

Materialized Photons

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Abstract

In this paper, a new hypothesis is proposed: when blackbody radiation interacts with matter, a photon can annihilate by capturing atoms or molecules, thereby generating a phonon-like quasiparticle—the materialized photon. This materialized photon then further couples to the blackbody radiation, producing a new quasiparticle—the condenson. We mainly take ^4He as the specific object of study and present a detailed exposition of this new condenson model. Based on the model, new explanations are provided for the quantization of circulation and magnetic flux. The phonon–roton dispersion curve (P–R curve) of the superfluid ^4He quasiparticles, which was conjectured by Landau, as well as the specific heat curve with singularities across the entire temperature range, are calculated and found to be in excellent agreement with the experimental data. More importantly, the condenson model provides the dynamical origin of the superfluidity of ^4He II. Later in this paper, the new model is applied to the study of superconductivity and the superfluidity of ^3He .

Keywords: ^4He , Materialized Photons, Condenson, Superfluidity

1 Introduction

As is well known, photons possess wave–particle duality. When interacting with matter, they may also exhibit new features. For example, a high-energy γ photon can annihilate, with the assistance of an atomic nucleus or an external field, to produce real particles such as an electron–positron pair. As another example, a phonon in the energy state $n\hbar\omega$ can transition to the state $(n-1)\hbar\omega$ by radiating a photon with energy $\hbar\omega$. We now ask whether the reverse process exists. In this paper, we assume

that the answer is affirmative; this is the so-called materialized photon hypothesis introduced below.

In general, interactions between photons can be neglected. However, the materialized photon is quite special in that it can couple to blackbody radiation (BBR). Remarkably, we find that this coupling leads to the emergence of a new condensed state (and even a superfluid state). In short, in addition to wave-particle duality, photons possess a third property—namely, the ability to become materialized.

2 Materialized Photon Hypothesis

Let us start from the cradle of quantum theory — BBR with a cavity. To study the photon gas in the cavity, the vector potential $\mathbf{A}$ and its canonically conjugate momentum $\mathbf{P} = m_p \dot{\mathbf{A}}$, (where $m_p \equiv 1/4\pi c^2$ is the formal mass of photons, c is the speed of light in vacuum) are chosen as the variables. The photon Hamiltonian H_p , expressed in terms of the creation operator ($a_{\mathbf{k}\sigma}^\dagger$) and the destruction operator ($a_{\mathbf{k}\sigma}$) of the $\mathbf{A}(\mathbf{r})$ and $\mathbf{P}(\mathbf{r})$, is given by [1, 2]:

$$\begin{aligned} H_p &= \sum_{\mathbf{k}\sigma} \left[\frac{1}{2m_p} \Pi_A(\mathbf{k}, \sigma) \Pi_A(-\mathbf{k}, \sigma) \right. \\ &\quad \left. + \frac{1}{2} m_p \omega_k^2 A(\mathbf{k}, \sigma) A(-\mathbf{k}, \sigma) \right] \\ &= \sum_{\mathbf{k}\sigma} \hbar \omega_k (a_{\mathbf{k}\sigma}^\dagger a_{\mathbf{k}\sigma} + \frac{1}{2}) \end{aligned} \quad (1)$$

where

$$\begin{aligned} A(\mathbf{k}, \sigma) &= \left(\frac{\hbar}{2m_p \omega_k} \right)^{1/2} (a_{\mathbf{k}\sigma} + a_{-\mathbf{k}\sigma}^\dagger) \\ \Pi_A(\mathbf{k}, \sigma) &= -i \left(\frac{\hbar m_p \omega_k}{2} \right)^{1/2} (a_{\mathbf{k}\sigma} - a_{-\mathbf{k}\sigma}^\dagger) \\ \mathbf{A}(\mathbf{r}) &= \frac{1}{\sqrt{V}} \sum_{\mathbf{k}\sigma} \mathbf{e}_{\mathbf{k}\sigma} A(\mathbf{k}, \sigma) e^{i\mathbf{k}\cdot\mathbf{r}} \\ \mathbf{P}(\mathbf{r}) &= \frac{1}{\sqrt{V}} \sum_{\mathbf{k}\sigma} \mathbf{e}_{\mathbf{k}\sigma} \Pi_A(\mathbf{k}, \sigma) e^{i\mathbf{k}\cdot\mathbf{r}} \end{aligned}$$

Where $\omega_k = c|\mathbf{k}| = ck$, the V is the volume of the cavity. The $\mathbf{e}_{\mathbf{k}\sigma}$ is the unit vector and $\mathbf{e}_{-\mathbf{k}\sigma}^* = \mathbf{e}_{\mathbf{k}\sigma}$. The $\sigma = (1, 2)$ denotes possible polarization directions of the photons. The commutation relations of the operators are $[A(\mathbf{k}, \sigma), \Pi_A(-\mathbf{k}', \sigma')] = i\hbar \delta_{\mathbf{k}\mathbf{k}'} \delta_{\sigma\sigma'}$, $[a_{\mathbf{k}\sigma}, a_{\mathbf{k}'\sigma'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'} \delta_{\sigma\sigma'}$.

