back,if above [12.69] is an equivalence,then $\mathbb{L}^{U/S} \simeq 0$ for étale f.Also étale morphism is unramified around [7.30],and by the lemma 29.35.13 in [16],the diagonal is open immersion,let $Y = \mathrm{Spec}(S)$ we have

$$\begin{tikzcd} U \arrow[r, "\mathrm{et}"] \arrow[rr, bend right, dotted, "\mathrm{smooth}"'] & \mathrm{Spec}(S) \arrow[r, "\mathrm{smooth}"] & \mathrm{Spec}(R) \\ U \times_{\mathrm{Spec}(S)} U \arrow[u, "\mathrm{pr}_1"', "\mathrm{op.im.}"] \arrow[r, "\mathrm{pr}_2"] & U \arrow[u] \arrow[r] & \mathbb{A}^n_R \arrow[u] \end{tikzcd} \tag{12.75}$$

because $\mathbb{L}^{U/S} \simeq_{\mathrm{weak}} 0, \mathrm{pr}_1^*\mathbb{L}^{U/Y} \oplus \mathrm{pr}_2^*\mathbb{L}^{U/Y} \simeq_{\mathrm{weak}} 0$,along $U \to U \times_{\mathrm{Spec}(S)} U \to \mathrm{Spec}(S)$ we have $\Delta^*_{U/Y}\mathbb{L}^{(U \times_{\mathrm{Spec}(S)} U)/S} \simeq_{\mathrm{weak}} \mathbb{L}^{U/S}$ and we recover the weak equivalence [12.73],which means we can replace $\mathrm{Spec}(S)$ with U because $U \to \mathrm{Spec}(R)$ also smooth,in this case,it is suffices to show $\mathbb{L}^{U/S} \simeq 0$ with $U \to \mathrm{Spec}(S)$ is open immersion.The simplest open immersion is from localization that is $k[x]_x \cong k[x, x^{-1}] \cong k[x,y]/(xy-1)$ inducing the open immersion $\mathrm{Spec}(k[x,y]/(xy-1)) \to \mathrm{Spec}(k[x])$,every open immersion can be conprised by such localizations,in this case,we have a cofibrant replacement over $B = k[x]$

$$\tilde{A}_\bullet = (k[x,y] \cdot t \xrightarrow{\partial} k[x,y]) \xrightarrow{\partial} A = k[x,y]/(xy-1) \qquad [12.76]$$

$\partial^2 t = 0$ means $\partial t = xy - 1 \in k[x,y]$ and we have $\Omega^1_{\tilde{A}_\bullet/B} = \tilde{A}_0(dy) \oplus \tilde{A}_1(dt)$ with $d\partial t = \partial dt = xdy \in \tilde{A}_0(dy)$.By [12.58],we have

$$\mathbb{L}^{A/B} = \Omega^1_{\tilde{A}_\bullet/B} \otimes_B \tilde{A}_\bullet = (\tilde{A}_0(dy) \oplus \tilde{A}_1(dt)) \otimes_{\tilde{A}_\bullet} A \simeq A(dy) \oplus A(dt) \qquad [12.77]$$

with $y, t \in A$ as they are units,we get $\mathbb{L}^{A/B} \simeq 0$ then it suffices to compute $\mathbb{L}^{A/B}$ by $B_\bullet \to A_\bullet$ is a composition of cofibration and smooth morphism.A morphism $f : A_\bullet \to B_\bullet$ is strong if each object is in a section of quasi-coherent complex i.e.$\mathscr{O}_{X_\bullet}$-mod$(V) = B_\bullet$ satisfying [12.13] on a derived scheme.And we say a morphism is homotopy-(...) if it is strong and $H_0(A_\bullet) \to H_0(B_\bullet)$ is (...).Also,f is called homotopy-étale if and only if $\mathbb{L}^{B/A} \simeq 0$.And for derived schemes $(f, \pi^0 f) : \pi^0 X_\bullet \to \pi^0 Y_\bullet$,define the presheaf $\mathbb{L}^{X/Y} = \mathbb{L}^{\mathscr{O}_{X_\bullet}/f^{-1}\mathscr{O}_{Y_\bullet}}$ and for any inclusion $U \to V \in \pi^0 X_\bullet$ the inducing restriction on sections of $H_0(\mathscr{O}_{X_\bullet})$ is an open-immersion see above [12.76],so $\mathscr{O}_{X_\bullet}(V) \to \mathscr{O}_{X_\bullet}(U)$ is homotopy-open immersion which is homotopy-étale and $\mathbb{L}^{\mathscr{O}_{X_\bullet}(U)/\mathscr{O}_{X_\bullet}(V)} \simeq f^{-1}\mathbb{L}^{\mathscr{O}_{X_\bullet}(U)/\mathscr{O}_{X_\bullet}(V)} \simeq 0$ and we can apply [12.69] along $f^{-1}\mathscr{O}_{Y_\bullet}(U) \to \mathscr{O}_{X_\bullet}(V) \to \mathscr{O}_{X_\bullet}(U)$,we get

$$\begin{aligned} \mathbb{L}^{\mathscr{O}_{X_\bullet}(U)/f^{-1}\mathscr{O}_{Y_\bullet}(U)} &\simeq \mathscr{O}_{X_\bullet}(U) \otimes_{\mathscr{O}_{X_\bullet}(V)} (\mathbb{L}^{\mathscr{O}_{X_\bullet}(V)/f^{-1}\mathscr{O}_{Y_\bullet}(U)}) \\ &\simeq_{\text{weak}[12.48]} \mathscr{O}_{X_\bullet}(U) \otimes^{\mathbf{L}}_{\mathscr{O}_{X_\bullet}(V)} (\mathbb{L}^{\mathscr{O}_{X_\bullet}(V)/f^{-1}\mathscr{O}_{Y_\bullet}(V)}) \end{aligned} \qquad [12.78]$$

where we took $\mathbb{L}^{\mathscr{O}_{X_\bullet}(V)/f^{-1}\mathscr{O}_{Y_\bullet}(V)} \to \mathbb{L}^{\mathscr{O}_{X_\bullet}(V)/f^{-1}\mathscr{O}_{Y_\bullet}(U)}$ which is a quasi-flat cofibrant replacement.And [12.78] is isomorphism after taking homology and gives [12.13],thus we see $\mathbb{L}^{X/Y} = \mathbb{L}^{\mathscr{O}_{X_\bullet}/f^{-1}\mathscr{O}_{Y_\bullet}}$ is a quasi-coherent complex.We can see a double-weak,firstly we retract to $\otimes^{\mathbf{L}}$ and secondly we quotient the open simplexes (with boundaries),this gives us diagram [12.85].

12.3 n-hypergroupoids and eigenbrane

To see clearly of the closed string with open string as a weak form [12.12] and to construct a good space for evolution with such property [12.19],we need to combine following math with physics.See figures of followings in 4.1 in [21].

A combinatorial simplex is $\Delta^n = \mathrm{Hom}_\Delta(-, [n])$,the geometric realization $|\Delta^n|$ gives us the standard n-simplex in topological space see [8.4].A category of simplicial set is $s\mathrm{Set}$ with objects are functor $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$,an example is $\mathrm{Sing}(|X|)_\Delta : |X| \to X = (\Delta^{\mathrm{op}} \to \mathrm{Hom}(|\Delta^-|, |X|))$,from Top to $s\mathrm{Set}$.A morphism $X \to Y$ in $s\mathrm{Set}$ is weak homotopy equivalence if $|X| \to |Y|$ is weak homotopy equivalence ($\pi_\bullet$-equivalence).And naturally,we have a model structure on sSet.We also have the operation for simplicial set $X_n = X([n])$

$$\begin{aligned} &\partial_i : X_n \to X_{n-1}, \quad \partial^i : |\Delta^{n-1}| \to |\Delta^n|, \quad \partial_i\partial_j = \partial_{j-1}\partial_i, \forall\, i < j \\ &\sigma_i : X_n \to X_{n+1}, \quad \sigma^i : |\Delta^{n+1}| \to |\Delta^n|, \quad \sigma_i\sigma_j = \sigma_{j+1}\sigma_i, \forall\, i \leq 0 \end{aligned} \qquad [12.79]$$

∂^i is to include i-th face and σ^i is to collapse $(i, i+1)$-th faces.And the condition is for keeping them in one $|\Delta^n|$ system.Define the boundary $\partial\Delta^n = \bigcup_i \partial^i\Delta^{n-1}$ and $|\partial\Delta^n|$ is the full boundary of $|\Delta^n|$ in topological space.Also,the k-th horn is $\Lambda^{n,k} = \bigcup_{i\neq k} \partial^i\Delta^{n-1}$,the $|\Lambda^{n,k}|$ is just removing the k-th face from the full boundary.A trivial Kan fibration and Kan fibration is the following diagrams

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & X \arrow[d, "\text{trivial fib.}"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd} \quad \begin{tikzcd} \Lambda^{n,k} \arrow[r] \arrow[d, hook] & X \arrow[d, "\text{fib.}"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd} \quad \begin{tikzcd} \Lambda^{n,k} \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd} \quad \forall n, k \qquad [12.80]$$

with $X, Y \in s$Set.A Kan complexis is a simplicial set X satisfying the third diagram.Now we want to deform the diagram

$$\begin{tikzcd} X & & \\ {|\Lambda^{n,k}|} \arrow[u, "\mathrm{Sh}_{\text{D-brane}}"] \arrow[r, hook] \arrow[u, phantom, "\Rightarrow"] & {|\partial\Delta^n|} \arrow[r] \arrow[ul] & {|\Delta^n|} \arrow[ull] \end{tikzcd} \qquad [12.81]$$

if we let $n = 2$,the $\partial\Delta^2$ can be view like closed string and a horn is an open string,because of X is homotopy enriched (∞-groupoid now),in X(RHS of [12.19]) the open string is a weak form of closed string and they are equivalent by 2-isomorphism,but in the LHS of [12.19],they are weak homotopy equivalent which means they are seen as in two different types of strings.Such gives us $n = 2$ case,but we want to see a full evolution degree by degree in this case,we need a Kan complex.In this case,DAG gives supports for Brane-cosmology.

For studying the RHS,we naturally have the following definitions.A n-th matching space is $M_{\partial\Delta^n}(X) = \mathrm{Hom}_{s\mathrm{Set}}(\partial\Delta^n, X)$,explicitly

$$M_{\partial\Delta^n}(X) = \{x^n \in \prod_{i=0}^{n} X_{n-1} | \partial_i x_j^n = \partial_{j-1} x_i^n, i < j, x \in X_n\} \qquad [12.82]$$

$X_n \to M_{\partial\Delta^n}(X), x \mapsto x^n = (\partial_0 x, ..., \partial^n x)$.A (n,k)-th partial matching space is

$$M_{\Lambda^{n,k}}(X) = \{x^{n-1}\ (i \neq k) \in \prod_{i=0}^{n} X_{n-1} | \partial_i x_j^{n-1} = \partial_{j-1} x_i^{n-1}, i < j, x \in X_n\} \qquad [12.83]$$

with $\mathrm{Hom}_{s\mathrm{Set}}(\Lambda^{n,k}, X)$ and $x \mapsto x^{n-1} = (\partial_0 x, ..., \partial_{k-1} x, \partial_{k+1} x, ..., \partial_n x)$.Thus

$$\begin{tikzcd} X_n \arrow[r, "\text{surj}"] & Y_n \times_{M_{\Lambda^{n,k}}(Y)} M_{\Lambda^{n,k}}(X) \arrow[r] \arrow[d] & M_{\Lambda^{n,k}}(X) \arrow[d] \\ & Y_n(\simeq \Delta^n/Y) \arrow[r] & M_{\Lambda^{n,k}}(Y) \end{tikzcd} \qquad [12.84]$$

the Kan fibration [12.80] follows from the diagram.Notice the double-weak

$$\begin{tikzcd} \text{homotopy-étale}(\simeq_{\text{weak}}) \arrow[r, "\mathbf{Ret}"] & \text{closed} =_{\text{et}} \arrow[r, "\mathbf{Ret}_*\mathbf{Ads/Cft}"] \arrow[r, bend right, leftrightarrow, "\mathscr{H}\text{-quotient}"'] & \text{closed}_{\text{weak}}(\cong \setminus =)_{\text{et}} \end{tikzcd} \quad [12.85]$$

our aim is to get a good space where we can only discuss étale closed string [12.12],such a space is for unified field! theory living on.In physics this is acquired by $\mathbf{Ret}_*\mathbf{Ads/Cft}$-fusion,and in math this is achieved by taking derived geometry.Now we back,$=\in\cong$,so we generally consider a Kan complex and say it trivial fibration [12.80] in RHS of [12.19].

$$\begin{tikzcd} & \text{Set} & \\ \Delta^{\text{op}} \arrow[ur] \arrow[rr] & & (\Delta\times\Delta)^{\text{op}} \arrow[ul] \end{tikzcd} \qquad \text{diag} : ss\text{Set} \to s\text{Set} \quad [12.86]$$

Then the diagonal morphism $\Delta \to \Delta \times \Delta$ contravariantly gives us a diagonal functor from bisimplicial category with objects $(\Delta \times \Delta)^{\text{op}} \to$ Set.Now,along the singular chain complex [8.7],we set $\partial = \sum(-1)^i\partial_i$ which is [8.6] and the corresponding sequence of singular homology groups is that of abelian groups.In this case,the simplicial abelian group $(A_\bullet, \partial)$ becomes a DG abelian group (chain complex).We have $N_mA = \{a \in A_m | \partial_i a = 0, \forall i > 0\}$ which is a complex,it gives us the normalization of simplicial abeilian group NA,thus we have (NA, ∂_0) by the [12.78] $\partial_0^2 a = \partial_0\partial_1 a = 0$ and by Hurewicz theorem we have $H_\bullet(NA) \cong \pi_\bullet(|A|, 0)$.Also,$N$ induces equivalence of category (Dold-Kan thm 4.17 in [21])

$$N : (\text{simplicial abelian groups}) \xrightarrow{\simeq} (\text{chain complexes}^{\geq 0}) \qquad [12.87]$$

So this gives us a way to get a double complex see [10.90].Follows from [12.85]

$$(\text{Eilenberg-Zilber}) \quad \nabla : \text{Tot}\underline{N}A \to N\text{diag}A \qquad [12.88]$$

with $\underline{N}$ for set $\partial_i^h a = \partial_i^v a = 0 \forall i > 0$ and $A \in ss$Set.

Next,we discuss simplicial mapping space,for a model category C and a object $Y \in C$,let simplicial diagram $\hat{Y} : \Delta^{\text{op}} \to C$ over $\Delta^{\text{op}} \to Y$,be the simplicial fibrant resolution of Y such that

$$(\text{i})\, \forall Y \simeq_{\text{weak}} \hat{Y} \quad (\text{ii})\, \hat{Y} \xrightarrow{\text{fib.}} M_{\partial\Delta^n}\, (n \geq 0, \hat{Y})(\hat{Y}_0 \text{ is a fibrant}) \qquad [12.89]$$

For a commuative ring R,we have category of simplicial commutative R-algebras $s\text{Alg}_R$ with objects are functors $\Delta^{\text{op}} \to \text{Alg}_R$ with ∂_i degree down and σ_i degree up.The [12.87] gives us an equivalence $N : s\text{Alg}_R \xrightarrow{\simeq} \text{dgAlg}_R^{\geq 0}$.They give equivalent homotopy theory ≥ 0 but $s\text{Alg}_R$ still work < 0.Also,for a simplicial ring A,we denote $s\text{Mod}_A$ for the category of simplicial A-modules.

Now,we are able to discuss higher groupoids for higher stacks.An ∞-groupoid is a Kan complex [12.80] and for $X, Y \in s$Set,a relative n-hypergroupoid over

Y is a Kan fibration $p: X \to Y$ [12.84] satisfying the following condition

$$X_m \to Y_m \times_{M_{\Lambda^{m,k}}(Y)} M_{\Lambda^{m,k}}(X), \quad \text{surj. for } m \leq n \text{ and isom. for } m > n \tag{12.90}$$

which means horn-filler [12.80] exists and $m > n$ it is unique.When $Y = *$ we say X is a n-hypergroupoid.A n-hypergroupoid is a simplicial set with $m \leq n$-isomorphism and $m > n$-identity.And we say [12.13] is trivial if we replace $\Lambda^{n,k}$ to $\partial\Delta^n$ [12.82].

Theorem13.13 The vibration along a dimension (quantum effect) of closed (no boundary) geometry of $\partial\Delta^n$ will generate that of Δ^n_{weak}.It is F-dual to gravity effect [12.68] which decrease the dimension of Δ^n to $\partial\Delta^n$ because of the black hole.We can see this in the following diagram see [12.95].The meaning of the identity is from the physics below.

$$\begin{tikzcd}
\Delta^n_{\text{weak}} = \underline{\Lambda^{n,k}} \coprod \text{D}n\text{-brane} \arrow[r, leftrightarrow] & \Delta^n \\
\underline{\Lambda^{n,k}} = \Lambda^{n,k} \coprod \text{D}(n-1)\text{-brane} \arrow[u, hook] \arrow[r, leftrightarrow, "\text{Ads/Cft}"] \arrow[ur, dotted, hook] & \partial\Delta^n \arrow[u, hook]
\end{tikzcd} \tag{12.91}$$

where we igonore the geometric realization $||$ for clarity.In this case,[12.91] it is really like gauge-fixing or uniqueness problem of generalized super relative properties below [9.124],because of [12.91],we have two ways to fill,so it is unique if and only if we fuse this duality.

Recall the quantum effect we discuss about homotopy [12.36] and combine the definition13.2 above [12.7] we get the corollary below.

Definition13.14 By the diagram [12.19],we need push things to retracted level and we call them eigenbrane

$$\begin{tikzcd}
\mathbf{D}n\text{-eigenbrane} \arrow[r, leftrightarrow] & \mathbf{P}n\text{-eigenbrane} \\
\text{D}n\text{-brane} \arrow[u, twoheadrightarrow, "\textbf{Ret}"] \arrow[r, leftrightarrow] & \text{P}n\text{-brane} \arrow[u, twoheadrightarrow, "\textbf{Ret}"]
\end{tikzcd} \tag{12.92}$$

Corollary13.15 A $\mathbf{D}n$-eigenbrane is a quantum algerbaic section (carrier of quantum effect) of algebraifold $\mathcal{A}$ and an ordinary $\mathbf{P}n$-eigenbrane is a gravity algebraic section (carrier of gravity effect) of F-dual algebraifold $\check{\mathcal{A}}$.And $\underline{\mathcal{A}} = \check{\mathcal{A}} \vee_{\bigcirc} \mathcal{A}$ which is equivalent to [11.2],so if we shrink these two algebraifold together $\check{\mathcal{A}} \oplus \mathcal{A} \to \underline{\mathcal{A}}$ we will get the uniqueness.

Theorem13.16 Quantum effect make D-brane always exist to weak equivalently compensate a geometry with boundary to a geometry without boundary with higher dimension.Gravity effect make P-brane exist always to no boundary geometry with lower dimension.

$$\begin{tikzcd}
\Delta^D, \Delta^D_{\text{weak}} \arrow[r] \arrow[rr, bend left, "\tilde{\mathscr{R}}^*\ \text{disorder}"] & \cdots \arrow[r] & \Delta^0 = \Delta^0_{\text{weak}} \arrow[ll, bend left, "\text{order}\ \tilde{\mathscr{R}}_*"] \Rightarrow_{[9.97]} \mathbf{\Pi}
\end{tikzcd} \tag{12.93}$$

The entropy increase principle is a functor $\tilde{\mathscr{R}}^*$

The dark energy is its left adjoint to decrease the entropy $\tilde{\mathscr{R}}_*$

For instance,D-brane exists to let a 2-sphere with boundary to 3-sphere without boundary,and in [12.68] the black hole (by limit of gravity) gives a hole to let it be a circle without boundary.

Definition13.17 The dark energy is the left adjoint of the functor entropy increase principle.It should be understood as a force to make universe tend to be orderly.

Theorem13.18 The ordinary entropy increase (i.e.to heat energy) is an effect of gravity bending the spacetime it is a tendency to disorder preserving the degree.And the limit of gravity (black hole) is a tendency to disorder to lower degree.The degree is for [12.93] representing the dimension of our universe.

