A new $3D$ Spherical Coordinates warp drive vector with Hodge Star over the y-axis and variable speeds using the Natario methodology

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Abstract

The Natario warp drive appeared for the first time in 2001.Although the idea of the warp dive as a spacetime distortion that allows a spaceship to travel faster than light predated the Natario work by 7 years Natario introduced in 2001 the new concept of a propulsion vector to define or to generate a warp drive spacetime.Natario defined a warp drive vector for constant speeds in Polar Coordinates over the x-axis but remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model so it must possesses variable speeds.We developed in this work a new warp drive vector for the y-axis in both Polar and Spherical coordinates that encompasses variable speeds.Also Polar Coordinates uses only two dimensions and we know that a real spaceship is a tridimensional $3D$ object inserted inside a tridimensional $3D$ warp bubble that must be defined in real $3D$ Spherical Coordinates.

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1 Introduction:

The Natario warp drive appeared for the first time in 2001.([1]).Although the idea of the warp dive as a spacetime distortion that allows a spaceship to travel faster than light predated the Natario work by 7 years Natario introduced in 2001 the new concept of a propulsion vector to define or to generate a warp drive spacetime.

This propulsion vector nX uses the form $nX = X^i e_i$ where X^i are the shift vectors responsible for the spaceship propulsion or speed and e_i are the Canonical Basis of the Coordinates System where the shift vectors are based or placed.

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Polar Coordinates(See pg 4 in [1]).(see Appendix D about Polar Coordinates).The final form of the original Natario warp drive vector is given by $nX = vs * d(r\cos\theta)$.However Polar Coordinates are not real tridimensional $3D$ coordinates since it uses only the two Canonical Basis $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

We introduced in this work a new warp drive vector $nY = vs * (dY)$ where vs is the constant speed of the warp bubble and $*(dy)$ is the Hodge Star taken over the y-axis of motion in Polar Coordinates.The final form of our new warp drive vector is given by $nY = vs * d(r\sin\theta)$.

The Hodge Star actually must be taken over the product (yvs) giving the expression $nY = *(yvs) = vs * (dy) + y * (dvs)$ but due to a constant speed vs the term $y * d(vs) = 0$.In this work we examine what happens with the new warp drive vector when the velocity is variable and then the term $y * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.

Natario used Polar Coordinates(See pg 4 in [1]) but for a real $3D$ Spherical Coordinates another warp drive vector must be calculated.Remember that a real spaceship is a tridimensional $3D$ object inserted inside a tridimensional $3D$ warp bubble that must be defined in real $3D$ Spherical Coordinates.The final form of the Hodge Star for this warp drive vector based over the y-axis is calculated no longer over $*d(r\sin\theta)$ but instead over $*d(r\sin\phi\sin\theta)$ since this form uses all the tridimensional $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.(see Appendix E about tridimensional $3D$ Spherical Coordinates).

In this work we present the new warp drive vector in tridimensional $3D$ Spherical Coordinates with the Hodge Star over the y-axis calculated for both constant $nY = vs * d(y)$ or variable speeds $nY = vs * (dy) + y * (dvs)$.

In order to fully understand the idea presented in this work(a new warp drive vector in tridimensional $3D$ Spherical Coordinates over the y-axis) acquaintance or familiarity with the Natario original warp drive paper is required but we provide all the mathematical demonstration QED(Quod Erad Demonstratum) in the Appendices.

This work is organized as follows:

- A)-Section 2 introduces the new Natario warp drive vector nY in Polar Coordinates $nY = vs * d(y)$ for constant speeds.
- B)-Section 3 introduces the new Natario warp drive vector nY in Polar Coordinates $nY = vs * d(y) + y * (dvs)$ for variable speeds.
- C)-Section 4 introduces the new warp drive vector nY in tridimensional $3D$ Spherical Coordinates $nY = vs * d(y)$ for constant speeds.
- D)-Section 5 introduces the new warp drive vector nY in tridimensional $3D$ Spherical Coordinates $nY = vs * d(y) + y * (dvs)$ for variable speeds.

We adopted in this work a pedagogical language and a presentation style that perhaps will be considered as tedious,monotonous, exhaustive or extensive by experienced or seasoned readers and we designated this work for novices,newcomers,beginners or intermediate students providing in our work all the mathematical background needed to understand the process Natario used to generate warp drive vectors.

As a matter of fact if a novice,newcomer,beginner or intermediate student not familiarized with the Natario techniques reads the Natario warp drive paper in first place he(or she) will perhaps feel some difficulties.

We hope our paper is suitable to fill this gap.

Although this work was designed to be independent,self-consistent and self-contained it may be regarded as a companion work to our works in [9],[16] and [17].

2 The equation of the new Natario warp drive vector nY in $2D$ polar coordinates over the y-axis with a constant speed vs

The equation of the new Natario warp drive vector nY is given by:

$$nY = Y^r e_r + Y^\theta e_\theta \tag{1}$$

With the contravariant shift vector components Y^{rs} and Y^θ given by:(see Appendix A for details)

$$Y^{rs} = 2v_s f(rs) \sin\theta \tag{2}$$

$$Y^\theta = v_s(2f(rs) + (rs)f'(rs)) \cos\theta \tag{3}$$

Considering a valid $f(rs)$ as a shape function being $f(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $f(rs) = 0$ for small rs(inside the warp bubble) while being $0 < f(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(see pg 5 in [1]):

We must demonstrate that the Natario warp drive vector given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nY generates a Natario warp drive spacetime if $nY = 0$ and $Y = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nY = vs(t)$ with $Y = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(see pg 4 in [1])(see also Appendices G and H).

Natario in its warp drive uses the polar coordinates rs and θ.In order to simplify our analysis we consider motion in the $y - axis$ only or the vertical plane rs where $\theta = 90$ $\sin(\theta) = 1$ and $\cos(\theta) = 0$.(see pgs 4,5 and 6 in [1]).

In a $1 + 1$ spacetime the vertical plane we get¿:

$$nY = Y^r e_r \tag{4}$$

The contravariant shift vector component Y^{rs} is then:

$$Y^{rs} = 2v_s f(rs) \tag{5}$$

Remember that we now defines the y axis as the axis of motion.Inside the bubble $f(rs) = 0$ resulting in a $Y^{rs} = 0$ and outside the bubble $f(rs) = \frac{1}{2}$ resulting in a $Y^{rs} = vs$ and this illustrates the Natario definition for a warp drive spacetime.(see pg 4 in [1]).(see Appendix D about Polar Coordinates).

3 The equation of the new Natario warp drive vector nY in $2D$ polar coordinates over the y-axis with a variable speed vs and a constant acceleration a

The equation of the new Natario vector nY is given by:

$$nY = Y^t e_t + Y^r e_r + Y^\theta e_\theta \tag{6}$$

The contravariant shift vector components Y^t,Y^{rs} and Y^θ of the Natario vector are defined by(see Appendices B and C for details):

$$Y^t = 2f(rs)rs\sin\theta a \tag{7}$$

$$Y^{rs} = 2[2f(rs)^2 + rsf'(rs)]at\sin\theta \tag{8}$$

$$Y^\theta = 2f(rs)at[2f(rs) + rsf'(rs)]\cos\theta \tag{9}$$

Considering a valid $f(rs)$ as a shape function being $f(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $f(rs) = 0$ for small rs(inside the warp bubble) while being $0 < f(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]):

We must demonstrate that the Natario warp drive vector given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nY generates a warp drive spacetime if $nY = 0$ and $Y = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nY = vs(t) * dy + y * dvs$ with $Y = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])(see Appendices G and H for an explanation about this statement)

Natario in its warp drive uses the polar coordinates rs and θ.In order to simplify our analysis we consider motion in the $y - axis$ or the vertical plane rs where $\theta = 90$ $\sin(\theta) = 1$ and $\cos(\theta) = 0$.(see pgs 4,5 and 6 in [1]).

In a $1 + 1$ spacetime the vertical plane we get¿:

$$nY = Y^t e_t + Y^r e_r \tag{10}$$

$$Y^t = 2f(rs)rsa \tag{11}$$

$$Y^{rs} = 2[2f(rs)^2 + rsf'(rs)]at \tag{12}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2f(rs)at \tag{13}$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.In this case y is now the axis of motion.Inside the bubble $f(rs) = 0$ resulting in a $vs = 0$ and outside the bubble $f(rs) = \frac{1}{2}$ resulting in a $vs = at$ as expected from a variable velocity vs in time t due to a constant acceleration a.Since inside and outside the bubble $f(rs)$ always possesses the same values of 0 or $\frac{1}{2}$ then the derivative $f'(rs)$ of the Natario shape function $f(rs)$ is zero and the shift vector $Y^{rs} = 2[2n(rs)^2]at$ with $Y^{rs} = 0$ inside the bubble and $Y^{rs} = 2[2f(rs)^2]at = 2[2\frac{1}{4}]at = at = vs$ outside the bubble and this illustrates the Natario definition for a warp drive spacetime.(see Appendix D about Polar Coordinates).

4 The equation of the new warp drive vector nY in tridimensional $3D$ spherical coordinates over the y-axis with a constant speed vs

The equation of the new warp drive vector in tridimensional $3D$ spherical coordinates with a constant speed vs nY is given by:

$$nY = Y^r e_r + Y^\theta e_\theta + Y^\phi e_\phi \tag{14}$$

The corresponding contravariant shift vectors are:(see Appendix J for details)

$$Y^r = [2f(r)]vs(t)sin\phi \sin\theta \tag{15}$$

$$Y^\theta = vs(t)sin\phi[2f(r) + rf'(r)]\cos\theta \tag{16}$$

$$Y^\phi = vs(t)[2f(r) + rf'(r)]cos\phi \tag{17}$$

Considering a valid $f(r)$ as a shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble) and $f(r) = 0$ for small rs(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the warped region(pg 5 in [1]):

We must demonstrate that our new warp drive vector satisfies the Natario criteria for a warp drive defined by:

any warp drive vector nY generates a warp drive spacetime if $nY = 0$ and $Y = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nY = vs(t) * dy$ with $Y = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])(see Appendices G and H for an explanation about this statement).

Natario in its warp drive uses the polar coordinates r and θ.In order to simplify our analysis we consider motion in the $y-axis$ the vertical plane $x-y$ in r where $\theta = 90$ $\sin(\theta) = 1$ and $\cos(\theta) = 0$.(see pgs 4 and 5 in [1]).Also the vertical plane $x-y$ makes an angle of 90 degrees with the $z-axis$ so $\sin\phi = 1$ and $cos\phi = 0$.

Then the contravariant components reduces to:

$$Y^r = vs(t)[\sin\phi][2f(r)\sin\theta] \rightarrow Y^r = vs(t)[2f(r)] \rightarrow \sin\phi = 1 \rightarrow \sin\theta = 1 \tag{18}$$

$$Y^\theta = vs(t)[\sin\phi][2f(r) + rf'(r)]\cos\theta] = 0 \rightarrow \sin\phi = 1 \rightarrow \cos\theta = 0 \tag{19}$$

$$Y^\phi = [vs(t)cos\phi][2(f(r)) + (rf'(r))] = 0 \rightarrow cos\phi = 0 \tag{20}$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.Now the y-axis is the axis of motion.Inside the bubble $f(r) = 0$ resulting in a $Y^r = 0$ and outside the bubble $f(r) = \frac{1}{2}$ resulting in a $Y^r = vs$ and this illustrates the Natario definition for a warp drive spacetime.(See Appendix E about Spherical Coordinates).