Suppose we inject N_0 ^4He atoms into the cavity. Under the influence of the vector potential, some ^4He atoms may become polarized. Our hypothesis concerning a new mechanism of blackbody radiation-matter interaction consists of two parts:

1. The Hamiltonian given by Equation (1) is not only mathematically identical to that of a regular phonon containing only transverse modes, but is also physically analogous to it. Corresponding to the “equilibrium points” of the vector potential, there exists a set of perfectly periodic lattice sites. The structure corresponding to each individual lattice site must be a non-strongly-degenerate fermion.

2. At the temperature T_λ and density ρ_λ , a photon (referred to as the induced photon) with momentum $\hbar\mathbf{k}_\lambda$ and energy $\hbar\omega_\lambda$ annihilates after capturing N_λ polarized ^4He atoms (out of the total N_0) to its “equilibrium points,” producing an electromagnetically coupled phonon with momentum $\hbar\mathbf{k}_\lambda$ and energy $\hbar\omega_\lambda$. Since this process can be regarded as the inverse of a phonon radiation transition, the new phonon is referred to as a materialized photon (MP). Unlike the induced photon, the MP possesses both transverse and longitudinal vibration modes. The Hamiltonian of the MP, H_{mp} , is expressed as

$$H_{mp} = \sum_j^{N_\lambda} \left(\frac{\mathbf{P}_j^2}{2m} + \frac{1}{2} m \omega_\lambda^2 \mathbf{B}_j^2 \right) \quad (2)$$

where m is the mass of ^4He , and $\mathbf{P}_j$ and $\mathbf{B}_j$ denote the momentum and amplitude of the j -th ^4He atom in the MP, respectively. Compared with regular phonons, the MP exhibits two distinct features. First, it is evident that the MP represents a long-range ordered subsystem, that is, a perfect crystal. However, since the MP is uniformly intermingled with non-MP ^4He atoms, the entire system still appears disordered. The nearest-neighbor ^4He atom of a ^4He atom within the MP does not necessarily belong to the MP, and the interaction between these two types of ^4He atoms transforms the motion of ^4He atoms in the MP from that of a harmonic oscillator into that of a damped harmonic oscillator. Let c_1 denote the damping coefficient, and define the damping ratio as $\zeta \equiv c_1/(2m\omega_\lambda)$. Then, the resonant frequency is given by $\omega_r = \omega_\lambda \sqrt{1 - 2\zeta^2}$. The resonant frequency ω_r has a critical point at $\zeta = 1/\sqrt{2}$. When $\zeta > 1/\sqrt{2}$, the MP should not appear. Second, since the ^4He atoms in the MP are polarized, they interact with BBR. Let the coupling constant between the ^4He atom and the vector potential be η . Because BBR couples only to the transverse mode of the MP, by hiding the longitudinal mode in Eq.(2), H_{mp} reduces to H_1 :

$$H_1 = \sum_j^{N_\lambda} \left[\frac{1}{2m} (\mathbf{P}_j - \eta \mathbf{A}_j)^2 + \frac{1}{2} m \omega_\lambda^2 \mathbf{B}_j^2 \right] \quad (3)$$

Using the following normal modes of the fields and their conjugate momenta,

$$\begin{aligned} B(\mathbf{k}, \sigma) &= \left(\frac{\hbar}{2m\omega_\lambda} \right)^{1/2} (b_{\mathbf{k}\sigma} + b_{-\mathbf{k}\sigma}^\dagger) \\ \Pi_B(\mathbf{k}, \sigma) &= -i \left(\frac{\hbar m \omega_\lambda}{2} \right)^{1/2} (b_{\mathbf{k}\sigma} - b_{-\mathbf{k}\sigma}^\dagger) \end{aligned}$$

$$B_j = \frac{1}{\sqrt{N_\lambda}} \sum_{\mathbf{k}\sigma} e_{\mathbf{k}\sigma} B(\mathbf{k}, \sigma) e^{i\mathbf{k} \cdot \mathbf{R}_j}$$

$$P_j = \frac{1}{\sqrt{N_\lambda}} \sum_{\mathbf{k}\sigma} e_{\mathbf{k}\sigma} \Pi_B(\mathbf{k}, \sigma) e^{i\mathbf{k} \cdot \mathbf{R}_j}$$

Where $\mathbf{R}_j$ is the Cartesian coordinate of j -th lattice point, the commutation relations of the operators are $[B(\mathbf{k}, \sigma), \Pi_B(-\mathbf{k}', \sigma')] = i\hbar \delta_{\mathbf{k}\mathbf{k}'} \delta_{\sigma\sigma'}$, $[b_{\mathbf{k}\sigma}, b_{\mathbf{k}'\sigma'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'} \delta_{\sigma\sigma'}$, ($\sigma = 1, 2$). The final Hamiltonian of the system combining Equation (1) and (3), $H = H_p + H_1$, is therefore

$$H = \sum_{\mathbf{k}\sigma} [\hbar\omega_k (a_{\mathbf{k}\sigma}^\dagger a_{\mathbf{k}\sigma} + \frac{1}{2}) + \hbar\omega_\lambda (b_{\mathbf{k}\sigma}^\dagger b_{\mathbf{k}\sigma} + \frac{1}{2})$$

$$+ \frac{i}{2} \hbar\omega_p \left(\frac{\omega_\lambda}{\omega_k} \right)^{1/2} (b_{\mathbf{k}\sigma} - b_{-\mathbf{k}\sigma}^\dagger) (a_{-\mathbf{k}\sigma} + a_{\mathbf{k}\sigma}^\dagger)$$

$$+ \frac{1}{4} \hbar\omega_p \frac{\omega_p}{\omega_k} (a_{\mathbf{k}\sigma} + a_{-\mathbf{k}\sigma}^\dagger) (a_{-\mathbf{k}\sigma} + a_{\mathbf{k}\sigma}^\dagger)] \quad (4)$$

The quantity $\omega_p \equiv c\eta\sqrt{4\pi N_\lambda/mV}$.