Now we go back,a 1-hypergroupoid is given from elements of groupoid $\mathscr{G}$

$$(N\mathscr{G})_n = \coprod_{x_0,...,x_n} \mathscr{G}(x_0,x_1)\times\mathscr{G}(x_1,x_2)\times...\times\mathscr{G}(x_{n-1},x_n) \qquad [12.94]$$

which are nerves of a groupoid $\mathscr{G}$.Focus on $(N\mathscr{G})_1$ we have a horn-filler

$$\Lambda^{2,3} \simeq \begin{tikzcd} & x_1 \arrow[dr, "2"] & \\ x_0 \arrow[ur, "1"] & & x_2 \end{tikzcd} \simeq_{(\cong\backslash=)_{\text{et}}} \begin{tikzcd} & x_1 \arrow[dr, "\text{string}"] \arrow[d, Rightarrow, "\text{D2-brane}"] & \\ x_0 \arrow[ur, "\text{string}"] \arrow[rr, "\text{D1-brane}"'] & {} & x_2 \end{tikzcd} \qquad [12.95]$$

which gives us a weak form of closed string (filling into closed diagram) and in the hypergroupoid such fillers always exist,which means we can use this to construct a good space for the RHS of the bottom line in [12.19].And we also have the properties for n-hypergroupoids X

$$\begin{aligned} &\text{(i)}\ \forall m>n, \pi_m X = 0 \\ &\text{(ii)}\ Y\in s\text{Set}, \pi_m Y = 0\ \forall m>n \Rightarrow Y\simeq_{\text{weak}} X \\ &\text{(iii)}\ \text{A } n\text{-hypergroupoid } X \text{ is completely determined by } X^{\leq n+1} \end{aligned} \qquad [12.96]$$

Also,we need to put étale of smooth morphism in,this gives us the followings.An algebraic (Deligne-Mumford) n-hypergroupoid to be a simplicial affine scheme X satsfying

$$* \times_{M_{\Lambda^{m,k}}(*)} M_{\Lambda^{m,k}}(X) \simeq M_{\Lambda^{m,k}}(X) \to X_m \qquad [12.97]$$

is a smooth (étale) presentation and for $m>n$ it is an isomorphism.For defining higher stack and derived stack,we need to study simplicial mapping spaces.For a model category C and $Y\in C$,a simplicial fibrant resolution of Y is a simplicial diagram $\hat{Y}:\Delta^{\text{op}}\to C$ with $Y\to\hat{Y}_0$ such that

$$\text{(i)}\ Y\simeq_{\text{weak}}\hat{Y}_n, \quad \text{(i)}\ \hat{Y}_n \to_{\text{fib.}} M_{\partial\Delta^n}(\hat{Y}) \qquad [12.98]$$

For instance,in $A\in \text{dgAlg}_k^{\geq 0}$,natually over $|\Delta^n|\to A, \hat{A}_n = \tau_{\geq 0}(A\otimes\Omega^\bullet(\Delta^n))$

$$\Omega^\bullet(\Delta^n) = k[x_0,...,x_n,dx_0,..,dx_n]/(\sum x_i - 1, d\sum x_i) \qquad [12.99]$$

By using [12.82],we can express the n-th matching space

$$\begin{aligned} M_{\partial\Delta^n}(\hat{A}) &= \{(x,dx)^n \in \prod_i^n \hat{A}_{n-1} | [12.82]\} \simeq_{\text{below}[7.29]} \sum_{i=0}^n r_i x_i, r_i \in k \\ &\simeq \tau_{\geq 0}(A \otimes \Omega^\bullet(\partial\Delta^n)), \Omega^\bullet(\partial\Delta^n) = \Omega^\bullet(\Delta^n)/(\prod_i x_i, d(\prod_i x_i)) \end{aligned} \quad [12.100]$$

Because $\Omega^\bullet(\Delta^n) \to \Omega^\bullet(\partial\Delta^n)$ is surjective,so $\hat{A} \to M_{\partial\Delta^n}(\hat{A})$ is surjective and by above [12.36],the surjective is automatically a fibration satisfying [12.95].

Let $X, Y \in (C, W)$ is a cofibrant and $\hat{Y}$ is simplicial fibrant resolution,the right derived function complex $\mathbf{R}\mathrm{Map}_\bullet(X,Y)$ on Y is given by $\mathrm{Hom}_W(X, \hat{Y}_n)$.

12.4 Derived geometric n-stack and consistency

For category of derived affine schemes (work étale locally [9.79]) $d\mathrm{Aff}_R$,that of simplicial derived affine schemes is $sd\mathrm{Aff} = (d\mathrm{Aff}_R)^{\Delta^{\mathrm{op}}}$ an object is

$$\begin{tikzcd} X_0 & X_1 \arrow[l, shift left] \arrow[l, shift right] & X_2 \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] & \vdots & X_3 \arrow[ll, bend right] \arrow[ll, bend left] & \cdots \end{tikzcd} \quad [12.101]$$

with each X_m a derived affine scheme.A homotopy derived algebraic (DM) n-hypergroupoid is a $X \in sd\mathrm{Aff}_R$ satisfying

$$\begin{aligned} &\text{(i) } \pi^0 X \text{ is an algebraic (DM) } n\text{-hypergroupoid [12.97]} \\ &\text{(ii) } H_0(\mathscr{O}_{X_m}) \otimes_{\partial_i^{-1} H_0(\mathscr{O}_{X_m})} \partial_i^{-1} H_j(\mathscr{O}_{X_m}) \cong H_j(\partial_i^{-1}\mathscr{O}_{X_m}) \cong H_j(\mathscr{O}_{\partial_i^{-1} X_m}), \end{aligned} \quad [12.102]$$

Where $\partial_i : X_{m+1} \to X_m \forall i, m, j$.Which says we need $\pi^0 X_m$ to form an étale site with the global descent for Sh [9.70] over this site,Also by below [12.77],∂_i is strong for all i, m.By below [12.12]

$$\begin{aligned} &(a)\, \mathscr{O}_{\pi^0 X_m} = H_0(\mathscr{O}_{X_m}), \text{Decent for Sh on } \pi^0 X \Leftrightarrow \text{Decent for QCoh on } X \\ &(b)\, \text{étale (smooth) on } \pi^0 X \Leftrightarrow \text{homotopy-étale (smooth) on } X \end{aligned} \quad [12.103]$$

the second condition in [12.102] preserves the [12.103].And for trivial case of [12.90],we just replace to trivial algebraic (DM) n-hypergroupoid.A homotopy derived algebraic (DM) n-hypergroupoid X over Y satisfies

$$\begin{aligned} &\text{(i) } \pi^0 X \to \pi^0 Y \text{ is an relative algebraic(DM) } n\text{-hypergroupoid} \\ &\text{(ii) preserve [12.102] for all } f_m : X_m \to Y_m \end{aligned} \quad [12.104]$$

where $f : X \to Y \in sd\mathrm{Aff}$.Theorem 6.11 in [21] tells us

$$\begin{aligned}&\infty\text{-Cat of strongly quasi-compact } (n-1)\text{-geometric derived algebraic stack}\\ &\simeq (\mathcal{C}, \mathcal{W}) \text{ with } \mathcal{C} \text{ of homotopy derived algebraic } n\text{-groupoid } X \in sd\mathrm{Aff}\\ &\quad \text{and } \mathcal{W} \text{ of homotopy trivial}\\ &\text{relative derived algebraic } n\text{-hypergroupoid } X \to Y \in \mathcal{C}\end{aligned} \tag{12.105}$$

such relative should be $(\mathcal{C}, \mathcal{W}) \simeq (\cong, =)$ in the double-weak diagram [12.85].We can give a further understand by the diagram

$$\begin{array}{ccc} \cong(\text{horn-filling}) & \longleftrightarrow & =(\text{simplex-filling}) \\ \downarrow{\scriptstyle [12.91]} & & \downarrow{\scriptstyle [12.91]} \\ \text{quasi-no boundary} & \longleftrightarrow & \text{no boundary} \\ \downarrow & & \downarrow \\ \text{discontinuous} & \underset{\text{quasi [14.10]}}{\longleftrightarrow} & \text{smooth} \end{array} \quad \Rightarrow (\cong \setminus =)_{\mathrm{et}} \text{ for qut.} \leftrightarrow =_{\mathrm{et}} \text{ for gr.} \tag{12.106}$$

This has two points,firstly,it is consistent with quantum (discontinuity) and gravity (smooth manifold) and give a mathematical support (description) of the difference of them which we discussed above [12.34].Secondly,the black hole which is a local discontinuity is an effect of quantum gravity and it is consistent with [12.67] and [12.68] where we combine quantum and gravity behaviors to explain the black hole.Thus,we find our theory combining the math and physics is highly consistent.And we want to give a further explanation based on [12.68] of [8.34] combining [12.92] by the diagram

$$\begin{tikzcd} \mathbf{P}0 \arrow[rr, "\mathbf{P}1\text{-eigenbrane}"] \arrow[dr, "\mathbf{P}1"'] & & \mathbf{P}0 \\ & \mathbf{P}0 \arrow[ur, "\mathbf{P}1", "f"'] & \end{tikzcd} \xrightarrow[{[12.67]}]{\text{flip } f} \begin{tikzcd} \mathbf{P}0 \arrow[rr, "\mathbf{P}1"] \arrow[dr, "\mathbf{P}1"'] & & \mathbf{P}0 \arrow[dl, "\text{localize}"', "\mathbf{P}1'"] \\ & \color{red}{\mathbf{D}0\text{-eigenbrane}} & \end{tikzcd} \tag{12.107}$$

By the local 2-nonexistence,localization is a property which will generate the property canceling **at the same time** from super generalized relative principle above [12.35] and along [12.19] $\mathbf{Ret}^*$ gives a black hole in our universe.

Let $d\mathrm{Alg}_R$ be category of derived R-algebras opposite to $d\mathrm{Aff}_R$.A derived ∞-stack over R is a category fibered in simplicial set $p : s\mathrm{Set} \to \mathrm{Et}d\mathrm{Aff}_R$ over $d\mathrm{Aff}_R$ with étale topology with global descent theory.Combining [9.26] and [12.105],

$$X^\sharp(A) = \mathbf{R}\mathrm{Map}_{\mathcal{C}}(\mathrm{Spec}(A), X) = HOM_{\mathcal{W}}(\mathrm{Spec}(A), \hat{X}), \quad A \in \mathrm{Et}sd\mathrm{Alg} \tag{12.108}$$

where we used the simplicial fibrant replacement $X \to \hat{X}$ see [12.98] with $\hat{X}$ a homotopy trivial derived algebraic (DM) n-hypergroupoid.Now the right derived functor gives us the second retract H-quotient in [12.85] and we extract the things from left of [12.106] to the right by quotient the weak homotopy equivalence [12.91].Which gives us the derived geometric algebraic (DM) $(n-1)$-stack.

For a homotopy derived algebraic n-hypergroupoid [12.101],each X_m is a derived affine scheme so we denote it as $X_{\bullet\bullet}$.On it,we have cochain complex $O(X)^{\bullet}_{\bullet}$ where we put superscript for the cochain complex index,which is the structure sheaf on the simplicial derived affine scheme,also we have $O(X)^{\bullet}_{\bullet}$-modules.A homotopy-cartesian module $\mathscr{F}$ on the homotopy derived algebraic n-hypergroupoid X consists of

$$\begin{aligned}
&\text{(i) } \mathscr{F}^m_{\bullet}\text{is a } O(X)^{\bullet}_{\bullet}\text{-module for each } m \\
&\text{(ii) } \partial^i : \partial_i^* \mathscr{F}^{m-1} \to \mathscr{F}^m_{\bullet} \quad \sigma^i : \sigma_i^* \mathscr{F}^{m+1}_{\bullet} \to \mathscr{F}^m_{\bullet} \quad \text{keeping [12.79]} \\
&\text{(iii) On each } \pi^0 X_{\bullet\bullet} \text{ the } H_j(O(X)^{\bullet}_{\bullet})\text{-module } H_j(\mathscr{F}^{\bullet}_{\bullet}) \text{ is cartesian,} \\
&\qquad \partial^i : (\pi^0\partial_i)^* H_j(\mathscr{F}^{m+1}_{\bullet}) \cong H_j(\mathscr{F}^m_{\bullet}) \quad \forall\, \pi^0\partial_i : \pi^0 X_{m\bullet} \to \pi^0 X_{m-1\bullet}
\end{aligned} \tag{12.109}$$

where by [12.13],the 2nd of [12.102] gives us $H_0(\mathscr{O}_{m\bullet})$-comodule structure

$$(\pi^0\partial_i)^* H_j(\mathscr{F}^{m+1}_{\bullet}) = H_0(O(X)^m_{\bullet}) \otimes_{\partial_i^{-1} H_0(O(X)^{m-1}_{\bullet})} \partial_i^{-1} H_j(\mathscr{F}^{m-1}_{\bullet}) \tag{12.110}$$

And the 1st of [12.102] gives us the cochain complex on $\pi^0 X_{\bullet}$

$$\begin{aligned}
&H_j(\mathscr{F}^0_{\bullet}) \rightrightarrows H_j(\mathscr{F}^1_{\bullet}) \Rrightarrow H_j(\mathscr{F}^2_{\bullet}) \cdots \qquad (\dagger)\text{-comodules} \\
&H_0(O(X)^0_{\bullet}) \rightrightarrows H_0(O(X)^1_{\bullet}) \Rrightarrow H_0(O(X)^2_{\bullet}) \cdots \qquad (\dagger)
\end{aligned} \tag{12.111}$$

on $\pi^0 X_{0\bullet} \leftleftarrows \pi^0 X_{1\bullet} \Lleftarrow \pi^0 X_{2\bullet} \cdots$.

Application in math For $F : d\mathrm{Alg}_R \to s\mathrm{Set}$ a derived geometric algebraic n-stack(representable sheaf version),a morphism $A \to B \leftarrow C \in d\mathrm{Alg}_R$ with $A \to B$ a nilpotent surjection above [12.55] gives us a weak equivalence in the fibers $F(A \otimes^{\mathbf{L}}_B C) \simeq_{\text{weak}} F(A) \times^h_{F(B)} F(C)$ and a functor satsifying this equivalence is called homotopy-homogeneous.Above [12.56],the representable functor is $\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dgAlg}^{\geq 0}_R)}(S_{\bullet}, -) \simeq \mathbf{R}\mathrm{Map}_{\mathrm{dgAlg}^{\geq 0}_R}$ by below [12.100] which preserving homotopy limits with surjective factorization on the path components

$$\begin{aligned}
&\pi_0 F(A_{\bullet} \otimes^{\mathbf{L}}_{B_{\bullet}} C_{\bullet}) \simeq \pi_0(F(A_{\bullet}) \times^h_{F(B_{\bullet})} F(C_{\bullet})) \\
&\quad \to_{\text{surj}} \pi_0 F(A_{\bullet}) \times_{\pi_0 F(B_{\bullet})} \pi_0 F(C_{\bullet}) \simeq \pi_0(F(A_{\bullet}) \times_{F(B_{\bullet})} F(C_{\bullet}))
\end{aligned} \tag{12.112}$$

which gives the surjective factorization for representable functor and the above weak equivalence.Conversely,if F is homotopy-homogeneous then we have the surjection over the $d\mathrm{Alg}_R$ of derived algebras [12.112].So we can apply derived stack in tangent space and obstruction around [12.56].By [12.108] and 2-Yoneda lemma [9.25],for a derived stack $U^\sharp \simeq \mathbf{R}\mathrm{Map}_{d\mathrm{Aff}}(-, U)$,it corresponds to the homotopy derived 0-hypergroupoid which is a derived affine scheme U which gives previous case for schemes.Now,let $A \in \mathrm{dgAlg}^{\geq 0}_R, M \in \mathrm{dgMod}_A$ with $F : \mathrm{dgAlg}_R \to s\mathrm{Set}$,$A \oplus M \in \mathrm{dgAlg}_R$ means $(A \oplus M) \times (A \oplus M) \to A \oplus M^2 \in \mathrm{dgAlg}_R$ so M is a dual number.For a $x \in F(A)$ with coefficient in M,the tangent space of F at x is $T_x(F, M) = F(A \oplus M) \times^h_{F(A)} \{x\}$ by the pullback.If

F is homotopy-homogeneous,we have an additive action for the tangent space $T_x(F,M) \in \mathrm{Ho}(s\mathrm{Set})$ with $A \oplus M \to A$ a nilpotent surjection.

$$F(A \oplus M) \times^h_{F(A)} F(A \oplus M) \simeq_{\text{weak}} F((A \oplus M) \times_A (A \oplus M)) \simeq F(A \oplus M) \quad [12.113]$$

Also,we form a short exact sequence $0 \to M \to \mathrm{cone}(M \to M) \to M[-1] \to 0$,$M \simeq \mathrm{cong}(M \to M)/M[-1] \simeq \mathrm{cone}(M \to M) \times_{M[-1]} 0$.

$$\begin{aligned} F(A \oplus M) &\simeq F(A \oplus (\mathrm{cone}(M \to M) \times_{M[-1]} 0)) \\ &= F(A \oplus \mathrm{cone}(M \to M) \times_{A \oplus M[-1]} A) \\ &\simeq_{\text{weak}} F(A \oplus \mathrm{cone}(M \to M)) \times^h_{F(A \oplus M[-1])} F(A) \end{aligned} \quad [12.114]$$

13 To Complete Einstein's Dream

13.1 The derived geometry of M-theory

We haved defined consistent string-Space,pre M-theory and the M-theory and their flows,we also want to see the evolutions in them dimension by dimension,and we do a summarization of the framework by diagrams at first.

$$\begin{tikzcd} \mathscr{M}^{\text{rep}}_\Lambda \vee \mathscr{M}_{-\Lambda} \arrow[rr, Rightarrow] \arrow[d, dotted] \arrow[drr, dotted] & & \boldsymbol{\mathcal{M}} \\ \mathrm{Ran}(\mathcal{S}^{\text{rep}}_{\text{cons.}}) \in \boldsymbol{\mathcal{M}}^{\text{pre.rep.}}_{\text{cons.}} \arrow[urr] \arrow[rr, leftrightarrow, "\mathbf{U}"] & & \mathrm{Ran}(\mathcal{S}_{\text{cons.}}) \in \boldsymbol{\mathcal{M}}^{\text{rep.}}_{\text{cons.}} \arrow[u] \end{tikzcd} \quad \text{[Step I]}$$

$$\begin{tikzcd} \boldsymbol{\mathcal{M}}\text{-flow} \arrow[r, leftrightarrow, "\mathbf{F}", "\text{twisted } \mathbf{U}"'] & \boldsymbol{\mathcal{M}} \\ \boldsymbol{\mathcal{M}}^{\text{pre.rep.}}_{\text{cons.}}\text{-flow} \arrow[u, "\mathrm{P}(\mathbf{U})"] \arrow[r, leftrightarrow] & \boldsymbol{\mathcal{M}}^{\text{pre.}}_{\text{cons.}} \arrow[u, "\mathrm{P}(\mathbf{U})"'] \\ \coprod_{\text{type}} \mathrm{Ran}(\mathcal{S}^{\text{rep}}_{\text{cons}}) \arrow[u, hook, "\textbf{live in}"] \arrow[r, leftrightarrow, "{[12.24]}"'] & \coprod_{\text{type}} \mathrm{Ran}(\mathcal{S}_{\text{cons.}}) \arrow[u, "\textbf{descent}"'] \end{tikzcd} \quad \begin{matrix}[11.82] \\ [11.85]\end{matrix} \Rightarrow \textbf{Unified field}^!\textbf{theory}$$

[Step II]

$$\begin{tikzcd} & \textbf{Unified field}^!\textbf{theory lives in} & \\ \boldsymbol{\mathcal{M}}\text{-flow} \arrow[ur, "\text{simp.derived}"] \arrow[dr] \arrow[rr, leftrightarrow, "\mathbf{F}"'] & & \boldsymbol{\mathcal{M}} \arrow[ul, "\text{simp.derived}"'] \arrow[dl] \arrow[l, phantom, "\Uparrow \text{shrink } \underline{\boldsymbol{\mathcal{A}}} \simeq \mathscr{H}\text{-quotient[12.85]}"] \\ & {}^{\text{Lie}}\mathrm{DGSch}^{\text{SupGen}}_{\text{eff,cons.}}(\mathscr{M}^{\text{rep}} \vee \mathscr{M}) & \end{tikzcd} \quad \text{[Step III]}$$

And to achieve the top of step III,we need to reconstruct our theory about stacks by derived stack.We can see our settings in section 8.1 is consistent with the simplicial setting in this case we first upgrade them.

