An elementary demonstration of the Goldbach'strong conjecture by the analysis of congruence rules in [0 – n] and [n – 2n] intervals

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Abstract.

This paper presents detailed analyses of congruences modulo in the case of even sum S = A + B. These analyses were performed in order to design a way to demonstrate GSC in an elementary logical way.
Even if we succeed with the rules of congruence in putting an even number in the sum of two prime numbers, this does not constitute a definitive mathematical proof, which is why the GSC remains unprovable. This is why we must resort to a logical reasoning which consists of eliminating false propositions and keeping only one which is true. The one which is true must lead us to the truth of the GSC and thus we succeed in demonstrating it mathematically.
This paper provides an elementary mathematical proof by deciding between four propositions such that the GSC is the only true one (logical reasoning by an indirect proof). This conclusion is reached by taking into account established facts in mathematics about prime numbers in [0 ▬ n] and [n ▬ 2n] intervals.

Keywords. Prime. Goldbach's strong conjecture. Euclidean division. Euclidean equation. Gap. Congruence. Modulo. Infinity. Exponential.

Abbreviations. GSC : Goldbach's strong cnjecture. P : prime. C : composite.

Introduction.

I have recently reported in several papers that for an even number denoted S Goldbach's strong conjecture (GSC) depends on the presence of two equidistant primes p and Q such that $p < S/2$ and $Q > S/2$ and such that $S/2 – p = Q – S/2$ therefore $S = p + Q$ [1-6]. In addition I have shown that GSC depends closely on the gaps between primes especially gaps = 6 or 4 [1,5]. I have also shown that GSC might hold true to infinity [4]. In the present paper, I study GSC in function of the remainders of euclidean divisions and rules of congruence. This is an attempt to demonstrate how Q primes are formed so that $Q + p = S$. I also study the impact of some gaps devoid of primes on the GSC. Finally, I analyse whether GSC might hold true to infinity. The paper paves a way for an elementary and basic understanding of the GSC or at least defines the critical elements that must be dealt with if one attempts to solve it. It provides an elementary mathematical demonstration and a mathematical proof of this conjecture.

The article is organized into three parts in the first one I provide simple examples of illustrations of the congruence rules; the second one contains a logical mathematical demonstration of the GSC and the third one a deductive demonstration of the GSC at infinity.

Results.

A. The rules that apply to the remainders of Euclidean divisions of sums S = A + B

Let us take an even number S like S = 100. There are only four possible ways of putting it into the sum of two odd terms A and B such that S = A + B. Either the even number S = 100 is the sum of two composite (C) odd numbers with which it shares a common factor, e.g. **S1 = 75 + 25** (*if one term shares a common prime factor with S so the other one does too*). Let 100 be the sum of two composite numbers with which it shares no common factor, e.g. **S2 = 49 + 51**. Let 100 be the sum of two odd numbers, one of which is composite and the other prime, e.g. **S3 = 67 + 33** and therefore not sharing a common factor. Or 100 is the sum of two primes, e.g. **S4 = 47 + 53** and therefore not sharing a common factor. These four types of sums will impose rules on the remainders of Euclidean divisions of S = 100 by the primes q < S/2 = 50 or q < S as divisors of the two terms A and B of the four sums **S = A + B**.

<u>Tables A1-4</u> are constructed using this method: i) The prime numbers denoted q < S/2 < S are determined. ii) Each of the 4 sums S = A + B such that B > A is taken and the terms A and B are divided by all the prime numbers q < B. iii) The remainder of the Euclidean division thus performed is noted each time. Let's note the remainders of the A terms as **r1** and those of the B terms as **r2** and those of S as **r3**. We have two cases
if r1 + r2 < q then r1 + r2 = r3. **If r1 + r2 > q then (r1 + r2) : q = r3**. In all cases, the sum S = A + B is such that S ≡ (r1 + r2) mod(q). Examples in the case of S2 = 49 + 51 we have 49 : 17 has r1 = 15 and 51 : 17 has r2 = 0 → r3 = 15 + 0 = 15 of S2 : 17 and therefore S2 ≡ (r1 + r2) mod(q) = 100 ≡ (15) mod(17). On the other hand, 49 : 29 has r1 = 20 and 51 : 29 has r2 = 22 and so r1 + r2 = 42 > 29 and therefore (r1 + r2) : q = 42 : 29 leads to r3 = 13. When S is divided by one of its prime factors denoted q the remainders r1 + r2 = q however r3 = 0. Example 100 = 49 + 51 such that 49 : 5 has r1 = 4 and 51 : 5 has r2 = 1 so that r1 + r2 = 5 but S = 100 : 5 has r3 = 0.

Table 1A-D**.** Remainders of euclidean divisions of S = A + B by q including S : q ; A : q and B : q. The q represents primes < B of S = A + B such that B > A. S1 ; S2 ; S3 and S4 are explained in the text. Highlighted 0s determine how many times A can be increased by a prime factor. Example in the case of S2 = 49 + 51 (**Table B**) we have S2 ≡ 51 mod (7) and therefore r1 = 0. This means that we can add 7 times 7 to get to 100 and so in this case GSC is not verified because 7 x 7 = 49 is composite and not prime. By contrast in **Table D**, in the case of S4 = 11 + 89 we have S4 ≡ 89 mod (11) and so r1= 0. However this time we can only add one 11 and so S4 = 11 + 89 therefore satisfying the GSC. GSC depends on the fact whether the gap between S4 and B can be filled with ONE prime factor.

A. S1

	A	B	S1 = A + B
q ≤ B ↓	25 (r1)	75 (r2)	100 (r3)
3	1	0	1
5	0	0	0
7	1	1	2
11	3	9	1
13	12	10	9
17	8	7	15
19	6	18	5
23	2	6	8
29	25	17	13
37	25	1	26
41	25	34	18
43	25	32	14
47	25	28	6
53	25	22	47
59	25	16	41
61	25	14	39
67	25	8	33
71	25	4	29
73	25	2	27

B. S2

	A	B	S2 = A + B
q ≤ B ↓	49 (r1)	51 (r2)	100 (r3)
3	1	0	1
5	4	1	0
7	0	2	2
11	5	7	1
13	10	12	9
17	15	0	15
19	11	13	5
23	3	5	8
29	20	22	13
37	12	14	26
41	8	10	18
43	6	8	14
47	2	4	6

C. S3

	A	B	S3 = A + B
q ≤ B ↓	33 (r2)	67 (r1)	100 (r3)
3	0	1	1
5	3	2	0
7	5	4	2
11	0	1	1
13	7	2	9
17	16	16	15
19	14	10	5
23	10	21	8
29	4	9	13
37	33	30	26
41	33	26	18
43	33	24	14
47	33	20	6
53	33	14	47
59	33	8	41
61	33	6	39
67	33	0	33

	A	B	S3 = A + B
q ≤ B ↓	23 (r1)	77 (r2)	100 (r3)
3	2	2	1
5	3	2	0
7	2	0	2
11	1	0	1
13	10	12	9
17	6	9	15
19	4	1	5
23	0	8	8
29	23	19	13
37	23	3	26
41	23	36	18
43	23	34	14
47	23	30	6
53	23	24	47
59	23	18	41
61	23	16	39
67	23	10	33
71	23	6	29
73	23	4	27

D.S4

	A	B	S4 = A + B
q ≤ B ↓	47 (r1)	53 (r2)	100 (r3)
3	2	2	1
5	2	3	0
7	5	4	2
11	3	9	1
13	8	1	9
17	13	2	15
19	9	15	5
23	1	7	8
29	18	24	13
37	10	16	26
41	6	12	18
43	4	10	14
47	0	6	6
53	47	0	47