3 Exact Solution of the Hamiltonian H

We will directly diagonalize H in four steps to obtain the eigenvectors and eigenvalues.

Step 1: We transform H into H_2 using the following scale transformations:

$$(a_{\mathbf{k}\sigma} + a_{-\mathbf{k}\sigma}^\dagger) = \omega_k^{1/2} (\Xi_{\mathbf{k}\sigma} + \Xi_{-\mathbf{k}\sigma}^\dagger)$$

$$(a_{\mathbf{k}\sigma} - a_{-\mathbf{k}\sigma}^\dagger) = \omega_k^{-1/2} (\Xi_{\mathbf{k}\sigma} - \Xi_{-\mathbf{k}\sigma}^\dagger)$$

$$(b_{\mathbf{k}\sigma} + b_{-\mathbf{k}\sigma}^\dagger) = \omega_\lambda^{1/2} (\Theta_{\mathbf{k}\sigma} + \Theta_{-\mathbf{k}\sigma}^\dagger)$$

$$(b_{\mathbf{k}\sigma} - b_{-\mathbf{k}\sigma}^\dagger) = \omega_\lambda^{-1/2} (\Theta_{\mathbf{k}\sigma} - \Theta_{-\mathbf{k}\sigma}^\dagger)$$

Since the commutation relations of Ξ 's are identical to those of a 's, and similarly for Θ 's and b 's, we can simplify notation by substituting $\Xi_{\mathbf{k}\sigma} \rightarrow a_{\mathbf{k}\sigma}$ and $\Theta_{\mathbf{k}\sigma} \rightarrow b_{\mathbf{k}\sigma}$. Thus,

$$H_2 = \frac{\hbar}{4} \sum_{\mathbf{k}\sigma} [-(b_{-\mathbf{k}\sigma} - b_{\mathbf{k}\sigma}^\dagger)(b_{\mathbf{k}\sigma} - b_{-\mathbf{k}\sigma}^\dagger)$$

$$+ \omega_\lambda^2 (b_{\mathbf{k}\sigma} + b_{-\mathbf{k}\sigma}^\dagger)(b_{-\mathbf{k}\sigma} + b_{\mathbf{k}\sigma}^\dagger)$$

$$- (a_{-\mathbf{k}\sigma} - a_{\mathbf{k}\sigma}^\dagger)(a_{\mathbf{k}\sigma} - a_{-\mathbf{k}\sigma}^\dagger)$$

$$+ (\omega_p^2 + \omega_k^2)(a_{\mathbf{k}\sigma} + a_{-\mathbf{k}\sigma}^\dagger)(a_{-\mathbf{k}\sigma} + a_{\mathbf{k}\sigma}^\dagger)$$

$$+ i2\omega_p (b_{\mathbf{k}\sigma} - b_{-\mathbf{k}\sigma}^\dagger)(a_{-\mathbf{k}\sigma} + a_{\mathbf{k}\sigma}^\dagger)] \quad (5)$$

In Step 2, we aim to find a canonical transformation to H_3 using the formula $H_3 = e^S H_2 e^{-S}$. We will determine S by

$$\begin{aligned}
S = & -\frac{i}{2} \sum_{\mathbf{k}\sigma} \frac{\omega_\lambda \omega_p}{\omega_\lambda - \omega_k} (b_{\mathbf{k}\sigma} a_{-\mathbf{k}\sigma} + b_{-\mathbf{k}\sigma}^\dagger a_{-\mathbf{k}\sigma} \\
& + b_{-\mathbf{k}\sigma}^\dagger a_{\mathbf{k}\sigma}^\dagger + b_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}^\dagger) \\
H_3 = & e^S H_2 e^{-S} \\
= & \frac{\hbar}{4} \sum_{\mathbf{k}\sigma} [-(b_{-\mathbf{k}\sigma} - b_{\mathbf{k}\sigma}^\dagger)(b_{\mathbf{k}\sigma} - b_{-\mathbf{k}\sigma}^\dagger) \\
& + \omega_B^2 (b_{\mathbf{k}\sigma} + b_{-\mathbf{k}\sigma}^\dagger)(b_{-\mathbf{k}\sigma} + b_{\mathbf{k}\sigma}^\dagger) \\
& - (a_{-\mathbf{k}\sigma} - a_{\mathbf{k}\sigma}^\dagger)(a_{\mathbf{k}\sigma} - a_{-\mathbf{k}\sigma}^\dagger) \\
& + \omega_A^2 (a_{\mathbf{k}\sigma} + a_{-\mathbf{k}\sigma}^\dagger)(a_{-\mathbf{k}\sigma} + a_{\mathbf{k}\sigma}^\dagger) \\
& - i \frac{2\omega_k \omega_p}{\omega_\lambda - \omega_k} (b_{\mathbf{k}\sigma} - b_{-\mathbf{k}\sigma}^\dagger)(a_{-\mathbf{k}\sigma} + a_{\mathbf{k}\sigma}^\dagger) \\
& - i \frac{2\omega_\lambda \omega_p}{\omega_\lambda - \omega_k} (b_{-\mathbf{k}\sigma} + b_{\mathbf{k}\sigma}^\dagger)(a_{\mathbf{k}\sigma} - a_{-\mathbf{k}\sigma}^\dagger)] \tag{6}
\end{aligned}$$