Definition14.1 A simplicial DG affine generalized superscheme is a simplicial set by using [8.11] and [12.45]

$$\underline{\mathscr{X}} : (\Delta_0 \oplus \Delta_1^*)^{\Leftarrow} \to (\mathrm{Spec}(N_A[\Delta^D]) \to \mathscr{N}), \quad \partial = \sum(-1)^i(\partial_0 \oplus \partial_1^*)_i \quad [13.1]$$

with $\underline{\mathscr{X}}([d,d]) = \underline{\mathscr{X}}_d$,we have a diagram with horn fillers

$$\begin{tikzcd}
\Lambda^{d,k} \arrow[r] \arrow[d, "\mathbf{Ret_*Ads/Cft}"'] \arrow[rr, bend left] & \underline{\mathscr{X}} \arrow[r, "\mathrm{P}(\mathbf{T})"] & \mathscr{M}^{\mathrm{rep}} \vee \mathscr{M} \arrow[r, Rightarrow, "\mathrm{smooth}"] & \boldsymbol{\mathcal{M}} \\
\Delta^d \arrow[ur, dotted] \arrow[urr, hook] \arrow[urrr, dotted] & & &
\end{tikzcd} \quad [13.2]$$

induced by the holographic duality,and the **T**-fusion lifts the diagram to the M-brane,but we know the derived obstruction theory [12.67] tells us the there are obstructions to **T**-fuse the closed boundary generated by closed string theory to $\mathscr{M}$.Also by the diagram [Step II] above,we see it has already lived in the M-brane,in this case,the U-dual pair of M-branes becomes a Kan complex and then by **U**-fusion the M-theory has a Kan complex structure.Then,we acting π_0 to get a derived M-brane

$$\pi^0\mathscr{M}^{\mathrm{rep}} \vee \pi^0\mathscr{M} \cong \mathscr{M}^{\mathrm{rep}} \vee \mathscr{M}, \mathscr{H}_i(\mathscr{O}_{\mathscr{M}^{\mathrm{rep}}}) \vee \mathscr{H}_i(\mathscr{O}_{\mathscr{M}}) \Rightarrow \mathscr{H}_i(O(\boldsymbol{\mathcal{M}})) \quad [13.3]$$

they have d-th matching space and (d,k)-th partial matching space respectively

$$\mathscr{M}_d^{\mathrm{rep}} \xrightarrow{\mathrm{et}} M_{\partial\Delta^d_{\mathbf{P}}}(\mathscr{M}^{\mathrm{rep}}), \quad \mathscr{M}_d \xrightarrow{\mathrm{et}} M_{\Lambda^{d,k}_{\mathbf{D}}}(\mathscr{M}) \quad [13.4]$$

Which are all étale surjective,which gives us a smooth presentation of $\boldsymbol{\mathcal{M}}$

$$M_{\partial\Delta^d_{\mathbf{P}}}(\mathscr{M}^{\mathrm{rep}}) \vee M_{\Lambda^{d,k}_{\mathbf{D}}}(\mathscr{M}) \Rightarrow M_{\partial\Delta^n_{\mathbf{P}} \vee \Lambda^{n,k}_{\mathbf{D}}}(\boldsymbol{\mathcal{M}}) \xrightarrow{\mathrm{smooth}} \boldsymbol{\mathcal{M}} \quad [13.5]$$

we know unclosed loop generated the same effect as the closed loop [12.92] by the enriched homotopy and our theory has $\mathbf{Ret_*Ads/Cft}$ duality,which means each matching [13.4] cannot be isomorphism $d \leq 11$ but after **F**-fusion the [13.5] have to be isomorphism,which means $d > 11$ the fillers are unique.Then,we get $\boldsymbol{\mathcal{M}}$ is an algebraic 11-hypergroupoid,explicitly

$$\begin{tikzcd}
\bigcirc \arrow[r, Leftarrow, "{[11.67]}", "{[12.29]}"'] & \mathscr{R}_{\mathscr{M}} \vee_{\bigcirc} \hat{\mathscr{R}}_{\mathscr{M}^{\mathrm{rep}}} \oplus \widetilde{\mathscr{R}}_{\mathscr{M}^{\mathrm{rep}}} \vee_{\bigcirc} \hat{\mathscr{R}}_{\mathscr{M}} \arrow[r, Leftarrow, "{[8.38]}"'] & \mathscr{R} = \mathscr{R}^{++}_{\mathscr{M}} + \mathscr{R}^{--}_{\mathscr{M}} + \widetilde{\mathscr{R}} \\
& {}^{\mathscr{P}}\mathbf{Ret}(\to)^{\leq 10}_{\mathrm{GenSup}} \arrow[ur] \arrow[ur, "\vdots"] \arrow[ur, bend right, "{[12.5]}"'] & \cdots
\end{tikzcd}$$

[13.6]

By [12.93],$\boldsymbol{\mathcal{M}}$ is completely determined by $d \leq 12$.And the algebraic stack is a representable sheaf,so it is a simplicial derived scheme with mixed matching map [13.5].If we work étale locally,we get a site [12.31],on this site we have QCoh($\boldsymbol{\mathcal{M}}$) with bounded cohomologies and by the descent theory [9.74] and [10.83] on the nerve,the homology sheaves are all Cartesian

$$\partial^i : \mathscr{H}_0((O_{\boldsymbol{\mathcal{M}}})_\bullet^{d+1}) \otimes_{\partial_i^{-1}\mathscr{H}_0(O(\boldsymbol{\mathcal{M}})_\bullet^d)} \partial_i^{-1}\mathscr{H}_j(O(\boldsymbol{\mathcal{M}})_\bullet^d) \cong \mathscr{H}_j((O_{\boldsymbol{\mathcal{M}}})_\bullet^{d+1}) \quad [13.7]$$

with cosimplicial condition in [12.79].And the cochain complex is given by duality breaking [11.35] which is,also see the [11.74]

$$\begin{tikzcd} \boldsymbol{\mathcal{M}} \arrow[r, "\mathrm{P}(\mathbf{F}\vee_{\bigcirc}\mathbf{U})^{-1}"] & \boldsymbol{\mathcal{M}}_{\mathbf{11}\bullet} \arrow[r, "\mathrm{P}(\mathbf{T}_{10}^{\delta})^{-1}"] \arrow[rr, bend left, dotted, "\tilde{\mathscr{R}}^*"] & \cdots \arrow[r] & \boldsymbol{\mathcal{M}}_{\mathbf{0}\bullet} \arrow[ll, bend left, "\tilde{\mathscr{R}}_*", "\mathrm{P}(\mathbf{T}_0^{\delta})"'] \end{tikzcd} \quad [13.8]$$

the closed loop is ganranteed by the universe evolution picture [9.97],now we have following answers to explain how $D = 10$ superstring theories are contained in $D + 1$ M-theory,which are equivalent in derived algebraic geometry.

(i) (Moduli)By [7.25],5 D-dim theories are solutions of a $D + 1$ theory
(ii) (Decent)By [9.93],5 D-dim theories descent to $D + 1$ theory
(iii) (Homotopy)By [12.17],5 D-dim theories are retracted to a point in 1-dim

[13.9]

Also,for each cochain degree d the $\boldsymbol{\mathcal{M}}_{\boldsymbol{d}\bullet}$ is a derived schemes,the DG grading is from our number counting fields

$$\begin{tikzcd} \dim\mathscr{H}_i(O(\boldsymbol{\mathcal{M}})) \arrow[r, Rightarrow] \arrow[d, "\simeq"] & \dim\mathscr{H}_i(O(\boldsymbol{\mathcal{M}})_\bullet^{11}) \arrow[r] \arrow[r, bend left] \arrow[r, bend right] \arrow[d, "\simeq", "{[9.17]}"'] & \dim\mathscr{H}_i(O(\boldsymbol{\mathcal{M}})_\bullet^{10}) \arrow[d, "\simeq"] \\ \bigcirc \arrow[r, Rightarrow, "{[11.7]}"'] & \mathrm{F}_{Q\boxtimes Q^*}^{++,--} \oplus \hat{\mathrm{F}}_{Q\boxtimes Q^*}^{++,--} \arrow[r, "{[9.18]}"'] \arrow[r, bend left] \arrow[r, bend right] & [\mathbb{Z} \oplus \mathbb{C} \oplus \mathbb{Q} \oplus \mathbb{R} \oplus (\mathbb{ZCQR})]_{\mathrm{DG,Lie}}^{+,-} \end{tikzcd}$$

[13.10]

with the transverse extension of the chain [13.8] by property evolution in each degree see [9.130] and [11.16].In detail,by [9.148] and [11.6],the sheaf $\mathscr{H}_i(O(\boldsymbol{\mathcal{M}}))$ has global $(-0) \vee_{\bigcirc} 0$ section.Now the $\boldsymbol{\mathcal{M}}$ becomes a trivial homotopy derived algebraic 11-hypergroupoid.Now,we get the good space for [Step III],which is a generalized super derived algebraic geometric 11-stack (for D=11),

$$\boldsymbol{\mathcal{M}}^! = \mathbf{R}\mathrm{Map}_{\mathbf{Ret}}(\mathrm{Spec}(-), \mathscr{M}^{\mathrm{rep}} \vee \mathscr{M}) : \mathrm{Et}sd\mathrm{Alg}_{\mathrm{GenSup}}^{\mathrm{DG,Lie}} \to s\mathrm{Set} \quad [13.11]$$

over the generalized super derived affine Lie algebras (i.e.$(\mathfrak{g} \otimes^{\mathbf{L}} \mathbb{C}[\mathbf{t}, \mathbf{t}^{-1}])^\Delta$) see [11.21],and the opposite category is that of simplicial derived DG Lie generalized super scheme.Explicitly,it is given by restriction

$$\boldsymbol{\mathcal{M}}^!(\tilde{\mathfrak{g}}^\Delta) = \mathbf{R}\mathrm{Map}_{\mathbf{Ret}}(\mathrm{Spec}(\tilde{\mathfrak{g}})^{\Delta^{\Leftarrow}}, \mathscr{M}^{\mathrm{rep}} \vee \mathscr{M}) = \boldsymbol{\mathcal{M}}|_{\mathrm{Spec}(\tilde{\mathfrak{g}})} \quad [13.12]$$

the restriction means the generalized super relative properties are made by this scheme for instance let $\mathcal{X} = \operatorname{Spec}(\tilde{\mathfrak{g}}^{\Delta})$ in [8.37].

On the $\boldsymbol{\mathcal{M}}^!$,naturally we have a topos $(\mathscr{H}_i(O(\boldsymbol{\mathcal{M}}))$-**Mod** of homotopy-cartesian sheaves [12.109] satisfying the diagram

$$\begin{tikzcd} (\mathscr{H}_i(O(\boldsymbol{\mathcal{M}})))\text{-}\mathbf{Mod} \cong \Gamma\underline{\mathcal{A}}\text{-}\mathbf{Mod} \\ \boldsymbol{\mathcal{M}}^! \arrow[u, "\underline{\mathcal{A}}\text{-}\mathbf{Mod}"] \end{tikzcd} \qquad \underline{\mathcal{A}}(\boldsymbol{\mathcal{M}}^!) = (\mathbf{D} \vee_{\bigcirc} \mathbf{P}\text{-eigenbranes}) \quad [13.13]$$

Now,we had finished the double-retract of the double-weak [12.85],and now an étale closed string $\mathfrak{s}^c_{\text{et}} = \mathbf{D} \vee_{\bigcirc} \mathbf{P}1$-eigenbrane see [12.11] and [12.12],they lives in this topos and the vibration of it gives effect of quantum gravity.The information of time evolutions is captured in the modules,there is no so called interaction on this level,the $\mathbf{D} \vee_{\bigcirc} \mathbf{P}$-eigenbranes smoothly evolve over the $\boldsymbol{\mathcal{M}}^!$ with the quantum gravity smoothly spread and be homotopy weakly projected out.

$$\textbf{Unified field}^! \textbf{ theory} = (\mathscr{H}_i(O(\boldsymbol{\mathcal{M}})))\text{-}\mathbf{Mod} \to \boldsymbol{\mathcal{M}}^! \qquad [13.14]$$

The first explanation is given by [8.8] and the second is given by the view of analytic ring in the theory of analytic stack below [14.28].Actually,the full diagram of [11.16] gives an evolution of $\mathbf{D} \vee_{\bigcirc} \mathbf{P}$-eigenbranes.We still have things need to be consdered,the first is our elegant theory is in the RHS of [12.19],which means our real world is a weak form of it,so we want to discuss more details about the connections.The second is we need to consider the ***Nonexpressibility*** below [12.36],and these motivates the next subsection.

13.2 $\bigcirc$-sense and math-physics duality

This starts at the discussion in [12.40],now we put physics in it and recall that we focus on the first retract [12.85] with model category $(\simeq_{\text{weak}}, \cong_{\text{et}})$

$$\begin{tikzcd} & \mathcal{X}_0 \arrow[dl, bend right, "h"'] \arrow[d] & \mathscr{M}/(\mathcal{X}_0 \times_{\mathscr{M}} \mathcal{X}_1^*) \arrow[l] \arrow[d] & \mathscr{M}/(\mathcal{X}_0 \times^{\mathbf{Ret}^*}_{\mathscr{M}} \mathcal{X}_1^*) \arrow[l, twoheadrightarrow, "\mathbf{Ret}"'] \arrow[ll, bend right] \arrow[dl, bend left] \\ \mathcal{X}_0 \times^{\mathbf{Ret}^*}_{\mathscr{M}} \mathcal{X}_1^* \arrow[r, twoheadrightarrow, "\mathbf{Ret}"] & \mathcal{X}_0 \times_{\mathscr{M}} \mathcal{X}_1^* & \mathcal{X}_1^* \arrow[l] \arrow[ll, bend left] & \end{tikzcd}$$

[13.15]

where we used [8.12] and [8.37].The D-brane is $\mathcal{X}_0 \times^{\mathbf{Ret}^*}_{\mathscr{M}} \mathcal{X}_1^*$ which is a weak relative property (homotopy type of properties) which is a weak homotopy étale equivalence which behaves like a path $P\mathcal{X}_1^*$ evolving in the universe with the evolution of relative property [9.130] or $\mathbf{D}$-eigenbrane.Also,we have a rep version of [13.2] for P-brane.In this case,we locally see the generalized super algebraifold

$\underline{\mathcal{A}} = \check{\mathcal{A}} \vee_{\bigcirc} \mathcal{A}$ is the generalized super algebraic space,locally for $\mathcal{A}$ in [13.1]

$$\mathscr{M}/(\mathcal{X}_0 \times_{\mathscr{M}} \mathcal{X}_1^*) \cong \mathcal{A}|_{\mathcal{X}}, \quad \mathscr{M}/(\mathcal{X}_0 \times_{\mathscr{M}}^{\mathbf{Ret}^*} \mathcal{X}_1^*) \cong \mathbf{Ret}^*\mathcal{A}|_{\mathcal{X}} \qquad [13.16]$$

And the weak homotopy equivalence induces the followings

$$(\mathfrak{s}^o, \mathcal{X}_0 \times_{\mathscr{M}}^{\mathbf{Ret}^*} \mathcal{X}_1^*) \cong_{\text{weak}} \mathfrak{s}^c, \quad ([\mathfrak{s}^o], \mathcal{X}_0 \times_{\mathscr{M}} \mathcal{X}_1^*) \cong_{\text{weak}} [\mathfrak{s}^c] \qquad [13.17]$$

Guided by [13.12],we have

$$\text{String Landscape} = (\mathscr{M}/(\mathcal{X}_0 \times_{\mathscr{M}}^{\mathbf{Ret}^*} \mathcal{X}_1^*) \oplus (\mathscr{M}/(\mathcal{X}_0 \times_{\mathscr{M}}^{\mathbf{Ret}^*} \mathcal{X}_1^*))^{\text{rep}})\text{-}\mathbf{Mod} \qquad [13.18]$$

By the super generalized relative principle,it must corresponds to the non-solvable theory to cancel the property of solvable theory in the RHS of [12.19].We know our real world is in LHS of [12.19],which explains why we cannot find SUSY in our world because we can regard SUSY as a relative property and it is stable in RHS because of the space is homotopy enriched,it becomes unstable in LHS so there is actually no SUSY in LHS,it only lives in RHS and by theorem13.3 and along the first weak in double-weak [12.85],SUSY breaks in LHS.

And for solving the problem of the ***Nonexpressibility***,recall the definition below [12.36],we find the problem on the bottom line about this is our definitions.The observation is our definitions are representatives of the things but not the things themselves,if we do not have definitions will not affect the existence of truth itself.Thus,we want to get rid of the definitions.

Definition14.2 The $\mathbf{Def}^{\Leftarrow !}$ is a functor from category of definitions to category of $\bigcirc$-senses,with no reverse functor

$$\mathbf{Def}^{\Leftarrow !} : (\text{definitions}) \to (\bigcirc\text{-senses}) \qquad [13.19]$$

governed by below [12.36].By [12.5],we have a $\bigcirc$-sense

$$\bigcirc\text{-sense} = \mathbf{Def}^{\Leftarrow ! \mathscr{P}}\mathbf{Ret}(\to)^{\leq 10}_{\text{GenSup}} \qquad [13.20]$$

Also we let $\mathbf{Math}$ = (definitions of math) and $\mathbf{Phys}$ = (definitions of physics)

$$\mathbf{Def}^{\Leftarrow !}\mathbf{Math} \simeq \bigcirc\text{-sense} \simeq \mathbf{Def}^{\Leftarrow !}\mathbf{phys} \qquad [13.21]$$

which is a $\bigcirc$-sense,we have it because we have removed definitions [13.19].

Definition14.3 The weak projection of the $\bigcirc$-sense [13.8] gives us duality between math and physics.

In the end,we get the theory of everything **TOE**

$$\textbf{the } \boldsymbol{\mathcal{M}}^!\textbf{-theory} \equiv \mathbf{Def}^{\Leftarrow !}\textbf{Unified field}^! \textbf{ theory} \qquad [13.22]$$

with $\mathbf{Def}^{\Leftarrow !}(\bigcirc\text{-flow}) \simeq \bigcirc\text{-sense} \simeq \mathbf{Def}^{\Leftarrow !}\bigcirc$.Thus,exactly [11.69] should be

$$\bigcirc\text{-sense} \Leftarrow \mathbf{F}, \quad \text{Left evolution is guaranteed by [13.19]} \qquad [13.23]$$

13.3 $\mathbf{T}^\delta$-fusion hierarchy and smoothification

Now we have finished **TOE** but actually we do not see it in detail.The first is the differential of cochain complex [13.8] induced by duality fusion and breaking,recall in [8.34] and [8.35] it should be based on **T**-fusion hierarchy.The second is if we have [13.21],every math should correspond to physics,thus we claim that solving the **T**-fusion hierarchy (combing [8.34] with [13.10]) is equivalent to solving the Goldbach conjecture (understanding the physics meaning).

$$\begin{tikzcd} \mathbf{T}\text{-fusion hierarchy} \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\mathbf{Ret}^*"] & \mathbf{Ret}_*\text{Goldbach conjecture} \\ \mathbf{Ret}^*\mathbf{T}\text{-fusion hierarchy} \arrow[r, "\simeq_{\text{weak}}"] & \text{Goldbach conjecture} \arrow[u, "\mathbf{Ret}_*"'] \end{tikzcd} \qquad [13.24]$$

we have 5 types of superstring theories,so we give a notation **T**-fusion$^{\text{type}}$ that means different type has different fusion homotopy weakly projecting out

$$\operatorname{Spec}(\mathbb{Z}_0)_{(*,0)} \oplus \operatorname{Spec}(\mathbb{Z}_1^*)_{(0,*)} \cong_{\mathrm{P}(\mathbf{T}_0^\delta)} \operatorname{Proj}(\mathbb{Z}[x]) \qquad [13.25]$$

guided by [8.12] and slightly abuse it for focusing on the fusion,also we used $\mathbb{Z}$ in [9.18] for one type of string-Space $\mathcal{S}^{\text{type}}$.We want go to LHS of [12.19],

$$\mathbf{Ret}^*(\operatorname{Proj}(\mathbb{Z}[x]) \cong \operatorname{Spec}(\mathbb{Z}_0) \boxtimes_{\mathbf{T}_0^\delta(\mathbb{Z})} \operatorname{Spec}(\mathbb{Z}_1^*)) \simeq \text{weak-additivity} \qquad [13.26]$$

Notice that we do not know the operation $\mathbb{Z}[x]$,it is algebraically described

$$\mathbb{Z}[x] \in \operatorname{Proj}(\mathbb{Z}[x], \simeq \mathbb{Z} \boxtimes_{\mathbf{T}_0^\delta(\mathbb{Z})} [x] \simeq \mathbb{Z}_0 \boxtimes_{\mathbf{T}_0^\delta(\mathbb{Z})} \{x \in \mathbb{Z}_1^*\} \qquad [13.27]$$

Then,before the next step we need to make a thing clear in [12.40] and [13.15],we know the RHS of [12.19] is homotopy enriched so why we denote the étale equivalence relation $\mathcal{X}_0 \times_S \mathcal{X}_1^*$ but not the derived case $\mathcal{X}_0 \times_S^h \mathcal{X}_1^*$ which based on tensor product of algebras in Ho,because we are in Ho or derived category

$$\mathcal{X}_0 \times_S^h \mathcal{X}_1^* \cong_{\text{stable}[12.18]} \mathcal{X}_0 \times_S \mathcal{X}_1^* \in {}^{\mathscr{P}}\mathbf{Ret}(\to)^{\leq 10}_{\text{GenSup}} \qquad [13.28]$$

but in LHS of [12.19],we have $\mathcal{X}_0 \times_S \mathcal{X}_1^* \simeq^{\text{unstable}}_{\text{weak}} \mathcal{X}_0 \times_S^h \mathcal{X}_1^*$.In this case,

$$\mathbf{Ret}_*((2)_0 \boxtimes_{\mathbf{T}_0^\delta(\mathbb{Z})} (2)_1^*) \simeq \{2\} \times^h_{\mathbb{A}^1} \{2\} \simeq_{\text{weak}} (2,2) \simeq_{\text{weak}} (4) \qquad [13.29]$$

which means in the ordinary space $2+2 \simeq 2(\boxtimes_{\mathbf{T}_0^\delta(\mathbb{Z})})_{\text{weak}} 2 \simeq_{\text{weak}} 4$,and

$$(+,-,\times,\div,+-\times\div) \to_{\text{retract}} (\boxtimes_{\mathbf{T}^\delta(\mathbb{Z})}, \boxtimes_{\mathbf{T}^\delta(\mathbb{C})}, \boxtimes_{\mathbf{T}^\delta(\mathbb{R})}, \boxtimes_{\mathbf{T}^\delta(\mathbb{Q})}, \boxtimes_{\mathbf{T}^\delta(\mathbb{Z}\mathbb{C}\mathbb{Q}\mathbb{R})}) \qquad [13.30]$$

which should be an explanation of the source of algorithm in our ordinary space.We claim that understand the Goldbach conjecture is about understand the algerithm for instance solving $1+1=2$,but by [13.30],+ is an unstable relative property.Thus,we can only find a quasi-proof of it in LHS but a proof in