	A	B	S4 = A + B
q ≤ B ↓	11 (r1)	89 (r2)	100 (r3)
3	2	2	1
5	1	4	0
7	4	5	2
11	0	1	1
13	11	11	9
17	11	4	15
19	11	13	5
23	11	20	8
29	11	2	13
37	11	15	26
41	11	7	18
43	11	3	14
47	11	42	6
53	11	36	47
59	11	30	41
61	11	28	39
67	11	22	33
71	11	18	29
73	11	16	27
79	11	10	21
83	11	6	17
89	11	0	11

B. The case of S = A + B with A and B being both primes

Let B be a prime < S and suppose we don't know whether A is prime or composite (C). Let q be any prime < S. **<u>If S = A + B such that B > A and S ≡ B mod(q) then A = nq</u>**. Let us pose S **= tq + r3** and **B = t'q + r2** we then have r3 = r2 because S ≡ B mod(q) therefore A = S – B = (t – t') q + (r3 – r2) = (t' – t)q = **n**q.

Given that A = nq The n factor is decisive for the GSC to be true.
If n = 1 and if S = tq + r then B = S – A = S – q = (t– 1)q + r. This means S – B = tq – (t – 1)q = q → S = B + q and because B is prime therefore S is sum of two primes B and q.
<u>This always applies every time the GSC is true</u>.

Why should this theorem be considered a critical element for understanding GSC? Let's take just two examples. The first example is **S3 = A + B = 33 + 67** such that A = 33 and B = 67. In this example we have 100 ≡ 67 mod(3) and therefore **A = n x 3**. If n = 1 then S3 = 3 + 67 = 70 and S3 < 100 and so to discard. If n = 2 then S3 = 73 < 100. If n = 3 then S3 = 76 < 100 ...and if n = 11 then S3 = 100 which is correct. Therefore **A = 11 x 3**. This means that 67 is far enough from 100 for A to be C = 3 x 11, given that 3 is the smallest distance between two odd or prime numbers. This can be seen in the tables by the highlighted 0s in blue. Because r2 (67 : 11) = r3 (100 : 11) we know then that r1(A : 11) = 0. Similarly A = n x 11 and if n = 1 then S = 11 + 67 = 78 < 100 to discard ; if n = 2 then S3 = 89 < 100 to discard and if n = 3 then S3 = 100 to keep. By contrast to S3 = 33 + 67, in the case of S4 = 100 = 11 + 89 we have A = **1 x 11** and B = 89. Indeed 100 ≡ 89 mod(11) and therefore A = n x 11. If n = 2 we have S4 = 22 + 89 = 111 > 100 to discard and if n = 3 we have S4 = 33 + 89 = 122 > 100 to discard. In this case we have certainly n = 1 and therefore A = 11 thus prime. Hence S4 is sum of two primes 11 and 89. This is because 89 is closer to 100 than 67 which we denote here this way 89 → 100. Following the theorem demonstrated above we have A = 11 and <u>B = (**8** x 11) + 1</u> and <u>S4 = 100 = (**9** x 11) + 1</u> and so we see that if <u>S4 = tq + r3 then B = (t – 1)q + r2 such that r3 = r2</u>. This means that GSC is true when B and S are separated by a gap equal to the value of a prime number. Since S4 = A + B with B > A means A < S/2 and B > S/2; this means that the GSC is true if S4 and B are separated by a gap that is equal to a prime number A < S/2. <u>For GSC to be true we need a prime number > S/2 noted **Q = (t – 1)q + r**.</u> Otherwise <u>S = (a + 1)q + r and B = aq + r</u> and then <u>Q = aq + r</u>.

Any even number S can be posed as S = tq + r with q any prime < S. If q > S/2 then t = 1 and r is either prime or composite. If q < S/2 t > 1 and we have as many Q = (t – 1)q + r odd numbers > S/2 as q primes < S/2. **<u>If ONE SINGLE Q = (t – 1)q + r is prime GSC is true</u>**. If we take any even S ; pose it as S = tq + r with q any prime < S and found one Q = (t – 1)q + r that is prime then GSC is verified.

C- How to determine if primes do exist in the [0 ▬ 2n] interval that verify the GSC ?

There is a symmetry in the GSC that we exploit to identify the occurrence of a prime number Q > S/2. *An even number is an interval [0 ▬ 2n] with n in the center. We will extrapolate the prime numbers from the interval [0 ▬ n] to [0 ▬ 2n] and see if these same positions will be occupied by prime numbers Q*. Let's take the example of 100 and suppose that we don't know the prime numbers between 50 and 100. Let's name the p's the prime numbers < 50, they are 3; 5 (excluded) ; 7; 11; 13; 17; 19; 23; 29; 31; 37; 41; 43 and 47... So we'll place them in the interval [50 ▬ 100]and we'll have 100 - 3 = X1; 100 - 7 = X2; 100 - 11 = X3; 100 - 13 = X4;...100 - 47 = X13. We'll apply the congruence rule by posing q any prime < p < S/2 and so if 100 ≡ p mod(q) then X is composite otherwise X is prime. For example 100 ≡ 97 mod(3) and 100 ≡ 3 mod(97) and therefore 97 is prime. 100 ≡ 7mod(3) and therefore 100 – 7 = X2 = 3n = 93 and therefore composite. Let's take 100 - 11 = X3 we see that 100 ≡ 89 mod(11) and 100 ≡ 11 mod(89) and therefore 89 is prime. In the same way 100 ≡ 83 mod(17) and 100 ≡ 17 mod(83). Hence 83 is prime. But 100 ≡ 19mod(3) and so 100 - 19 = 81 = 3^4 which is composite. 100 ≡ 23mod(11) and so 100 - 23 = 77 = 7 x 11 which is composite. **A prime number that appears in the interval [n ▬ 2n] at a position equivalent to that of [0 ▬ n] satisfies the rule of GSC**. Example 89 such that 100 - 11 = 89. We know 11 is prime but if we determine that 100 – 11 is prime this means that 11 and 100 – 11 = 89 are two equidistant primes at 50 which verify the GSC. *This method of extrapolating p from the interval [0 ▬ n] to [n ▬ 2n] allows us to focus on the key positions in the interval [0 ▬ 2n] that verify the GSC through congruence rules*. This is the only way to predict whether a number is prime at a key position where GSC is verified. Because if we analyze primeness of all odd numbers in the interval [n ▬ 2n] except 3n and 5n that we recognize, we quickly realize that the task is extremely tedious. Note that some primes (more or less numerous) are absolutely useless for GSC. I name them here Qh because 2n – Qh = C (composite). These primes can be avoided by this method of extrapolating the key positions of primes from [0 ▬ n] to [n ▬ 2n].

Let us suppose we have an even S = 2n. Let us suppose there are primes p < n and primes Q > n.Let us suppose there are primes q < p (q 1 ; q1 ; q3, ...q_n < p) .

