Bessel functions and Pi

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ABSTRACT: We give two sequences for Pi

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I. Introduction. Bessel function of the first kind

The Bessel function of the first kind is defined as

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)} \tag{1}$$

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin(u) - n u)\, du \quad , \quad n \in \mathbb{Z} \tag{2}$$

In particular

$$J_0(1) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-2n}}{(n!)^2} \tag{3}$$

$$J_0(1) = \frac{1}{\pi} \int_0^\pi \cos(\sin(u))\, du \tag{4}$$

Remark: $\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right)$.

II. $J_0(1)$ and Pi

Define

$$F(\alpha) = \int_0^\alpha \cos(\sin(u))\, du \tag{5}$$

$$\alpha_{n+1} = \frac{\alpha_n - F(\alpha_n)}{1 - J_0(1)} \quad , \quad \alpha_1 = 3 \, , \quad n = 1, 2, 3, \ldots \tag{6}$$

we have

$$\alpha_n \longrightarrow \pi \tag{7}$$

n	$\lvert \pi - \alpha_n \rvert$
1	0.141593
2	0.00200491
3	5.72042×10^{-9}
4	1.32871×10^{-25}
5	1.66509×10^{-75}
6	3.27688×10^{-225}

$$|\pi - \alpha_{n+1}| \approx \frac{1}{6(1 - J_0(1))} |\pi - \alpha_n|^3 \quad , \quad n = 1, 2, 3, \ldots \tag{8}$$

III. Modified Bessel function of the first kind

The modified Bessel function of the first kind is defined as

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)} \tag{9}$$

$$I_n(z) = \frac{1}{\pi} \int_0^{\pi} e^{z\cos(u)} \cos(n\,u)\, du \quad , \quad n \in \mathbb{Z} \tag{10}$$

In particular

$$I_0(1) = \sum_{n=0}^{\infty} \frac{2^{-2n}}{(n!)^2} \tag{11}$$

$$I_0(1) = \frac{1}{\pi} \int_0^{\pi} \cosh(\sin(u))\, du \tag{12}$$

IV. $I_0(1)$ and Pi

Define

$$F(\alpha) = \int_0^{\alpha} \cosh(\sin(u))\, du \tag{13}$$

$$\alpha_{n+1} = \frac{\alpha_n - F(\alpha_n)}{1 - I_0(1)} \quad , \; \alpha_1 = 3 \,, \; n = 1,\, 2,\, 3,\, \ldots \tag{14}$$

we have

$$\alpha_n \longrightarrow \pi \tag{15}$$

n	$\lvert \pi - \alpha_n \rvert$
1	0.141593
2	0.00177285
3	3.49043×10^{-9}
4	2.66376×10^{-26}
5	1.18399×10^{-77}
6	1.03968×10^{-231}

$$|\pi - \alpha_{n+1}| \approx \frac{1}{6\,(I_0(1) - 1)} \, |\pi - \alpha_n|^3 \quad , \; n = 1,\, 2,\, 3,\, \ldots \tag{16}$$

V. End note

$$J_0(1) = \frac{2}{\pi} \int_0^{\pi/2} \cos(\sin(u))\, du \tag{17}$$

$$J_0(1) = \frac{2}{\pi} \int_0^{\pi/2} \cos(\cos(u))\, du \tag{18}$$

$$J_0(1) = \frac{2}{\pi} \int_0^{1} \frac{\cos(u)}{\sqrt{1-u^2}}\, du \tag{19}$$

$$J_0(1) = \frac{2}{\pi} \int_0^{1} \frac{\cos\left(\sqrt{1-u^2}\right)}{\sqrt{1-u^2}}\, du \tag{20}$$

$$J_0(1) = \frac{1}{\pi} \int_{-1}^{1} \frac{\cos(u)}{\sqrt{1-u^2}}\, du \tag{21}$$

$$J_0(1) = 1 - \frac{2}{\pi} \int_{\cos(1)}^{1} \cos^{-1}\left(\cos^{-1}(u)\right) du \tag{22}$$

$$J_0(1) = \cos(1) + \frac{2}{\pi} \int_{\cos(1)}^{1} \sin^{-1}\left(\cos^{-1}(u)\right) du \tag{23}$$

$$J_0(1) = \frac{2\cos(1)}{\pi} \int_0^1 \frac{\cos(u)}{\sqrt{u(2-u)}} du + \frac{2\sin(1)}{\pi} \int_0^1 \frac{\sin(u)}{\sqrt{u(2-u)}} du \tag{24}$$

$$J_0(1) = 1 - \frac{2}{\pi} \int_0^{\pi/2} u \sin(u) \sin(\cos(u)) du \tag{25}$$

$$J_0(1) = \frac{2}{\pi} \int_0^{\infty} \operatorname{sech}(u) \cos(\operatorname{sech}(u)) du \tag{26}$$

$$J_0(1) = \frac{2}{\pi} \int_0^{\infty} \operatorname{sech}(u) \cos(\tanh(u)) du \tag{27}$$

$$J_0(1) = \frac{2}{\pi} \int_0^1 \frac{\cos\left(2u\sqrt{1-u^2}\right)}{\sqrt{1-u^2}} du \tag{28}$$

$$J_0(1) = \frac{4}{\pi} \int_0^{1/\sqrt{2}} \frac{\cos\left(2u\sqrt{1-u^2}\right)}{\sqrt{1-u^2}} du \tag{29}$$

$$J_0(1) = \frac{2}{\pi} \int_0^{1/\sqrt{2}} \left(\cos(u) + \cos\left(\frac{u+\sqrt{1-u^2}}{\sqrt{2}}\right)\right) \frac{1}{\sqrt{1-u^2}} du \tag{30}$$

$$J_0(1) = \frac{4}{\pi} \int_0^{\pi/4} \cos\left(\frac{\cos(u)}{\sqrt{2}}\right) \cos\left(\frac{\sin(u)}{\sqrt{2}}\right) du \tag{31}$$

$$J_0(1) = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \cos\left(\frac{\cos(u)}{\sqrt{2}}\right) \cos\left(\frac{\sin(u)}{\sqrt{2}}\right) du \tag{32}$$

for $0 \le \theta \le \pi/2$ we have

$$\frac{\pi}{2} J_0(1) = \theta \cos(\sin(\theta)) + \int_{\theta}^{\pi/2} \cos(\sin(u)) du + \int_{\cos(\sin(\theta))}^{1} \sin^{-1}\left(\cos^{-1}(u)\right) du \tag{33}$$

$$\frac{\pi}{2} J_0(1) = \frac{\pi}{2} \cos(1) - \theta \cos(\sin(\theta)) + \int_0^{\theta} \cos(\sin(u)) du + \int_{\cos(1)}^{\cos(\sin(\theta))} \sin^{-1}\left(\cos^{-1}(u)\right) du \tag{34}$$

VI. References

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