RHS of [12.19].And it is a natural behavior guaranteed by the $\mathbf{T}$-fusion$_{\text{type}}$ hierarchy.For giving a systematic description of the evolution of relative properties over the hierarchy,we first define the following $\mathbf{Ret}_*$algorithm

$$\begin{aligned}
&\boxtimes_{\mathbf{T}^\delta(\mathbb{Z})} : \text{number of relative properties increase}\\
&\boxtimes_{\mathbf{T}^\delta(\mathbb{C})} : \text{number of relative properties decrease}\\
&\boxtimes_{\mathbf{T}^\delta(\mathbb{Q})} : \text{number of relative properties no change}\\
&\boxtimes_{\mathbf{T}^\delta(\mathbb{R})} : \text{change to opposite relative property over } \bigcirc\\
&\boxtimes_{\mathbf{T}^\delta(\mathbb{Z})} \times_{\mathscr{M}} \boxtimes_{\mathbf{T}^\delta(\mathbb{C})} \simeq \boxtimes_{\mathbf{T}^\delta(\mathbb{Z}) \times_{\mathscr{M}} \mathbf{T}^\delta(\mathbb{C})} \in \boxtimes_{\mathbf{T}^\delta(\mathbb{ZCQR})}
\end{aligned} \qquad [13.31]$$

Now,we can see the meaning of weird setting in algebraifold [8.15]

$$\begin{tikzcd}
 & \tilde{P}^{--} \arrow[dl] & \cdots & P^{++} \arrow[dr] & \\
-0_{Q\boxtimes Q^*} \arrow[ddd, bend right] & {} \arrow[l, Rightarrow, "\boxtimes_{\mathbf{T}^\delta(\mathbb{C})}"'] & P^{++} \arrow[ul, "\tilde{\mathscr{R}}"'] \arrow[ur] \arrow[l, Rightarrow, "\boxtimes_{\mathbf{T}^\delta(\mathbb{Z})}"'] \arrow[r, Rightarrow, "\boxtimes_{\mathbf{T}^\delta(\mathbb{Z})}"] \arrow[dl] \arrow[dr] \arrow[d, Rightarrow, "\boxtimes_{\mathbf{T}^\delta(\mathbb{R})}"] & {} \arrow[r, Rightarrow, "\boxtimes_{\mathbf{T}^\delta(\mathbb{C})}"] & P^{++} \arrow[dd, bend left] \\
 & P^{--} \arrow[ul] \arrow[dl, Rightarrow, "\boxtimes_{\mathbf{T}^\delta(\mathbb{C})}"'] \arrow[dr] & {} \arrow[d, Rightarrow, "\boxtimes_{\mathbf{T}^\delta(\mathbb{C})}"] & P^{++} \arrow[ur] \arrow[dl] \arrow[dr, Rightarrow, "\boxtimes_{\mathbf{T}^\delta(\mathbb{Q})}"] & \\
0 & & 0_{Q\boxtimes Q^*} \arrow[ll] \arrow[rr] & & P^{++}
\end{tikzcd} \qquad [13.32]$$

And we call this the $\mathbf{T}^\delta$-fusion hierarchy with $-0_{Q\boxtimes Q^*} \times_0 0_{Q\boxtimes Q^*} \Rightarrow 0$.For further discussion,we need to introduce p-adic field,we can write a prime number as $p = (x/x')^{-v_p(x)}, p \nmid x', x \in \mathbb{Z}$ with $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ and for $x = \mathbb{Q}$ we have $v_p(x = a/b) = v_p(a) - v_p(b)$,and the p-adic norm is $|x|_p = p^{-v_p(x)}, x \neq 0$,in this case in the p-adic metric space,a convergence happens when $v_p(x) \to \infty, p > 1$,so a divergent sequence in ordinary $(\mathbb{Q}, |\,|)$ corresponds to convergent sequence in $(\mathbb{Q}_p, |\,|_p)$ with $\mathbb{Q}_p = \mathbb{Q}_{\hat{p}} \cong \{\sum_n^\infty c_n p^n | 0 \leq c_n \leq p-1\}$ see below [14.29].Each sequence is convergent in $\mathbb{Q}_p$,p-adic completion behaves like a smoothification of $\mathbb{Q}$ (filling the horn in $\mathrm{Spec}(\mathbb{Q})^{\Delta^{\mathrm{op}}}$ [12.106]),so we can combine analytic approach with functorial approach for studying the dynamics of **TOE**,

$$\begin{tikzcd}
\boldsymbol{\mathcal{M}}\text{-flow} \arrow[r, leftrightarrow, "\mathbf{F}\,\text{global}\ \mathbf{GLC}\,\text{global}\ \bigcirc"] \arrow[d] & \boldsymbol{\mathcal{M}} \arrow[d] \\
\text{functorial approach} \arrow[r, leftrightarrow, "\text{smoothification}"] & \text{analytic approach}
\end{tikzcd} \qquad [13.33]$$

Then,combing [11.85] and [Step III] above [13.1],these inspire us to view

$$\textbf{Unified field}^{!}\ \textbf{theory} \longrightarrow \text{smoothification (analytic + functorial)} \qquad [13.34]$$

and it corresponds to local **GLC** and local $\bigcirc$.

Definition14.4 A quantum gravity effect is a smoothfication.This means if we want to see it in detail,we should have a formalism to study the process of smoothing but not the two sides of [13.30].

Corollary14.5 The **Unified field$^!$ theory** is a theory with spreading of smoothifications.And this motivates us to develop analytic stack in section 14.

Now,we back if we cannot get a reason of $p+p=2p$ can we shift it to $p-p=0$ which is guaranteed by the local $\bigcirc$,which means similarly to [12.107]

$$
\begin{tikzcd}
N=2 \arrow[rr, "\text{string}"] \arrow[dr, "\text{string}"] & & N=2 & \arrow[r, "\text{local}\,\bigcirc"] & & N=2 \arrow[rr, "\text{string}"] \arrow[dr, "\text{string}"] & & N=2 \arrow[dl, "\text{string}"] \\
& N=4 \arrow[ur, "\text{string}"'] & & & & & {\color{red}N=4} &
\end{tikzcd} \qquad [13.35]
$$

where N is oscillation level of string,which means $2+2 \neq 4 = (N, \tilde{N}) = (2,2)$ because the information of red 4 is for **D**0-eigenbrane in quasi-discontinuity.Similar to [Step III] above [13.1] that **Unified field$^!$ theory** lives in the good space,the proof of Goldbach conjecture also lives in the good space but we need to see it in detail (combing functorial and analytic).

13.4 Prism and DG Lie adic space

Combing below [13.22],the non-perturbation property of the flows and p-adic completion below [13.32],we have the following diagram

$$
\begin{tikzcd}
\boldsymbol{\mathcal{M}} \arrow[r, shift left, Rightarrow] \arrow[r, Rightarrow, "\mathbf{Def}^{\Leftarrow !}"'] & \bigcirc\text{-sense} & \boldsymbol{\mathcal{M}}\text{-flow} \arrow[l, shift right, Rightarrow] \arrow[l, Rightarrow, "\mathbf{Def}^{\Leftarrow !}"] \arrow[r, "p\text{-adic com.}"] & \boldsymbol{\mathcal{M}}\text{-flow}_{p\text{-adic completed}} \arrow[lll, bend right, dashed, "\triangle_{\text{prism}}"', "\simeq"] \arrow[ll, bend right, red] \arrow[ll, bend right, yellow] \arrow[ll, bend right, blue]
\end{tikzcd} \qquad [13.36]
$$

In the **UFT** [13.13],we have (co)homology on the $\boldsymbol{\mathcal{M}}$ with simplicial derived setting [13.8],actually the $\bigcirc$-sense gives a effect of prism and along this prism the co(homology) in LHS is scattered to various different (co)homology in the RHS of [13.36].And this gives us the prismatic cohomology in math.Also,we can form the following diagram by [14.23]

$$
\begin{tikzcd}
{\color{red}\mathscr{H}(\boldsymbol{O}(\boldsymbol{\mathcal{M}})^{\boldsymbol{:}})} & & \\
& {\color{blue}\check{O}(\boldsymbol{\mathcal{M}})^{\boldsymbol{:}} \Leftrightarrow O(\boldsymbol{\mathcal{M}})^{\boldsymbol{:}}} \arrow[ul, Rightarrow, red, "\mathscr{H}\text{-quotient}"', "\text{unique}"] & O(\boldsymbol{\mathcal{M}}) \arrow[l] \arrow[ull, bend right] \arrow[dl, Rightarrow, blue, "\text{global } \mathbf{GLC}"] \\
& O(\boldsymbol{\mathcal{M}}\text{-flow}) \arrow[u] \arrow[uul, bend left] & \boldsymbol{\mathcal{M}}^! \arrow[l, "\check{\mathcal{A}}"] \arrow[u, "\mathcal{A}"'] \arrow[ul, Rightarrow, "\bigcirc\text{-localizing}"]
\end{tikzcd} \qquad [13.37]
$$

where we used the notation below [12.92],and combing [13.36] with [14.26],

$$\begin{tikzcd} O(\mathcal{M})_{\bullet}^{\bullet} \arrow[r, "!"] \arrow[rr, bend left, "\text{not }!"] & \boldsymbol{O(\mathcal{M})_{\bullet}^{\bullet}} & \check{O}(\mathcal{M})_{\bullet}^{\bullet} \arrow[l, "!"'] \arrow[r, Rightarrow, "\mathbf{Def}^{\Leftarrow !}"] & \bigcirc\text{-sense} \end{tikzcd} \qquad [13.38]$$

such diagram gives an explanation of prismatic cohomology in physics.We can see in [23] for a rough framework of prismatic cohomology.Which also means,the top left sheaf of (co)homology in [13.37] is global,and locally it splits into several different types,and by [13.10] they are given by different types of superstring theories,the different types of (co)homologies are to count different types of LEEs (generalized super relative properties) of $\bigcirc$,also notice below [13.22].

We have seen below [13.32],the $|\,|_p$ is non-archimedean,so we cannot define Zariski topology for p-adic scheme.Similarly to the metic topology with metic as a valuation,the additive and multiplicative valuations give ring a topology,called topological ring.For a toplogical ring A and $I \subset A$ an ideal,we can define I-adic topology on M which is an A-module,generated by $\{x+I^nM|x \in M, n \in \mathbb{Z}^+\}$.It is completion because the metric is $d(a_m - b_n) = 2^{-\sup\{n|(a_m-b_n)\in I^nM\}}$ and $d(a_m - a_n) \to 0$ means $(a_m - b_n) \in I^\infty = \cap_\infty I^\infty = \{0\}$ with filtered ring $I = \cup_n I^n, I^n \subset I^{n-1}, I^0 = I$ see below [7.19].A Huber ring is a topological ring A admitting open subring $A_0 \subset A$ and for A_0-ideal I,A_0 has p-adic topology (let ideal I prime).A Huber pair (A, A^+) is a relative pair with A^+ is an open and integrally closed subring of integral elements in A.And the $\mathrm{Spv}(A, A^+)$ is the set of equivalence classes of valuations on A with $|A^+| \leq 1$.

Now,for a ring with characteristic p (i.e.$R/(p)$),we have a homomorphism $\varphi : A \to A$ with $a \mapsto a^p$ because $p|C_n^p, p\,\mathrm{or}\,n \neq 0$,so the binary expansion $(a+b)^p = a^p + b^p$ in this ring.If there is a Frobenius isomorphism on it,we call it the perfect ring.Recall that we use $\mathbb{Z}/n\mathbb{Z}$ graded ring for our generalized super setting,which has characteristic 2,for $a, b \in A_0 \oplus A_1^*$,if it has Frobenius equivalence,we find $(a+b)^2 = a^2 + b^2$ which behaving like we mod fermionic states.And the superalgebra is naturally Huber with a nilpotent unit ϖ generated by odd elements see above [7.11],so it is Tate which is not Lie so it is complete,satsfying

$$A_1^* \in \varpi^{-n} A_1^* \text{is bounded}, 2/\varpi^2 \in A_1^*, A_*^1/\varpi \xrightarrow{\text{hom}} (A_0 \oplus A_1^*)/(\varpi^2) \qquad [13.39]$$

This let us see in detail of $\mathbf{T}^\delta$-fusion in detail that is

$$A_0 \oplus A_1^* \cong_{\mathrm{P}(\mathbf{T})} A_{\text{Perfectoid}} = \lim_{a \mapsto a^2} A_0 \oplus A_1^* \qquad [13.40]$$

So we get $\mathcal{X}_0 \oplus \mathcal{X}_1^* \cong_{\mathrm{P}(\mathbf{T})} \mathcal{X}_{\text{Perf(ectoid)}} \subset \mathscr{M}$

Definition14.6 A DG Lie scheme is a DG Lie adic space covered by the adic space of huber ring which is generalized super affine Lie algebra

$$\mathrm{Spv}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}^+)^{\Delta^{\Leftarrow}}_{\text{Perf}} \subset \mathscr{M}^{\text{pre}} \vee_{\bigcirc} \mathscr{M} \qquad [13.41]$$

motivating us to study the algebraic geometry of (Tate) adic spaces.To understand [13.41],the open subsets are rational opens see [14.60],recall [2.12] the conformal symmetry contained in string theory preserves angle but not length,also by below [8.1] [11.18] and [12.19],so we remove length and remain the equivalence relation about bigness and the equivalence is reflected by [14.34].

14 Modern super algebraic geometry IV

Motivation

Based on the the discussion in section 13.3 as one motivation of this section which we want to introduce analytic stack and we based on YouTube videos [22] and [25].On the other hand,the key problem to achieve unification is the divergence from quantization of gravity or non-perturbation,we see in the diagram [12.19],we can construct M-theory because we put things into a good space,so we naturally assign the solution to that of finding a setting on the space to let it good in order to get rid of the divergence problem as we indeed have higher isomorphisms see [14.8] and [14.16].Geometrically,the compactness means every covering has a truncation to finite cardinality and analytically,the completeness means every power series is convergent to a point,the divergence means there is not a point for a series to get closed to.Thus,the compactness and completeness actually are equivalent settings extra for the good space and this is a naive view that there is a correspondence between algebraic geometry and analytic geometry see [14.45].We will see in [14.47],completing the $\mathrm{RHS}^{[12.19]}$ and taking to derived category relative to it let us go into the **UFT** following from this analytic AG theory which also helps us to further study it based on [13.14].

Theorem14.7 The existence of solution of quantization of gravity is equivalent to that of compactness (topological invariant) of $\mathrm{RHS}^{[12.19]}_{\mathrm{Solid}_{\mathbb{Z}[T]}}$.

Corollary14.8 Along the [12.19],$\mathbf{Ret}^*$compactness $\simeq$ quasi-compactness which gives an explanation why we have compactifications in string theory.In detail,the compactness can be seen in Ran space [11.52] in the RHS,explained by solidification [14.20] and the quasi-compactness is given by $susy^{\mathrm{c}}$ below [7.34] in the LHS of [12.19],the quasi is reflected by the physical compactification.

14.1 Quantisation of gravity (analytic setting in UFT)

A light profinite set is a countable inverse limit of finite set,with Grothendieck topology [9.1] generated by finite disjoint unions and surjective maps.So we get a site of profinite sets $\mathrm{Lightprof}^{\mathrm{op}}$,a light condensed set is a sheaf X : $\mathrm{Lightprof}^{\mathrm{op}} \to \mathrm{Set}$ with sheaf condition [9.2].A condensed ring $R^{\triangleright}$ is a light condensed simplicial set of rings see above [12.79].An analytic ring is a relative pair $R = (R^{\triangleright}, D(R))$,the category with strong equivalence is $D(R) \subseteq D(R^{\triangleright})$,s.t.

(i) Any limits and colimits has a finite or initial objects in $D(R)$

(ii) $\mathbf{R}\mathrm{Hom}_{D(R^{\triangleright})}(M,N) \in D(R)$, D(R) is homotopy enriched relatively [14.1]

(iii) $\widehat{N}_R \in D(R)^{\geq 0}, \forall N \in D(R^{\triangleright})^{\geq 0}$ (iv) $R^{\triangleright} \in D(R)$

In this case,we can assign a nontrivial behavior (singularity) a weak-regularity in $D(R^{\triangleright})$,this is just a analytic version of derived settings in [12.106].We have

$$\mathrm{Pro(Fin)}, \mathrm{Hom}_{\mathrm{Pro(Fin)}}(\lim_i S_i, \lim_j T_j) = \operatorname*{colim}_i \mathrm{Hom}_{\mathrm{Pro(Fin)}}(S_i, \lim_j T_j)$$

$$\text{totally disconnected compact Hausdorff spaces} \subset \mathrm{Top}, \quad (\mathrm{BooleanAlgebras})^{\mathrm{op}} \tag{14.2}$$

the third is $\forall x \in R, x^2 = x$ and these are equivalence of categories.Also,we need to consider the bigness of profinite set,for $S = \lim_{i \in \mathbb{N}} S_i$,the size is $\kappa = |S|$ and weight is $\lambda = |\mathrm{Cont}(S, \mathbb{F}_2)| = |\operatorname*{colim}_i (S_i)/\mathbb{F}_2|$,if $\lambda \leq \omega$ we call it is light.A proposition tells us if S is a light profinite set then there exist a surjection $\{0,1\}^{\mathbb{N}} \to S$ from the set of functions.