Since the mode with momentum $\hbar \mathbf{k}_\lambda$ and energy $\hbar \omega_\lambda$ corresponds to the MP rather than to the BBR, the summation over $\mathbf{k}$ in the above equations must exclude $\mathbf{k}_\lambda$, because including it would yield $\omega_k = \omega_\lambda$, producing a zero denominator and a singularity. The same exclusion rule applies to analogous situations throughout this paper. Where

$$\omega_A^2 = \omega_k^2 [1 + \frac{\omega_p^2}{(\omega_\lambda - \omega_k)^2}], \quad \omega_B^2 = \omega_\lambda^2 [1 + \frac{\omega_p^2}{(\omega_\lambda - \omega_k)^2}]$$

In the third step, we apply the scale transformation described in Step 1 to the Hamiltonian H_3 , resulting in H_4 . A scale transformation, analogous to the one performed earlier, is then applied to H_3

$$\begin{aligned}
(a_{\mathbf{k}\sigma} + a_{-\mathbf{k}\sigma}^\dagger) &= \omega_A^{-1/2} (\Xi_{\mathbf{k}\sigma} + \Xi_{-\mathbf{k}\sigma}^\dagger) \\
(a_{\mathbf{k}\sigma} - a_{-\mathbf{k}\sigma}^\dagger) &= \omega_A^{1/2} (\Xi_{\mathbf{k}\sigma} - \Xi_{-\mathbf{k}\sigma}^\dagger) \\
(b_{\mathbf{k}\sigma} + b_{-\mathbf{k}\sigma}^\dagger) &= \omega_B^{-1/2} (\Theta_{\mathbf{k}\sigma} + \Theta_{-\mathbf{k}\sigma}^\dagger) \\
(b_{\mathbf{k}\sigma} - b_{-\mathbf{k}\sigma}^\dagger) &= \omega_B^{1/2} (\Theta_{\mathbf{k}\sigma} - \Theta_{-\mathbf{k}\sigma}^\dagger)
\end{aligned}$$

therefore

$$H_4 = \frac{\hbar}{4} \sum_{\mathbf{k}\sigma} \{ \omega_B [-(b_{-\mathbf{k}\sigma} - b_{\mathbf{k}\sigma}^\dagger)(b_{\mathbf{k}\sigma} - b_{-\mathbf{k}\sigma}^\dagger)$$

$$\begin{aligned}
& + (b_{\mathbf{k}\sigma} + b_{-\mathbf{k}\sigma}^\dagger)(b_{-\mathbf{k}\sigma} + b_{\mathbf{k}\sigma}^\dagger)] \\
& + \omega_A [-(a_{-\mathbf{k}\sigma} - a_{\mathbf{k}\sigma}^\dagger)(a_{\mathbf{k}\sigma} - a_{-\mathbf{k}\sigma}^\dagger) \\
& + (a_{\mathbf{k}\sigma} + a_{-\mathbf{k}\sigma}^\dagger)(a_{-\mathbf{k}\sigma} + a_{\mathbf{k}\sigma}^\dagger)] \\
& - i \frac{2\omega_p \sqrt{\omega_\lambda \omega_k}}{\omega_\lambda - \omega_k} [(b_{\mathbf{k}\sigma} - b_{-\mathbf{k}\sigma}^\dagger)(a_{-\mathbf{k}\sigma} + a_{\mathbf{k}\sigma}^\dagger) \\
& + (b_{-\mathbf{k}\sigma} + b_{\mathbf{k}\sigma}^\dagger)(a_{\mathbf{k}\sigma} - a_{-\mathbf{k}\sigma}^\dagger)] \} \\
& = \sum_{\mathbf{k}\sigma} [\hbar\omega_B (b_{\mathbf{k}\sigma}^\dagger b_{\mathbf{k}\sigma} + \frac{1}{2}) + \hbar\omega_A (a_{\mathbf{k}\sigma}^\dagger a_{\mathbf{k}\sigma} + \frac{1}{2}) \\
& - i \frac{\hbar\omega_p \sqrt{\omega_\lambda \omega_k}}{\omega_\lambda - \omega_k} (b_{\mathbf{k}\sigma} a_{-\mathbf{k}\sigma} - b_{-\mathbf{k}\sigma}^\dagger a_{\mathbf{k}\sigma}^\dagger)] \tag{7}
\end{aligned}$$