5 The equation of the new warp drive vector nY in tridimensional $3D$ spherical coordinates over the y-axis with a variable speed vs due to a constant acceleration a

The equation of the new warp drive vector in tridimensional $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a nY is given by:

$$nY = Y^t e_t + Y^r e_r + Y^\theta e_\theta + Y^\phi e_\phi \tag{21}$$

With the contravariant shift vector components Y^t,Y^{rs}, Y^θ and Y^ϕ given by:(see Appendices K and L for details)

$$Y^t = 2(rf(r)a))(\sin\phi)(\sin\theta) \tag{22}$$

$$Y^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\sin\theta) \tag{23}$$

$$Y^\theta = (2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\cos\theta) \tag{24}$$

$$Y^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi) \tag{25}$$

Considering a valid $f(r)$ as a shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble) and $f(r) = 0$ for small rs(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the warped region(pg 5 in [1]):

We must demonstrate that our warp drive vector satisfies the Natario criteria for a warp drive defined by:

any warp drive vector nY generates a warp drive spacetime if $nY = 0$ and $Y = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nY = vs(t) * dy + y * dvs(t)$ with $Y = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])(see Appendices G and H for an explanation about this statement)

Natario in its warp drive uses the polar coordinates r and θ.In order to simplify our analysis we consider motion in the $y-axis$ or the vertical plane $x-y$ in r where $\theta = 90$ $\sin(\theta) = 1$ and $\cos(\theta) = 0$.(see pgs 4,5 and 6 in [1]).Also the vertical plane $x - y$ makes an angle of 90 degrees with the $z - axis$ so $\sin\phi = 1$ and $cos\phi = 0$.Then the contravariant components reduces to:

$$Y^t = 2(rf(r)a))(\sin\phi)(\sin\theta) \rightarrow Y^t = 2(rf(r)a)) \rightarrow \sin\phi = 1 \rightarrow \sin\theta = 1 \tag{26}$$

$$Y^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\sin\theta) \rightarrow Y^r = (2at)[2f(r)^2 + (rf'(r))] \rightarrow \sin\phi = 1 \rightarrow \sin\theta = 1 \tag{27}$$

$$Y^\theta = (2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\cos\theta) = 0 \rightarrow \sin\phi = 1 \rightarrow \cos\theta = 0 \tag{28}$$

$$Y^{\phi} = (2f(r)at)[2f(r) + (rf'(r))](cos\phi) = 0 \rightarrow \cos\phi = 0 \tag{29}$$

The remaining contravariant components are:

$$Y^{t} = 2(rf(r)a))(\sin\phi)(\sin\theta) \rightarrow Y^{t} = 2(rf(r)a)) \rightarrow \sin\phi = 1 \rightarrow \sin\theta = 1 \tag{30}$$

$$Y^{r} = (2at)[2f(r)^{2} + (rf'(r))](\sin\phi)(\sin\theta) \rightarrow Y^{r} = (2at)[2f(r)^{2} + (rf'(r))] \rightarrow \sin\phi = 1 \rightarrow \sin\theta = 1 \tag{31}$$

$$nY = Y^{t}e_{t} + Y^{r}e_{r} \tag{32}$$

$$Y^{t} = 2rf(r)a \tag{33}$$

$$Y^{rs} = 2[2f(r)^{2} + rf'(r)]at \tag{34}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2f(r)at \tag{35}$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.Now the axis of motion is y.Inside the bubble $f = 0$ resulting in a $vs = 0$ and outside the bubble $f = \frac{1}{2}$ resulting in a $vs = at$ as expected from a variable velocity vs in time t due to a constant acceleration a.Since inside and outside the bubble $f(r)$ always possesses the same values of 0 or $\frac{1}{2}$ then the derivative $f'(r)$ of the shape function $f(r)$ is zero and the shift vector $Y^{rs} = 2[2f(r)^{2}]at$ with $Y^{r} = 0$ inside the bubble and $Y^{rs} = 2[2f(r)^{2}]at = 2[2\frac{1}{4}]at = at = vs$ outside the bubble and this illustrates the Natario definition for a warp drive spacetime.See Appendix E about Spherical Coordinates.

6 Conclusion

In this work we introduced a new tridimensional $3D$ spherical coordinates warp drive vector nY with the Hodge Star based over the y-axis using the Natario mathematical techniques.We focused ourselves in the application of the Hodge Star in $3D$ spherical coordinates for both constant and variable speeds.

Natario used Polar Coordinates(See pg 4 in [1]) and a Hodge Star based over the x-axis and in this work we computed Hodge Stars in both Polar and Spherical Coordinates for the y-axis.

For a real $3D$ Spherical Coordinates a new warp drive vector must be calculated.Remember that a real spaceship is a tridimensional $3D$ object inserted inside a tridimensional $3D$ warp bubble that must be defined using all the tridimensional $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.(see Appendix E about tridimensional $3D$ Spherical Coordinates).

Polar Coordinates are not real tridimensional $3D$ coordinates since it uses only the two Canonical Basis $\mathbf{e}_r$ and $\mathbf{e}_\theta$.(see Appendix D about $2D$ Polar Coordinates).

The Hodge Star actually must be taken considering variable speeds.In this work we examine what happens with the warp drive vector when the velocity is variable.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.

Our focus was concentrated in the Natario methods to obtain a warp drive vector.We know that we used a language and a presentation method or style that may be regarded as exhaustive tedious and monotonous for experienced or seasoned readers but we are concerned about beginners,newcomers,novices or intermediate students not familiarized with the techniques Natario used to develop warp drive vectors so our extensive mathematical demonstrations QED Quod Erad Demonstratum will benefit this audience at least we hope.We gave our best efforts trying to accomplish this goal but only this audience will tell in the future if we succeeded (or not).

The application of the new tridimensional $3D$ spherical coordinates warp drive vector wether in constant or variable speeds to the ADM(Arnowitt-Dresner-Misner) formalism equations in General Relativity using the approach of MTW(Misner-Thorne-Wheeler)resembling the works [10],[11][12] and [13] will appear in a future work.

The Natario warp drive is probably the best candidate(known until now) for an interstellar space travel considering the fact that a spaceship in a real superluminal spaceflight will encounter(or collide against) hazardous objects(asteroids,comets,interstellar dust and debris etc) and the Natario spacetime offers an excellent protection to the crew members as depicted in the works [18],[19],[20] and [21].

7 Appendix A:differential forms,Hodge star and the mathematical demonstration of the Natario vector $nY = vs * dy$ for a constant speed vs over the y-axis in Polar Coordinates in a R^3 space basis

This appendix is also being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods we used to arrive at the final expression of the Natario Vector nY

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eq 3.72 pg 69$(a)(b)$ in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{36}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \tag{37}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{38}$$

From above we get the following results

$$dr \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{39}$$

$$rd\theta \sim r\sin\theta(d\varphi \wedge dr) \tag{40}$$

$$r\sin\theta d\varphi \sim r(dr \wedge d\theta) \tag{41}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(see eq 3.72 pg 69$(a)(b)$ in [2]):

$$*dr = r^2 \sin\theta(d\theta \wedge d\varphi) \tag{42}$$

$$*rd\theta = r\sin\theta(d\varphi \wedge dr) \tag{43}$$

$$*r\sin\theta d\varphi = r(dr \wedge d\theta) \tag{44}$$

Look that

$$dy = d(r\sin\theta) = \sin\theta dr + r\cos\theta d\theta \tag{45}$$

Or

$$dy = d(r\sin\theta) = \sin\theta dr + \cos\theta r d\theta \tag{46}$$

Applying the Hodge Star operator * to the above expression:

$$*dy = *d(r\sin\theta) = \sin\theta(*dr) + \cos\theta(*rd\theta) \tag{47}$$

From

$$*dr = r^2\sin\theta(d\theta \wedge d\varphi) \tag{48}$$

$$*rd\theta = r\sin\theta(d\varphi \wedge dr) \tag{49}$$

We have:

$$*dy = *d(r\sin\theta) = \sin\theta[r^2\sin\theta(d\theta \wedge d\varphi)] + \cos\theta[r\sin\theta(d\varphi \wedge dr)] \tag{50}$$

$$*dy = *d(r\sin\theta) = [r^2\sin^2\theta(d\theta \wedge d\varphi)] + [r\sin\theta\cos\theta(d\varphi \wedge dr)] \tag{51}$$

$$*dy = *d(r\sin\theta) = \sin\theta e_r + \cos\theta e_\theta \tag{52}$$

Now examining the expression:

$$d\left(\frac{1}{2}\sin^2\theta d\varphi\right) \tag{53}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}\sin^2\theta d\varphi\right) \tag{54}$$

$$*d\left(\frac{1}{2}\sin^2\theta d\varphi\right) \sim \frac{1}{2} * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [dd\varphi] \tag{55}$$

According to eq 3.90 pg $74(a)(b)$ in [2] the term $\frac{1}{2}\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2} * d[(\sin^2\theta)d\varphi] \sim \frac{1}{2}2\sin\theta\cos\theta(d\theta \wedge d\varphi) \tag{56}$$

$$\frac{1}{2} * d[(\sin^2\theta)d\varphi] \sim \frac{1}{2} 2\sin\theta\cos\theta(d\theta\wedge d\varphi) \tag{57}$$

$$\frac{1}{2} * d[(\sin^2\theta)d\varphi] \sim \sin\theta\cos\theta(d\theta\wedge d\varphi) \tag{58}$$

Because and according to eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.2 pg 68$(a)(b)$ in [2]:

$$*d(\alpha+\beta) = d\alpha + d\beta \tag{59}$$

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \tag{60}$$

$$*d(dx) = d(dy) = d(dz) = 0 \tag{61}$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = d(\sin^2\theta)\wedge d\varphi + \sin^2\theta\wedge dd\varphi = 2sin\theta\cos\theta(d\theta\wedge d\varphi) \tag{62}$$

Now examining the expression:

$$[(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] \tag{63}$$

$$[(r^2)(\tan\theta)][\sin\theta\cos\theta(d\theta\wedge d\varphi)] = [(r^2)(\frac{\sin\theta}{\cos\theta})][\sin\theta\cos\theta(d\theta\wedge d\varphi)] \tag{64}$$

$$[(r^2)(\frac{\sin\theta}{\cos\theta})][\sin\theta\cos\theta(d\theta\wedge d\varphi)] = [(r^2)][\sin^2\theta(d\theta\wedge d\varphi)] = \sin\theta e_r \tag{65}$$

Now examining the expression:

$$d\left(\frac{1}{2}r^2 d\varphi\right) \tag{66}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2 d\varphi\right) \tag{67}$$

$$*d\left(\frac{1}{2}r^2 d\varphi\right) \sim \frac{1}{2} * [d(r^2)d\varphi] + \frac{1}{2}r^2 * d[(d\varphi)] \tag{68}$$

According to eq 3.90 pg 74$(a)(b)$ in [2] the term $\frac{1}{2}r^2 * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2} * [d(r^2)d\varphi] \sim \frac{1}{2} 2r(dr\wedge d\varphi) \tag{69}$$

Because and according to eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.2 pg 68$(a)(b)$ in [2]:

$$*d(\alpha + \beta) = d\alpha + d\beta \tag{70}$$

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df \wedge \alpha + f \wedge d\alpha \tag{71}$$

$$*d(dx) = d(dy) = d(dz) = 0 \tag{72}$$

From above we can see for example that

$$*[d(r^2)d\varphi] = 2rdr \wedge d\varphi + r^2 \wedge dd\varphi = 2r(dr \wedge d\varphi) \tag{73}$$

$$\frac{1}{2} * [d(r^2)d\varphi] \sim \frac{1}{2}2r(dr \wedge d\varphi) \sim r(dr \wedge d\varphi) = r(dr \wedge d\varphi) = -r(d\varphi \wedge dr) \tag{74}$$

We know that the following expression holds true(see eq 3.79 pg 70$(a)(b)$ in [2]):

$$d\varphi \wedge dr = -dr \wedge d\varphi \tag{75}$$

$$\frac{1}{2} * [d(r^2)d\varphi] \sim -r(d\varphi \wedge dr) \tag{76}$$

Now examining the expression:

$$(-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] = (-1)(\sin\theta)(\cos\theta)[-r(d\varphi \wedge dr)] \tag{77}$$

$$(-1)(\sin\theta)(\cos\theta)[-r(d\varphi \wedge dr)] = [r\sin\theta\cos\theta(d\varphi \wedge dr)] = \cos\theta e_\theta \tag{78}$$

Combining the expressions:

$$[(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] = \sin\theta e_r \tag{79}$$

and

$$(-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] = \cos\theta e_\theta \tag{80}$$

As being

$$[(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] + (-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] \tag{81}$$

We obtain the same result of the Hodge Star for the y-axis

$$*dy = *d(r\sin\theta) = [r^2\sin^2\theta(d\theta \wedge d\varphi)] + [r\sin\theta\cos\theta(d\varphi \wedge dr)] \tag{82}$$

$$*dy = *d(r\sin\theta) = \sin\theta e_r + \cos\theta e_\theta \tag{83}$$

Then we have:

$$*dy = *d(r\sin\theta) = \sin\theta e_r + \cos\theta e_\theta \tag{84}$$

$$*dy = [(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] + (-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] \tag{85}$$

Now using the following expression:

$$[2f(r)][(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] + [2f(r) + rf'(r)](-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] \tag{86}$$

With these ones:

$$[(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] = \sin\theta e_r \tag{87}$$

$$(-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] = \cos\theta e_\theta \tag{88}$$