Combing above,a light condensed set as a funcor $X : \mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})^{\mathrm{op}} \to \mathrm{Set}$

$$\mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin}) \cong (\text{metrizable totally disconnected compact } T_2 \text{ spaces}) \tag{14.3}$$

$X \in \mathrm{CondSet}^{\mathrm{light}} = \mathrm{Sh}(\mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin}))$,with $X(\varnothing)$,$X(S_1 \coprod S_2) \cong X(S_1) \times X(S_2)$ which means ΓX is a factorization category and $X(S) \cong \mathrm{Eq}(X(T) \rightrightarrows X(T \times_S T)), \forall T \to_{\mathrm{surj}} S$,for example,a representable sheaf is light condensed set,$\underline{A}$: $\mathrm{Cont}(-, A)$ with $\underline{A}(*) = A$ the underlying set and $\underline{A}(\mathbb{N} \cup \{\infty\})$ = convergent sequences in A,so being metrizablly compactly generated = sequential (continuity from preserving convergence on level of sequences).And another reason for condensed set is let the topos $\mathrm{Sh}(-)$ below [14.3] be of compact and of being Hausdorff.Based on this,we can discuss light condensed abelian group,recall below [7.28],we can have a Grothendieck abelian category of sheaves of abelian groups denoted as $\mathrm{CondAb}^{\mathrm{light}}$,for an inclusion $\mathbb{Q} \hookrightarrow \mathbb{R}$ to the sheaf level we can form a relative sheaf $(\underline{\mathbb{R}}/\underline{\mathbb{Q}})$ with condensed setting,$(\underline{\mathbb{R}}/\underline{\mathbb{Q}})(*) = \mathbb{R}/\mathbb{Q}$,$(\underline{\mathbb{R}}/\underline{\mathbb{Q}})(S) = \mathrm{Cont}(S, \mathbb{R})/\mathrm{Cont}(S, \underline{\mathbb{Q}})$ with $\mathbb{Q}$ is relative discrete so $\mathrm{Cont}(S, \mathbb{Q})$ is a set of locally constant maps.To see clearly,we have a left Kan extension [10.79]

$$\begin{tikzcd}
& X \in \mathrm{Top} \arrow[dr] & \\
S \in \mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin}) \arrow[ur, hook, "\mathrm{Cov}"] \arrow[rr, hook] & & \mathrm{CondSet}^{\mathrm{light}}
\end{tikzcd} \tag{14.4}$$

Similarly to [10.80],we have for $\mathrm{Hom}_{\mathrm{Top}}(-, X) \to \underline{X} \cong \underline{X}(X)$

$$\underline{X}(X) \cong \operatorname*{colim}_{S \to X} \underline{X}(S) \cong \operatorname*{colim}_{S \to \underline{X}(*)} \underline{X}(S) \cong \underline{X}(\lim_{S \to \underline{X}(*)} S) \tag{14.5}$$

So $\underline{X}(*) \in \mathrm{Top}$ with quotient topology $(\sim = S \times_{X(*)} S)$ from

$$\coprod S \to \underline{X}(*) \cong \coprod \text{Cantor set} \to \underline{X}(*) \quad \text{with Cantor set} \cong \operatorname*{colim}_{Z_{\text{co.cl.}} \in \mathrm{Top}} Z \tag{14.6}$$

where co.cl.denotes for countable closed subsets Z with a sequential presentation $\coprod(\mathbb{N} \cup \{\infty\}) \to$ Cantor set with $\{0,1\}^{\mathbb{N}} \in$ (Cantor set),in this case we get a sequential set $\coprod(\mathbb{N}\cup\{\infty\}) \to \underline{X} \in \mathrm{CondSet}^{\mathrm{light}}$ with comparison to a simplicial set.From below [14.3],we get the following equivalence

$$\text{(qs (light) condesed sets)} \cong \mathrm{Ind}_{\mathrm{inj}}(\text{(metrizable) compact Hausdorff spaces}) \tag{14.7}$$

where qs means quasi-seperated,notice that the $\mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})^{\mathrm{op}}$ gives us Ind.For $\mathrm{CondAb}^{\mathrm{light}}$,we have unit object $\underline{\mathbb{Z}}$ with glued global tensor product $\otimes$ from assignment $S \mapsto M(S)\otimes N(S)$ and left adjoint of forgetfull functor $\mathrm{CondAb}^{\mathrm{light}} \to \mathrm{CondSet}^{\mathrm{light}}$ is given by $\underline{X} \to \mathbb{Z}[\underline{X}]$.Now,we want to discuss physics from [3.43]

$$\begin{tikzcd}
\mathcal{O}^1(z) \arrow[r, Leftrightarrow, "\sim"] \arrow[d] & \mathcal{O}^1_{-h} \arrow[r] \arrow[d] & \frac{\mathcal{O}^1_{-h+1}}{z} \arrow[r] \arrow[d] \arrow[dl, Rightarrow, red, "\simeq"'] & \frac{\mathcal{O}^1_{-h+2}}{z^2} \arrow[d] \arrow[dll, Rightarrow, red] & \cdots \arrow[dl, Rightarrow, red, "\simeq"] \\
\mathcal{O}^2(z) \arrow[r, Leftrightarrow, "\sim"] & \mathcal{O}^2_{-h} \arrow[r] & \frac{\mathcal{O}^2_{-h+1}}{z} \arrow[r] \arrow[ul, Rightarrow, red] & \frac{\mathcal{O}^2_{-h+2}}{z^2} \arrow[ul, Rightarrow, red, "\simeq"'] & \cdots
\end{tikzcd} \tag{14.8}$$

which are Laurent series of quantum fields,when we open the locality $z \to 0$,then we find the further to the right,the greater the divergence.Recall in Ho,we have retractions [12.17] which let us deform objects to another.But,we want to ask,is there an analogue in [14.8] that we can deform the divergence to convergence,and we call such thing a n-smoothification (red n-isomorphisms) which should be over $\boldsymbol{\mathcal{M}}^!$ [13.14] and this gives a concrete description of the no interaction below [13.13].And by the theorem14.4,these are from the quantum gravity effects in the **Unified field$^!$ theory** (**UFT**).In [13.8] and [13.13],actually we have a smooth hyper presentation with hyper descent of category of $\underline{\mathcal{A}}$-modules

$$\boldsymbol{\mathcal{M}}_{\bullet\bullet} \to \boldsymbol{\mathcal{M}}, \quad \underline{\mathcal{A}}\text{-}\mathbf{Mod}(\{\boldsymbol{\mathcal{M}}_{\bullet\bullet} \to \boldsymbol{\mathcal{M}}\}) \cong \underline{\mathcal{A}}\text{-}\mathbf{Mod}(\boldsymbol{\mathcal{M}}) \tag{14.9}$$

Exactly in [12.106],that should be quasi-discontinuity in RHS[12.19]

$$\begin{tikzcd}
n\text{-smoothification} \arrow[r] \arrow[d] & \text{quasi-continuity (get closed to each point)} \arrow[d] \\
\mathbf{P}\vee_{\bigcirc}\mathbf{D}n\text{-eigenbrane} \arrow[r] & \text{continuity}
\end{tikzcd} \tag{14.10}$$

This means we only have real continuity in **UFT**,otherwise the continuity is relative depending on we based on D-brane or P-brane,so it is quasi.And getting closed to each point means we can express each point as a convergent sequence [14.8] in RHS[12.19].For $T,U \in \mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})$,the local operation $\mathbb{Z}[[T]]\otimes_{\mathrm{Ab}}\mathbb{Z}[[U]] \neq \mathbb{Z}[[T,U]]$ corresponds to locality in physics [3.39] (cannot get closed to a local point,commutator does not vanish [3.31],open interaction in non-abelian case)

$$\begin{tikzcd}
\mathbb{Z}[[T]] \otimes_{\mathrm{Ab}} \mathbb{Z}[[U]] \arrow[r, "\simeq", "\text{locality induced by } \otimes_{\mathrm{Ab}}"'] \arrow[d, "\mathbf{UFT}"] & \overbracket{\mathbb{Z}[[T]]}\mathbb{Z}[[U]] \arrow[d, "\mathbf{UFT}"] \\
\mathrm{Smo}(\mathbb{Z}[[T]] \otimes_{\mathrm{Ab}} \mathbb{Z}[[U]]) \arrow[r, "\simeq"] & \mathrm{Smo}(\overbracket{\mathbb{Z}[[T]]}\mathbb{Z}[[U]]) \simeq \mathrm{Smo}(\mathbb{Z}[[T,U]])
\end{tikzcd} \tag{14.11}$$

where Smo$(-)$ means action of smoothification.We also have

$$\mathbb{Z}[\underline{\mathbb{R}}](*) = \left\{ \sum_{\delta x=[x,x+\epsilon]\subset\mathbb{R}} n_x \delta x | n_x \in \mathbb{Z}, \text{almost all } 0 \right\} = \underset{I=[-c,c]\subset\mathbb{R}}{\operatorname{colim}} \cup_{n\in\mathbb{N}} \mathbb{Z}[I]_{\leq n} \tag{14.12}$$

with $\sum |n_x| \leq n$ and we see that in condensed set a point of sections can be covered by a sequence and combining with physics we get

$$\begin{tikzcd}
\text{convergent sequence} \arrow[r, leftrightarrow, "{[11.33]}"] & \text{perturbative quantum effect} \, (\cong \setminus =)_{\text{et}} \\
\text{divergent sequence} \arrow[u, "\text{smoothification}"] \arrow[r, leftrightarrow, "{[12.67]}"] \arrow[ur, phantom, "\overset{\textbf{UFT}}{\Longleftrightarrow}"] & \text{non-perturbation in black hole} =_{\text{et}} \arrow[u, "\text{smoothification}"']
\end{tikzcd} \tag{14.13}$$

Recall we only can quantize the gravity in the RHS of [12.19],so similarly there is no solution of it in our real world.Combining with [14.8],this gives us a clear description of quantisation of gravity in **UFT**.

Theorem14.9 We say a gravity is quantised is equivalent to say a non-perturbation behaving like a divergence in gravity dominant region is smoothified to a perturbation behaving like a convergence in quantum dominant region.

By Yoneda lemma [9.7],the internal Hom (hom-tensor adjunction) is

$$\begin{aligned}
&\underline{\mathrm{Hom}}_{\mathrm{CondAb}^{\mathrm{light}}}(X,-)(S) \\
&\quad \cong \mathrm{Hom}_{\mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})^{\mathrm{light}}}(S, \underline{\mathrm{Hom}}_{\mathrm{CondAb}^{\mathrm{light}}}(X,-)(-)) \\
&\quad \cong \mathrm{Hom}_{\mathrm{CondAb}^{\mathrm{light}}}(\mathbb{Z}[S], \underline{\mathrm{Hom}}_{\mathrm{CondAb}^{\mathrm{light}}}(X,-)) \\
&\quad \cong \mathrm{Hom}_{\mathrm{CondAb}^{\mathrm{light}}}(X \otimes \mathbb{Z}[S], -)
\end{aligned} \tag{14.14}$$

where we used below [14.7].And for solving the problem and to get a completed tensor product [14.11],we perform the free collection [7.21] and things happens like the lifting in [10.110],which gives us an example of analytic ring

$$\mathrm{Gr}(\mathrm{CondAb}^{\mathrm{light}}, \otimes)_{\mathbb{Z}} \simeq (\mathrm{CondAb}^{\mathrm{light}}_{\mathrm{SymMon}}, \otimes^{\square})_{\mathbb{Z}} \text{ used in bottom of [14.11]} \tag{14.15}$$

Notice that this does not mean we come into classical physics,n-smoothifications from **UFT** [14.17] let us retract things to convergent level [14.8].In [14.24],we have $M \otimes^{\square} N = (M \otimes N)^{\square} = (M^{\square} \otimes N^{\square})^{\square}$ from physics [14.11].We can see

$$\begin{tikzcd}
\mathbf{P}0 \arrow[rr, "\mathbf{P}1"] \arrow[dr, "\mathbf{P}1"'] & & \mathbf{P}0 \arrow[dl, "\mathbf{P}1'"] \arrow[r, "\mathscr{H}\text{-quotient}", "{[13.13]}"'] & \mathbf{P}\vee\mathbf{D}0 \arrow[rr] \arrow[dr, "\mathbf{P}1"'] & & \mathbf{P}\vee\mathbf{D}0 \\
& \mathbf{D}0 \arrow[u, phantom, "\Uparrow \text{2-smoothification}"] & & & \mathbf{P}\vee\mathbf{D}0 \arrow[ur] \arrow[u, phantom, "\Uparrow \simeq"] &
\end{tikzcd} \tag{14.16}$$

where we used [12.107],also we need to know we are still in RHS of [12.19].

$$n\text{-smoothification RHS}^{[12.19]} \cong (\mathscr{H}\text{-quotient})^{*}\mathbf{P}\vee_{\bigcirc}\mathbf{D}n\text{-eigenbrane } \mathbf{UFT} \tag{14.17}$$

Now,we get a relative category ($\mathrm{RHS}^{[12.19]}$,**UFT**).

Remark14.7 So called quantum gravity effect below [13.13] [13.31],means the weak projection from **UFT** to the relative weak category $\mathrm{RHS}^{[12.19]}$.

Also,the quantum gravity is governed by the local 2-nonexistence see above [12.34],we have locality and globality **at the same time**,this is reflected by

$$f^*: \underline{\mathrm{Hom}}(P, M) \xrightarrow{\simeq^{\mathrm{RHS}^{[12.19]}}} \underline{\mathrm{Hom}}(P, M),\ (m_0, m_1, ...) \mapsto (m_0 - m_1, m_1 - m_2, ...) \quad [14.18]$$

in math in $\mathrm{CondAb}^{\mathrm{light}}$,this is a non-archimedean property of summability of null sequence $P = \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/\mathbb{Z}[\infty]$ and the quasi-isomorphism follows from $f: P \to P, [n] \mapsto [n] - [n+1]$.In this case,$M$ is solid and $D(\mathrm{Solid}^{\mathrm{GenSup}}) \subset$ **UFT**.And in the derived category,$A \in D(\mathrm{CondAb}^{\mathrm{light}})$ is solid if it satisfies the following,it is solid if it satisfies

$$\mathbf{R}\underline{\mathrm{Hom}}(P, A) \xrightarrow{\simeq} \mathbf{R}\underline{\mathrm{Hom}}(P, A) \Leftrightarrow \underline{\mathrm{Hom}}(P, H^i(A)) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(P, H^i(A)) \quad [14.19]$$

which means all $H^i(A)$ is solid (in relative strong category) and this gives an explanation of [13.14] that why we define **UFT** after $\mathscr{H}$-quotient.Also

$$\begin{tikzcd} & \text{non-archimedean} \arrow[r, "\sim"] \arrow[d] & \text{discontinuity in qut.dominance} \arrow[dl, "\text{solidification}"', "\text{quasi-continuity}"] \arrow[d, "\text{localizing}"] \\ & 1 + z(\text{qut.dominance}) & z(\text{gr.dominance}) \arrow[l, "="', "\text{non-localizing}"] \end{tikzcd} \quad [14.20]$$

So the solidification is a smoothification and a quantization of gravity if we put physics in.Notice that the bottom line in [14.20] says non-archimedean getting closed to 0 from 1 and getting closed to 1 happen **at the same time**,for $(\underline{\mathbb{Z}} \to P \to \underline{\mathbb{R}}) \to \underline{\mathbb{R}}^{\square}, 1 = 0 \in \underline{\mathbb{R}}^{\square} = 0 = 1, \subset$ étale closed local $\bigcirc$,giving [14.8].

$$z = \sum_{n=1}^{\infty} n\frac{1}{n} = \sum_{n+1}^{\infty} n\frac{1}{n} = 1 + \sum_{n=2}^{\infty} n\frac{1}{n} = 1 + z, \quad \mathbb{Z} \in \mathrm{Solid} \quad [14.21]$$

The derived solidification of $\otimes_{\mathrm{Ab}}$ is $\otimes^{\mathbf{L}\square}$.Based on the double-weak [12.85]

$$\begin{tikzcd} & \text{étale closed local} \bigcirc & \\ \text{closed} =_{\mathrm{et}} \arrow[ur] \arrow[rr, "\mathbf{Ret_*Ads/Cft}"'] & {\color{red}\Uparrow \mathscr{H}\text{-quotient}} & \text{closed}_{\mathrm{weak}}(\cong \setminus =)_{\mathrm{et}} \arrow[ul] \end{tikzcd} \quad [14.22]$$

where local $\bigcirc \simeq =_{\mathrm{et}} \vee_{\bigcirc} (\cong \setminus =)_{\mathrm{et}}$,and this gives us a clear description of the relative pair below [14.26] based on the relativity of the double-weak,with

$$\mathscr{H}\text{-quotient}_*(-) \circ \mathrm{Smo}(-) = D(-) \circ \mathrm{Smo}(-) : \mathrm{RHS}^{[12.19]} \to \mathbf{UFT} \quad [14.23]$$

Also,we want to push all things to scheme level with Y-duality see [9.106]

$$(\mathrm{Ind}_{\mathbb{N}}(\mathrm{Sch}_{\mathbf{Fin}}), \mathbf{Y}) \cong (\text{Corresponding representables sheaves}) \qquad [14.24]$$

recall we have the representability only over at least fppf site (smoothness),which means we need give the relative strong category **UFT** such a structure,that is

$$\operatorname*{colim}_{[n]\in\Delta^{\Leftarrow}, n\in\mathbb{N}\cup\{\infty\}} \mathbf{P} \vee_{\bigcirc} \mathbf{D}n\text{-eigenbranes} \in \mathrm{Ind}_{\mathbb{N}}(\mathbf{P} \vee_{\bigcirc} \mathbf{D}\text{-eigenbranes}^{\Delta^{\Leftarrow}}) \quad [14.25]$$

notice that we have self T-duality,which means $\mathbb{N}\cup\{\infty\} \cong \mathbb{N}$,so $\mathrm{Ind}_{\mathbb{N}} \cong \mathrm{Ind}_{\Delta^{\Leftarrow}}$,In this case,a simplical sequence is a convergence sequence.Also,we need to notice that we need an orientation (by tangent vector) to get closed to a point,but now $\{0\} \cong \{\infty\}$ we lost the orientation,meaning that we need point it out,we need

$$=_{\mathrm{et}} \vee_{\bigcirc} (\cong \setminus =)_{\mathrm{et}} \simeq \bigcirc\text{-flow} \vee_{\bigcirc\text{-sense}} \bigcirc, \quad \mathbf{UFT} \rightsquigarrow_{\mathbf{Def}^{\Leftarrow !}} \bigcirc\text{-sense} \qquad [14.26]$$

14.2 Structure of Solid and behaviors on $\mathbf{RHS}^{[12.19]}_{\mathbf{Solid}_{\mathbb{Z}[T]}}$

The solidification Solid $\hookrightarrow$ CondAb$^{\text{light}}$ is to find a class of complete objects which means in Solid all sequences are convergent,to study over $\mathbb{Z}$ because in math it is hard to let $\mathbb{R}$ (archimedean) be complete,also by [14.20] the non-archimedean corresponds to discontinuity in physics.Also,by a theorem 5.13 in [25],the Solid is abelian,stable under (co)limit and has a single compact projective generator $\prod_{\mathbb{N}} \mathbb{Z}$,so we work over $\mathbb{Z}$.In detail,the $P^{\square} \cong \mathbb{Z}[S]^{\square}$ below [14.18] with $\mathbb{Z}[S]^{\square} = \operatorname*{colim}_i (C(S_i, \mathbb{Z}), \mathbb{Z}) = \mathrm{Hom}(C(S, \mathbb{Z}), \mathbb{Z}) = \mathrm{Hom}(\bigoplus_{\mathbb{N}} \mathbb{Z}, \mathbb{Z}) = \prod_{\mathbb{N}} \mathbb{Z}$.

Notice that [14.15] gives us $(-)^{\square} : \mathrm{Cond}^{\text{light}}_{\mathbb{Z}} \to \mathrm{Solid}_{\mathbb{Z}}$,to the abelian subcategory (free collected,quantised [14.8] in RHS$^{[12.19]}$ to **UFT**).By the definition of being finitely generated below [7.29],for $\underline{M}$ and $\mathrm{Solid}_{\mathbb{Z}} \cong \mathrm{Ind}((\underline{M} \in)\mathrm{Solid}^{\text{fin.pres.}}_{\mathbb{Z}})$

$$\begin{aligned} &0 \to \prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{Z} \to M \to 0, \mathrm{Hom}(\prod_{\mathbb{N}} \mathbb{Z}, \mathbb{Z}) = \bigoplus_{\mathbb{N}} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \bigoplus_{\mathbb{N}} \mathbb{Z} \\ &0 \to \mathrm{Hom}(M, \mathbb{Z}) \to \bigoplus_{\mathbb{N}} \mathbb{Z} \xrightarrow{h} \bigoplus_{\mathbb{N}} \mathbb{Z} \to \mathrm{Ext}^1(M, \mathbb{Z}) \to 0 \end{aligned} \qquad [14.27]$$

where we let h to be injective and now $\mathrm{Hom}(M, \mathbb{Z}) = 0$.To the sheaf level,we act $\underline{\mathrm{Hom}}(-, \mathbb{Z})$,we get with $\underline{\mathrm{Hom}}(M, \mathbb{Z})$ dual to quotient

$$0 \to \underline{\mathrm{Hom}}(M, \mathbb{Z}) \hookrightarrow \prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{Z} \to \underline{\mathrm{Ext}}^1(M, \mathbb{Z}) \to 0 \qquad [14.28]$$

By using [14.14],we have

$$\underline{\mathrm{Hom}}(M, \mathbb{Z})(S) = \mathrm{Hom}(M \otimes \mathbb{Z}[S], \mathbb{Z}) = \mathrm{Hom}(M, \mathrm{Cont}(S, \mathbb{Z})) = 0 \qquad [14.29]$$

where we work over free module,it generated by $\mathrm{Hom}(M, \mathbb{Z})$ over S.Because any finitely presented submodule is isomorphic to a product of copies of $\mathbb{Z}$,so for any

$\underline{M}$,it is in the form of sheaf of cohomology $\underline{\mathrm{Ext}}^1(M,\mathbb{Z})$,which is also the cokernal of the short exact sequence [14.28] and this gives us an analytic understanding of definition **UFT** in [13.14].Also,$\prod_{\mathbb{N}}\mathbb{Z}$ is flat under $\otimes^{\square}$

$$(0 \to \prod_{\mathbb{N}}\mathbb{Z} \xrightarrow{g} \prod_{\mathbb{N}}\mathbb{Z} \to \underline{M} \to 0) \otimes^{\square} \prod_{\mathbb{N}}\mathbb{Z}, \quad \prod_{\mathbb{N}} \otimes^{\square} \prod_{\mathbb{N}}\mathbb{Z} = \prod_{\mathbb{N}\times\mathbb{N}}\mathbb{Z} \qquad [14.30]$$

as g always keep injective and $\otimes^{\mathbf{L}}$ is the $\otimes^{\mathbf{L}\square}$ with degree 0 which commutes with filtered colimit in homotopy enriched category and we have $\underline{M} \otimes^{\square} \prod_{\mathbb{N}}\mathbb{Z} = \operatorname*{colim}_i \prod_{\mathbb{N}}(\underline{M}_i \otimes^{\square}\mathbb{Z}) = \prod_{\mathbb{N}}\underline{M}$.For $M \in \mathrm{Ab}$,the derived p-adic completion of M is $M_{\hat{p}} = \mathbf{R}\lim_i M/^{\mathbf{L}}p^n$,we can choose $M/^{\mathbf{L}}p^n = (M \to M/p^n)$ which is a cofibrant replacement see [12.76].If $N, M \in D(\mathrm{Solid})^{\geq 0}$ are derived p-complete,then $M \otimes^{\mathbf{L}\square} N$ is derived p-complete,$(\bigoplus_{\mathbb{N}}\mathbb{Z})_{\hat{p}} \otimes^{\mathbf{L}\square} (\bigoplus_{\mathbb{N}}\mathbb{Z})_{\hat{p}} = (\bigoplus_{\mathbb{N}\times\mathbb{N}}\mathbb{Z})_{\hat{p}}$.To see this,from [14.11],we get $(\mathbb{Z}[[T]] \otimes^{\mathbf{L}\square} \mathbb{Z}[[U]] = \mathbb{Z}[[T,U]])/(T-p, U-p)$,we get a simple example $\mathbb{Z}_p \otimes^{\mathbf{L}\square} \mathbb{Z}_p = \mathbb{Z}_p$ and we let $\hat{\otimes}$ to be p-adic completion of local $\otimes$.We want to focus on $M = N = \hat{\bigoplus}_{\mathbb{N}}\mathbb{Z}_p$,first we have an injection

$$\operatorname*{colim}_{f:\mathbb{N}\to\mathbb{N}} \prod_{\mathbb{N}} p^{f(n)}\mathbb{Z}_p \to \hat{\bigoplus}_{\mathbb{N}}\mathbb{Z}_p = \left(\bigoplus_{\mathbb{N}}\mathbb{Z}\right)_{\hat{p}} = \lim_n \left(\bigoplus_{\mathbb{N}}\mathbb{Z}/p^n\right), \ker = 0 \quad [14.31]$$

where $\bigoplus_{m\leq f(n)}\mathbb{Z}/p^n$ with $f: n \mapsto m_{\max}$ so it is also surjective.In this case,

$$\begin{aligned} &M \otimes^{\mathbf{L}\square} N \cong \left(\operatorname*{colim}_{f:\mathbb{N}\to\mathbb{N}} \prod_{\mathbb{N}} p^{f(n_1)}\mathbb{Z}_p\right) \otimes^{\mathbf{L}\square} \left(\operatorname*{colim}_{g:\mathbb{N}\to\mathbb{N}} \prod_{\mathbb{N}} p^{g(n_2)}\mathbb{Z}_p\right) \\ &\cong \operatorname*{colim}_{f,g:\mathbb{N}\to\mathbb{N}} \prod_{\mathbb{N}} p^{f(n_1)+g(n_2)}\mathbb{Z}_p \cong \operatorname*{colim}_{h:\mathbb{N}\times\mathbb{N}\to\mathbb{N}\times\mathbb{N}} \prod_{\mathbb{N}} p^{h(n_1,n_2)}\mathbb{Z} \cong \hat{\bigoplus}_{\mathbb{N}\times\mathbb{N}}\mathbb{Z}_p \end{aligned} \qquad [14.32]$$

where we used [14.31].Next,we want to connect the solidification with rational opens in Spv to study physics in [13.39].