In the fourth step, Bogoliubov transformations are applied to the Hamiltonian H_4 . We introduce two new complex bosonic operators $\alpha_{\mathbf{k}\sigma}$ and $\beta_{\mathbf{k}\sigma}$, representing the new α and β elementary excitations, respectively. These operators are defined as:

$$\alpha_{\mathbf{k}\sigma} \equiv c_1 b_{\mathbf{k}\sigma} + ic_2 a_{-\mathbf{k}\sigma}^\dagger, \tag{8}$$

$$\beta_{\mathbf{k}\sigma} \equiv c_1 a_{\mathbf{k}\sigma} + ic_2 b_{-\mathbf{k}\sigma}^\dagger, \tag{9}$$

$$c_1^2 = \frac{1}{2} \left[1 + \frac{(\omega_\lambda + \omega_k) I_1}{(\omega_\lambda - \omega_k) I_2} \right],$$

$$c_2^2 = \frac{1}{2} \left[-1 + \frac{(\omega_\lambda + \omega_k) I_1}{(\omega_\lambda - \omega_k) I_2} \right],$$

$$I_1 = \sqrt{(\omega_\lambda - \omega_k)^2 + \omega_p^2} = \sqrt{(\omega_\lambda - vk)^2 + \omega_p^2},$$

$$I_2 = \sqrt{(\omega_\lambda + \omega_k)^2 + \omega_p^2} = \sqrt{(\omega_\lambda + vk)^2 + \omega_p^2}$$

where v denotes the speed of the bare photons that constitute the α and β elementary excitations. The commutation relations satisfied by the operators $\alpha_{\mathbf{k}\sigma}$ and $\beta_{\mathbf{k}\sigma}$ are given by: $[\alpha_{\mathbf{k}\sigma}, \alpha_{\mathbf{k}'\sigma'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'} \delta_{\sigma\sigma'}$, $[\beta_{\mathbf{k}\sigma}, \beta_{\mathbf{k}'\sigma'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'} \delta_{\sigma\sigma'}$, $[\alpha_{\mathbf{k}\sigma}, \alpha_{\mathbf{k}'\sigma'}] = [\alpha_{\mathbf{k}\sigma}, \beta_{\mathbf{k}'\sigma'}] = [\alpha_{\mathbf{k}\sigma}, \beta_{\mathbf{k}'\sigma'}^\dagger] = [\beta_{\mathbf{k}\sigma}, \beta_{\mathbf{k}'\sigma'}] = 0$. Hence, the final diagonalized Hamiltonian H_f is

$$H_f = \sum_{\mathbf{k}\sigma} \left[\hbar\omega_\alpha (\alpha_{\mathbf{k}\sigma}^\dagger \alpha_{\mathbf{k}\sigma} + 1/2) + \hbar\omega_\beta (\beta_{\mathbf{k}\sigma}^\dagger \beta_{\mathbf{k}\sigma} + 1/2) \right] \tag{10}$$

$$\hbar\omega_\alpha = (\hbar/2)(I_2 + I_1) \tag{11}$$

$$\hbar\omega_\beta = (\hbar/2)(I_2 - I_1) \tag{12}$$

Therefore, the exact eigenvalues $E_{n_{\alpha\mathbf{k}\sigma} n_{\beta\mathbf{k}\sigma}}$ and Fock states $|n_{\alpha\mathbf{k}\sigma} n_{\beta\mathbf{k}\sigma}\rangle$ for H_f are as follows:

$$E_{n_{\alpha\mathbf{k}\sigma} n_{\beta\mathbf{k}\sigma}} = \hbar\omega_\alpha (n_{\alpha\mathbf{k}\sigma} + 1/2) + \hbar\omega_\beta (n_{\beta\mathbf{k}\sigma} + 1/2) \tag{13}$$

$$|n_{\alpha_{\mathbf{k}\sigma}}\rangle = \frac{(\alpha_{\mathbf{k}\sigma}^\dagger)^{n_{\alpha_{\mathbf{k}\sigma}}}}{\sqrt{n_{\alpha_{\mathbf{k}\sigma}}!}} |0\rangle, |n_{\beta_{\mathbf{k}\sigma}}\rangle = \frac{(\beta_{\mathbf{k}\sigma}^\dagger)^{n_{\beta_{\mathbf{k}\sigma}}}}{\sqrt{n_{\beta_{\mathbf{k}\sigma}}!}} |0\rangle \quad (14)$$

$$|n_{\alpha_{\mathbf{k}\sigma}} n_{\beta_{\mathbf{k}\sigma}}\rangle = |n_{\alpha_{\mathbf{k}\sigma}}\rangle \otimes |n_{\beta_{\mathbf{k}\sigma}}\rangle, n_{\alpha_{\mathbf{k}\sigma}}, n_{\beta_{\mathbf{k}\sigma}} = 0, 1, 2, \dots \quad (15)$$

Mathematically, H is identical to Hopfield's Hamiltonian[3]. However, the new elementary excitation operators we obtained, as shown in Equations (8) and (9), are formed solely through the coupling of two operators originating from the MP and the bare photons, whereas Hopfield's new elementary excitation operator is composed of the coupling of four operators, namely[3]

$$\alpha_{\mathbf{k}\sigma} \equiv w a_{\mathbf{k}\sigma} + x b_{\mathbf{k}\sigma} + y a_{-\mathbf{k}\sigma}^\dagger + z b_{-\mathbf{k}\sigma}^\dagger$$

From Equations (11) or (12), it can be seen that the α and β elementary excitations have their respective spectral ranges: $0 \leq \omega_\beta(k) < \omega_\lambda$, $\omega_\alpha(k) \geq \omega_d$, where $\omega_d^2 \equiv \omega_\lambda^2 + \omega_p^2$. Therefore, does $\omega_\lambda < \omega < \omega_d$ correspond to the energy gap? If not, what is the relationship between these energy states and the α and β elementary excitations?