If n : p = ap + r1 then 2n = 2ap + 2r1. Given that 2n : p = 2ap + 2r1, we have two cases either 2r1 < p then 2r1 is the remainder of 2n ; and if 2r1 > p then 2r1 – p = is the remainder of 2n. Therefore r1≠2r1 in all cases and so n ≇ 2n mod(q) for every q < p < n. Therefore if n ≡ p mod(q) ↔ 2n ≇ p mod(q) and so if n – p = C then 2n – p = Q (prime) ; or 2n – p = C' such that C and C'have no common prime factor. Let us pose n = aq1 + r1 ; n= a'q2 + r2 ; n = a''q3 + r3 ;... n = $a_n q_n + r_n$. Then 2n = bq1 + r2 or b'q2 + r3... $b_n q_n + r_n$. Therefore if n ≡ p mod(q1) then n – p = C (multiple of q1) and therefore 2n ≡ p mod (q2 or q3 or...qn) and then 2n – p = C' (multiple of other q but not q1).

If **n – p1 = p2 with p2 > p1**. In all cases **n ≇ p1 mod(q)** for every q < p but in all cases **n ≡ p2 mod(p1)** because if n = ap1 + r1 and p2 = dp1 + r3 < then d = a – 1 and r1 = r3. If n – p1 = p2 with p2 > p1 → **n ≡ p2 mod(p1)** and **n ≡ p1 mod(p2).** Reciprocal congruence occurs only in this case.

The difference between $n – p = C$ and $n – p1 = p2$ is that if the former $n \equiv p$ but $n \ncong C$ while by contrast in the latter $n \equiv p2 \bmod(p1)$ and $n \equiv p1 \bmod (p2)$. If $n – p1 = p2$ then $2n – p1 = p2 + n$ but $p2 + n$ is either prime or composite. Although these congruence rules can help determining whether $p2 + n$ is prime or not,**$n \equiv p2 \bmod(p1)$** they require performing calculation of remainders which is the same as factoring.
We have therefore $2n \ncong n \bmod(p)$ with $p < n$ and for every $q < p$.

The table 2 shows by simple visual examination of the sums of two primes that some same primes are involved in those sums in both n and 4n on one hand ; and in both 2n and 8n in the other hand. By contrast no common primes are seen in n compared to 2n ; in 2n compared to 4n ; and in 4n compared to 8n. The only prime shared by all these is 3. This shows that congruence rules change at each 2n and return back at 4n.

D. The different categories of 2n numbers important for the application of GSC

On the other hand as I have previously reported [3,5] there are three categories of 2n numbers inlcuding 6x ; 6x + 2 and 6x + 4. Primers are either 6x + 1 or 6x – 1. A number 2n (6X) = (6x – 1) + (6x' + 1) ; 2n (6X + 2) = (6x' + 1) + (6x + 1) ; 2n (6x' - 2) = (6x - 1) + (6x – 1). **Tables 3A-C** show that each of these even numbers has its own configuration to produce primes or composites according to S - P1 = C; or S - P1 = P2 which only satisfies GSC. To illustrate this with examples, I chose an even number 2n from each category and then performed the subtraction 2n - p such that p < 2n.
We have **6X - (6x - 1) = 6X + 1 and 6X - (6x + 1) = 6X - 1** each of which can be prime or composite **but by no means 3n** (Table 3A; 2n = 120) . Whereas an even number 6X + 2 is as follows: **(6X + 2) - (6x - 1) = 6X and (6X + 2) - (6x + 1) = 6X + 1** (**Table 3B; 2n = 50**). Finally; an even number 6X - 2 is as follows: **(6X - 2) - (6x - 1) = 6X - 1 and (6x - 2) - (6x + 1) = 6X** (**Table 3C; 2n = 76**). **Unlike an even number 6x, even numbers 6X + 2 and 6X - 2 will always produce multiples of 3 or 3n which might be the most numerous in [n — 2n].** Note that 50 is 6X + 2 whereas 76 is 6X - 2. The three categories of the Even numbers obey specific congruence rules depeding on 6x ± 1 equations ; for example, one even number cannot be congruent to all primes at once < S/2, or to all composite numbers < S/2 at once. In conclusion, E - P = C and E - P1 = P2 depends on the type of even numbers according to the 6x ± 1 equations.

E. Elementary demonstration by an indirect proof or reductio ad absurdum of Goldbach's strong conjecture (GSC)

Even if we succeed with these rules of congruence in putting an even number in the sum of two prime numbers, this does not constitute a definitive mathematical proof, which is why the GSC remains unprovable. This is why we must resort to a logical reasoning which consists of eliminating false propositions and keeping only one which is true. The one which is true must lead us to the truth of the GSC and thus we succeed in demonstrating it mathematically.

Be n any even ≥ 8. Be p any prime < n and q any prime < p (depending on p value we have a variable number of q such q1 ; q2 ; q3 ; ...q_n < p). Be Q any prime > n and < 2n. **Prime factors of the even 2n are excluded**. Let note **c** any composite < n and **C** any composite > n and < 2n.

1. 2n – p = **Qg** such that Qg > n. Therefore if 2n = (a + 1)p + r then **Qg = ap + r**. **This kind of Qg primes are required for the GSC to be true**.
2. 2n – p = **C** then 2n ≡ p mod(q) and if 2n = (a + 1)p + r then **C = ap + r** although C is a multiple of q.
3. 2n – c = C then 2n ≡ c mod(q) such that C is a multiple of q.
4. 2n – c = Qh then 2n ≡ Qh mod(q) such that c is a multiple of q. **This kind of Qh prime is NOT required for the GSC to be true**.

Therefore only if 2n = (a + 1)p + r and Qg = ap + r prime, GSC can be true. However ap + r = C in the case 2n – p = C does not make GSC true. We must then decide between these two opposing cases.

We only have four propositions one of which is true if the others are false or contradictory :

1. All supposed Qg = 2n – p (p < S/2) are composite such that Qg = ap + r = C in [n ▬ 2n] ; *therefore there would be only primes Qh = 2n - c that do not verify GSC → GSC untrue*. This is impossible because as we saw above an even number produces primes according to 6x ± 1 equation and cannot be congruent to all primes < S/2 at once. Evens 6X + 2 and 6X – 1 produce 3N composites while 6X evens do not produce 3N composites which show that evens obey to different congruence rules in [0 ▬ n] interval. What's more, the composite numbers C in the interval [n ▬ 2n] come from the c's in the interval [0 ▬ n] , and we've seen that n and 2n cannot be congruent to the same q < p < S/2 and therefore can in no way produce the same prime factors of the same composite number. That 2n - p = C every time is impossible, so there is at least one P1 such that 2n - P1 = P2. This is true ad infinitum whether there are long or short gaps between primes and whatever their density.
2. All ap + r = Qh Prime and therefore there would be more primes (Qg + Qh) in [n ▬ 2n] than [0 ▬ n] which contradicts the well-known fact that [0 ▬ n] contains more primes. Although GSC is true in this case, it cannot be accepted due to the contradiction. Let us remember that Qg in [n ▬ 2n] interval are equidistant at n to p in [0 ▬ n] because 2n – p = Qg and this is why if all Qg are primes, there would be more primes in [n ▬ 2n].
3. At least One ap + r = Qg is prime → GSC is TRUE. Because prime factors of S = 2n are excluded in GSC in addition to 3 (for 3n evens) and 5 for 5n ; primes Qg density in [n ▬ 2n] is < than that of [0 ▬ n] in this case which is what expected.
4. All ap + r = C and all 2n – c = C which means no primes at all in [n ▬ 2n] which absolutely would contradict Bertands's postulate.

Among the four propositions 3 of them are subject to contradictions including the first ; second and fourth. **Only the third is correct and therefore GSC is true**.

Example

50 – 11 = 39 = 3 x 13 ↔ 50 ≡ 11 mod (3) ↔ 50 – 11 = 3n = 3 x 13.