Next,we want to give $\mathbb{Z}[\underline{T}]$ [14.12] a geometric interpretation,it behaves like $\mathrm{Spec}(\mathbb{Z}[T])$,where the $\mathbb{N}\cup\{\infty\}$-points give us $\mathbb{Z}[[T]]$ which behaves like a subspace,but it breaks the property of functorial structure,we see this by

$$\mathbb{Q}_p \otimes^{\square}_{\mathbb{Z}} \mathbb{Z}[[T]] = \mathbb{Z}_p\left[\frac{1}{p}\right] \otimes^{\square}_{\mathbb{Z}} \mathbb{Z}[[T]][1] = (\mathbb{Z}_p \otimes \mathbb{Z}[[T]])\left[\frac{1}{p}\right] = \mathbb{Z}_p[[T]]\left[\frac{1}{p}\right] \quad [14.33]$$

which should be isomorphic to $\mathbb{Q}_p[[T]]$,but $\mathbb{Z}_p[1/p]$ is not isomorphic to it.Recall the definition above [13.31] with $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \| x|_p \leq 1\}$,the p-adic norm is for measuring the length of the functions in this vector space,so actually $\mathbb{Z}_p[[T]][1/p]$ should be understood as the ring of bounded functions,and taking Spec,we get an open unit disc in $\mathrm{Spec}(\mathbb{Q}_p[T])$.This suggests that $\mathbb{Z}[[T]] \hookrightarrow \mathbb{Z}[T]$ as the open unit disc,and it gives a constraint on the local operation in [14.32] that is $|x|_p \leq 1$ for $\mathbb{Q}_p$,then we preserve the the functorial structure.Getting the closed unit disc bases on (6.1.1) of [2], explained in 6.3 of [4].We need to know the conformal map above [3.39] shift invariantly from the additive lower cylinder C_2 to multiplicative unit disc and upper C_2 to outside of unit disc.

$$C_2 \quad w \leftrightarrow_{\cong} -w \xrightarrow{z=e^{-iw}} D_2 \quad (z \to \infty) \leftrightarrow_{\cong} (\tfrac{1}{z} \to 0) \qquad [14.34]$$

where the outside and inside is equivalent and we can cover any geometric object by closed unit discs.Then,we form a 2-cover $(S^1 \cong C_2)$,$\mathbb{Z}[[T]] \otimes^{\square} \mathbb{Z}[[T^{-1}]] = \mathbb{Z}[[T, T^{-1}]]$ and we want to localized at the $w > 0$ (mapping to open unit disc),$\mathbb{Z}((T^{-1})) = \mathbb{Z}[[T^{-1}]] \otimes_{\mathbb{Z}[[T^{-1}]]} \mathbb{Z}[T, T^{-1}]$,killing it (getting $w \leq 0$) is equivalent to mod the equivalence relation in [14.33] (restrict to $w \leq 0$).

$$\text{killing}\, \mathbb{Z}((T^{-1})) \cong (\mathbb{Z}[[T]][T] \cong \mathbb{Z}[[T]][T]/(UT-1)) \qquad [14.35]$$

where we mod the equivalence relation of 2-cover formed by T-valued and $(1/U)$-valued discs by the isomorphism.We can try to equip with physics,we can form an eigen S_2 from [12.92] in ($\mathrm{RHS}^{[12.19]}$, $\mathbf{UFT}$) see below [14.17],

$$\mathscr{H}\text{-quotient}_*(\mathbf{P}D_2 \oplus \mathbf{D}D_2\text{-eigenbrane}) \cong \mathbf{P} \vee_{\bigcirc} \mathbf{D}D_2\text{-eigenbrane} \in \mathbf{UFT} \qquad [14.36]$$

If we let $\mathbb{Z}((T^{-1})) = \mathbf{P}D_2$-eigenbrane,we get in $\mathbf{UFT}$

$$\mathbf{R}\underline{\mathrm{Hom}}(\mathbf{P}D_2\text{-eigenbrane}, D(M)) = 0, D(M) \in D\big((\mathrm{Mod}_{\mathbb{Z}[T]}(\mathrm{Solid}_{\mathbb{Z}}))^{\mathrm{RHS}^{[12.19]}}\big) \qquad [14.37]$$

And this gives us an explanation of the modules in [13.14] where

$$D\big((\mathrm{Mod}_{\mathbb{Z}[T]}(\mathrm{Solid}_{\mathbb{Z}}))^{\mathrm{RHS}^{[12.19]}}\big) \subset (\mathscr{H}_i(O(\boldsymbol{\mathcal{M}})))\text{-}\mathbf{Mod} \qquad [14.38]$$

Back to math,M is $\mathbb{Z}[T]$-solid,if and only if

$$\begin{aligned}&\underline{\mathrm{Hom}}_{\mathbb{Z}}(P, M) \cong \underline{\mathrm{Hom}}_{\mathbb{Z}}(P, M/(UT-1)) \Leftrightarrow [14.37] \Leftrightarrow \\ &\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(P \otimes_{\mathbb{Z}} \mathbb{Z}[T], M) \cong \underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(P \otimes_{\mathbb{Z}} \mathbb{Z}[T], M/(UT-1))\end{aligned} \qquad [14.39]$$

We have group $(\mathbb{Z}[T_M], \mathbb{Z}[T_N])$-actions on $D(\mathbb{Z}[T])$-modules (M, N) by the derived tensor product,see below [12.46],by quasi-flat cofibrant replacement of $\mathbb{Z}[T]$

$$\begin{aligned}M &\cong M \otimes^{\mathbf{L}}_{\mathbb{Z}[T]} \mathbb{Z}[T] \cong M \otimes^{\mathbf{L}} \mathrm{cofib}(\mathbb{Z}[T_N] \cdot T_M \to \mathbb{Z}[T_N]) \\ &\cong \mathrm{cofib}(M \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Z}[T_N] \to M \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Z}[T_N]/(T_M - T_N))\end{aligned} \qquad [14.40]$$

Then,we put if into the derived sheaf of Hom,we get

$$\begin{aligned}&\mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(M, N) \\ &\cong \mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathrm{cofib}(M \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Z}[T_N] \to M \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Z}[T_N]/(T_M - T_N)), N) \\ &\cong \mathrm{fib}(\mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T_N]}(M \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Z}[T_N], N) \to_{/(T_M - T_N)} \mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T_N]}(M \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Z}[T_N], N)) \\ &\cong \mathrm{fib}(\mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T_N]}(M, \mathbf{R}\underline{\mathrm{Hom}}(\mathbb{Z}[T_N], N) \to_{/(T_M - T_N)} \mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T_N]}(M, \\ &\mathbf{R}\underline{\mathrm{Hom}}(\mathbb{Z}[T_N], N)) \cong \mathrm{fib}(\mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(M, N) \to_{/(T_M - T_N)} \mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(M, N))\end{aligned} \qquad [14.41]$$

where we used [14.14].Now,for a $\mathbb{Z}[S]$-module with action $f : X \to X$,

$$\begin{aligned} X &\cong \operatorname{cofib}(X \otimes^{\mathbf{L}}_{\mathbb{Z}[f]} \mathbb{Z}[f] \cdot T \to_{/(T-f)} X \otimes^{\mathbf{L}}_{\mathbb{Z}[f]} \mathbb{Z}[f] \cdot T) \\ &\cong \operatorname{cofib}(X[T] \to_{/(T-f)} X[T]) \end{aligned} \tag{14.42}$$

where T is a quasi-flat S-module.We want to make several things clear,by below [14.29],we will have a form of $\prod_{\mathbb{N}}(-)$ in $\mathrm{Solid}_{\mathbb{Z}}^{\text{fin.pres.}}$ and 2-categorical notion

$$\mathrm{Mod}_{\mathbb{Z}[T]}(M)_{[9.25]} \in \mathrm{Solid}_{\mathbb{Z}}\text{-}\mathrm{Mod}_{\mathbb{Z}[T]} = \mathrm{Mod}_{\mathbb{Z}[T]}(M \in \mathrm{Solid}_{\mathbb{Z}}) \tag{14.43}$$

Thus,we have inclusions with adjoint $(-)^{\square}$ which is left to the 2nd above [14.27].

$$\mathrm{Solid}_{\mathbb{Z}[T]} \subset \mathrm{Solid}_{\mathbb{Z}}\text{-}\mathrm{Mod}_{\mathbb{Z}[T]} \subset \mathrm{CondAb}^{\mathrm{light}}\text{-}\mathrm{Mod}_{\mathbb{Z}[T]} \tag{14.44}$$

and the first inclusion should also have a left adjoint $(-)^{T\square}$,such functor should have a property $((\prod_{\mathbb{N}} \mathbb{Z})[T])^{T\square} = \prod_{\mathbb{N}} \mathbb{Z}[T]$.Before we see more properties,we want to give a summary of solidification by diagrams,in the math side

$$\begin{tikzcd} \text{(series of algebraic structures)} \arrow[r, "\simeq", "{[14.7]}"'] \arrow[d, "{[14.8]}"', "\simeq"] \arrow[dr, "\text{to complete}"] & \text{(series of geometric structures)} \arrow[d, "\text{to complete}"] \\ \text{(series of analytic structures)} \arrow[r, "\text{to complete}"] & \mathrm{RHS}^{[12.19]}_{\mathrm{Solid}_{\mathbb{Z}[T]}} \end{tikzcd} \tag{14.45}$$

where because $\bigcirc$-sense [13.21],we do not distinguish math with physics.

$$\mathrm{RHS}^{[12.19]}_{\mathrm{Solid}_{\mathbb{Z}[T]}} \cong \text{(spaces covered by } \mathbf{P} \oplus \mathbf{D}D_2\text{-eigenbranes)} \tag{14.46}$$

where $\mathbf{P}D_2 \oplus \mathbf{D}D_2 \cong \mathbf{P} \oplus \mathbf{D}D_2$ should be understood as the closed unit disc in $\mathrm{RHS}^{[12.19]}_{\mathrm{Solid}_{\mathbb{Z}[T]}}$ and combing with [14.23] and [14.4] we have the diagram

$$\begin{tikzcd} \mathrm{RHS}^{[12.19]} \arrow[r, hookleftarrow] \arrow[r, bend left, "\mathrm{Smo}(-)"] & \mathrm{RHS}^{[12.19]}_{\mathrm{Solid}_{\mathbb{Z}[T]}} \arrow[r, bend left, "\mathscr{H}\text{-quotient}_*(-)"] & \mathbf{UFT} \arrow[l, "\mathrm{P}(\mathbf{F})^{-1}"] \end{tikzcd} \tag{14.47}$$

And the reason we do not consider $\mathrm{Solid}_{\mathbb{Z}} \neq \mathrm{Solid}_{\mathbb{Z}[T]}$ is in [14.16]

$$\begin{tikzcd} \mathbf{P}0 \vee_{\bigcirc} \mathbf{D}0 \arrow[r, "\simeq"] \arrow[r, bend left, "\mathbf{D}0\text{-localizing}"] \arrow[r, bend right, "\mathbf{P}0\text{-localizing}"'] & \mathbf{P}0 \oplus \mathbf{D}0 \arrow[r, hook] & \mathrm{RHS}^{[12.19]}_{\mathrm{Solid}_{\mathbb{Z}}} \end{tikzcd} \tag{14.48}$$

We want $\mathrm{RHS}^{[12.19]}_{\mathrm{Solid}_{\mathbb{Z}[T]}} \cap \mathbf{UFT} = \varnothing$ and [14.37] is preserved in it.

$$\mathbf{P}D_2\text{-eigenbrane} \to_{\text{cannot retract}} \mathbf{P} \vee_{\bigcirc} \mathbf{D}0\text{-eigenbrane} \in \mathbf{UFT} \tag{14.49}$$

Back to math,let $M \in \mathrm{Mod}_{\mathbb{Z}[T]}(\mathrm{Solid}_{\mathbb{Z}})$,so $M \simeq \underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[T], M)$,by the [14.37] we should have $\mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1})), M^{\mathbf{L}T\square}) = 0$ which helps us to find an expression of $M^{\mathbf{L}T\square}$,it just to give a modification of $\mathbb{Z}[T]$.We find

$$\begin{aligned}0 =&\mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1})), \mathrm{cofib}(\mathbb{Z}((T^{-1})) \to \mathbb{Z}((T^{-1}))) \\ &\cong \mathrm{cofib}(\mathbb{Z}((T^{-1})) \otimes^{\mathbf{L}\square}_{\mathbb{Z}[T]} \mathbb{Z}[T] \to \mathbb{Z}((T^{-1})) \otimes^{\mathbf{L}\square}_{\mathbb{Z}[T]} \mathbb{Z}((T^{-1}))) \\ &\cong \mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1})), \mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1], M))\end{aligned} \quad [14.50]$$

so $M^{\mathbf{L}T\square} \simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1], M)$ which should be understood as killing $\mathbb{Z}((T^{-1}))$ so $\mathbb{Z}((T^{-1}))$ derived hom to give zero.To connect with physics,

$$\mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1] \simeq_{[14.35]} \Gamma(\mathbb{Z}[T]/(\mathbb{Z}[[T]][T] \cong \mathbb{Z}[[T]][T]/(UT-1))) \quad [14.51]$$

Then we open the generalized super setting by below [12.92]

$$\begin{aligned}&\mathbb{Z}[T]/\mathbb{Z}[[T]][T] \in \check{\mathcal{A}}, \mathbb{Z}[T] \setminus \mathbb{Z}[[T]][T]/(UT-1) \in \mathcal{A} \\ &\mathbb{Z}[T]/(\mathbf{P}\mathbb{Z}[[T]][T] \cong \mathbf{D}\mathbb{Z}[[T]][T]/(UT-1)) \\ &\simeq \mathbb{Z}[T]/((\mathbb{Z}[[T]][T])^{\mathrm{rep}} \times \mathbb{Z}[[T]][T]) \in \check{\mathcal{A}} \oplus \mathcal{A}\end{aligned} \quad [14.52]$$

In this case,by [13.13],[14.23] and [14.46] we have

$$\begin{aligned}&\Gamma(\mathbb{Z}[T]/((\mathbb{Z}[[T]][T])^{\mathrm{rep}} \times \mathbb{Z}[[T]][T])) \simeq \mathbf{P} \oplus \mathbf{D}D_2\text{-eigenbrane} \\ &D(\Gamma(\mathbb{Z}[T]/((\mathbb{Z}[[T]][T])^{\mathrm{rep}} \times \mathbb{Z}[[T]][T]))) \simeq \mathbf{P} \vee_{\bigcirc} \mathbf{D}D_2\text{-eigenbrane}\end{aligned} \quad [14.53]$$

which gives us a way to understand the object in **UFT** that is

$$(\mathbf{P}\,\mathrm{or}\,\mathbf{D}D_2 \subset \mathbf{P} \oplus \mathbf{D}D_2 \not\subset \mathbf{P} \vee_{\bigcirc} \mathbf{D}D_2)\text{-eigenbrane} \quad [14.54]$$

The next property is for $D(\mathrm{Solid}_{\mathbb{Z}}) \to D(\mathrm{Solid}_{\mathbb{Z}[T]}), M^{\mathbf{L}\square} \mapsto (M \otimes_{\mathbb{Z}} \mathbb{Z}[T])^{\mathbf{L}T\square}$

$$\begin{aligned}&(M \otimes \mathbb{Z}[T])^{\mathbf{L}T\square} \cong (M[T])^{\mathbf{L}T\square} \simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1], M[T]) \\ &\mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[[U]][-1], M[T]) \cong \mathrm{fib}(\mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[[U]][-1], M[T]) \\ &\to_{/(T-f)} \mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[[U]][-1], M[T])) \\ &\cong \mathrm{fib}(\mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(U\mathbb{Z}[[U]][-1], M) \otimes^{\mathbf{L}T\square}_{\mathbb{Z}[T]} \mathbb{Z}[T] \\ &\to_{/(T-f)} \mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(U\mathbb{Z}[[U]][-1], M) \otimes^{\mathbf{L}T\square}_{\mathbb{Z}[T]} \mathbb{Z}[T]) \\ &\cong \mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathrm{cofib}(U\mathbb{Z}[[U]][T][-1] \to_{/(1/U-f)} U\mathbb{Z}[[U]][T][-1]), \\ &\mathrm{cofib}(M[T] \to_{/(T-f)} M[T])) \cong \mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}}(U\mathbb{Z}[[U]][-1], M) \in D(\mathrm{Solid}_{\mathbb{Z}})\end{aligned} \quad [14.55]$$

where we used [14.35] in the first line,so acting $\mathbb{Z}[T]$ is equivalent to act $\mathbb{Z}[1/U]$ and used [14.41] in the second line,then used [14.42] in the last line.And the isomorphism tells us the functor below [14.54] is t-exact.Which means we have a t-structure on **UFT** based on the t-structure of the M-theory see [11.66].And this t-structure gives the orientation [14.26],and it should be

$$\mathbf{UFT}^{\heartsuit} = \bigcirc\text{-sense} \hookrightarrow (\bigcirc\text{-senses}) \equiv \bigcirc\text{-}\mathbf{Sense} \quad [14.56]$$

Loosely speaking,$\mathbf{UFT}^{\heartsuit}$ is a point of sense in **UFT**,of absolute nonexistence,it can be "seen" if we do meditation at that level.We need to realize that the **TOE** is a theory of everything (math,physics,philosophy,religion etc.),but we mainly discuss philosophical and scientific views here and we put the religious correspondence to our theory at the very end.And combing with [14.26],we get

$$\mathbf{Def}^{\Leftarrow !}\mathbf{UFT} \simeq \mathbf{UFT}^{\heartsuit} =_{[13.22]} \mathbf{TOE} \hookrightarrow \bigcirc\textbf{-Sense} \qquad [14.57]$$

and we need to notice that compared to [13.20] and [13.21] this is the only way we can get (quasi-define) to the $\bigcirc$-sense.