We have finally

$$[2f(r)][(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] + [2f(r) + rf'(r)](-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] \tag{89}$$

$$[2f(r)]\sin\theta e_r + [2f(r) + rf'(r)]\cos\theta e_\theta \tag{90}$$

Defining the new Natario vector nY with the Hodge Star operator * explicitly resolved :

$$nY = vs(t)[2f(r)]\sin\theta e_r + vs[2f(r) + rf'(r)]\cos\theta e_\theta \tag{91}$$

$$nY = 2vs(t)f(r)\ sin\theta e_r + vs(t)[2f(r) + rf'(r)]\cos\theta e_\theta \tag{92}$$

compare the new Natario vector nY with the original Natario vector nX pg 5 in [1]:

$$nX = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{93}$$

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{94}$$

$$nY = 2vs(t)f(r)\ sin\theta e_r + vs(t)[2f(r) + rf'(r)]\cos\theta e_\theta \tag{95}$$

Do they look familiar ?

$$nY = Y^r e_r + Y^\theta e_\theta \tag{96}$$

$$Y^{rs} = 2v_s f(rs)\sin\theta \tag{97}$$

$$Y^\theta = +v_s(2f(rs) + (rs)f'(rs))\cos\theta \tag{98}$$

8 Appendix B:differential forms,Hodge star and the mathematical demonstration of the Natario vector $nY = vs * dy$ for a constant speed vs or for the first term $vs*dy$ from the Natario vector $nY = vs*dy+y*dvs$(a variable speed) in a R^4 space basis-Polar Coordinates

This appendix is also being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods we used to arrive at the final expression of the Natario Vector nY

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eqs 3.135 and 3.137 pg 82(a)(b) in [2],eq 3.74 pg 69(a)(b) in [2])(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3])(see also eqs 4.55 and 4.56 pg 179 in [8]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim dt \wedge (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) \tag{99}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim dt \wedge (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(dt \wedge d\varphi \wedge dr) \tag{100}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dt \wedge dr \wedge (rd\theta) \sim r(dt \wedge dr \wedge d\theta) \tag{101}$$

From above we get the following results

$$dr \sim r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) \tag{102}$$

$$rd\theta \sim r\sin\theta(dt \wedge d\varphi \wedge dr) \tag{103}$$

$$r\sin\theta d\varphi \sim r(dt \wedge dr \wedge d\theta) \tag{104}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(see eq 3.74 pg 69(a)(b) in [2])(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3])(see also eqs 4.55 and 4.56 pg 179 in [8]):

$$*dr = r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) \tag{105}$$

$$*rd\theta = r\sin\theta(dt \wedge d\varphi \wedge dr) \tag{106}$$

$$*r\sin\theta d\varphi = r(dt \wedge dr \wedge d\theta) \tag{107}$$

Look that

$$dy = d(r \sin\theta) = \sin\theta dr + r\cos\theta d\theta \tag{108}$$

Or

$$dy = d(r \sin\theta) = \sin\theta dr + \cos\theta r d\theta \tag{109}$$

Applying the Hodge Star operator * to the above expression:

$$*dy = *d(r \sin\theta) = \sin\theta(*dr) + \cos\theta(*rd\theta) \tag{110}$$

From

$$*dr = r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) \tag{111}$$

$$*rd\theta = r\sin\theta(dt \wedge d\varphi \wedge dr) \tag{112}$$

We have:

$$*dy = *d(r \sin\theta) = \sin\theta[r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi)] + \cos\theta[r\sin\theta(dt \wedge d\varphi \wedge dr)] \tag{113}$$

$$*dy = *d(r \sin\theta) = [r^2 \sin^2\theta(dt \wedge d\theta \wedge d\varphi)] + [r\sin\theta\cos\theta(dt \wedge d\varphi \wedge dr)] \tag{114}$$

$$*dy = *d(r \sin\theta) = \sin\theta e_r + \cos\theta e_\theta \tag{115}$$

Now examining the expression:

$$d\left(\frac{1}{2}\sin^2\theta d\varphi\right) \tag{116}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}\sin^2\theta d\varphi\right) \tag{117}$$

$$*d\left(\frac{1}{2}\sin^2\theta d\varphi\right) \sim \frac{1}{2} * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [dd\varphi] \tag{118}$$

According to eq 3.90 pg 74$(a)(b)$ in [2] the term $\frac{1}{2}\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2} * d[(\sin^2\theta)d\varphi] \sim \frac{1}{2}2\sin\theta\cos\theta(dt \wedge d\theta \wedge d\varphi) \tag{119}$$

$$\frac{1}{2} * d[(\sin^2\theta)d\varphi] \sim \frac{1}{2} 2\sin\theta\cos\theta(dt \wedge d\theta \wedge d\varphi) \tag{120}$$

$$\frac{1}{2} * d[(\sin^2\theta)d\varphi] \sim \sin\theta\cos\theta(dt \wedge d\theta \wedge d\varphi) \tag{121}$$

Because and according to eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.2 pg 68$(a)(b)$ in [2]:

$$*d(\alpha + \beta) = d\alpha + d\beta \tag{122}$$

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 3 \dashrightarrow *d(f\alpha) = df \wedge \alpha - f \wedge d\alpha \tag{123}$$

$$*d(dx) = d(dy) = d(dz) = 0 \tag{124}$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = dt \wedge d(\sin^2\theta) \wedge d\varphi - \sin^2\theta dt \wedge dd\varphi = 2sin\theta\cos\theta(dt \wedge d\theta \wedge d\varphi) \tag{125}$$

Now examining the expression:

$$[(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] \tag{126}$$

$$[(r^2)(\tan\theta)][\sin\theta\cos\theta(dt \wedge d\theta \wedge d\varphi)] = [(r^2)(\frac{\sin\theta}{\cos\theta})][\sin\theta\cos\theta(dt \wedge d\theta \wedge d\varphi)] \tag{127}$$

$$[(r^2)(\frac{\sin\theta}{\cos\theta})][\sin\theta\cos\theta(dt \wedge d\theta \wedge d\varphi)] = [(r^2)][\sin^2\theta(dt \wedge d\theta \wedge d\varphi)] = \sin\theta e_r \tag{128}$$

Now examining the expression:

$$d\left(\frac{1}{2}r^2 d\varphi\right) \tag{129}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2 d\varphi\right) \tag{130}$$

$$*d\left(\frac{1}{2}r^2 d\varphi\right) \sim \frac{1}{2} * [d(r^2)d\varphi] + \frac{1}{2}r^2 * d[(d\varphi)] \tag{131}$$

According to eq 3.90 pg 74$(a)(b)$ in [2] the term $\frac{1}{2}r^2 * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2} * [d(r^2)d\varphi] \sim \frac{1}{2} 2r(dt \wedge dr \wedge d\varphi) \tag{132}$$

Because and according to eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.2 pg 68$(a)(b)$ in [2]:

$$*d(\alpha+\beta)=d\alpha+d\beta \tag{133}$$

$$*d(f\alpha)=df\wedge\alpha+(-1)^p f\wedge d\alpha \dashrightarrow p=3 \dashrightarrow *d(f\alpha)=df\wedge\alpha-f\wedge d\alpha \tag{134}$$

$$*d(dx)=d(dy)=d(dz)=0 \tag{135}$$

From above we can see for example that

$$*[d(r^2)d\varphi]=2rdt\wedge dr\wedge d\varphi-r^2dt\wedge dd\varphi=2r(dt\wedge dr\wedge d\varphi) \tag{136}$$

$$\frac{1}{2}*[d(r^2)d\varphi]\sim\frac{1}{2}2r(dt\wedge dr\wedge d\varphi)\sim r(dt\wedge dr\wedge d\varphi)=r(dt\wedge dr\wedge d\varphi)=-r(dt\wedge d\varphi\wedge dr) \tag{137}$$

We know that the following expression holds true(see eq 3.79 pg 70$(a)(b)$ in [2]):

$$dt\wedge d\varphi\wedge dr=-dt\wedge dr\wedge d\varphi \tag{138}$$

$$\frac{1}{2}*[d(r^2)d\varphi]\sim -r(dt\wedge d\varphi\wedge dr) \tag{139}$$

Now examining the expression:

$$(-1)(\sin\theta)(\cos\theta)\frac{1}{2}*[d(r^2)d\varphi]=(-1)(\sin\theta)(\cos\theta)[-r(dt\wedge d\varphi\wedge dr)] \tag{140}$$

$$(-1)(\sin\theta)(\cos\theta)[-r(dt\wedge d\varphi\wedge dr)]=[r\sin\theta\cos\theta(dt\wedge d\varphi\wedge dr)]=\cos\theta e_\theta \tag{141}$$

Combining the expressions:

$$[(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)]=\sin\theta e_r \tag{142}$$

and

$$(-1)(\sin\theta)(\cos\theta)\frac{1}{2}*[d(r^2)d\varphi]=\cos\theta e_\theta \tag{143}$$

As being

$$[(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)]+(-1)(\sin\theta)(\cos\theta)\frac{1}{2}*[d(r^2)d\varphi] \tag{144}$$

We obtain the same result of the Hodge Star for the y-axis

$$*dy=*d(r\sin\theta)=[r^2\sin^2\theta(dt\wedge d\theta\wedge d\varphi)]+[r\sin\theta\cos\theta(dt\wedge d\varphi\wedge dr)] \tag{145}$$

$$*dy=*d(r\sin\theta)=\sin\theta e_r+\cos\theta e_\theta \tag{146}$$

Then we have:

$$*dy = *d(r\sin\theta) = \sin\theta e_r + \cos\theta e_\theta \tag{147}$$

$$*dy = [(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] + (-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] \tag{148}$$

Now using the following expression:

$$[2f(r)][(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] + [2f(r) + rf'(r)](-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] \tag{149}$$

With these ones:

$$[(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] = \sin\theta e_r \tag{150}$$

$$(-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] = \cos\theta e_\theta \tag{151}$$

We have finally

$$[2f(r)][(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] + [2f(r) + rf'(r)](-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] \tag{152}$$

$$[2f(r)]\sin\theta e_r + [2f(r) + rf'(r)]\cos\theta e_\theta \tag{153}$$

Defining the new Natario vector nY with the Hodge Star operator * explicitly resolved :

$$nY = vs(t)[2f(r)]\sin\theta e_r + vs[2f(r) + rf'(r)]\cos\theta e_\theta \tag{154}$$

$$nY = 2vs(t)f(r)\ sin\theta e_r + vs(t)[2f(r) + rf'(r)]\cos\theta e_\theta \tag{155}$$

compare the new Natario vector nY with the original Natario vector nX pg 5 in [1]:

$$nX = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{156}$$

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{157}$$

$$nY = 2vs(t)f(r)\ sin\theta e_r + vs(t)[2f(r) + rf'(r)]\cos\theta e_\theta \tag{158}$$

Do they look familiar ?

$$nY = Y^r e_r + Y^\theta e_\theta \tag{159}$$

$$Y^{rs} = 2v_s f(rs)\sin\theta \tag{160}$$

$$Y^\theta = +v_s(2f(rs) + (rs)f'(rs))\cos\theta \tag{161}$$

9 Appendix C:differential forms,Hodge star and the mathematical demonstration of the Natario vector $nY = *(vsy) = vs*dy+y*dvs$ for a variable speed vs and a constant acceleration a in Polar Coordinates

any Natario vector nY generates a warp drive spacetime if $nY = 0$ and $Y = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nY = vs(t) * dy$ with $Y = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])(see Appendix G for an explanation about this statement)

In the Appendices A and B we gave the mathematical demonstration of the Natario vector $nY = vs * dy$ in the R^3 and R^4 space basis when the velocity vs is constant.Hence the complete expression of the Hodge star that generates the Natario vector nY for a constant velocity vs is given by:

$$nY = *(vsy) = vs * (dy) \tag{162}$$

The equation of the Natario vector nY for a constant velocity vs is given by:

$$nY = 2vs(t)f(r)\ sin\theta e_r + vs(t)[2f(r) + rf'(r)]\cos\theta e_\theta \tag{163}$$

$$nY = Y^r e_r + Y^\theta e_\theta \tag{164}$$

With the contravariant shift vector components explicitly given by:

$$Y^{rs} = 2v_s f(rs)\sin\theta \tag{165}$$

$$Y^\theta = +v_s(2f(rs) + (rs)f'(rs))\cos\theta \tag{166}$$

Because due to a constant speed vs the term $y * d(vs) = 0$.Now we must examine what happens when the velocity is variable and then the term $y * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.The complete expression of the Hodge star that generates the Natario vector nY for a variable velocity vs is now given by:

$$nY = *(vsy) = vs * (dy) + y * (dvs) \tag{167}$$

In order to study the term $y * d(vs)$ we must introduce a new Canonical Basis for the coordinate time in the R^4 space basis defined as follows:(see eqs 10.102 and 10.103 pgs 363$(a)(b)$ and 364$(a)(b)$ in [2] with the terms $S = u = 1$[1],eq 3.74 pg 69$(a)(b)$ in [2],eqs 11.131 and 11.133 with the term $m = 0$[2] pg 417$(a)(b)$ in [2].)(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3])(see also eqs 4.55 and 4.56 pg 179 in [8]):

$$e_t \equiv \frac{\partial}{\partial t} \sim dt \sim dr \wedge (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(dr \wedge d\theta \wedge d\varphi) \tag{168}$$

$$dt \sim r^2 \sin\theta(dr \wedge d\theta \wedge d\varphi) \tag{169}$$

[1]These terms are needed to deal with the Robertson-Walker equation in Cosmology using differential forms.We dont need these terms here and we can make $S = u = 1$

[2]This term is needed to describe the Dirac equation in the Schwarzschild spacetime we dont need the term here so we can make $m = 1$.Remember also that here we consider geometrized units in which $c = 1$

The Hodge star operator defined for the coordinate time is given by:(see eq 3.74 pg 69(a)(b) in [2])(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3])(see also eqs 4.55 and 4.56 pg 179 in [8]):

$$*dt = r^2 \sin\theta(dr \wedge d\theta \wedge d\varphi) \tag{170}$$

The valid expression for a variable velocity $vs(t)$ in the Natario warp drive spacetime due to a constant acceleration a must be given by:

$$vs = 2f(r)at \tag{171}$$

Because and considering a valid $f(r)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble where $Y = vs(t)$ and $nY = vs(t) * dy + y * d(vs(t))$) and $f(r) = 0$ for small r(inside the warp bubble where $Y = 0$ and $nY = 0$) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pgs 4 and 5 in [1]) and considering also that the Natario warp drive is a ship-frame based coordinates system(a reference frame placed in the center of the warp bubble where the ship resides-or must reside!!) then an observer in the ship inside the bubble sees every point inside the bubble at the rest with respect to him because inside the bubble $vs(t) = 0$ because $f(r) = 0$.