Therefore 100 ≇ 11 mod (3) ↔ 100 – 11 cannot be composite and 100 – 11 = 89. **Of note** 100 ≇ 11 mod (7) and so 100 ≇ 11 for every q < p = 11.

50 – 13 = 37 ↔ 50 ≡ 37 mod (13) because 37 = (**2 x 13**) + 11 and 50 = (**3 x 13**) + 11 and 50 – 37 = 11. Therefore 100 ≇ 37 (mod 13) but 100 ≡ 37 mod (3) and 100 ≡ 37 mod (7) → 100 – 37 = (3 x 7)n = 63 = 3 x 21 = 3^2 x 7.

50 – 17 = 33 ↔ 50 ≡ 17 mod (11) and 50 ≡ 17 mod (3) and therefore 50 – 17 = (3 x 11)n = 33. By contrast 100 ≇ 17 mod (11) ; 100 ≇ 17 mod (3) ; and 100 ≇ 17 mod (3). In addition 100 ≇ 17 mod (7) ; 100 ≇ 17 mod (13). Hence 100 – 17 = Q prime = 83.

Let us take another even number like 2 = 200 ad n = 100.

We have 100 – 37 = 63 because 100 ≡ 37 mod (3) and 100 ≡ 37 mod (7) and thus 100 – 37 = (3 x 7) x 3 = 63. Therefore, 200≇ 37 mod(3) and 200≇ 37 mod(7) ; in addition, 200≇ 37 mod(q) for any q < 37. Given all that we can expect 200 – 37 = Q ptime = 163.

Table 2 **: Congruence rules mean that the SAME prime numbers don't add up to form the even numbers n and 2n**. The table shows 50 (n); 100 (2n). 100 (n); 200 (2n). 200 (n) 400 (2n). 400 (n) 800 (2n). 800 (n) 1600 (2n). 1600 (n) 3200 (2n). Then, for example, 50 (n) 200 (4n) and so on.The table shows data highlighted in yellow and green. Yellow indicates n and green 2n. The underlined primes are common to n and 4n. No common primes between n and 2n.

50	100	200	400	800	1600	3200
3+47	3+97	3+197	3+397	3+797	3+1597	13+3187
7+43	11+89	7+193	11+389	13+787	17+1583	19+3181
13+37	17+83	19+181	17+383	31+769	29+1571	31+3169
19+31	29+71	37+163	41+359	43+757	41+1559	37+3163
	41+59	43+157	47+353	61+739	47+1553	79+3121
	47+53	61+139	53+347	67+733	89+1511	139+3061
		73+127	83+317	73+727	101+1499	151+3049
		97+103	89+311	109+691	107+1493	163+3037
			107+293	127+673	113+1487	181+3019
			131+269	139+661	149+1451	199+3001
			137+263	157+643	167+1433	229+2971
			149+251	181+619	173+1427	283+2917
			167+233	193+607	191+1409	313+2887
			173+227	199+601	227+1373	349+2851
				223+577	233+1367	367+2833
				229+571	239+1361	397+2803
				277+523	281+1319	409+2791
				313+487	293+1307	433+2767
				337+463	311+1289	487+2713
				367+433	317+1283	523+2677
				379+421	383+1217	541+2659
					419+1181	607+2593
					449+1151	643+2557
					491+1109	661+2539
					503+1097	727+2473
					509+1091	733+2467
					569+1031	811+2389
					587+1013	823+2377
					617+983	829+2371
					647+953	853+2347
					653+947	859+2341
					659+941	907+2293
					719+881	919+2281
					743+857	997+2203
					761+839	1021+2179
					773+827	1039+2161
						1063+2137
						1069+2131
						1087+2113
						1117+2083
						1171+2029
						1201+1999
						1213+1987
						1249+1951
						1321+1879
						1327+1873
						1399+1801
						1423+1777
						1447+1753
						1453+1747
						1459+1741
						1531+1669
						1543+1657
						1579+1621

Table 3 : **There are three types of even numbers 6x.** The table shows illustrative examples. 6X = 120 ; 6X + 2 = 50 and 6X – 2 = 76. The 6x + 1 primes are highlighted. Evens 6x + 2 and 6x – 2 always produce some composite numbers 3n that might be the most numerous while 6X evens produce composite numbers (C) that are not 3n. These three categories of evens do produce pirme numbers (P) in a sdifferent way and therefore GSC although verified with all of them involves different kind of primes.This shows that evens numbers obey different congruence rules depending on 6x ± 1 equations. A 6X ; B 6X + 2 ; C 6X – 2.

A 120 6x

P	120 – P	P or C
7	113	P
11	109	P
13	107	P
17	103	P
19	101	P
23	97	P
31	89	P
37	83	P
41	79	P
43	77	C
47	73	P
53	67	P
59	61	P
61	59	P
67	53	P
71	49	C
73	47	P
79	43	P
83	37	P
89	31	P
97	33	C
101	19	P
103	17	P
107	13	P
109	11	P

B 50 6x + 2

P	50 – P = X	P or C	3n or not
7	43	P	
11	39	C	3n
13	47	P	
17	33	C	3n
19	31	P	
23	27	C	3n
29	21	C	3n
31	29	P	
37	33		
41	9	C	3n
43	7	P	
47	3	P	3n

C76 6x – 2

P	76 – P	P or C	3n or not
7	69	C	
11	65	C	3n
13	63	C	
17	59	P	
19	57	C	3n
23	53	P	
29	47	P	
31	45	C	3n
37	39	C	3n
41	35	C	
43	33	C	3n
47	29	P	
53	23	P	
59	17	P	
61	15	C	3n
67	9	C	3n
71	5	P	

F. The Ultimate-Goldbach-Gap-of-a-Prime-Value (UGGPV)
Let S be an even number that can verify the GSC. Let q be any prime < S. Among q, we have the primes P < S/2 and Q > S/2. If we subtract S - Q, we'll obtain numbers X that are either prime P < S/2 or composites C < S/2. So we perform these subtractions in series S - Q1 = X1; S - Q2 = X2; S - Q3 = X3; S - Q4 = X4...S - Qn = Xn with Qn...>Q4>Q3>Q2>Q1. We'll obtain a sequence of prime and composite numbers in reverse order Xn...<X4<X3<X2<X1. Assume that X4; X3; X2; and X1 are primes and therefore S - Qn = Xn = Pn then Pn is the UGGPV. **The UGGPV is the prime number P that separates a prime number Q > S/2 from S.** For an ,even number that is not a multiple of 3, the minimum value of a UGGPV is 3. On the other hand, the UGGPV has a minimum value of 7 for even numbers that are multiples of 3. We need to exclude prime numbers P that are prime factors of S. If we set S = tq + r, t**hen UGGPV = S - Qn = tPn + r - ((t -1)Pn + r) = Pn, provided that S ≡ Qn mod(Pn)**. The smallest UGGPV depends upon the gap between the even number S and the last prime number Qn that precedes it. The more the gap is larger the higher is the value of the UGGPV.
In **Tables 5**, we consider the case of S = 100. We take the prime numbers < 100 and subdivide them into P < S/2 = 50 and those > S/2, which we denote Q. Then divide all the Qs by a P. The table shows the quotients. **We can see that each UGGPV is deduced from the subtraction of two quotients that differ by a single unit**. Example 100 = 11 x 9 with r = 1 ; and 89 : 11 = 8 with r = 1. We see 100 ≡ 89 mod (11) and 9 - 8 = 1. When the difference > 1, the number is ccomposite denoted by C e.g. 100 - 73 = 27 with 100 : 3 = 33 and r = 1and 73 : 3 = 24 and r = 1 and so 33 - 24 = 9 such that 9 x 3 = 27 = C (**Table 4A**).