To address this, we first determine the wave vector of an elementary excitation based on Equations (11) or (12):

$$k^2 = \frac{\omega^2(\omega^2 - \omega_d^2)}{v^2(\omega^2 - \omega_\lambda^2)} \quad (16)$$

When $\omega_\lambda < \omega < \omega_d$, the wave vector k becomes imaginary. This indicates that the energy states within this spectral range are unstable. Substituting an imaginary wave vector ik into Equations (11) and (12), we find that $\hbar\omega_\beta(ik)$ is imaginary and can be neglected. The remaining term, $\hbar\omega_\alpha(ik)$, is real, and it is defined as $\hbar\omega_\gamma(k)$, i.e., $\hbar\omega_\gamma(k) \equiv \hbar\omega_\alpha(ik)$, and is given by:

$$\begin{aligned} \hbar\omega_\gamma(k) &= \frac{\hbar}{2} \left(\sqrt{(\omega_\lambda + iv_\gamma k)^2 + \omega_p^2} + \sqrt{(\omega_\lambda - iv_\gamma k)^2 + \omega_p^2} \right) \\ &= \frac{\hbar}{\sqrt{2}} \left(\omega_d^2 - v_\gamma^2 k^2 + \sqrt{(\omega_d^2 - v_\gamma^2 k^2)^2 + 4v_\gamma^2 k^2 \omega_\lambda^2} \right)^{1/2} \end{aligned} \quad (17)$$

The corresponding energy states are referred to as γ elementary excitations. Alternatively, the γ elementary excitation can be seen as the α elementary excitation with an imaginary wave vector. Here, v_γ denotes the speed of the bare photon that contributes to the formation of the γ elementary excitation. In summary, $\omega_\beta(k)$, $\omega_\gamma(k)$ and $\omega_\alpha(k)$ have their respective spectral ranges: $0 \leq \omega_\beta(k) < \omega_\lambda$, $\omega_\lambda < \omega_\gamma(k) \leq \omega_d$ and $\omega_\alpha(k) \geq \omega_d$. As shown in Fig. 1.

As shown in Fig. 1, the β elementary excitation can be regarded as a kind of condensed state; therefore, we refer to it as the β condenson. The γ and α elementary excitations correspond to its first and second excited states, and are accordingly called the γ and α condensons, respectively. To emphasize their relation to the λ transition

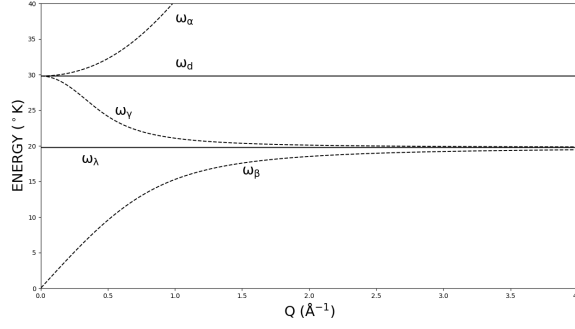


Fig. 1 Shows the dispersion curves of $\alpha(\lambda)$, $\gamma(\lambda)$, $\beta(\lambda)$ condensons. The parameters used here are $\hbar\omega_\lambda = 39$ K , $\hbar\omega_p = 44$ K , $v = 61.5$ KÅ = 805.2 m/s , $v_\gamma = 120$ KÅ = 1571.1 m/s.

point, the α , β , and γ condensons are respectively denoted as $\alpha(\lambda)$, $\beta(\lambda)$, and $\gamma(\lambda)$ condensons. The abbreviation is λ -condensons.

4 Key Characteristics of the α , β and γ Condensons

4.1 The Dispersion Relations of Condensons Should Depend on the Temperature of the System

In the previous section, we obtained the exact solution of H at $T = T_\lambda$. Therefore, the dispersion relations of the condensons can be explicitly written as $\omega_\beta(T_\lambda, k)$, $\omega_\gamma(T_\lambda, k)$, and $\omega_\alpha(T_\lambda, k)$. The question now is how the dispersion relations of the condensons change when $T \neq T_\lambda$. From the results obtained above, it is known that $\omega_\beta(T_\lambda, k)$, $\omega_\gamma(T_\lambda, k)$, and $\omega_\alpha(T_\lambda, k)$ each possess a finite spectral range, which implies that they cannot be in thermal equilibrium with the bare photon system at arbitrary temperatures. We find a way to resolve this problem by examining the Planck distribution satisfied by the bare photon system. The idea is to transform the Planck distribution at temperature T into that at temperature T_λ , with the correspondence $\omega(T, k)/T = \omega(T_\lambda, k)/T_\lambda$ that is,

$$\omega(T, k) = (T/T_\lambda) \omega(T_\lambda, k) \quad (18)$$

Assuming a similar scaling relations for condensons, we obtain:

$$\begin{aligned} \omega_\beta(T, k) &= (T/T_\lambda) \omega_\beta(T_\lambda, k) \\ \omega_\gamma(T, k) &= (T/T_\lambda) \omega_\gamma(T_\lambda, k) \\ \omega_\alpha(T, k) &= (T/T_\lambda) \omega_\alpha(T_\lambda, k) \end{aligned} \quad (19)$$

From Equation (19), we can then derive the respective temperature ranges over which the β , γ , and α condensons can reach thermal equilibrium with the blackbody radiation (BBR). For $\beta(\lambda)$ condensons, since the variation of the wavevector $k \in [0, \infty)$ does

not change the range of the spectrum, i.e., $0 \leq \omega_\beta(T, k) < \omega_\lambda$ and $0 \leq \omega_\beta(T_\lambda, k) < \omega_\lambda$, combined with equation (19), this implies that the $\beta(\lambda)$ condensons have their own temperature range: $\beta(\lambda)$ condensons: $0 \leq T < T_\lambda$. Similarly, $\alpha(\lambda)$ condensons: Imaginary wave vector part ($\gamma(\lambda)$ condensons): $T_\lambda < T \leq T_{\lambda d}$, where $T_{\lambda d} \equiv (\omega_d/\omega_\lambda)T_\lambda$; Real wave vector part: $T \geq T_{\lambda d}$.