To illustrate the statement pointed above imagine a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The stream varies its velocity with time.The warp bubble in this case is the aquarium and the walls of the aquarium are the walls of the warp bubble-Natario warped region.An observer in the margin of the river would see the aquarium passing by him at a large speed considering a coordinates system(a reference frame) placed in the margin of the river but inside the aquarium the fish is at the rest with respect to his local neighborhoods.Then for the fish any point inside the aquarium is at the rest with respect to him because inside the aquarium $vs = 2f(r)at$ with $f(r) = 0$ and consequently giving a $vs(t) = 0$.Again with respect to the fish the fish "sees" the margin passing by him with a large relative velocity.The margin in this case is the region outside the bubble "seen" by the fish with a variable velocity $vs(t) = v1$ in the time $t1$ and $vs(t) = v2$ in the time $t2$ because outside the bubble the generic expression for a variable velocity vs is given by $vs = 2f(r)at$ and outside the bubble $f(r) = \frac{1}{2}$ giving a generic expression for a variable velocity vs as $vs(t) = at$ and consequently a $v1 = at1$ in the time $t1$ and a $v2 = at2$ in the time $t2$.Then the variable velocity in not only a function of time alone but must consider also the position of the bubble where the measure is being taken wether inside or outside the bubble.So the velocity must also be a function of r.Its total differential is then given by:

$$dvs = 2[atf'(r)dr + f(r)tda + f(r)adt] \tag{172}$$

Applying the Hodge star to the total differential dvs we get:

$$*dvs = 2[atf'(r) * dr + f(r)t * da + f(r)a * dt] \tag{173}$$

But we consider here the acceleration a a constant.Then the term $f(r)tda = 0$ and in consequence $f(r)t * da = 0$.This leaves us with:

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] \tag{174}$$

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] = 2[atf'(r)r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) + f(r)ar^2 \sin\theta(dr \wedge d\theta \wedge d\varphi)] \tag{175}$$

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] = 2[atf'(r)e_r + f(r)ae_t] \tag{176}$$

The complete expression of the Hodge star that generates the Natario vector nY for a variable velocity vs is given by:

$$nY = *(vsy) = vs * (dy) + y * d(vs) \tag{177}$$

The term $vs * dy$ was obtained in the Appendices A and B as follows:

$$vs * (dy) = 2vs(t)f(r)\ sin\theta e_r + vs(t)[2f(r) + rf'(r)] \cos\theta e_\theta \tag{178}$$

$$*dy = 2f(r)\ sin\theta e_r + [2f(r) + rf'(r)] \cos\theta e_\theta \tag{179}$$

The complete expression of the Hodge star that generates the Natario vector nY for a variable velocity vs is now given by:

$$nY = *(vsy) = vs(2f(r)\ sin\theta e_r + [2f(r) + rf'(r)] \cos\theta e_\theta) + y(2[atf'(r)e_r + f(r)ae_t]) \tag{180}$$

But remember that we are in polar coordinates(pg 4 in [1]) in which $y = rsin\theta$(see pg 5 in [1]) (see also Appendix D)and this leaves us with:

$$nY = *(vsy) = vs(2f(r)\sin\theta e_r + [2f(r) + rf'(r)] \cos\theta e_\theta) + rsin\theta(2[atf'(r)e_r + f(r)ae_t]) \tag{181}$$

But we know that $vs = 2f(r)at$.Hence we get:

$$nY = *(vsy) = 2f(r)at(2f(r)\sin\theta e_r + [2f(r) + rf'(r)] \cos\theta e_\theta) + rsin\theta(2[atf'(r)e_r + f(r)ae_t]) \tag{182}$$

Then we can start with a warp bubble initially at the rest using the Natario vector shown above and accelerate the bubble to a desired speed of 200 times faster than light.When we achieve the desired speed we turn off the acceleration and keep the speed constant.The terms due to the acceleration now disappears and we are left again with the Natario vector for constant speeds shown below:

$$nY = 2vs(t)f(r)\sin\theta e_r + vs(t)[2f(r) + rf'(r)] \cos\theta e_\theta \tag{183}$$

Working some algebra with the Natario vector for variable velocities we get:

$$nY = *(vsy) = 2f(r)at(2f(r)\sin\theta e_r + [2f(r) + rf'(r)]\cos\theta e_\theta) + r\sin\theta(2[atf'(r)e_r + f(r)ae_t]) \quad (184)$$

$$nY = 4f(r)^2 at\sin\theta e_r + 2f(r)at[2f(r) + rf'(r)]\cos\theta e_\theta + 2atf'(r)r\sin\theta e_r + 2f(r)r\sin\theta ae_t \quad (185)$$

$$nY = 2f(r)r\sin\theta ae_t + 4f(r)^2 at\sin\theta e_r + 2atf'(r)r\sin\theta e_r + 2f(r)at[2f(r) + rf'(r)]\cos\theta e_\theta \quad (186)$$

$$nY = 2f(r)r\sin\theta ae_t + 2[2f(r)^2 + rf'(r)]at\sin\theta e_r + 2f(r)at[2f(r) + rf'(r)]\cos\theta e_\theta \quad (187)$$

Then the Natario vector for variable velocities defined using contravariant shift vector components is given by the following expressions:

$$nY = Y^t e_t + Y^r e_r + Y^\theta e_\theta \quad (188)$$

Or being:

$$nY = 2f(r)r\sin\theta ae_t + 2[2f(r)^2 + rf'(r)]at\sin\theta e_r + 2f(r)at[2f(r) + rf'(r)]\cos\theta e_\theta \quad (189)$$

The contravariant shift vector components are respectively given by the following expressions:

$$Y^t = 2f(r)r\sin\theta a \quad (190)$$

$$Y^r = 2[2f(r)^2 + rf'(r)]at\sin\theta \quad (191)$$

$$Y^\theta = +2f(r)at[2f(r) + rf'(r)]\cos\theta \quad (192)$$

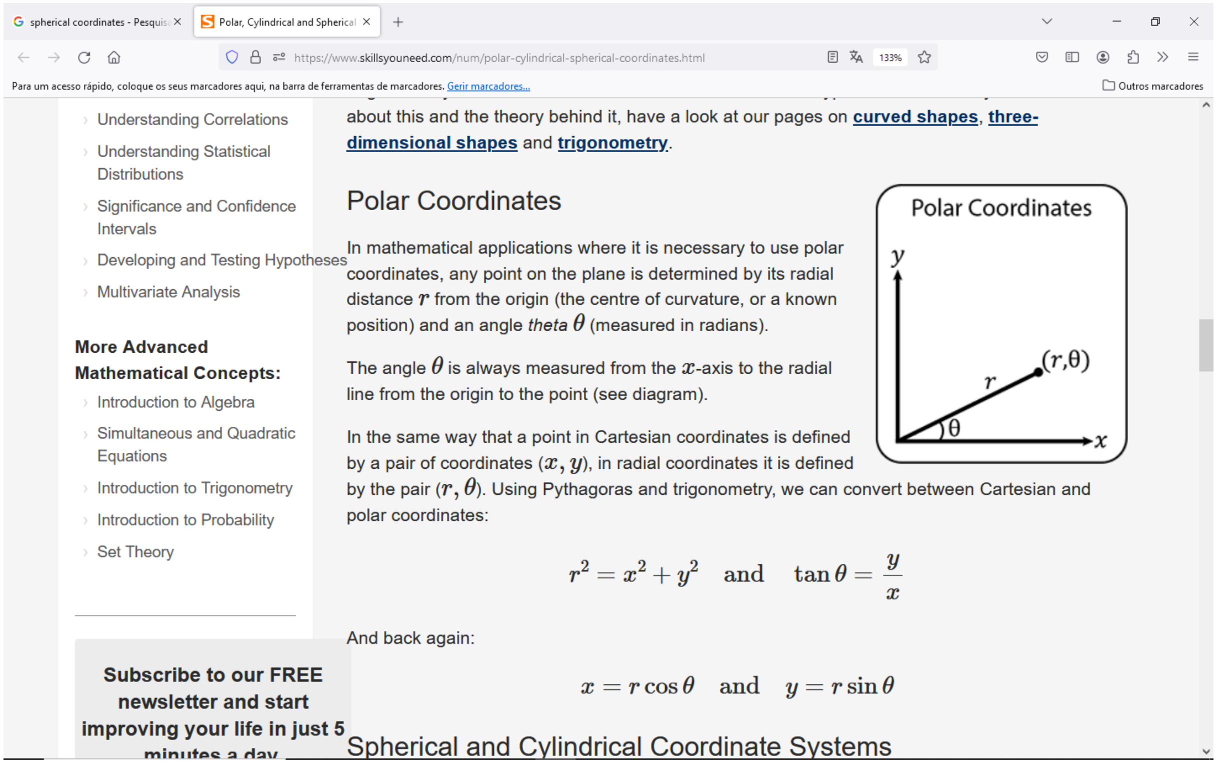

Figure 1: Polar Coordinates.(Source:Internet)

10 Appendix D:Polar Coordinates

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the **constant** speed of the warp bubble and $*(dx) = *d(r\cos\theta)$ is the Hodge Star taken over the x-axis of motion in **Polar Coordinates**(See pg 4 in [1]).(See also Appendices A and B in [9] for the detailed calculations).

We defined a warp drive vector $nY = vs * (dy)$ where vs is the **constant** speed of the warp bubble and $*(dy) = *d(r\sin\theta)$ is the Hodge Star taken over the y-axis of motion in **Polar Coordinates**.(See Appendices A and B for the detailed calculations).

Due to a **constant** speed vs the term $y * d(vs) = 0$.We examined what happens when the velocity is **variable** and then the term $y * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.

The complete expression of the Hodge star that generates the Natario vector nY for a variable velocity vs is now given by $nY = *(vsy) = vs * (dy) + y * (dvs)$(see Appendix C for detailed calculations).