In **Table 4B** we see 100 : 7 = 14 with r = 2 and 79 : 7 = 11 with r = 2 so that 14 - 11 = 3 and therefore 3 x 7 = 21 = C. On the contrary, 100 : 11 = 9 and r = 1 and 89 : 11 = 8 and r = 1 and therefore 9 - 8 = 1. This means that between 11 and 100 we have the odd numbers 11 x 2 + 1, 11 x 4 + 1; 11 x 6 + 1; 11 x 8 + 1 alternating with the even numbers 11 x 3 + 1; 11 x 5 + 1; 11 x 7 + 1 and 11 x 9 + 1. Note that two of these are < S/2 = 50 and the other two are > 50. Between two successive even and odd numbers there are 2 x 11 and between an odd and an even number there are 11. If the odd before S is prime, the GSC proves true. So if S = tP + r, the equation Q = (t - 1)P + r will generate an infinite number of possible primes. The primes Q > S/2 follow from each prime P < S/2 so that the even number S = tP + r and Q = (t - n)P + r. If n = 1 GSC is proved, and if n > 1 we have a composite number and so GSC does not apply. Other examples are shown in **Table 4C – E**.
The GSC can be proved using this method. Take all prime numbers P < S/2 and calculate S = tP + r. If only one number is prime of type Q = (t - 1)P + r which is > S/2, the GSC is true. For small even numbers S (relative to infinity), the primes are dense enough that at least one number Q = (t - 1)P + r is prime. For infinitely large S, there are infinitely many numbers P and therefore infinitely many possible primes of type Q = (t - 1)P + r.

An odd number is not only constructed by the Euclidean path of a multiple of prime factors, but also by the Euclidean equation ax + r, and can be composite or prime. The GSC means that an even number S = (a + 1)x + r is always preceded by one or many Q = ax + r prime such that S – Q = x with x any prime < S/2 and Q any prime > S/2.

The distance between x and ax + r is (ax + r) - x = (a - 1)x + r and therefore the distance of x and ax + r from S/2 is ((a - 1)x + r)/2. Example (89 - 11)/2 = 39 and therefore 11 + 39 = 50 and 50 + 39 = 81 and therefore 11 and 89 are equidistant from S/2. In fact 89 = 8 x 11 + 1 and so 89 – 11 = (8 – 1) x 11 + 1 = 7 x 11 + 1 = 78 and then 78/2 = 39.

Goldbach then sees even numbers in the form of the Euclidean linear equation (a + 1)x + r and prime numbers as ax + 1 with a gap = x between them. GSC can then be used to find new primes to infinity, starting from an even number S. Prime numbers multiply with each other to generate even or odd natural numbers; or follow the Euclidean equation ax + r to generate odd numbers, including odd prime and composite numbers and even numbers. We can conclude that an even number S of type (a + 1)x + r is an interval in which at least one prime Q of type ax + r >S/2 is formed with x any prime < S/2 such that S≡Q mod(x) and S≡x mod(Q). **The number 8 would be the first even number that satisfies this interval rule, since we have 8 = (2 x 3) + 2 and it is preceded by the prime number Q = (1 x 3) + 2 = 5. And so (5 - 3)/2 = 1 and so 5 and 3 are one unit away from 8/2 = 4. And so 8 would be the smallest interval in the whole set N that obeys this GSC rule (the case P1 = P2 is excluded)**.

Table 4 : Verification of the GSC by calculating the UGGPV or a prime gap between the even number S = 100 and the prime numbers Q > S/2 preceding it. The primes Q are all divided by one prime < S/2 as shown. The congruence rules required for the GSC to be true are shown in the tables. Some Composite numbers not satisfying The GSC are shown on the right of the table.

4A

Q→	53	59	61	67	71	73	79	83	89	97	100
Q/3	17	19	20	22	23	24	26	27	29	32	33
S – Q	47	41	39	33	29	27	21	17	11	3	
mod(3)	≇	≇	≡	≡	≇	≡	≡	≇	≇	≡	

4B

Q→	53	59	61	67	71	73	79	83	89	97	100	93 = 3 x 31
Q/7	7	8	8	9	10	10	11	11	12	13	14	13
S – Q	47	41	39	33	29	27	21	17	11	3		
mod(7)	≇	≇	≇	≇	≇	≇	≡	≇	≇	≇		≡

4C

Q→	53	59	61	67	71	73	79	83	89	97	100
Q/11	4	5	5	6	6	6	7	7	8	8	9
S – Q	47	41	39	33	29	27	21	17	11	3	
Mod(11)	≇	≇	≇	≡	≇	≇	≇	≇	≡	≇	

4D

Q→	53	59	61	67	71	73	79	83	89	97	100	87 = 3 x 29
Q/13	4	5	5	5	5	5	6	6	6	7	7	6
S – Q	47	41	39	33	29	27	21	17	11	3		9
Mod(13)	≇	≇	≡	≇	≇	≇	≇	≇	≇	≇		≡

4E

Q→	53	59	61	67	71	73	79	83	89	97	100
Q/17	3	3	3	3	4	4	4	4	5	5	5
S – Q	47	41	39	33	29	27	21	17	11	3	
Mod(17)	≇	≇	≇	≇	≇	≇	≇	≡	≇	≇	

G. GSC representation in a table or graph based on the remainders of Euclidean divisions

G1. A Table to test GSC

First, the example is the even number S = 74. We take prime numbers close to and less than 74 and prime numbers close to 0; and we divide the first by the second and then we note the remainders of the divisions thus carried out (**Table 5**). We compare all the remainders obtained with the prime numbers to those obtained with S = 74 and when they are identical we subtract them from 74. The GSC is true when the difference has a value of a prime number or what is called here UGGPV. Exampe 74 : 3 has a remainder r = 2 identical to that of 71 and 74 – 71 = 3 which is prime and so GSC is verified. Also 74 : 7 has r = 4 which is identical to that of 67 : 7 and so 74 – 67 = 7 another UGGPV that verifies GSC. Although 74 : 11 has the same r = 8 than 41 : 11 we have 74 – 41 = 33 which is not an UGGPV and therefore GSC is not verified in this case. Other divisors such that 13 and 31 are UGGPV that verify GSC.

Such tables can be therefore useful to test if an even number S is preceded very closely by primes such that the difference between them and S has values of primes and so verifying GSC.