Using Equations (16) and (19), the density of state of λ -condensons $g(k)$ is

$$\begin{aligned} g(k) &= \frac{2V}{h^3} 4\pi \hbar^3 k^2 = \frac{V\omega^2(T, k)}{\pi^2 v^2} \frac{\omega^2(T, k) - \omega_d^2}{\omega^2(T, k) - \omega_\lambda^2} \\ &= \frac{Vb^2\omega^2(T_\lambda, k)}{\pi^2 v^2} \frac{b^2\omega^2(T_\lambda, k) - \omega_d^2}{b^2\omega^2(T_\lambda, k) - \omega_\lambda^2} \end{aligned} \quad (20)$$

where h is Planck constant, $\hbar = h/2\pi$, $b \equiv T/T_\lambda$. Depending on the temperature range, $\omega(T_\lambda, k)$ can be expressed as $\omega_\beta(T_\lambda, k)$, $\omega_\gamma(T_\lambda, k)$ or $\omega_\alpha(T_\lambda, k)$. In the case of λ -condensons, the density of states $g(k)$ tends to infinity only when $T \rightarrow T_\lambda$.

4.2 Conditions for the Superfluidity of Condensons

We know that Landau[4] once established the critical velocity criterion for superfluidity. The dispersion relations of the transverse modes $\alpha(\lambda)$, $\gamma(\lambda)$, and $\beta(\lambda)$ of condensons are shown in Fig. 1. The $\gamma(\lambda)$ and $\alpha(\lambda)$ condensons are formally the first and second excited bands of the $\beta(\lambda)$ condenson, respectively. According to Landau's criterion, Fig. 1 shows that only $\partial\omega_\beta(T, k)/\partial k > 0$ in the long-wavelength limit. This implies that only $\beta(\lambda)$ condensons may exhibit superfluidity.

Below, we analyze the conditions for the superfluidity of ^4He based on our model. In order to obtain a clearer physical picture of the condenson operators defined by Eqs.(8) and (9), it is necessary to derive the canonical coordinates and canonical momenta of the condensons. The corresponding canonical coordinates $Q_\alpha(\mathbf{k}, \sigma)$, $Q_\beta(\mathbf{k}, \sigma)$ and momenta $\Pi_\alpha(\mathbf{k}, \sigma)$, $\Pi_\beta(\mathbf{k}, \sigma)$ are given by

$$\begin{aligned} Q_\alpha(\mathbf{k}, \sigma) &= \left(\frac{\hbar}{2m\omega_\alpha} \right)^{1/2} (\alpha_{\mathbf{k}\sigma} + \alpha_{-\mathbf{k}\sigma}^\dagger) \\ &= c_1 \sqrt{\frac{\omega_\lambda}{\omega_\alpha}} B(\mathbf{k}, \sigma) + \frac{c_2}{\sqrt{mm_p\omega_\alpha\omega_k}} \Pi_A(\mathbf{k}, \sigma) \\ \Pi_\alpha(\mathbf{k}, \sigma) &= -i \left(\frac{\hbar m\omega_\alpha}{2} \right)^{1/2} (\alpha_{\mathbf{k}\sigma} - \alpha_{-\mathbf{k}\sigma}^\dagger) \\ &= c_1 \sqrt{\frac{\omega_\alpha}{\omega_\lambda}} \Pi_B(\mathbf{k}, \sigma) + c_2 \sqrt{mm_p\omega_\alpha\omega_k} A(\mathbf{k}, \sigma) \\ Q_\beta(\mathbf{k}, \sigma) &= \left(\frac{\hbar}{2m\omega_\beta} \right)^{1/2} (\beta_{\mathbf{k}\sigma} + \beta_{-\mathbf{k}\sigma}^\dagger) \\ &= c_1 \sqrt{\frac{m_p\omega_k}{m\omega_\beta}} A(\mathbf{k}, \sigma) + \frac{c_2}{\sqrt{m^2\omega_\beta\omega_\lambda}} \Pi_B(\mathbf{k}, \sigma) \end{aligned} \quad (21)$$

$$\begin{aligned}
\Pi_\beta(\mathbf{k}, \sigma) &= -i \left(\frac{\hbar m \omega_\beta}{2} \right)^{1/2} (\beta_{\mathbf{k}\sigma} - \beta_{-\mathbf{k}\sigma}^\dagger) \\
&= c_1 \sqrt{\frac{m \omega_\beta}{m_p \omega_k}} \Pi_A(\mathbf{k}, \sigma) + c_2 \sqrt{m^2 \omega_\beta \omega_\lambda} B(\mathbf{k}, \sigma)
\end{aligned} \tag{22}$$