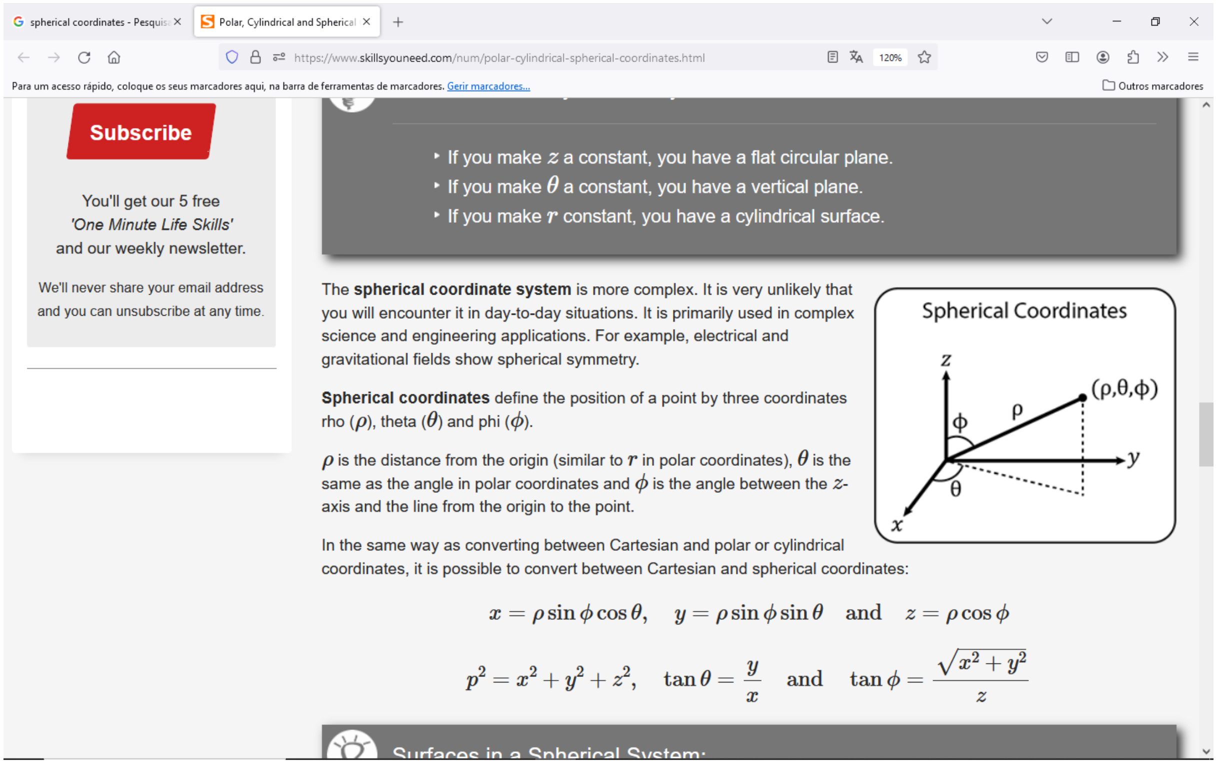

Figure 2: Tridimensional $3D$ Spherical Coordinates.(Source:Internet)

11 Appendix E:Tridimensional $3D$ Spherical Coordinates

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the **constant** speed of the warp bubble and $*(dx) = *d(r\cos\theta)$ is the Hodge Star taken over the x-axis of motion in **Polar Coordinates**(See pg 4 in [1].(See also Appendices D and F).

Note that in this case of Tridimensional $3D$ **Spherical Coordinates** the Hodge Star must be taken no longer over $d(r\cos\theta)$ but instead over $d(\rho\sin\phi\cos\theta)$ and this demands more calculations.Replacing ρ by r we have the following expression for the Hodge Star $*dx = *d(r\sin\phi\cos\theta)$:(see Appendices J and K in [9] for details)

We in this case of Tridimensional $3D$ **Spherical Coordinates** defined the Hodge Star no longer over the x-axis of motion but instead we took the Hodge Star over the y-axis and this means a Hodge Star taken over $d(\rho\sin\phi\sin\theta)$ and this demands more calculations.Replacing ρ by r we have the following expression for the Hodge Star $*dy = *d(r\sin\phi\sin\theta)$.

Our new tridimensional $3D$ **spherical coordinates** warp drive vector in R^3 with **constant speed** vs or in R^4 with **constant speed** vs is given by: $nY = vs * dy = vs * d(r\sin\phi\sin\theta)$.(see Appendices J and K for details)

Due to a **constant** speed vs the term $y * d(vs) = 0$.Now we must examine what happens when the velocity is **variable** and then the term $y * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.The complete expression of the Hodge star that generates the warp drive vector nY in tridimensional $3D$ **spherical coordinates** for a **variable velocity** vs is now given by $nY = vs * dy + y * dvs$(see Appendix L for detailed calculations):

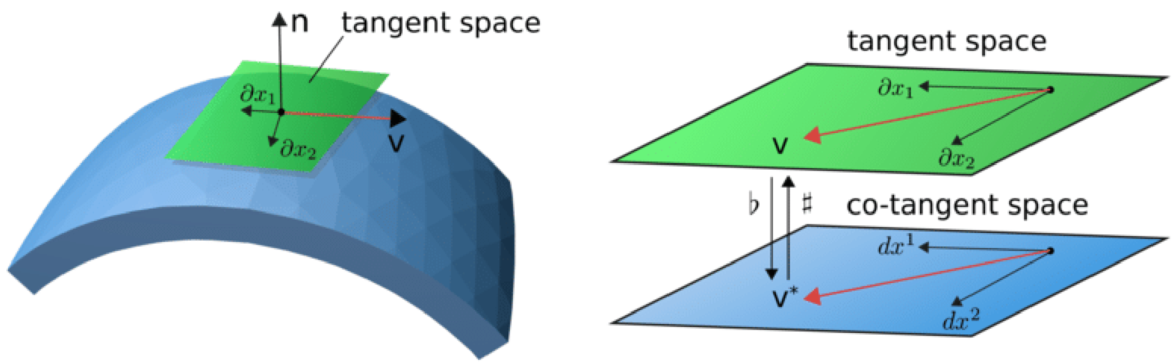

Figure 3: Artistic Presentation of Tangent and Cotangent Spaces I.(Source:Internet)

12 Appendix F:Tangent and Cotangent Spaces I

The Canonical Basis of the Hodge Star * in spherical coordinates in R^3 can be defined as follows(see pg 4 in [1],eq 3.72 pg 69(a)(b) in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r \sin\theta d\varphi) \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{193}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial \theta} \sim rd\theta \sim (r \sin\theta d\varphi) \wedge dr \sim r \sin\theta (d\varphi \wedge dr) \tag{194}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{195}$$

The Canonical Basis of the Hodge Star * in spherical coordinates in R^4 can be defined as follows(see pg 4 in [1],eqs 3.135 and 3.137 pg 82(a)(b) in [2],eq 3.74 pg 69(a)(b) in [2])(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3])(see also eqs 4.55 and 4.56 pg 179 in [8]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim dt \wedge (rd\theta) \wedge (r \sin\theta d\varphi) \sim r^2 \sin\theta (dt \wedge d\theta \wedge d\varphi) \tag{196}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial \theta} \sim rd\theta \sim dt \wedge (r \sin\theta d\varphi) \wedge dr \sim r \sin\theta (dt \wedge d\varphi \wedge dr) \tag{197}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \sim r\sin\theta d\varphi \sim dt \wedge dr \wedge (rd\theta) \sim r(dt \wedge dr \wedge d\theta) \tag{198}$$

In order to study the term $y * d(vs)$ we must introduce a new Canonical Basis for the coordinate time in the R^4 space basis defined as follows:(see eqs 10.102 and 10.103 pgs 363$(a)(b)$ and 364$(a)(b)$ in [2] with the terms $S = u = 1$[3],eq 3.74 pg 69$(a)(b)$ in [2],eqs 11.131 and 11.133 with the term $m = 0$[4] pg 417$(a)(b)$ in [2].)(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3])(see also eqs 4.55 and 4.56 pg 179 in [8]):

$$e_t \equiv \frac{\partial}{\partial t} \sim dt \sim dr \wedge (rd\theta) \wedge (r \sin\theta d\varphi) \sim r^2 \sin\theta (dr \wedge d\theta \wedge d\varphi) \tag{199}$$

As a matter of fact we have for the Canonical Basis and the Hodge Star * in R^4 the following equations (see pg 47 eqs 2.67 to 2.70 in [3]):

$$*e_0 = e_1 \wedge e_2 \wedge e_3 \tag{200}$$

$$*e_1 = e_0 \wedge e_2 \wedge e_3 \tag{201}$$

$$*e_2 = e_0 \wedge e_3 \wedge e_1 \tag{202}$$

$$*e_3 = e_0 \wedge e_1 \wedge e_2 \tag{203}$$

In R^3 the corresponding equations are:(see pg 55 in [5])(see also pg 54 fig 4.2 in [5] for a graphical presentation of the Hodge Star * in R^3)(see pg 18 eq 1.55 in [6]):

$$*e_1 = e_2 \wedge e_3 \tag{204}$$

$$*e_2 = e_3 \wedge e_1 = -e_1 \wedge e_3 \tag{205}$$

$$*e_3 = e_1 \wedge e_2 \tag{206}$$

The Canonical Basis e_i are related to the partial derivatives $\frac{\partial}{\partial x_i}$ or simplifying related to ∂x_i wether in R^3 or R^4 and are graphically represented by the partial derivatives ∂x_i included in the tangent space of the picture given in the beginning of this section.

[3]These terms are needed to deal with the Robertson-Walker equation in Cosmology using differential forms.We dont need these terms here and we can make $S = u = 1$

[4]This term is needed to describe the Dirac equation in the Schwarzschild spacetime we dont need the term here so we can make $m = 1$.Remember also that here we consider geometrized units in which $c = 1$

On the other hand in R^4 we also have the following relations for the Hodge Star *:(see pg 92 in [3])(see also eqs 4.55 and 4.56 pg 179 in [8])

$$*dt = dx \wedge dy \wedge dz \tag{207}$$

$$*dx = dt \wedge dy \wedge dz \tag{208}$$

$$*dy = dt \wedge dz \wedge dx \tag{209}$$

$$*dz = dt \wedge dx \wedge dy \tag{210}$$

Also for R^4 considering the $((w,v)(\epsilon\Lambda^3_p)(R^{1,3}))$ formalism we may have the following relations:(see pg 382 in [4])($x^1 = x$,$x^2 = y$,$x^3 = z$)

$$*dt = dx^1 \wedge dx^2 \wedge dx^3 \tag{211}$$

$$*dx^1 = dt \wedge dx^2 \wedge dx^3 \tag{212}$$

$$*dx^2 = dt \wedge dx^3 \wedge dx^1 \tag{213}$$

$$*dx^3 = dt \wedge dx^1 \wedge dx^2 \tag{214}$$

In R^3 we would have the following relations:(see pg 117 eqs 4.6 and 4.7 in [7])(see pg 298 in [4])

$$*dx = dy \wedge dz \tag{215}$$

$$*dy = dz \wedge dx \tag{216}$$

$$*dz = dx \wedge dy \tag{217}$$

The differentials dx,dy,dz or dx^1, dx^2 and dx^3 are related to the cotangent space differentials included in the picture given in the beginning of this section.

See the graphical presentations of the relations between tangent and cotangent spaces in pg 55 fig 2.28 and pg 70 fig 3.1 in [4].See pg 168 fig 5.19 for a graphical presentation of $dx \wedge dy$,pg 169 fig 5.20 for a graphical presentation of $dy \wedge dz$ and pg 170 fig 5.21 for a graphical presentation of $dz \wedge dx$ all in [4].

Useful relations to deal with the Hodge Star * are given by eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.3 pg 68$(a)(b)$ in [2]:See also pg 89 in [3],pg 112 in [4],pg 97 in [5],pg 36 eqs 2.21 and 2.22 in [6],pg 70 eq 3.3 in [7].

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 3 \dashrightarrow *d(f\alpha) = df \wedge \alpha - f \wedge d\alpha \tag{218}$$

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df \wedge \alpha + f \wedge d\alpha \tag{219}$$

$$*d(dx) = *d(dy) = *d(dz) = 0 \tag{220}$$

$p = 3$ stands for the R^4 and $p = 2$ stands for the R^3.

See also Appendix I.