Now,we have a structure of Six-Functor formalism see [24] on [14.44].

$$\begin{tikzcd} D(\mathrm{Mod}_{\mathbb{Z}((T^{-1}))}(\mathrm{Solid}_{\mathbb{Z}})) \arrow[r, "\subset = i_*"'] & D(\mathrm{Mod}_{\mathbb{Z}[T]}(\mathrm{Solid}_{\mathbb{Z}})) \arrow[l, bend right=40, "(-)\otimes_{\mathbb{Z}[T]}\mathbb{Z}((T^{-1}))=i^*"'] \arrow[l, bend left=40, "i^! \cong i^*"] \arrow[r, "(-)^{\mathbf{L}T\square}=j^*", "{[14.50]}"'] & D(\mathrm{Solid}_{\mathbb{Z}[T]}) \arrow[l, bend right=40, "M\mapsto [M\to M\otimes_{\mathbb{Z}}\mathbb{Z}[T]]=j_!"'] \arrow[l, bend left=40, "\subset = j_*"] \end{tikzcd}$$

[14.58]

where ! is for local functor and $*$ is for global functor,inclusion i is proper and we can get j_* by 2-sheafification of $j_!$.By [9.26] and [14.43],it should gives a Six-Functor formalism on the corresponding sheaf level

$$\begin{tikzcd} D(\mathrm{Sh}(Z;\mathbb{Z})) \arrow[r, "i_*"'] & D(\mathrm{Sh}(X;\mathbb{Z})) \arrow[l, bend right=40, "i^*"'] \arrow[l, bend left=40, "i^!"] \arrow[r, "j^*"'] & D(\mathrm{Sh}(U;\mathbb{Z})) \arrow[l, bend right=40, "j_!"'] \arrow[l, bend left=40, "j_*"] \end{tikzcd} \qquad [14.59]$$

In [14.58],the left is about the open unit disc $\mathbb{Z}((T^{-1}))$ and the right is about killing it,so we can let $Z \in X$ be open and $U = X \setminus Z$ be closed.The local to global property preserved by Six-Functor formalism follows from [14.58],gives a global descent theory on [14.59] for the derived stack [12.104] and [12.108] or on the category of sheaves of (co)homologies on the derived stack [12.102].

The conformal map [14.34] and [14.58] gives us enough reasons to restrict the middle global sheaf in [14.58] along the rational opens in valuation spectrum

$$X\left(\frac{f_1,...,f_n}{g}\right) \mapsto \{M \in D(\mathrm{Mod}_R(\mathrm{Solid}_{\mathbb{Z}}))\}, \quad R \in \mathrm{Alg}(\mathrm{Solid}_{\mathbb{Z}}, \otimes) \qquad [14.60]$$

where $g \neq 0, v(f_i/g) \leq 1$,such that $M \simeq M \otimes_{\mathbb{Z}} R, f, g_i \in R(*)$ see below [14.5],and for all i we have [14.9] which means to go outside is equivalent to go inside **at the same time**,that is $\mathbf{R}\underline{\mathrm{Hom}}(P,M) \cong \mathbf{R}\underline{\mathrm{Hom}}(P,M)/(T-g/f_i), T \in M$.And we glue [14.60] to get the structure sheaf on $\mathrm{Spv}(R(*))$.

By the discussion of compactness of topological space below [13.41] and [14.4],if $R \in \mathrm{Solid}_{\mathbb{Z}[T]}$,$f \in R(*)$ is power-bounded which is $\underline{\mathrm{Hom}}(P,R) \cong_{f\to 1/f}$

$\underline{\mathrm{Hom}}(P, R)$.For instance,we naturally have condensed ring structure

$$\begin{aligned}&\lim_d \mathcal{O}^{[d]} = \sum_{m+h=0}^{D} \left(\mathcal{O}^{[m+h]} = \frac{\mathcal{O}_m}{z^{m+h}}\right), \mathrm{Alg}(\mathrm{Solid}_{\mathbb{Z}})^{T\square} \subset \mathrm{Solid}_{\mathbb{Z}[T]} \\ &\mathfrak{L}^+(g \otimes_{\mathbb{Z}} \mathbb{C}[\mathbf{t}, \mathbf{t}^{-1}]) : (\lim_d \mathcal{O}^{[d]})^{\mathrm{op}} \to (h_{\lim_d \mathcal{O}^{[d]}})\end{aligned} \qquad [14.61]$$

where $(\lim_d \mathcal{O}^{[d]})^{\mathrm{op}} \subset \mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})^{\mathrm{op}}, 0 \leq d \leq D$ and h see above [3.58] it is given by the tensor transformation to the unit disc induced by the conformal map [14.34].$\mathfrak{L}^+(g \otimes_{\mathbb{Z}} \mathbb{C}[\mathbf{t}, \mathbf{t}^{-1}])^{T\square} \in \mathrm{Solid}_{\mathbb{Z}[T]}$,by the definition above [14.67],it is equivalent to swap $1/z$ to z in it and the solidified condensed affine Lie algebra is over the good space which preserving the smoothification [14.8].

Definition14.10 By [3.13] [11.32] theorem14.9 below [14.13] and [14.47],a current in $\mathfrak{L}^+(\tilde{\mathfrak{g}}_{\mathrm{Perf}})^{\mathbf{L}T\square} \in D(\mathrm{RHS}^{[12.19]}_{\mathrm{Solid}_{\mathbb{Z}[T]}}) \subset \mathbf{UFT}$ is a field generating the quantum gravity and we call it an unified field,and see perfectoid ring in [13.41].

Back to math,f is called topological nilpotent if $\mathbb{Z}[T] \to R, T \mapsto f$ factors through $P^{\square} = \prod_{\mathbb{N}} \mathbb{Z} = (1, ..., 1, 0, ..., 0)$ see above [14.27],with $\mathbb{Z}[T](P^{\square}) \simeq \mathbb{Z}\langle T\rangle$.Let R° be set of power-bounded elements in $R(*)$,we have $e_{R(*)} \in R^{\circ}$ and for $f, g \in R^{\circ} \subset R(*)$ we have the diagram along $\otimes^{\square}$ by [14.11]

$$\begin{tikzcd} \mathbb{Z}[T] \otimes^{\square} \mathbb{Z}[T] = \mathbb{Z}[T, U] \arrow[rr] \arrow[dr] & & R \\ & \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Z} \arrow[ur] \arrow[ul, bend left, "\mathbb{Z}\langle T,U\rangle"] & \end{tikzcd} \qquad [14.62]$$

It gives us the closure axiom of R° and $R^{\circ\circ} \subset R^{\circ}$,which is the set of topological nilpotent elements,and R° is a subring of R.We also have

$$\mathbb{Z}[T]^{\square} \xrightarrow{\simeq} \frac{\mathbb{Z}[x_0, ..., x_{n-1}][T]}{(T^n + x_{n-1}T^{n-1} + \cdots + x_0)} \to R^{\circ} \qquad [14.63]$$

where we used above [14.61],every monic polynomial in $\mathrm{Solid}_{\mathbb{Z}[T]}$ can be convergent,which also make R° integrally closed.And we can combine [14.62] and [14.63],$R^{\circ\circ} \subseteq R^{\circ}$ is a radical ideal.Guided by [14.63],we have in [14.60]

$$\mathcal{O}(\pi^0 X\left(\frac{f_1, ..., f_n}{g}\right)) = H_0(\frac{R[x_1, ..., x_n]^{T\square}}{(gx_1 - f_1, ..., gx_n - f_n)}\left[\frac{1}{g}\right]) \qquad [14.64]$$

such that $g \to 1/g$ gives equivalence and every f_i/g is power-bounded and we used derived AG because [14.17],and see the derived scheme in [12.9].Combing [13.41] and [14.61],we see the structure sheaf on DG Lie adic space is

$$\,^{\mathrm{DG,Lie}}_{\mathrm{Perf}}\mathscr{X}\left(\frac{z\mathcal{O}^{[0]}, ..., z\mathcal{O}^{[D]}}{z}\right) \mapsto \frac{\mathfrak{L}^+(\mathfrak{g}_{\mathrm{Perf}})[x_0, ..., x_D]^{T\square}}{(zx_0 - z\mathcal{O}_{-h}, ..., zx^D - \mathcal{O}_{D-h}/z^{D-1})}\left[\frac{1}{z}\right] \qquad [14.65]$$

where $z \to 1/z$ inducing $\mathcal{O}_{-d} \to \mathcal{O}_d$ gives equivalence,also see [3.61] which gives us commutativity here,and $\mathcal{O}_{D-h}/z^D$ is bounded by the spacetime dimension with $v(\mathcal{O}_{D-h}/z^D) \leq 1$ as the normalization of the quantum fields.The $\mathfrak{g}_{\mathrm{Perf}}$ means we mod the $\mathbb{Z}_2$ generated by $\mathbf{T}^{\delta}$ and we glue the rational opens to get $\mathrm{Spv}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}^{+})_{\mathrm{Perf}}$.Such things are the building blocks of $\mathrm{RHS}^{[12.19]}_{\mathrm{Solid}_{\mathbb{Z}[T]}}$.

14.3 Analytic rings and propertification$^{\mathbf{an}}$

Now,we want to use the analytic stack to formalize things we discussed above and to study the derived category **UFT** on the derived stack [13.13].For a light condensed ring $A^{\triangleright}$,an analytic ring structure is a abelian full subcategory

$$\mathrm{Mod}_A \subset \mathrm{Cond}_{A^{\triangleright}} = \{\text{light condensed } A^{\triangleright}\text{-modules}, A^{\triangleright} \otimes M \to M\} \qquad [14.66]$$

The property on it is stable under all (co)limit,extensions and $\underline{\mathrm{Ext}}^i_{A^{\triangleright}}$.There exists a left adjoint $\mathrm{Cond}_{A^{\triangleright}} \to \mathrm{Mod}_A, M \mapsto M \otimes_{A^{\triangleright}} A$ the kernel is stable under $\otimes$,which is a $\otimes$-ideal.To see this,for $M \otimes_{A^{\triangleright}} A = 0$,we have

$$\begin{aligned} &\underline{\mathrm{Hom}}(N \otimes_{A^{\triangleright}} M, A) \cong \underline{\mathrm{Hom}}(N \otimes_{A^{\triangleright}} M, \underline{\mathrm{Hom}}(N, A \otimes_{A^{\triangleright}} A) \\ &\cong \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(A, M), \underline{\mathrm{Hom}}(N, A)) \cong \underline{\mathrm{Hom}}(A, M \otimes_{A^{\triangleright}} A) \otimes_{A^{\triangleright}} N = 0 \end{aligned} \qquad [14.67]$$

which gives us the associativity and an unique symmetric monoidal structure making $(-) \otimes_{A^{\triangleright}} A$ which is given by the hom-tensor adjunction (H-T duality) [14.14],we need to see more information about it.A trivial case is,$1 \otimes_{A^{\triangleright}} A \simeq \underline{\mathrm{Hom}}_{A^{\triangleright}}(A, -)$,it actually correlates with the Y-duality [9.106],and combing discussions below [11.51],[11.66],[14.45] and [14.47],we have

$$\begin{tikzcd} \mathbf{F} \arrow[rr] & & \mathbf{UFT} \\ \mathbf{Y}^{\square} = \mathbf{Y}^{\mathrm{self}} \oplus \mathbf{Y}^{\mathrm{unself}} \arrow[u, "\mathscr{H}\text{-quotient}_*"] \arrow[rr] & & \mathrm{RHS}^{[12.19]}_{\mathrm{Solid}_{\mathbb{Z}[T]}} \arrow[u] \\ \mathrm{H} \leftrightarrow \mathrm{T} \arrow[u, hook] \arrow[rr] & & h^A \leftrightarrow A \arrow[u, hook] \end{tikzcd} \qquad [14.68]$$

where $\mathbf{Y}^{\square}$ is a 2-representability,which makes two definitions [11.85] and [13.14] in consistency.Next,we study the analytic ring structure in derived category.

We can write an analytic ring as a pair $A = (A^{\triangleright}, \mathrm{Mod}_A)$,we define $D(A)$ is a full subcategory of $D(\mathrm{Cond}_{A^{\triangleright}})$ such that $M \in \mathrm{Mod}_A$ with $H_i(M) \in D(A)$ which is triangulated.The derived category is triangulated,that is

$$\mathbf{R}^i \prod_{n \in I} M'_n \to \mathbf{R}^i \prod_{n \in I} M_n \to \mathbf{R}^i \prod_{n \in I} M''_n \to \mathbf{R}^i \prod_{n \in I} M'_n[1] \qquad [14.69]$$

with a fibrant replacement $M \to (M_n)_{n \in I}$ which is also a series expansion in Mod_A and we can forget the derived structure and leave triangulated Mod_A by

$$\mathbf{R}^i \prod_{n \in I} M_n \simeq \mathbf{R}^i \prod_{n \in I} \underline{\mathrm{Hom}}_{A^{\triangleright}}(A^{\triangleright}, M) \cong \underline{\mathrm{Ext}}^i_{A^{\triangleright}}(\bigoplus_{n \in I} A^{n \in I}, M) \simeq \mathbf{R}^i M \qquad [14.70]$$

where we used the stability below [14.66].Next is $\otimes$-ideal,for $M_\bullet \in D(\mathrm{Cond}_{A^\triangleright})$

$$\begin{aligned} &M \otimes^{\mathbf{L}}_{A^\triangleright} N \simeq \mathrm{Hom}_{D(\mathrm{Cond}_{A^\triangleright})}(M_\bullet, \mathbf{R}\underline{\mathrm{Hom}}_{D(A)}(N_\bullet, A_\bullet)) \\ &A_\bullet \in D(A), \quad (N_\bullet = \mathbf{R}\lim_n \tau_{\le n} N \xrightarrow{\simeq} N) \in D(A) \end{aligned} \qquad [14.71]$$

where we used Postnikov limit as the fibrant replacement making things into $D(A)$ as we can use the H-T duality in [14.68],and this gives us a spectral sequence,so we first calculate it by $\underline{\mathrm{Ext}}^q(N_p, A_\bullet) \Rightarrow \underline{\mathrm{Ext}}^{p+q}(N, A_\bullet)$ see 2.4.25 in [12],then the $\otimes$-ideal on the derived category is given by that of Mod_A [14.67] through [14.70].And in the $D(A)$,$(- \otimes^{\mathbf{L}}_A -) \cong (-) \otimes^{\mathbf{L}}_{A^\triangleright} A$ gives the symmetric monoidal structure.The t-structure is stable under derived limit and the truncation $\tau_{\le n}$ so the inclusion $D(A) \hookrightarrow D(\mathrm{Cond}_{A^\triangleright})$ is and the its left adjoint $(-) \otimes^{\mathbf{L}}_{A^\triangleright} A$ are t-exact which gives $D(A)$ a nature t-structure with

$$D(A)^\heartsuit = \mathrm{Mod}_A, \quad (- \otimes^{\mathbf{L}}_{A^\triangleright} A)^\heartsuit = (-) \otimes_{A^\triangleright} A \quad (- \otimes^{\mathbf{L}}_A -)^\heartsuit = (- \otimes_A -) \quad [14.72]$$

Similarly to [14.4],we have the following left Kan extension

$$\begin{tikzcd} & \mathrm{SolidRings} \arrow[dr] & \\ A^\triangleright(*) \in \mathrm{HuberRings} \arrow[ur, hook] \arrow[rr, hook] & & A^\triangleright \in \mathrm{CondRings} \end{tikzcd} \qquad [14.73]$$

where $A^{\circ\circ} \subset A^\circ \subset A^\triangleright(*)$ see around [14.62]

$$A^\circ = \{A^\triangleright(*) | A^\triangleright \in \mathrm{Solid}_{\mathbb{Z}[T]}\}, \quad A^{\circ\circ} = \{A^\triangleright(*) \in A^\circ | [14.62]\} \qquad [14.74]$$

By [14.18],an analytic ring $A = (A^\triangleright, \mathrm{Mod}_A)$ is solid if and only if all $M \in \mathrm{Mod}_A$ are solid that is every $f^* : P \otimes_{\mathbb{Z}} M \to P \otimes_{\mathbb{Z}} M$ gives an equivalence.For an analytic ring structure A on a solid condensed ring $A^\triangleright$,we can define

$$\begin{aligned} A^+(*) &= \{g \otimes^{\square}_{A^\triangleright} f | (\mathbb{Z}[T], \mathbb{Z}\langle T\rangle) \to A, T \mapsto (g, g \otimes_{A^\triangleright} f), g \in A^{\circ\circ}\} \\ &= \{g \in A^\triangleright(*) | P^\square \otimes_{\mathbb{Z}} A \cong_{g \to 1/g} P^\square \otimes_{\mathbb{Z}} A\}, P^\square \cong \prod_{\mathbb{N}} \mathbb{Z} \end{aligned} \qquad [14.75]$$

Now,by $\mathbb{Z}\langle T\rangle \subset \mathbb{Z}[[T]]$ and [14.58],we get a category of relative Huber pairs

$$\begin{aligned} &(A^\triangleright, A^+) \in (\mathrm{Alg}(\mathrm{Solid}_{\mathbb{Z}[T]})^{T\square}, \mathrm{Mod}_{\mathbb{Z}[[T]]}(\mathbf{Mod_A})),\ A^{\circ\circ} \subset A^+(*) \subset A^\circ \\ &\mathrm{Mod}_{\mathbb{Z}[[T]]}(\mathrm{Mod}_A) \hookrightarrow_{h_*} \mathrm{Mod}_{\mathbb{Z}[[T]]}(\mathrm{Solid}_{\mathbb{Z}}) \to_{(-)^{T\square}} \mathrm{Solid}_{\mathbb{Z}[T]} \hookrightarrow_{k_*} \mathrm{Solid}_{\mathbb{Z}} \end{aligned} \qquad [14.76]$$

with $\mathrm{Mod}_A \hookrightarrow \mathrm{Solid}_{\mathbb{Z}}$ and we have $A_{\mathrm{sol}} = k_*((h_*(A^\triangleright, A^+))^{T\square})$ with

$$\mathrm{Mod}_{A_{\mathrm{sol}}} = k_*((h_* \mathrm{Mod}_{\mathbb{Z}[[T]]}(\mathrm{Mod}_A))^{T\square}) = \mathrm{Mod}_{\mathbb{Z}[[T]]}(\mathrm{Cond}_{A^\triangleright}) \cap \mathrm{Solid}_{\mathbb{Z}} \quad [14.77]$$

which is a relative solid analytic ring.And we denote $(-)^{\triangleright\square} = k_* \circ (-)^{T\square} \circ h_*$,

$$(\mathrm{Alg}(\mathrm{Solid}_{\mathbb{Z}[T]})^{T\square}, \mathrm{Mod}_{\mathbb{Z}[[T]]}(\mathbf{Mod_A}))^{\triangleright\square} = (\mathrm{Alg}(\mathrm{Solid}_{\mathbb{Z}[T]})^{T\square}, \mathbf{Solid_A}(\mathrm{Solid}_{\mathbb{Z}})) \qquad [14.78]$$

where the bold symbol for category of Mod_As.In this case,we can summaries them by the following diagram

$$\begin{tikzcd}
\text{(analytic rings on solid condensed rings)} \arrow[r, "\mathrm{id}\times(-)^+"] \arrow[d, hook] & \text{(relative Huber pairs)} \arrow[d, "(-)^{\triangleright\square}"] \\
(\mathrm{Alg}(\mathrm{Solid}_{\mathbb{Z}[T]})^{T\square}, \mathrm{Mod}_{\mathbb{Z}[[T]]}(\mathbf{Mod_A})) & \text{(solid relative analytic rings)} \arrow[l]
\end{tikzcd}$$

[14.79]

with solid relative analytic ring denoted as $(A^{\triangleright}, \mathrm{Mod}_{(A^{\triangleright},A^+)^{\triangleright\square}})$.Now,we put physics in,for the condensed affine Lie algebra,we have a relative pair

$$(\mathfrak{L}^+(\tilde{\mathfrak{g}}_{\mathrm{Perf}})^{T\square}, \mathfrak{L}^+(\tilde{\mathfrak{g}}_{\mathrm{Perf}})^+)^{\triangleright\square} \in \mathbf{Ret}^*(\mathrm{RHS}^{[12.19]}_{\mathrm{Solid}_{\mathbb{Z}[T]}}, \mathbf{UFT}) \qquad [14.80]$$

which let us put the solid analytic ring structure in.