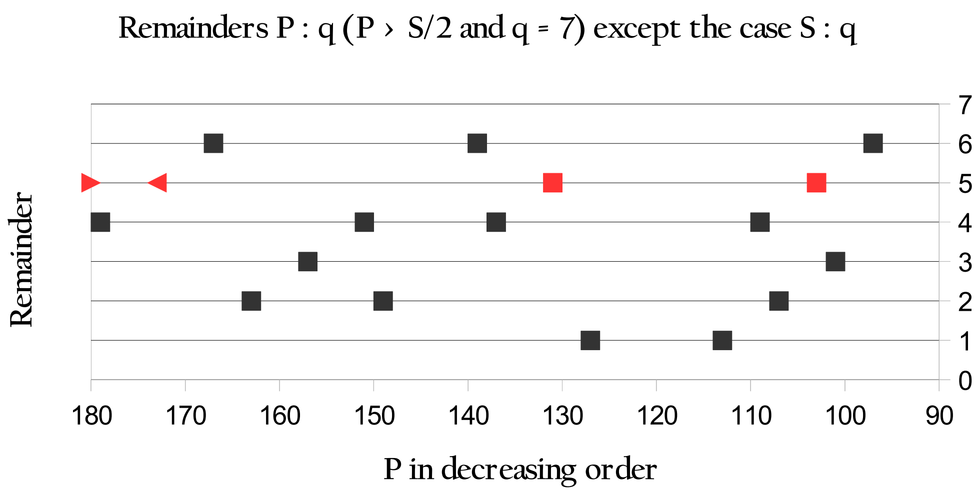
Table 5. Remainders of the Euclidean divisions of numbers in the first column by the numbers in the first line. All numbers are prime (close to 74 in the column) or closer to 0 in the line. Identical remainders obtained with a same prime divisor are highlighted. GSC is true depending on how far is the congruent prime from S = 74 in the column. In Green GSC satisfied with an UGGPV but not in blue.

	3	5	7	11	13	17	19	23	29	31
74	2	4	4	8	9	6	17	5	16	12
73	1	3	3	7	8	5	16	4	15	11
71	2	1	1	5	6	3	14	2	13	9
67	1	2	4	1	2	16	10	21	9	5
61	1	1	5	6	9	10	4	15	3	30
59	2	4	3	4	7	8	2	13	1	28
53	2	3	4	9	1	2	15	7	24	22
47	2	2	5	3	8	13	9	1	18	16
43	1	3	1	10	4	9	5	20	14	12
41	2	1	6	8	2	7	3	18	12	10

G2. GSC in graphics of remainders

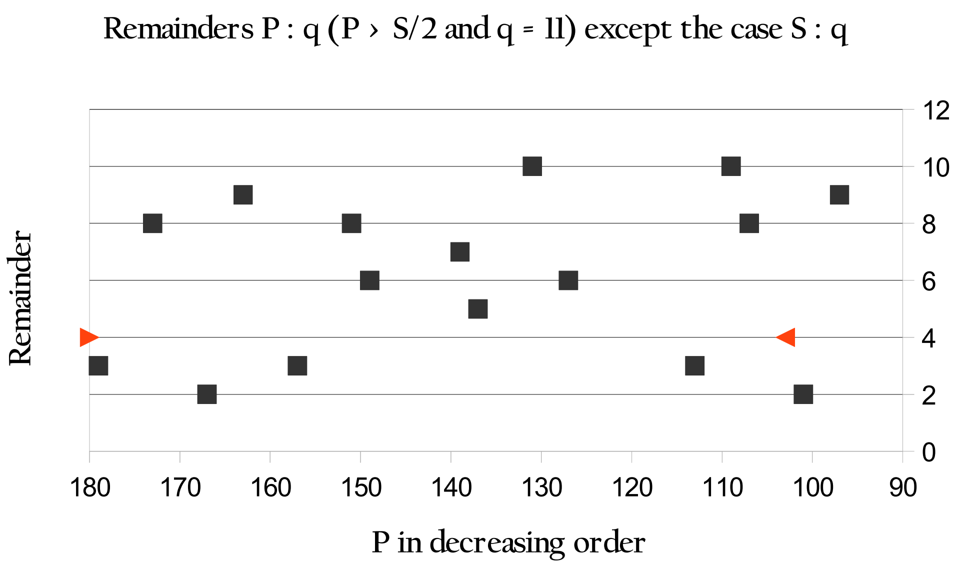
Using the same method as in Table 5 here for the even number S = 180. Primes P close but lower than S = 180 are divided by primes denoted q close to 0 (3 ; 7 ; 11 ; …).
In the **Graphic 1A**, the remainder of S is shown by the red arrow at the left. We see for example that 180 – 173 = 7 which is an UGGPV that verifies GSC. The prime number 173 is indicated by the red arrow at the right. Note how close are the arrows because the gap = 7 is too small. The square correspond to composites.

<u>Graphic 1A.</u>



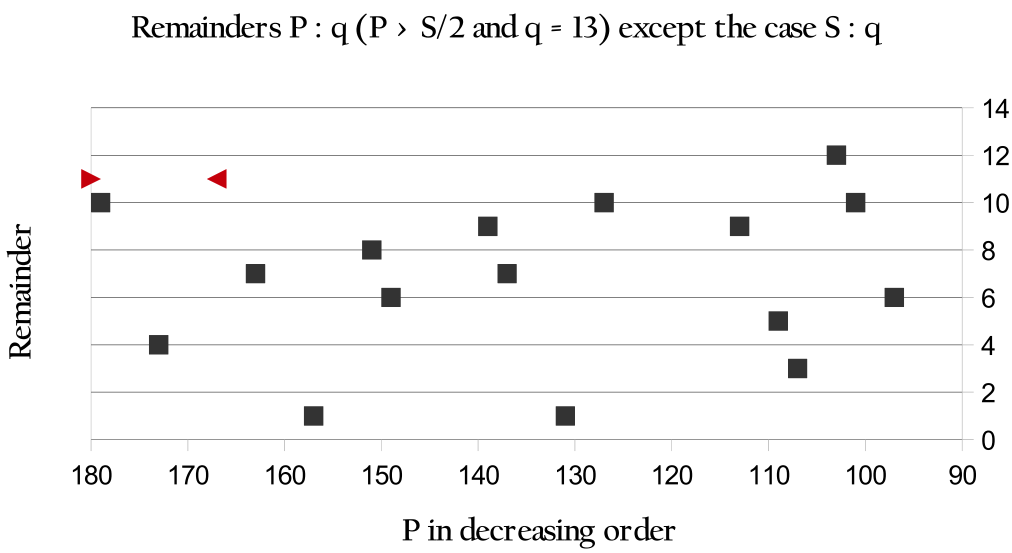
By contrast a larger gap is seen with 11 (**Graphic 1B**) because 180 – 103 = 77 as deduced from the remainders, which is not an UGGPV that verifies the GSC . We see the two arrows are more distant from each other because the gap = 77. Therefore 11 is not an UGGPV.

Graphic 1B.



In **Graphic C** we see that 13 is an UGGPV because 180 – 167 = 13 as deduced from the remainders and again the two arrows are closer to each other. **We can have large gaps between the two arrows which nevertheless verify the GSC for example 180 - 83 = 97 which is therefore the largest UGGPV gap for this number**.

Graphic 1C.



H. GSC remains true despite large gaps between even numbers and the prime numbers preceding them

H1. Example of an even after a gap = 35.

Large known gaps between primes are shown in https://t5k.org/notes/GapsTable.html by Chris Caldwell, et al. Suppose we have two primes p and q between which there is a larger gap. If we take the even S = q + 1 then S will be as distant from p as q which allows to determine how gaps can impact the GSC. In **Table 6**, I take the prime number 9551 after which there is a gap = 35 before finding the next prime and therefore I take the even S = 9551 + 35 = 9586. The same method as above is used to analyze remainders of euclidean divisions of 9586 and close primes lower than it divided by primes closer to 0 (7 ; 11 ; 13 ; 17 ;...47). In **Table 6** we see that 7 does not verify the GSC with the even number S = 5986 because there is the gap of 35 between 5986 and the prime number that precedes it 9551 and therefore 9586 - 9551 = 35 = 5 **x 7**. This is also the case with 13 because we have 9586 - 9547 = 39 = 3 x 13; or with 19 because we have 9586 - 9491 = 95 = 19 x 5. We must go up to the prime number 47 so that the GSC is verified with the number 9586 - 9539 = 47. We see that the initial gap of 35 between 9586 and 9551 eliminates the prime numbers from 7 to 43 before the GSC is verified correctly at 47. Note that 9586 is congruent with prime numbers whose remainders are highlighted mod(7) ; mod(13) ; mod(19) and mod(47).