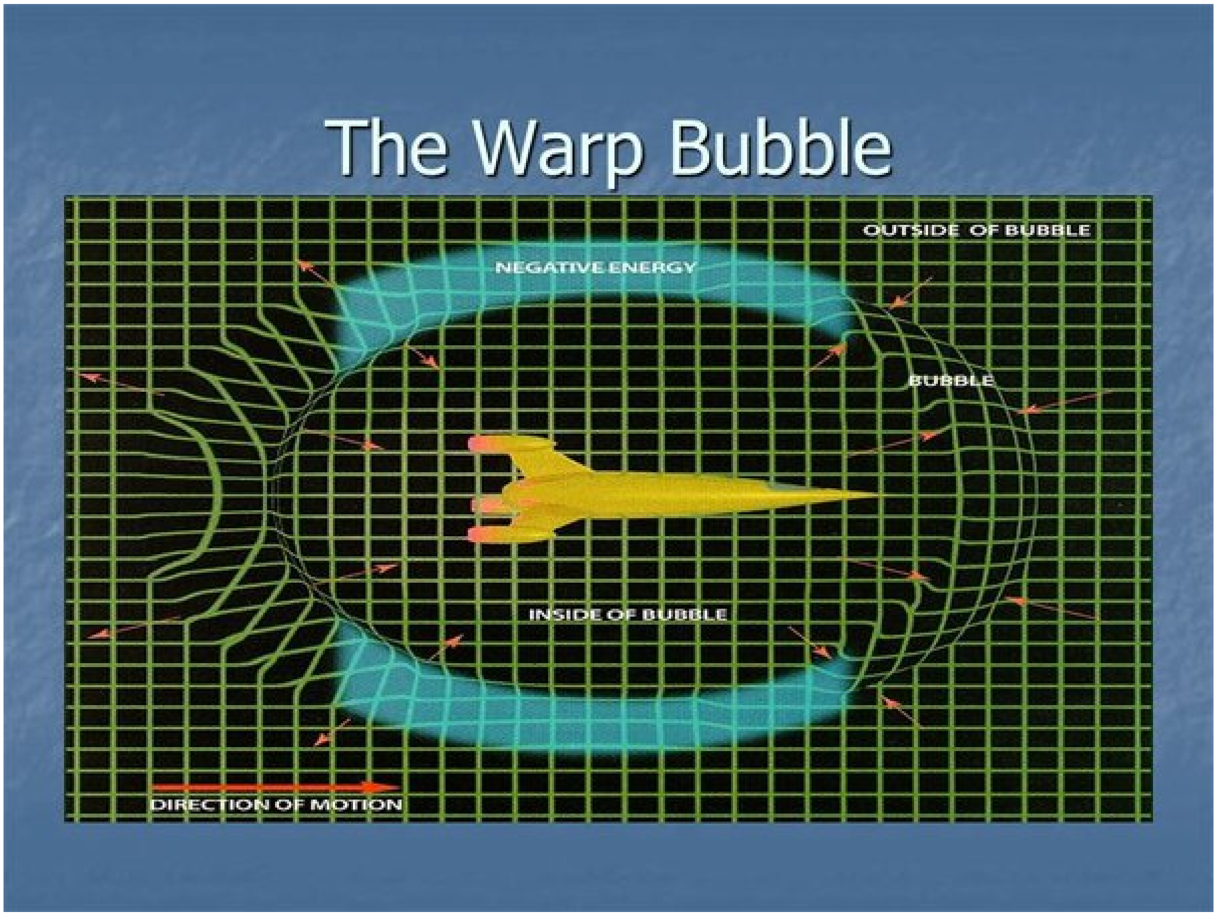

Figure 4: Artistic Presentation of a Warp Bubble.(Source:Internet)

13 Appendix G:Artistic Presentation of a Warp Bubble

In 2001 the Natario warp drive appeared.([1]).This warp drive deals with the spacetime as a "strain" tensor of Fluid Mechanics(pg 5 in [1]). Imagine a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The warp bubble in this case is the aquarium.An observer at the rest in the margin of the river would see the aquarium passing by him at a large speed but inside the aquarium the fish is at the rest with respect to his local neighborhoods.Since the fish is at the rest inside the aquarium the fish would see the observer in the margin passing by him with a large relative speed since for the fish is the margin that moves with a large relative velocity

any Natario vector nY generates a warp drive spacetime if $nY = 0$ and $Y = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nY = vs(t) * dy$ with $Y = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])

Lets explain better this statement:Natario considered in this case a coordinates reference frame placed inside the bubble where the fish inside the aquarium or the astronaut in a spaceship inside the bubble depicted above are at the rest with respect to their local neighborhoods.Then any Natario vector must be zero inside the bubble or the aquarium or the spaceship.

On the other hand since the fish sees the margin passing by him with a large relative velocity or the astronaut would see a stationary observer in outer space outside the bubble passing by him with a large relative velocity then any Natario vector outside the bubble must have a value equal to the relative velocity seen by both the fish and the astronaut.

Considering a valid f as a Natario shape function being $f = \frac{1}{2}$ for large r(outside the warp bubble) and $f = 0$ for small r(inside the warp bubble) while being $0 < f < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]):The walls of the bubble the Natario warped region corresponds to the distorted region in the picture depicted in this Appendix.

See also Appendix H.

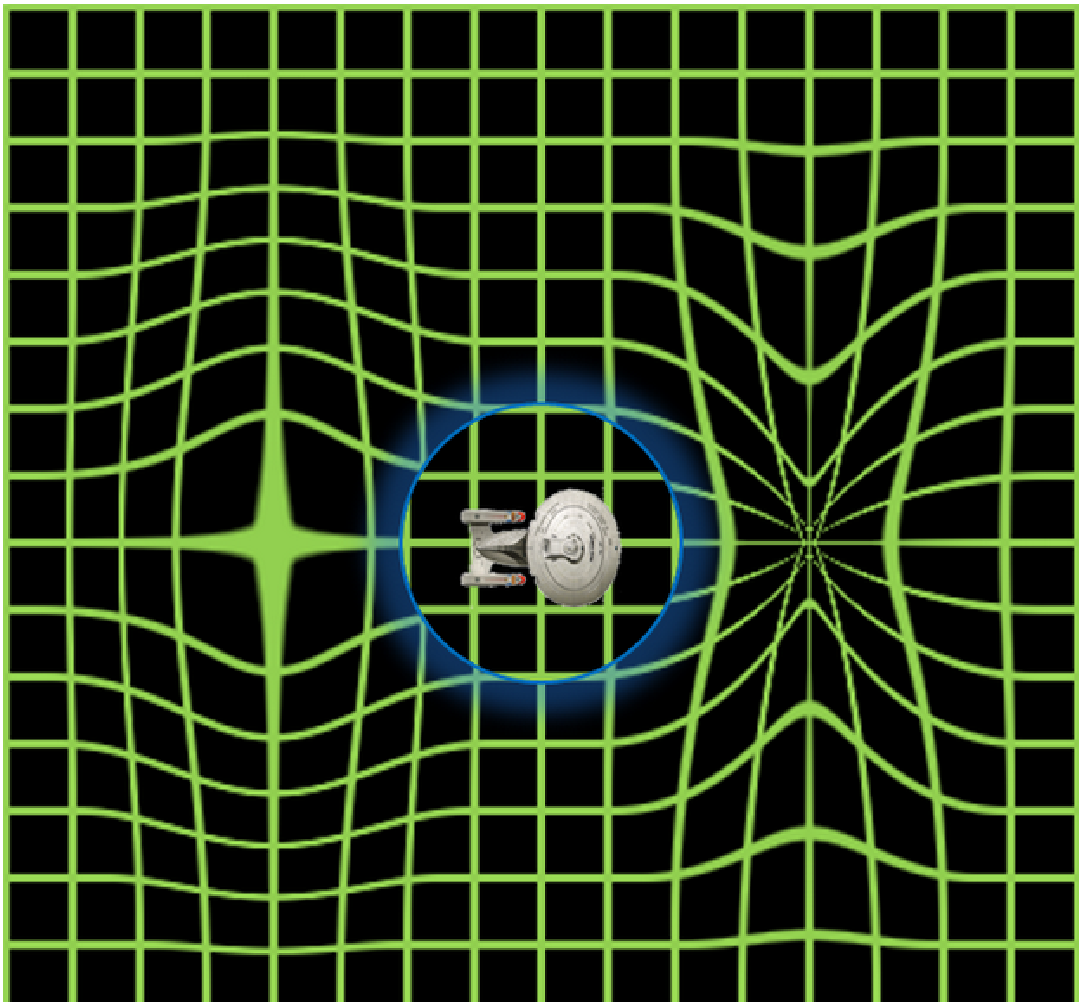

Figure 5: Another Artistic Presentation of a Warp Bubble.(Source:Internet)

14 Appendix H:Another Artistic Presentation of a Warp Bubble

Natario considered a coordinates reference frame placed inside the bubble.Now we must consider a coordinates reference frame placed outside the bubble:In this case the observer at the rest in the margin of the river would see the aquarium passing by him with a large velocity with the fish inside.Also a stationary observer at the rest in outer space would see the spaceship depicted in the picture above passing by him with a large velocity with the astronaut inside.

Now the rules originally defined by Natario are interchanged:

Since the observer in the margin and the observer in outer space are at the rest any Natario vector in this case must be zero outside the bubble.

But since the fish and the spaceship are being seen by the observer at the rest in the margin and the observer at the rest in outer space both fish and spaceship with a large velocity then the Natario vector inside the bubble must have a value equal to the velocity seen by both observers.

Considering a valid f as a Natario shape function being $f = 0$ for large r(outside the warp bubble) and $f = \frac{1}{2}$ for small r(inside the warp bubble) while being $0 < f < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region:The walls of the bubble the Natario warped region corresponds to the distorted region the "blue circle" in the picture depicted in this Appendix.

For an introductory explanation about remote frames outside the bubble or ship frames inside the bubble or comoving coordinates frames see pg 8 in [22].

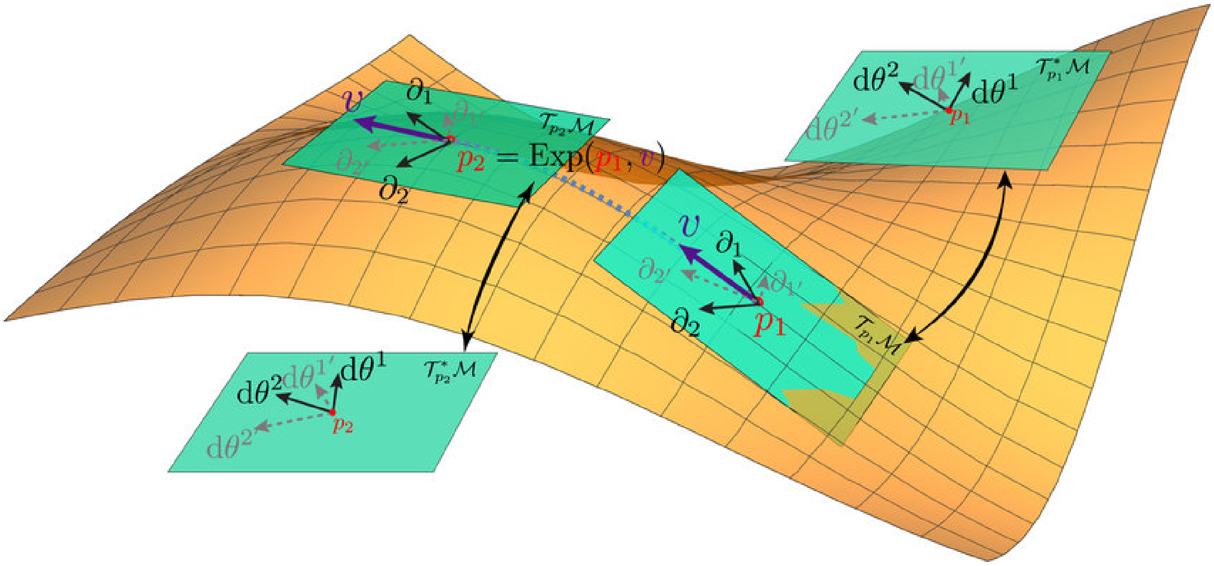

Figure 6: Artistic Presentation of Tangent and Cotangent Spaces II.(Source:Internet)

15 Appendix I:Tangent and Cotangent Spaces II

Consider a curve R in R^4 defined in function of a given set of coordinates u^0,u^1,u^2 and u^3 as being $R = R(u^0, u^1, u^2, u^3)$.

A total derivative of R is given by:

$$dR = \frac{\partial R}{\partial u^0} du^0 + \frac{\partial R}{\partial u^1} du^1 + \frac{\partial R}{\partial u^2} du^2 + \frac{\partial R}{\partial u^3} du^3 \tag{221}$$

Applying the Einstein summing convention:

$$dR = \frac{\partial R}{\partial u^i} du^i = e_i du^i \tag{222}$$

or

$$dR = \frac{\partial R}{\partial u^j} du^j = e_j du^j \tag{223}$$

With $i, j = 0, 1, 2, 3$ as the coordinates,$\frac{\partial R}{\partial u^i}$ and $\frac{\partial R}{\partial u^j}$ as the directional partial derivatives of R with respect to each coordinate and e_i and e_j are the respective Canonical Basis.

Defining $ds^2 = dR \bigotimes dR$ we have:

$$ds^2 = dR \bigotimes dR = \frac{\partial R}{\partial u^i} du^i \bigotimes \frac{\partial R}{\partial u^j} du^j = e_i du^i \bigotimes e_j du^j \tag{224}$$

$$ds^2 = \frac{\partial R}{\partial u^i} \frac{\partial R}{\partial u^j} du^i du^j = e_i e_j du^i du^j = g_{ij} du^i du^j \tag{225}$$

$$g_{ij} = \frac{\partial R}{\partial u^i} \frac{\partial R}{\partial u^j} = e_i e_j \tag{226}$$

The directional partial derivatives of R and their respective Canonical Basis are related to the ∂_i and ∂_j tangent spaces of the picture depicted in the beginning of this section while the differentials du^i and du^j are related to the respective cotangent spaces.See pg 148 problem 17 in [14],pg 132 eq 10.12 pg 133 eqs 10.14a,10.14b and 10.15 in [15].