Next,we want to put things into RHS of [12.19].For a commutative ring R,the $D(R)$ is fibered over $\mathrm{Spec}(R)$,with $\mathscr{B}$-sheaf $U \in \mathrm{Spec}(R), \mapsto D(\mathcal{O}(U))$ glued to structure sheaf of ∞-category.So,we need to let $D((A^{\triangleright}, A^+)^{\triangleright\square})$ fibered over $\mathrm{Spv}(A^{\triangleright}, \mathrm{Mod}_{(A^{\triangleright},A^+)^{\triangleright\square}})$.For achieving this and combining with physics,we need to make things clear.The first thing is a ring structure $(R, +, \times)$ is in LHS of [12.19],so actually it should be acted by $\mathbf{Ret}_*$ to RHS we need

$$\mathbf{Ret}_*(R, +, \times) = (\mathbf{Ret}_* R, \boxtimes_{\mathbf{T}^\delta(\mathbb{Z})}, \boxtimes_{\mathbf{T}^\delta(\mathbb{R})}) \qquad [14.81]$$

where with $\mathbf{Ret}_*$algorithm [13.30].For a $\mathbf{Ret}_*$algebra,the only nontrivial functions should be valuations to number counting fields,so we have

$$\begin{aligned}
&\mathbf{Ret}_*\mathrm{Spv}^0(R, R^+) = \{v : \mathbf{Ret}_* R \to [\mathbb{Z} \oplus \mathbb{C} \oplus \mathbb{Q} \oplus \mathbb{R} \oplus (\mathbb{ZCQR})]^+ \\
&|v(P_1 \boxtimes_{\mathbf{T}^\delta(\mathbb{R})} P_2) = v(P) \boxtimes_{\mathbf{T}^\delta(\mathbb{R})} v(P_2), v(P_1 \boxtimes_{\mathbf{T}^\delta(\mathbb{Z})} P_2) \leq \\
&\max\{v(P_1), v(P_2)\}, v(0) = 0, v(1) = 0_{Q\boxtimes Q^*}, v(\mathbf{Ret}_* R^+) \leq 0_{Q\boxtimes Q^*}\}/\sim
\end{aligned} \qquad [14.82]$$

for a Huber pair (R, R^+) and we need to mod the equivalence relation generated by dualities.Thus,we need the derived case of [14.80] to be fibered over

$$\mathbf{Ret}_*\mathrm{Spv}^{\bigcirc}((\mathfrak{L}^+(\tilde{\mathfrak{g}}_{\mathrm{Perf}})^{T\square}, \mathfrak{L}^+(\tilde{\mathfrak{g}}_{\mathrm{Perf}})^+)^{\triangleright\square}) \qquad [14.83]$$

and notice that the double-weak [14.22].Now,back to ordinary Spv,the rational opens gives a basis of quasi-compact opens,closed under finite intersections

$$\begin{aligned}
&U\left(\frac{f_1, ..., f_n}{g}\right) = \{v | v(g) \neq 0, v(f_i/g) \leq 1 \,\forall\, i\} \\
&\mathcal{O}\left(U\left(\frac{f_1, ..., f_n}{g}\right)\right) = R\left[\frac{1}{g}\right], \mathcal{O}^+\left(U\left(\frac{f_1, ..., f_n}{g}\right)\right) = \overline{R^+\left[\frac{f_1}{g}, ..., \frac{f_n}{g}\right]}
\end{aligned} \qquad [14.84]$$

the overline denotes for integral closure.The rational opens in $\mathrm{Spv}(R, R^+)$ give us a Zariski site,from [14.34] we have a pair of covering

$$(\{R[f], R[1/f]\}, \{R^+[f], R^+[1-f]\}) \to (R, R^+) \qquad [14.85]$$

For R,we naturally have the the global descent theory to let us glue and localize along the Zariski site $D(\mathrm{Sh}(X \to R)) \cong D(\mathrm{Sh}(R))$ which is an equivalence of derived category [9.70].but for the R^+,the localization process is

$$\text{on}\, U_f : \mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1], -) \circ (-) \otimes_{\mathbb{Z}} \mathbb{Z}[T](R) \qquad [14.86]$$

with the condition [14.39],and this is a T-solidification with killing the $\mathbb{Z}((T^{-1}))$ see [14.35] and below [14.50].And we use $\mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[[T]]/\mathbb{Z}[T][-1], -)$ to localize the derived module on $U_{1/f}$.Thus,for the Huber pair (R, R^+),we need to use (Zariski descent [14.59],!-descent [14.58]) guaranteed by the Six-Functor formalism and H-T duality that we can form [14.86].Notice that along [14.47] a solidification $\mathrm{Smo}(-)$ is to restrict the algebraic object on $\mathbb{Z}[[T]] \oplus \mathbb{Z}((T^{-1}))$,and $D(-)$ shrinks them [14.23],the shrink is understood in [Step III] above [13.1].We can understand this only if we combine math with physics as [14.46] and [14.49].

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1], -) \cong \mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[[T]]/\mathbb{Z}[T][-1], -) \qquad [14.87]$$

with the killing is understood as shrinking,so such formalism help us understand the behaviors on **UFT**,if it is still hard to understand,see the foundation of Taoism.Also,we can do solidification [14.8] in the RHS of [12.19],for [14.82]

$$\mathbf{Ret}_* \mathrm{Spv}(R^{\square}, R^+) = \mathrm{Spv}((\mathbf{Ret}_* R)^{\square}, (\mathbf{Ret}_* R)^+) \qquad [14.88]$$

And for solid ring R,$D(R, R^+)$ is fibered over $\mathrm{Spv}(R, R^+)$ by the $\mathscr{B}$-sheaf $U \mapsto \mathrm{Mod}_R(D(\mathcal{O}(U)), \mathcal{O}^+(U))$.

A Tate adic space is an analytic space that is covered by $\mathrm{Spa}(R, R^+)$ with Tate R that we for $I \in R^0 \subset R$ with I-adic topology.And for a Tate $A^{\triangleright}$,we can analyze propertification by analytification in the diagram

$$\begin{tikzcd} A^{\triangleright}(*) \arrow[rr, "\text{propertification}"] \arrow[d] & & \mathrm{Spec}(A^{\triangleright}(*)) \arrow[dll, "\text{analytification}"'] \arrow[d, "\text{propertification}^{\text{an}}"] \\ \mathrm{Spa}((A^{\triangleright}(*), A^+(*))^{\triangleright\square}) \arrow[rr] & & \mathbf{Ret}_* \mathrm{Spa}((A^{\triangleright}(*), A^+(*))^{\triangleright\square}) \end{tikzcd} \qquad [14.89]$$

The GAGA theorem tells us for a R-algebra $A \simeq A \otimes_{\mathbb{Z}} R$,if $\mathrm{Spec}(A)$ is proper over $\mathrm{Spec}(R)$,then the $\mathrm{Spec}(A)^{\text{an}} \cong \mathrm{Spec}(A)$,which means the analytification induces equivalence,in this case,the analytic propertification $\text{propertification}^{\text{an}} = \mathbf{Ret}_*\text{analytification}$ induces an equivalence.Thus,we can give an analytic setting for **UFT** [13.14] over the derived stack $\boldsymbol{\mathcal{M}}^!$,our goal is not only to study **UFT**,but also to achieve the **TOE**.For category of analytic rings AnRing based on the category of condensed rings below [14.3],we have analytic stack in the form $\mathrm{Sh}(\mathrm{AnRing}^{\mathrm{op}})$ with $D(\mathrm{Sh}(\mathrm{AnRing}^{\mathrm{op}})) = \mathrm{Sh}(D(\mathrm{AnRing}^{\mathrm{op}}))$ which is an analytic derived stack.By the $\text{propertification}^{\text{an}}$ we have

$$\mathbf{UFT}(\boldsymbol{\mathcal{M}}^!) \xrightarrow{\simeq} \mathrm{Sh}(D(\mathbf{Ret_*}^{\mathrm{GenSup,Lie}}\mathrm{AnRing}^{\mathrm{op}}_{\mathrm{sol}})) \qquad [14.90]$$

where it is over the opposite category of that of [14.83].By [14.26] [14.47] [14.48] [14.56] [14.77] and [14.80],we want it to be the pair $(\mathbf{UFT}, \mathbf{TOE})$.

14.4 6-functors,!-descent and the Unima

The stakified condition for the derived analytic stack need we have the global descent theory given by below [14.86],that we want to make it clear.In the diagram [14.58],we have six functors $(\otimes, \mathbf{R}\underline{\mathrm{Hom}}), (f^*, f_*), (f_!, f^!)$.By [10.68],the global descent for Sh follows from that of underlying stack,so we discuss the six functors for $D(\mathrm{AnRing}^{\mathrm{op}})$.The Six-functor formalism for a derived category $D(X)$ follows from the following axioms

$$\begin{gathered}\text{(1) Each } D(X) \text{ is a closed symmetric monoidal } \infty\text{-category with symmetric } \otimes \\ \text{dual to } \mathbf{R}\underline{\mathrm{Hom}} \quad \text{(2) We have global adjoint functors } f^*, f_* \text{ for } f: Y \to X \\ \text{with } f^*: D(X) \xrightarrow{\otimes} D(Y) \quad \text{(3) We have a local functor } f_!: D(Y) \to D(X) \\ \text{which is commutative with } g^* \text{ that is } g^* f_! \cong f_! g^* \\ \text{for } A \in D(X), B \in D(Y), f_! f^* A \cong f_! B \text{ with } B\text{-module } f^*A, A\text{-module } f_! B\end{gathered} \tag{14.91}$$

Below [14.58],we have seen that for a proper map$f$, $f_* \cong f_!$,this is because proper morphism is affine,we can glue affine morphisms by [10.7].We call a map is !-able if $f: Y \to X$ can be factorized to open immersion $j: Y \to \overline{Y}$ and proper map $\overline{f}: \overline{Y} \to X$ so $f_! = \overline{f}_* \circ j_!$.The derived category is enriched of !-able maps with $\overline{Y}$ a compactification of Y which is uniquely determined.And this things determine an abstract Six-Functor formalisms for derived category.Then,we want to apply this to analytic stack,an affine analytic stack is AffAnStk $\cong$ $\mathrm{AnRing}^{\mathrm{op}}$ with analytic spectrums $\mathrm{AnSpec}(A) \equiv \mathrm{Spv}(A)$ as objects and morphims are proper if $f_*: D(B) \to D(A)$ over $f: \mathrm{AnSpec}(B) \to \mathrm{AnSpec}(B)$,satisfies (3) in [14.91].This is equivalent to say the morphism of analytic rings factors through

$$\begin{tikzcd}[column sep=small] A = (A^{\triangleright}, \mathrm{Mod}_A) \arrow[rr] \arrow[dr, shift left, "\text{induced}"] & & B = (B^{\triangleright}, \mathrm{Mod}_B) \\ & (A \otimes_{A^{\triangleright}} B^{\triangleright}, \mathrm{Mod}_{A \otimes^{\mathbf{L}}_{A^{\triangleright}} B^{\triangleright}}) \arrow[ul, shift left, "(-)^{(\mathbf{L})T^{\square}}"] \arrow[ur, "\text{localization}"'] & \end{tikzcd} \tag{14.92}$$

$\mathrm{Mod}_A \cong D(A)^{\geq 0}$ see above [14.69],that is an induced analytic ring with A as the compactification of it.A map $j: \mathrm{AnSpec}(B) \to \mathrm{AnSpec}(A)$ is an open immersion if j^* admits a left adjoint $j_!$ satisfying (3) in [14.91].For instance,let $j: \mathrm{AnSpec}(\mathbb{Z}[T]^{\square}, \mathbb{Z}[T]^{\square}) \to \mathrm{AnSpec}(\mathbb{Z}[T]^{\square}, \mathbb{Z})$,recall the meaning of solidification above [14.87],this should be covered by the analytic spectrums of the following

$$j: (\mathbb{Z}[[T]] \oplus \mathbb{Z}((T^{-1})))^2 \to (\mathbb{Z}[[T]] \oplus \mathbb{Z}((T^{-1})), \mathbb{Z}[[T]] \vee \mathbb{Z}((T^{-1}))) \tag{14.93}$$

which are algebraic open unit disc for solid case and taking AnSpec$(-)$ gives us the geometric open unit disc,so $j_* j^* = \mathbf{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1], -)$ see [14.87] and for $M \in D((\mathbb{Z}[T]^{\square}, \mathbb{Z}))$,$M \otimes^{\mathbf{L}}_{\mathbb{Z}[T]} \mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1]$ is a compactification of M,that gives the diagram

$$\begin{tikzcd} D(\mathrm{Mod}_{\mathbb{Z}[T]}(\mathrm{Solid}_{\mathbb{Z}})) \arrow[r, bend left, "\text{compactification}"] & D(\mathrm{Solid}_{\mathbb{Z}[T]}) \arrow[l, bend left, "\text{proper}"] \end{tikzcd} \qquad [14.94]$$

see [14.58].So we get an assignment $M \mapsto j_! j^* M$.

Proposition15.1 By [14.23],The functor $\mathscr{H}$-quotient$_*(-)$ is a category of **F**-fusions with [14.47] can be understood as a process of compactification.

Back to math,[14.93] corresponds to idempotent (co)algebra after solidification which is a compactification in [14.94],that is $j_! = j_! \otimes j_!$ to $\mathbb{Z}[[T]] \oplus \mathbb{Z}((T^{-1}))$

$$(j_! \otimes j_!) e_{D(\mathbb{Z})} = j_! e_{D(\mathbb{Z})} \otimes j_! e_{D(\mathbb{Z})} \cong j_!(j^* j_! e_{D(\mathbb{Z})} \otimes e_{D(\mathbb{Z})}) = j_! e_{D(\mathbb{Z})} \qquad [14.95]$$

where we used (3) in [14.91],which is called projection formula.This also says that $j_! j^* M = j_! e_{D(\mathbb{Z})} \otimes_A M \simeq \mathbf{R}\underline{\mathrm{Hom}}_A((j_! \otimes j_!) e_{D(\mathbb{Z})}, M)$ which also gives an explanation of [14.48] also see below [14.86] and derives to over $\mathbb{Z}[[T]] \vee \mathbb{Z}((T^{-1}))$.

Now,we call the ∞-Grp anima $\mathrm{DSh}(\mathrm{AnRing}^{\mathrm{op}}) = \mathrm{Anima}(\mathrm{AnRing}^{\mathrm{op}})$,in analytic derived stack,an object is a homotopy-enriched ∞-groupoid and

$$\text{anima is (homotopy,!-map)-enriched with ($*$-descent,!-descent)} \qquad [14.96]$$

from the naive discussion below [14.86] with $*$-descent the global ordinary derived descent see [10.83].So a map of stacks $f : Y \to X$ satisfies $*$-descent if

$$f^* : D(X) \xrightarrow{\simeq} \lim_{\Delta}(D(Y) \rightrightarrows D(Y \times_X Y) \Rrightarrow \cdots) \qquad [14.97]$$

which is a derived gluing property of [9.2].And for a !-map $f : Y \to X$

$$f^! : D(X) \xrightarrow{\simeq} \lim_{\Delta}(D(Y) \rightrightarrows D(Y \times_X Y) \Rrightarrow \cdots) \qquad [14.98]$$

with !-descent in math lets us glue derived solidifications (compactification [14.94]) and we need pair of global descents for analytic derived stack.

Definition15.2 An analytic duality fusion is analytic propertification of duality fusion.Thus,for the analytic derived stack [14.90],we need

$$\mathbf{Ret}_*(\text{!-descent}) = \mathscr{H}\text{-quotient}_*(\text{propertification}^{\mathrm{an}}[8.53]) \qquad [14.99]$$

Thus,we need $\mathbf{Ret}_*$(!-descent) to glue the $\mathbf{Def}^{\Leftarrow !}\mathbf{F}$-fusions

$$\begin{tikzcd} \mathbf{Ret}_*\mathrm{AnSpec}^{\bigcirc}(-, (\mathfrak{L}^+(\tilde{\mathfrak{g}}_{\mathrm{Perf}})^{T\square}) \arrow[r, bend left, "\mathbf{Def}^{\Leftarrow !}\mathbf{F}\text{-fusion}"] & \mathbf{Ret}_*\mathrm{AnSpec}^{\bigcirc}(-, \mathfrak{L}^+(\tilde{\mathfrak{g}}_{\mathrm{Perf}})^+)^{\triangleright\square}) \arrow[l, dotted] \end{tikzcd} \qquad [14.100]$$

Then,we get $\mathbf{Def}^{\Leftarrow !}\mathbf{Ret}_*$(!-descent) to glue $\bigcirc$-senses in **TOE**.

Definition15.3 Similarly to analytic derived stack above [14.96],we call the category of ∞-groupoids in the RHS of [14.90],the Unima,and that becomes

$$\text{Unima}(\mathbf{Ret}_*{}^{\text{GenSup,Lie}}\text{AnRing}^{\text{op}}_{\text{sol,Perf}}) \qquad [14.101]$$

where we want the Unima denotes for the unification and the unimaginable.From [14.56] and [14.57],the global descent below [14.100] lets us glue the $\bigcirc$-senses.

$$\text{Unima} = (\mathbf{UFT}, \mathbf{TOE}) = \bigcirc\textbf{-Sense} \Leftarrow \mathbf{UFT} \qquad [14.102]$$

with left evolution and the no definition fusions in [14.100] which is proper and already !-ed.Now,back to math,we get a !-site with Grothendieck topolpogy generated by !-coverings !-able $\{f^! : X_i \to Y\}_{i\in I}$ and the globalization $\coprod_i X_i \to Y$ which is a !-able surjection.Combined with !-able factorization of [14.47].

$$\begin{tikzcd}
\mathbf{TOE} \arrow[rrrd, dotted, bend right] & & \mathbf{UFT} \arrow[ll, Rightarrow, "\mathrm{P}(\mathbf{Def}^{\Leftarrow !}\mathbf{F})"'] & & \\
& \mathrm{RHS}^{[12.19]}_{\mathrm{Solid}_{\mathbb{Z}[T]}} \arrow[ul, "\mathbf{Def}^{\Leftarrow !}\mathbf{Ret}_*!\text{-able}"'] \arrow[ur, "\mathscr{H}\text{-quotient}_*(-)"] & & \mathrm{RHS}^{[12.19]} \arrow[ul, "\mathbf{Ret}_*!\text{-able}"'] \arrow[ll, "\mathrm{Smo}(-)"] & \\
& & \mathrm{RHS}^{[12.19]}_{\mathbf{P}[13.32]} \arrow[ul, "\mathbf{Ret}_*!\text{-able}"] \arrow[ur, "\mathrm{P}(\mathbf{T}^{\delta\,\mathrm{self}})\ \mathrm{P}(\mathbf{T}^{\delta\,\mathrm{unself}})"] \arrow[rr, leftrightarrow, "\mathbf{Ret}_*\mathbf{Ads/Cft}"] \arrow[rd, leftrightarrow, bend right] \arrow[d] & & \mathrm{RHS}^{[12.19]}_{\mathbf{D}[13.32]} \arrow[ul] \arrow[uull, Rightarrow, bend right, "\text{[Step II,III]}"'] \arrow[ll, leftrightarrow, bend left] \arrow[uullll, bend left=60, "\mathbf{Ret}_*!\text{-able}"] \arrow[dll, bend left] \\
\text{Unima} & & \text{Our world in LHS}^{[12.19]} & &
\end{tikzcd}$$

[14.103]

with universe evolution picture $\mathbf{\Pi}$ in [12.93],property evolution in [11.16] and the ***Nonexpressability*** below [12.36],also with homotopy weak projection in [13.15].This is an upgraded [13.6] of derived stack,of analytic derived stack and we see things in details and achieve **TOE**.The diagram [14.103] collects all information we developed, for instance,[8.28] [8.34] [8.37] [12.25] [12.92] and above [13.1].Back to math,see below [10.94],we have $\text{AnRing} \to \text{ComAlg}(\text{pr}^{\text{L}})$ with $(R^{\triangleright}, D(R)^{\geq 0}) \mapsto D(R)^{\geq 0}$ under $D(\text{CondAb})$.

This theory is aimed at realizing the pursuit of perfection by physicists and mathematicians,but so-called perfection does not exist relative to our existing state.We have completed the complete framework,but we do not want to supplement some details in this paper.We can see it in the following.

Definition15.4 A so-called perfection is a property or a false vacuum in the string landscape [13.18].So this theory without so-called perfection is complete.

15 the $\mathcal{M}^!$-theory

Definition16.1 We define a forgettable functor **fSci**,which let us forget the scientific structure.Acting on it,we get true stances without biases.

$$\begin{aligned}\mathbf{fSci}(\mathrm{LHS}^{[12.19]}) &= \text{The world in our brain,out of our heart}\\ \mathbf{fSci}(\mathrm{RHS}^{[12.19]}) &= \text{Spiritual space in our heart}\end{aligned} \qquad [14.104]$$

Thus,in the end we get the ultimate theory [11.86],that is

$$\textbf{the } \boldsymbol{\mathcal{M}^!}\textbf{-theory} = \mathbf{fSci}\big([14.101],[14.102],[14.103] \text{ and all settings below}\big) \qquad [14.105]$$

Then,we want to make a thing clear that is the stability of relative properties [12.18],the $\mathrm{RHS}^{[12.19]}$ has smoothness everywhere,all singularities or holes in LHS are filled by [12.106] in RHS,we need to understand that a singularity or hole is an obstruction of retraction induced by the homotopies.But with the existence of higher dimensions and taking to derived case of LHS to RHS,there is no any obstruction of n-homotopies,so the relative properties are keeping retracted by the dark energy [12.93] in RHS.

The last thing is discuss the application,we see in [14.104],the application is Intangible mental guidance but not corporeal production and living.

Theorem16.2 The [14.105] is an unification of science philosophy and religion.For instance,Buddhism tells us that all spiritual origins are one as the global descent theory below [14.100].The Yin and Yang in Taoism corresponds to [14.48],and the [11.85] gives us Tai chi symbol and the 5 superstring theories corresponds to the five elements.

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