Table 6. Gaps can delay GSC to be true depending on prime sequence after the gap. Example of the even 9586 is preceded by a prime number 9551 at a gap = 35.

	7	11	13	17	19	23	29	31	37	41	43	47
9586	3	5	5	15	10	18	16	x	3	x	40	45
9551	3	3	9	14	13	12	10	x	5	x	5	10
9547	x	10	5	10	9	2	6	x	1	x	1	6
9539	x	2	x	2	1	17	27	x	30	x	36	45
9533	x	7	x	13	14	11	21	x	24	x	30	x
9521	x	6	x	1	2	22	9	x	12	x	18	x
9511	x	7	x	8	11	12	28	x	2	x	8	x
9497	x	4	x	11	16	21	14	x	25	x	37	x
9491	x	9	x	5	10	15	8	x	19	x	31	x
9479	x	8	x	10	x	3	25	x	7	x	19	x
9473	x	2	x	4	x	20	19	x	1	x	13	x

H2. Exponential shift between the values of even numbers after a gap and those of their first prime numbers that satisfy the GSC

Table 7 shows a sample of the first even numbers that occur just after a prime number before which there are gaps in ascending order. **While the even numbers increase from 96 to 2,010,880, the gaps vary only from 7 to 147**. The even numbers have increased 20,946 times, while the gaps have only lengthened 7 times. In this sample of numbers in **Table 7**, the even numbers are growing almost 3,000 times faster than the primes preceded by increasing gaps, and their growth has an exponential tendency.

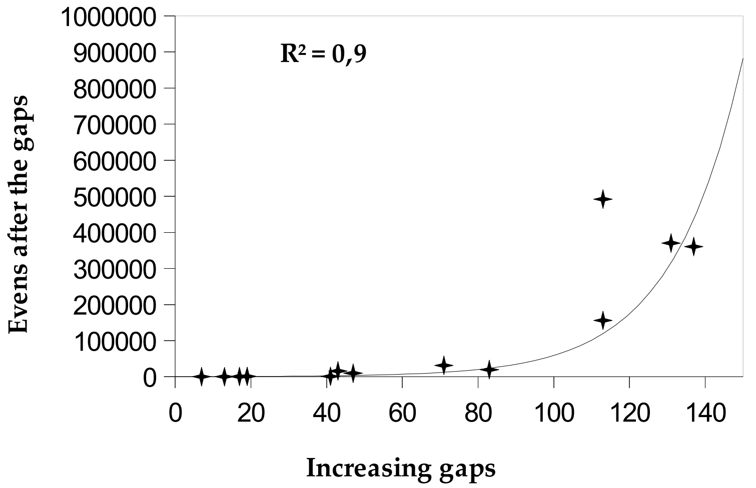
***Table 7*.** Even numbers after a gap devoid of primes grow much faster than the gaps thand primes that surround them. They still verify the GSC with their primes closer to 0. For instance the even number 360,748 occurrig after a gap = 95 verifies the GSC with a prime number as small as 137 (meaning 360,748 – 137 = 360,611 is prime).

S	gap	S – Q = q	Order
96	**7**	**7**	1
126	13	13	2
540	17	17	3
906	19	19	4
1150	21	41	5
1360	33	41	6
9586	35	47	7
15726	43	43	8
19660	51	83	9
31468	71	71	10
156006	85	113	11
360748	95	137	12
370372	111	131	13
492226	113	113	14
1349650	117	179	15
1357332	131	131	16
2010880	**147**	**191**	17

Graphic 2A shows an exponntial acceleration in the increase in even-numbered values following an empty prime gap.
On the other hand, for each even number S, we note their first prime < S/2 that verifies the GSC. For example, for the number 540, the GSC is verified from 17 onwards, while for the number 2,010,880, the GSC is verified from 191 onwards (**Table 7**). **Graphic 2B**, on the other hand, shows **a very slow increase in the value of prime numbers verifying the GSC** after empty prime number gaps. Correlation coefficient values approach 0.9.

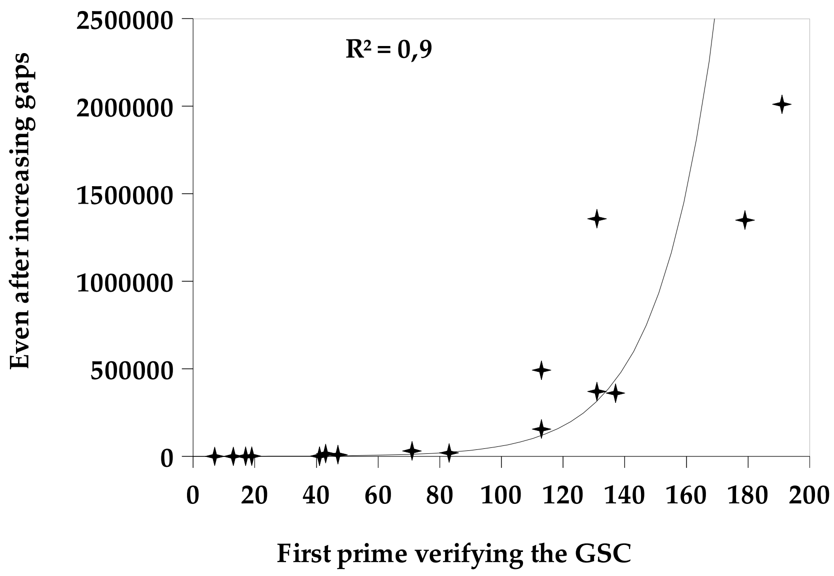
Graphic 2A

Exponential trends in increase of evens after a gap devoid of primes



Graphic 2B

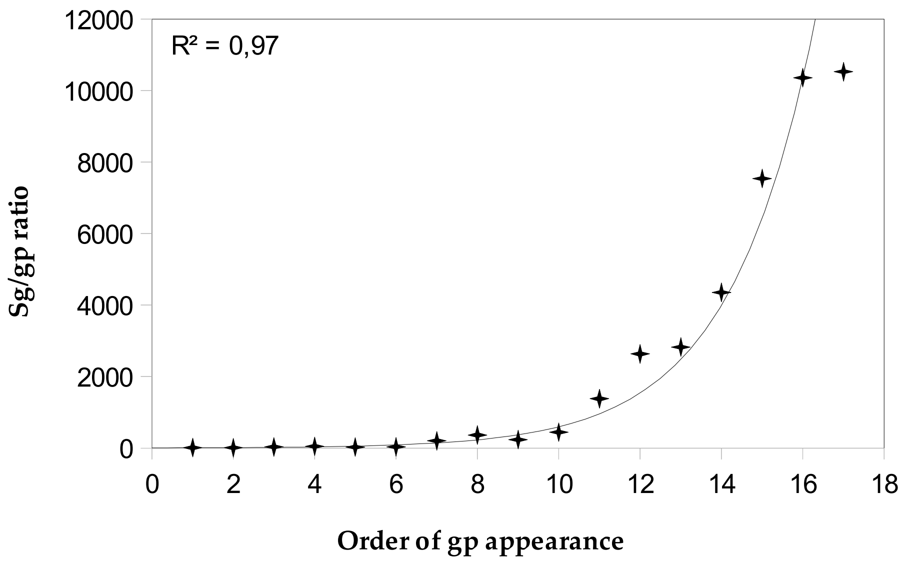
Verification of the GSC after increasing gaps