$g_{ij} = \frac{\partial R}{\partial u^i}\frac{\partial R}{\partial u^j} = e_i e_j$ is the spacetime metric tensor of General Relativity.

16 Appendix J:differential forms,Hodge star and the mathematical demonstration of the new warp drive vector $nY = vs * dy$ for a constant speed vs in a R^3 space basis-$3D$ Spherical Coordinates

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eq 3.72 pg 69$(a)(b)$ in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{227}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \tag{228}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{229}$$

Back again to the equivalence between $3D$ spherical and cartezian coordinates $d(\rho\sin\phi\sin\theta)$:(See Appendix E)

We will replace ρ by r and φ by ϕ.Then we have:

$$d(r\sin\phi\sin\theta) = sin\phi[d(r\sin\theta)] + (r\sin\theta)d(\sin\phi) \tag{230}$$

$$d(r\sin\phi\sin\theta) = sin\phi[sin\theta dr + r(d\sin\theta)] + (r\sin\theta)(cos\phi d\phi) \tag{231}$$

$$d(r\sin\phi\sin\theta) = sin\phi[sin\theta(dr) + rcos\theta(d\theta)] + (r\sin\theta)[cos\phi(d\phi)] \tag{232}$$

$$d(r\sin\phi\sin\theta) = sin\phi[sin\theta(dr) + cos\theta(rd\theta)] + cos\phi[(r\sin\theta)(d\phi)] \tag{233}$$

Applying the Hodge Star * to the term $[sin\theta(dr)+cos\theta(rd\theta)]$ we will get the same results already shown in the Appendix A and the first part of the $3D$ spherical warp drive vector is the one of the Appendix A multiplied by $\sin\phi$ .Then we must concern ourselves with the term $cos\phi[(r\sin\theta)(d\phi)]$ and the following Canonical Basis for the Hodge Star * since the other two were covered in the Appendix A.

$$e_\phi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \sim r\sin\theta d\phi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{234}$$

Now applying the Hodge Star * to the term $d(r\sin\phi\sin\theta)$ we have:

$$*d(r\sin\phi\sin\theta) = sin\phi[sin\theta * (dr) + cos\theta * (rd\theta)] + cos\phi[*(r\sin\theta)(d\phi)] \tag{235}$$

$$*d(r\sin\phi\sin\theta) = sin\phi[(sin\theta)e_r + (cos\theta)e_\theta] + (cos\phi)e_\phi \tag{236}$$

$$*d(r\sin\phi\sin\theta) = sin\phi[sin\theta e_r + cos\theta e_\theta] + cos\phi e_\phi \tag{237}$$

In Appendix A we computed the Hodge Star $*dy$ in $2D$ Polar Coordinates as being $*dy = *d(r\sin\theta)$ as being:

$$*dy = *d(r\sin\theta) = \sin\theta e_r + \cos\theta e_\theta \tag{238}$$

$$*dy = [(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] + (-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] \tag{239}$$

We used the following expression:

$$[2f(r)][(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] + [2f(r) + rf'(r)](-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] \tag{240}$$

With these ones:

$$[(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] = \sin\theta e_r \tag{241}$$

$$(-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] = \cos\theta e_\theta \tag{242}$$

We arrived finally at:

$$[2f(r)][(r^2)(\tan\theta)][*d\left(\frac{1}{2}\sin^2\theta d\varphi\right)] + [2f(r) + rf'(r)](-1)(\sin\theta)(\cos\theta)\frac{1}{2} * [d(r^2)d\varphi] \tag{243}$$

$$[2f(r)]\sin\theta e_r + [2f(r) + rf'(r)]\cos\theta e_\theta \tag{244}$$

This is the new Natario vector nY with the Hodge Star operator * explicitly resolved in $2D$ Polar Coordinates:

$$nY = vs(t)[2f(r)]\sin\theta e_r + vs[2f(r) + rf'(r)]\cos\theta e_\theta \tag{245}$$

But in $3D$ Spherical Coordinates the Hodge Star $*dy$ is given by $*dy = *d(r\sin\phi\sin\theta)$

$$*d(r\sin\phi\sin\theta) = sin\phi[sin\theta e_r + cos\theta e_\theta] + cos\phi e_\phi \tag{246}$$

The term $sin\theta e_r + cos\theta e_\theta$ above will be replaced by $[2f(r)]\sin\theta e_r + [2f(r) + rf'(r)]\cos\theta e_\theta$ and the term $cos\phi e_\phi$ will be replaced by $[2f(r) + rf'(r)]cos\phi e_\phi$

And finally we arrived at the final expression for the new warp drive vector nY with the Hodge Star operator * explicitly resolved in $3D$ Spherical Coordinates for a constant speed vs:

$$nY = vs(t)sin\phi[[2f(r)]\sin\theta e_r + [2f(r) + rf'(r)]\cos\theta e_\theta] + vs(t)[2f(r) + rf'(r)]cos\phi e_\phi \tag{247}$$

$$nY = vs(t)sin\phi[[2f(r)]\sin\theta e_r + [2f(r) + rf'(r)]\cos\theta e_\theta] + vs(t)[2f(r) + rf'(r)]cos\phi e_\phi \tag{248}$$

$$nY = [2f(r)]vs(t)sin\phi\sin\theta e_r + vs(t)sin\phi[2f(r) + rf'(r)]\cos\theta e_\theta + vs(t)[2f(r) + rf'(r)]cos\phi e_\phi \tag{249}$$

This is the final form of our new tridimensional $3D$ spherical warp drive vector nY with the Hodge Star over the y-axis for a constant speed vs.Note that Natario in pg 4 in [1] defined the x-axis as the polar axis but now the y-axis is the polar axis.If the motion occurs only in the y-axis in polar coordinates then the angle between the x-y plane and the z-axis is 90 degrees and in this case $\sin\phi = 1$ and $\cos\phi = 0$ and our new warp drive vector nY in tridimensional $3D$ spherical coordinates reduces to the original Natario warp drive vector nY in polar coordinates.(see Appendix A).

For our new tridimensional $3D$ spherical coordinates warp drive vector nY with a constant speed vs and Hodge Star over the y-axis:

$$nY = Y^r e_r + Y^\theta e_\theta + Y^\phi e_\phi \tag{250}$$

The corresponding shift vectors are:

$$Y^r = [2f(r)]vs(t)sin\phi\sin\theta \tag{251}$$

$$Y^\theta = vs(t)sin\phi[2f(r) + rf'(r)]\cos\theta \tag{252}$$

$$Y^\phi = vs(t)[2f(r) + rf'(r)]cos\phi \tag{253}$$

Compare with the equation of the new warp drive vector in tridimensional $3D$ spherical coordinates with a constant speed vs nX and the Hodge Star over the x-axis given by:

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{254}$$

With the contravariant shift vector components X^{rs}, X^θ and X^ϕ given by:(see Appendix J in [9] for details)

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{255}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{256}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{257}$$

Note that Natario in pg 4 in [1] defined the x-axis as the polar axis.If the motion occurs only in the x-axis in polar coordinates then the angle between the x-y plane and the z-axis is 90 degrees and in this case $\sin\phi = 1$ and $\cos\phi = 0$ and our new warp drive vector nX in tridimensional $3D$ spherical coordinates reduces to the original Natario warp drive vector nX in polar coordinates.(see Appendix A in [9]).

17 Appendix K:differential forms,Hodge star and the mathematical demonstration of the warp drive vector $nY = vs * dy$ for a constant speed vs or for the first term $vs * dy$ from the warp drive vector $nY = vs * dy + y * dvs$(a variable speed) in a R^4 space basis-Tridimensional $3D$ Spherical Coordinates

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eqs 3.135 and 3.137 pg 82(a)(b) in [2],eq 3.74 pg 69(a)(b) in [2])(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3])(see also eqs 4.55 and 4.56 pg 179 in [8]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim dt \wedge (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) \tag{258}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim dt \wedge (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(dt \wedge d\varphi \wedge dr) \tag{259}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dt \wedge dr \wedge (rd\theta) \sim r(dt \wedge dr \wedge d\theta) \tag{260}$$

Useful relations to deal with the Hodge Star * are given by eqs 3.90 and 3.91 pg 74(a)(b) in [2],tb 3.3 pg 68(a)(b) in [2]:See also pg 89 in [3],pg 112 in [4],pg 97 in [5],pg 36 eqs 2.21 and 2.22 in [6],pg 70 eq 3.3 in [7].

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 3 \dashrightarrow *d(f\alpha) = df \wedge \alpha - f \wedge d\alpha \tag{261}$$

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df \wedge \alpha + f \wedge d\alpha \tag{262}$$

$$*d(dx) = *d(dy) = *d(dz) = 0 \tag{263}$$

$p = 3$ stands for the R^4 and $p = 2$ stands for the R^3.

Back again to the equivalence between $3D$ spherical and cartezian coordinates $d(\rho\sin\phi\sin\theta)$:(See Appendix E)

We will replace ρ by r and φ by ϕ.Then we have:

$$d(r\sin\phi\sin\theta) = sin\phi[d(r\sin\theta)] + (r\sin\theta)d(\sin\phi) \tag{264}$$

$$d(r\sin\phi\sin\theta) = sin\phi[sin\theta(dr) + cos\theta(rd\theta)] + cos\phi[(r\sin\theta)(d\phi)] \tag{265}$$

Applying the Hodge Star * to the terms above we will get the same results already shown in the Appendix J.As a matter of fact comparing the Appendices A and B the given final result is the same in both Appendices except for the fact that in Appendix A the Hodge Star is taken over R^3 and in Appendix B the Hodge Star is taken over R^4.

$$*d(r\sin\phi\sin\theta) = sin\phi[sin\theta * (dr) + cos\theta * (rd\theta)] + cos\phi * [(r\sin\theta)(d\phi)] \tag{266}$$

The final result is the same of Appendix J:

$$nY = Y^r e_r + Y^\theta e_\theta + Y^\phi e_\phi \tag{267}$$

The corresponding shift vectors are:

$$Y^r = [2f(r)]vs(t)sin\phi \sin\theta \tag{268}$$

$$Y^\theta = vs(t)sin\phi[2f(r) + rf'(r)] \cos\theta \tag{269}$$

$$Y^\phi = vs(t)[2f(r) + rf'(r)]cos\phi \tag{270}$$

18 Appendix L:differential forms,Hodge star and the mathematical demonstration of the new warp drive vector $nY = *(vsy) = vs * dy + y * dvs$ for a variable speed vs and a constant acceleration a in Tridimensional $3D$ Spherical Coordinates

any warp drive vector nY generates a warp drive spacetime if $nY = 0$ and $Y = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nY = vs(t) * dy$ with $Y = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])(see Appendix G for an explanation about this statement)

In the Appendices J and K we gave the mathematical demonstration of the new warp drive vector nY in the R^3 and R^4 space basis in tridimensional $3D$ spherical coordinates where the velocity vs is constant.Hence the complete expression of the Hodge star that generates the warp drive vector $nY = vs * dy$ for a constant velocity vs is given by:

$$nY = *(vsy) = vs * (dy) \tag{271}$$

Our new tridimensional $3D$ spherical coordinates warp drive vector in R^4 with constant speed vs $nY = vs * dy$ or for the first term $vs * dy$ of the new tridimensional $3D$ spherical coordinates warp drive vector in R^4 with variable speed vs $nY = vs * dy + y * dvs$ is given by:

$$nY = vs(t)sin\phi[[2f(r)]\sin\theta e_r + [2f(r) + rf'(r)]\cos\theta e_\theta] + vs(t)[2f(r) + rf'(r)]cos\phi e_\phi \tag{272}$$

$$nY = Y^r e_r + Y^\theta e_\theta + Y^\phi e_\phi \tag{273}$$

The corresponding shift vectors are:

$$Y^r = [2f(r)]vs(t)sin\phi \sin\theta \tag{274}$$

$$Y^\theta = vs(t)sin\phi[2f(r) + rf'(r)]\cos\theta \tag{275}$$

$$Y^\phi = vs(t)[2f(r) + rf'(r)]cos\phi \tag{276}$$

Because due to a constant speed vs the term $y * d(vs) = 0$.Now we must examine what happens when the velocity is variable and then the term $y * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.The complete expression of the Hodge star that generates the warp drive vector nY for a variable velocity vs is now given by:

$$nY = *(vsy) = vs * (dy) + y * (dvs) \tag{277}$$

In order to study the term $y * d(vs)$ we must introduce a new Canonical Basis for the coordinate time in the R^4 space basis defined as follows:(see eqs 10.102 and 10.103 pgs 363$(a)(b)$ and 364$(a)(b)$ in [2] with the terms $S = u = 1$[5],eq 3.74 pg 69$(a)(b)$ in [2],eqs 11.131 and 11.133 with the term $m = 0$[6] pg 417$(a)(b)$ in [2].)(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3])(see also eqs 4.55 and 4.56 pg 179 in [8]):

$$e_t \equiv \frac{\partial}{\partial t} \sim dt \sim dr \wedge (rd\theta) \wedge (r \sin\theta d\phi) \sim r^2 \sin\theta(dr \wedge d\theta \wedge d\phi) \quad (278)$$

The Hodge star operator defined for the coordinate time is given by:(see eq 3.74 pg 69$(a)(b)$ in [2](see also eqs 4.55 and 4.56 pg 179 in [8])):

$$*dt = r^2 \sin\theta(dr \wedge d\theta \wedge d\phi) \quad (279)$$

The valid expression for a variable velocity $vs(t)$ in the Natario warp drive spacetime due to a constant acceleration a must be given by:

$$vs = 2f(r)at \quad (280)$$

Because and considering a valid $f(r)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble where $Y = vs(t)$ and $nY = vs(t) * dy + y * d(vs(t)))$ and $f(r) = 0$ for small r(inside the warp bubble where $Y = 0$ and $nY = 0$) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pgs 4 and 5 in [1]) and considering also that the Natario warp drive is a ship-frame based coordinates system(a reference frame placed in the center of the warp bubble where the ship resides-or must reside!!) then an observer in the ship inside the bubble sees every point inside the bubble at the rest with respect to him because inside the bubble $vs(t) = 0$ because $f(r) = 0$.