Then, for each gap that appears, we assign its order of appearance, i.e. the first gap, then the second, third and nth. We then calculate the ratio between the even number and the gap before it. The even number occurring after a gap is noted **Sg** and the gap is noted **gp**. Each Sg shown verifies the GSC (**see Table 7**). **Graphic 2C** shows the **Sg/gp** ratios as a function of the order of appearance of the gaps; and it shows that Sg even numbers verifying the GSC go 12,000 times faster towards infinity, while the lengths of gaps devoid of primes, on the contrary, increase remarkably slowly. Even if an even number Sg goes very far to infinity, it will still verify the GSC with prime numbers < Sg/2 much closer to 0 or going very slowly to infinity. The correlation coefficient shows a much higher value of 0.97, proving that Sg go to infinity exponentially, while their GSC-verifying primes still remain close to 0.The occurrence of an empty gap of primes before an even number does not equivocally mean that the GSC might not be verified, for two reasons: the length of the gap is always much lesser than the value of the even number after it ; and increases very slowly. Whereas by contrast the even numbers soar exponentially; and the primes verifying the GSC still remain closer to 0 and increase much more slowly.

Graphic 2C

Infinite exponential shift between Sg = p + q and gp



Conclusion

<u>This paper shows for the first time a detailed elementary demonstration of the Goldbach's strong conjecture (GSC)</u>.

If we set an even number as Sg = (a + 1)p + r (with p prime < S/2), then there are infinitely many prime numbers Q = at + r > S/2 and < S, such that S - Q = p and therefore S = p + Q according to the GSC. The first number that satisfies this equation is 8 = (2 x 3) + 2, preceded by the prime number 5 = (1 x 3) + 2. In this equation, S ≡ Q mod(p) such like 8 ≡ **5** mod(3) and also 8 ≡ **3** mod(5). The Euclidean equation will likely generate infinitely many possible prime numbers. For each even number S, there are as many possible prime numbers Q as there are as many prime numbers p < S/2.

Even numbers are not only formed by the Sieve of Erastothene, that is, by the product of prime factors with 2, but also by the equation of Euclidean division ap + r. If we have a prime number of the form Q = ap + r, then the even number S is of the form (a + 1)p + r. We increase the number of p by 1. This process makes the even number S obtained congruent to Q mod (p). If we increase the number of p following an odd progression (3; 5; 7... times), the even number S increases in parallel but remains congruent to Q mod (p). Example (3 x 11) + 4 = 37 and S = (4 x 11) + 4 = 48. Or (4 x 7) + 3 = 31 and (5 x 7) + 3 = 38. However, the equation can also give composite odd numbers, but an even number S has as many p < S/2 as it is large and therefore there are several chances that a number Q = at + r is prime. Therefore, the analysis of the remainders of the Euclidean divisions of S : p and Q : p is crucial for the verification of the GSC.

Indeed, an even number S can be written in the form of a Euclidean equation with all p < S/2 and this is also the case for prime numbers Q > S/2 and therefore S follows the progression of Q as a function of p. We have in general S = (a + n)p + r and Q = ap + r with n ≥ 1. Only if n = 1 does the GSC prove to be exact because S – Q = p and therefore S = p + Q. There are ways to twist these Euclidean equations. For example, 7 = (1 x 5) + 2 and the resulting even is 12 = (2 x 5) + 2 and therefore 12 = 5 + 7. However 3 cannot be used neither 2. That starts with 5. The integers form a tree whose trunk is the Sieve of Erastothene but the branches follow the Euclidean equations ap + r. Example 5 x 7 = 35. But 5 x 7 + 2 = 37 (prime) or 5 x 7 + 4 = 39 = 3 x 13 (composite). Now 5 x 7 - 2 = 33 = 3 x 11 (composite) and 5 x 7 - 4 = 31 (prime). **The prime numbers follow from the Sieve of Erastothene to which we add or subtract remainders**. The prime number is then a branch but if the equation ap + r gives a composite then we back to the trunk. **An even number S is continuous with the prime numbers Q > S/2 which precede it, some of which share the same remainder with it when divided by the prime numbers < S/2**. We therefore have S ≡ Qmod (p) and S - Q = X. X will be prime depending on the distance which separates Q from S and depending on the value of the prime number p (is it repeated n times or once?) Only if S - Q = p does the GSC holds true. However, **if we change our point of view and look at evens S in the form of Euclidean equations (a + 1)p + r and similarly at Q = ap + r, we will see that GSC is natural and occurs for every even**.

To demonstrate GSC, we really need to set aside the concept that an integer is always a multiple of prime factors, and its multiples align with the Sieve of Erasthotene. We must now recognize that an even number is also in the form (a + 1)p + r, which relates it to prime numbers Q of the form ap + r. Odd numbers in general are also of the form ax + r, the most classic of which is the equation 2x + 1. Bearing this in mind, GSC holds naturally true.
After each prime number of form ap + r will give an even number of form (a + 1)p + r to infinity. Either the even numbers follow the trunk of Erasthotene by multiplying prime factors by 2 or they follow the branches by deriving from the prime numbers of type ap + r which precede them. It is in this last case that the GSC is verified and finds its meaning.
It follows that an even number S of form (a + 1)p + r is always preceded by a prime number Q of form ap + r. However, the prime number Q might be very far before the even number S. In fact, the growth of even numbers does not follow that of prime numbers Q; but it is much faster and follows an exponential trend.
GSC means that an even number S is an interval where there exists at least one pair of primes (p, Q) equidistant from S/2 whose sum p + Q = S. But S ≡ Q mod(p) and this means that an even number that tends to infinity will have an infinity of possible primes Q. This makes empty gaps of primes not contradict GSC because the growth of even numbers is infinitely greater than that of the primes that precede them. But since the primes Q can in turn give primes in the form tp + r then the primes continue to be present as far as the even numbers go.
The Bertand's postulate indicate that there exists at least one prime in [n — 2n] interval **but what is if this postulate is true in two opposite symmetric directions?** We have an [n — 2n] interval and a [0 — n] interval of the same length. Therefore, a prime number Q is between n and 2n, but at the same time, another symmetric number p is present between n and 0. The two prime numbers are equidistant from n. For example, between 5 and 10 there is 7, and between 5 and 0 there is 3. Or between 7 and 14 there is 11, and between 7 and 0 there is 3. This Bertrand postulate does not hold only in one [n — 2n] interval, but in two symmetrically spaced intervals, [0 — n] and [n — 2n] at the same time. **The GSC seems to signify a doubling of Bertrand's postulate in two intervals of the same length**. This also means that primes in [n — 2n] interval are related to those in [0 — n] interval by the ap + r equation. **Prime numbers follow a mirror symmetry rule such that a prime number never appears alone out of nowhere but occupies a specific position in the [n — 2n] interval that is mirror symmetric to another one in the [0 — n] interval as if prime numbers appear in pairs at a time**. The Euclidean equation exhibits this mirror symmetry because we have **Q = ap + r** or **Q = ap - r**. This idea deserves future research.
In a whole, this article shows that GSC is true at infinity and follow euclidean equation ap + r. Primes numbers Q préceding an even are related to it by congruence rules and the gap between them.

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