To illustrate the statement pointed above imagine a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The stream varies its velocity with time.The warp bubble in this case is the aquarium and the walls of the aquarium are the walls of the warp bubble-Natario warped region.An observer in the margin of the river would see the aquarium passing by him at a large speed considering a coordinates system(a reference frame) placed in the margin of the river but inside the aquarium the fish is at the rest with respect to his local neighborhoods.Then for the fish any point inside the aquarium is at the rest with respect to him because inside the aquarium $vs = 2f(r)at$ with $f(r) = 0$ and consequently giving a $vs(t) = 0$.Again with respect to the fish the fish "sees" the margin passing by him with a large relative velocity.The margin in this case is the region outside the bubble "seen" by the fish with a variable velocity $vs(t) = v1$ in the time $t1$ and $vs(t) = v2$ in the time $t2$ because outside the bubble the generic expression for a variable velocity vs is given by $vs = 2f(r)at$ and outside the bubble $f(r) = \frac{1}{2}$ giving a generic expression for a variable velocity vs as $vs(t) = at$ and consequently a $v1 = at1$ in the time $t1$ and a $v2 = at2$ in the time $t2$.Then the variable velocity in not only a function of time alone but must consider also the position of the bubble where the measure is being taken wether inside or outside the bubble.So the velocity must also be a function of r.Its total differential is then given by:

$$dvs = 2[atf'(r)dr + f(r)tda + f(r)adt] \quad (281)$$

[5]These terms are needed to deal with the Robertson-Walker equation in Cosmology using differential forms.We dont need these terms here and we can make $S = u = 1$

[6]This term is needed to describe the Dirac equation in the Schwarzschild spacetime we dont need the term here so we can make $m = 1$.Remember also that here we consider geometrized units in which $c = 1$

Applying the Hodge star to the total differential dvs we get:

$$*dvs = 2[atf'(r) * dr + f(r)t * da + f(r)a * dt] \tag{282}$$

But we consider here the acceleration a a constant.Then the term $f(r)tda = 0$ and in consequence $f(r)t * da = 0$.This leaves us with:

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] \tag{283}$$

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] = 2[atf'(r)r^2 \sin\theta(dt \wedge d\theta \wedge d\phi) + f(r)ar^2 \sin\theta(dr \wedge d\theta \wedge d\phi)] \tag{284}$$

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] = 2[atf'(r)e_r + f(r)ae_t] \tag{285}$$

The complete expression of the Hodge star that generates the warp drive vector nX for a variable velocity vs is given by:

$$nY = *(vsy) = vs * (dy) + y * d(vs) \tag{286}$$

The term $vs * dy$ was obtained in the Appendices J and K as follows:

$$nY = vs(t)sin\phi[[2f(r)] \sin\theta e_r + [2f(r) + rf'(r)] \cos\theta e_\theta] + vs(t)[2f(r) + rf'(r)]cos\phi e_\phi \tag{287}$$

The complete expression of the Hodge star that generates the warp drive vector nY for a variable velocity vs is now given by:

$$nY = vs(t)sin\phi[[2f(r)] \sin\theta e_r + [2f(r) + rf'(r)] \cos\theta e_\theta] + vs(t)[2f(r) + rf'(r)]cos\phi e_\phi + y(2[atf'(r)e_r + f(r)ae_t]) \tag{288}$$

But remember that we are in tridimensional $3D$ spherical coordinates(see Appendix E) in which $y = r\sin\phi\sin\theta$ and this leaves us with:

$$nY = A + B \rightarrow A = vs * dy \rightarrow B = y * dvs \tag{289}$$

$$A = vs(t)sin\phi[[2f(r)] \sin\theta e_r + [2f(r) + rf'(r)] \cos\theta e_\theta] + vs(t)[2f(r) + rf'(r)]cos\phi e_\phi \tag{290}$$

$$B = (r\sin\phi\sin\theta)(2[atf'(r)e_r + f(r)ae_t]) \tag{291}$$

But we know that $vs = 2f(r)at$.Hence we get:

$$nY = A + B \rightarrow A = vs * dy \rightarrow B = y * dvs \tag{292}$$

$$A = [2f(r)at]sin\phi[[2f(r)]\sin\theta e_r + [2f(r) + rf'(r)]\cos\theta e_\theta] + [2f(r)at][2f(r) + rf'(r)]cos\phi e_\phi \tag{293}$$

$$B = (r\sin\phi\sin\theta)(2[atf'(r)e_r + f(r)ae_t]) \tag{294}$$

Then we can start with a warp bubble initially at the rest using the warp drive vector shown above and accelerate the bubble to a desired speed of 200 times faster than light.When we achieve the desired speed we turn off the acceleration and keep the speed vs constant.The term B due to the acceleration $y * (dvs)$ now disappears the speed vs is no longer $vs = 2f(r)at$ and we are left again with the warp drive vector for constant speeds shown below:

$$nY = A \rightarrow A = vs * dy \tag{295}$$

$$A = vs(t)sin\phi[[2f(r)]\sin\theta e_r + [2f(r) + rf'(r)]\cos\theta e_\theta] + vs(t)[2f(r) + rf'(r)]cos\phi e_\phi \tag{296}$$

Working some algebra with the new warp drive vector for variable velocities we get:[7]

$$nY = A + B \rightarrow A = vs * dy \rightarrow B = y * dvs \tag{297}$$

$$A = [2f(r)at]sin\phi[[2f(r)]\sin\theta e_r + [2f(r) + rf'(r)]\cos\theta e_\theta] + [2f(r)at][2f(r) + rf'(r)]cos\phi e_\phi \tag{298}$$

$$B = (r\sin\phi\sin\theta)(2[atf'(r)e_r + f(r)ae_t]) \tag{299}$$

$$A = (2f(r)at)\sin\phi[2f(r)\sin\theta e_r] + (2f(r)at)\sin\phi[2f(r) + rf'(r)]\cos\theta e_\theta + (2f(r)at)cos\phi[[2(f(r)) + (rf'(r))]e_\phi]] \tag{300}$$

$$B = 2(r\sin\phi\sin\theta)atf'(r)e_r + 2(r\sin\phi\sin\theta)f(r)ae_t \tag{301}$$

$$A = 4(f(r)^2at)(\sin\phi)(\sin\theta)e_r + (2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\cos\theta)e_\theta + (2f(r)at)[2(f(r)) + (rf'(r))](cos\phi)e_\phi \tag{302}$$

$$B = 2(at)(rf'(r))(\sin\phi)(\sin\theta)e_r + 2(rf(r)a))(\sin\phi)(\sin\theta)e_t \tag{303}$$

[7] we know that we are being tedious monotonous and repetitive but we are writing this mainly for beginners or introductory students

Rearranging the terms we have:

$$A = 4(f(r)^2 at)(\sin\phi)(\sin\theta)e_r + (2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta)e_\theta + (2f(r)at)[2(f(r)) + (rf'(r))](cos\phi)e_\phi \tag{304}$$

$$A = (2f(r)at)\sin\phi[2f(r)\sin\theta e_r] + (2f(r)at)\sin\phi[2f(r) + rf'(r)]\cos\theta e_\theta + (2f(r)at)cos\phi[[2(f(r)) + (rf'(r))]e_\phi]] \tag{305}$$

$$(2f(r)at)[2f(r)](\sin\phi)(\sin\theta)e_r + (2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\cos\theta)e_\theta + (2f(r)at)[2f(r) + (rf'(r))](cos\phi)e_\phi \tag{306}$$

$$B = 2(at)(rf'(r))(\sin\phi)(\sin\theta)e_r + 2(rf(r)a))(\sin\phi)(\sin\theta)e_t \tag{307}$$

Working the terms with e_r

$$(2f(r)at)\sin\phi[2f(r)\sin\theta e_r] + 2(at)(rf'(r))(\sin\phi)(\sin\theta)e_r \tag{308}$$

$$(2f(r)at)[2f(r)](\sin\phi)(\sin\theta)e_r + 2(at)(rf'(r))(\sin\phi)(\sin\theta)e_r \tag{309}$$

$$(2at)[2f(r)^2](\sin\phi)(\sin\theta)e_r + 2(at)(rf'(r))(\sin\phi)(\sin\theta)e_r \tag{310}$$

$$(2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\sin\theta)e_r \tag{311}$$

At last we can give now the new warp drive vector for variable velocities in real tridimwensional $3D$ spherical coordinates using its respective contravariant shift vector components:[8]

$$nY = Y^t e_t + Y^r e_r + Y^\theta e_\theta + Y^\phi e_\phi \tag{312}$$

$$Y^t = 2(rf(r)a))(\sin\phi)(\sin\theta) \tag{313}$$

$$Y^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\sin\theta) \tag{314}$$

$$Y^\theta = (2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\cos\theta) \tag{315}$$

$$Y^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi) \tag{316}$$

[8] again:the section is extensive but a beginner needs all these QED Quod Erad Demonstratum mathematical demonstrations

Comparing the new warp drive vector for variable velocities in real tridimensional $3D$ spherical coordinates with the Natario polar coordinates warp drive vector counterpart:

$$nY = Y^t e_t + Y^r e_r + Y^\theta e_\theta + Y^\phi e_\phi \tag{317}$$

$$Y^t = 2(rf(r)a))(\sin\phi)(\sin\theta) \tag{318}$$

$$Y^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\sin\theta) \tag{319}$$

$$Y^\theta = (2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\cos\theta) \tag{320}$$

$$Y^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi) \tag{321}$$

$$nY = Y^t e_t + Y^r e_r + Y^\theta e_\theta \tag{322}$$

$$Y^t = 2f(r)r\sin\theta a \tag{323}$$

$$Y^r = 2[2f(r)^2 + rf'(r)]at\sin\theta \tag{324}$$

$$Y^\theta = +2f(r)at[2f(r) + rf'(r)]\cos\theta \tag{325}$$

Natario defined a motion in the $x - axis$ of polar coordinates (pgs 4 and 5 in [1]) but we considered the motion in the $y - axis$ then the polar plane $x - y$ makes an angle of 90 degrees with the $z - axis$ and since $\sin\phi = 1$ and $\cos\phi = 0$ it is easy to see that in this case the new warp drive vector for variable velocities in real tridimensional $3D$ spherical coordinates reduces itself to the Natario polar coordinates warp drive vector counterpart:

The difference occurs only in a real tridimensional motion.

Comparing the new warp drive vector for variable velocities in real tridimensional $3D$ spherical coordinates and Hodge Star over the y-axis nY with the the new warp drive vector nX in tridimensional $3D$ spherical coordinates with a variable speed vs and Hodge Star over the x-axis:

$$nY = Y^t e_t + Y^r e_r + Y^\theta e_\theta + Y^\phi e_\phi \tag{326}$$

$$Y^t = 2(rf(r)a))(\sin\phi)(\sin\theta) \tag{327}$$

$$Y^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\sin\theta) \tag{328}$$

$$Y^\theta = (2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\cos\theta) \tag{329}$$

$$Y^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi) \tag{330}$$

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{331}$$

With the contravariant shift vector components X^t,X^{rs}, X^θ and X^ϕ given by:
(see Appendices K and L in [9] for details)

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \tag{332}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \tag{333}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{334}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \tag{335}$$

Do they look familiar ?

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