And we have $[Q_\alpha(\mathbf{k}, \sigma), \Pi_\alpha(-\mathbf{k}', \sigma')] = i\hbar \delta_{\mathbf{k}\mathbf{k}'} \delta_{\sigma\sigma'}$, $[Q_\beta(\mathbf{k}, \sigma), \Pi_\beta(-\mathbf{k}', \sigma')] = i\hbar \delta_{\mathbf{k}\mathbf{k}'} \delta_{\sigma\sigma'}$. Interestingly, the canonical coordinates and momenta of $\alpha(\lambda)$ and $\beta(\lambda)$ condensons exhibit a chimeric nature, being crossed linear combinations of MP and BBR components. Their expectation values under the exact solution $|n_{\alpha_{\mathbf{k}\sigma}}, n_{\beta_{\mathbf{k}\sigma}}\rangle$ obtained above are all zero.

Considering that the ^4He atoms in an MP (referred to as superfluid ^4He atoms) are modulated by photons, they form a coherent whole—namely, a condenson. That is, a condenson consists of an MP plus blackbody radiation. At the same time, they interact with the ^4He atoms outside the MP. Such interactions usually hinder the motion of condensons in the system. From Equation (9), the annihilation of a $\beta(\lambda)$ condenson corresponds to the simultaneous occurrence of the following two processes: the annihilation of a photon in the state $(\mathbf{k}, \sigma)$ and the creation of an MP in the state $(-\mathbf{k}, \sigma)$. The photon momentum $\hbar\mathbf{k}$ acts, from the viewpoint of the MP, as an external force that drives the transport of the MP mass, and this is precisely the dynamical reason why the MP can possess superfluidity. This situation is analogous to the principle of rocket propulsion. Only when the driving force provided by photons exceeds this resistance can the superfluidity of condensons be observed. At the λ point of ^4He ($T_\lambda = 2.17\text{ K}$), condensons with superfluidity are generated, whereas in most other cases the condensons that form do not possess superfluidity. The critical transition of ^4He atoms at the temperature $T_c = 5.19\text{ K}$ is precisely one example of such a situation. Here we would like to emphasize explicitly that the emergence of critical points and λ points is based on the same physical mechanism—the formation of condensons. Their common feature is that the specific heat diverges at these transition points, which reflects the divergence of the corresponding density of states of the condensons at these points.

4.3 Composition of ^4He in Different Temperature Ranges

First, several characteristic temperatures of ^4He are marked on the temperature axis, as shown in Fig. 2. T_c , T_b and T_λ represent the critical, boiling and lambda temperatures, respectively, with values of 5.18 K, 4.22 K and 2.17 K. T_{cd} denotes the temperature corresponding to $T_{\lambda d}$.

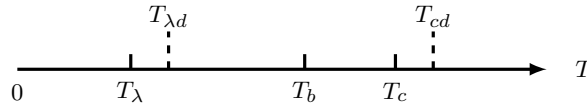


Fig. 2 Schematic diagram of several characteristic temperatures of ^4He .

Table 1 The composition of ^4He as a function of temperature.

Temperature range	Components	Notes
$[T_c, \infty)$	classical ideal gas ^4He	$(T_c, T_{cd}]$ $\gamma(c)$ condensons while heating across T_c
$[T_b, T_c)$ $(T_\lambda, T_b]$	$\beta(c)$ condensons, gas ^4He $\beta(c)$ condensons, normal liquid ^4He	$(T_\lambda, T_{\lambda d}]$ $\gamma(\lambda)$ condensons while heating across T_λ
$[0, T_\lambda)$	$\beta(c)$ and $\beta(\lambda)$ condensons, normal liquid ^4He	

Starting from temperatures above T_c , we gradually lower the temperature in sequence to describe the evolution of different structural components in the ^4He system. When the density is ρ_c and the temperature is reduced to T_c , non-superfluid (damped) $\beta(c)$ condensons are generated. Conversely, if the system is heated from below T_c up to T_{cd} , then $\gamma(c)$ condensons should exist in the interval $T_c < T \leq T_{cd}$.

It's important to note that when heating or cooling liquid helium across the critical point, the microscopic structures on either side of T_c are asymmetric. This asymmetry leads to a difference between the transition temperatures during heating, $T_c(\text{heating})$, and cooling, $T_c(\text{cooling})$. This difference is known as thermal hysteresis. Generally, the transition temperature T_c is higher during heating than during cooling. Near the critical point of CO_2 , this kind of thermal hysteresis has indeed been observed[5]. This hysteresis should be also observed at the lambda point.

For the lambda point, the situation is the same as that at the critical point, except that the $\beta(\lambda)$ condensons are superfluid. The properties of the $\alpha(c)$ and $\alpha(\lambda)$ condensons are very similar to those of bare photons and are therefore not listed separately in Table 1.

5 Applications of the Condenson Model to Helium-4

5.1 Temperature Range with an Imaginary Refractive Index

Based on Equations (16) and (19), the transverse dielectric function, ε_t (or equivalently, the index of refraction, n) can be determined.

$$n^2 = \varepsilon_t(\omega) \equiv \frac{k^2}{\omega^2} = \frac{1}{v^2} \frac{b^2 \omega^2(T_\lambda, k) - \omega_d^2}{b^2 \omega^2(T_\lambda, k) - \omega_\lambda^2} \quad (23)$$

When $T_\lambda < T \leq T_{\lambda d}$ or $T_c < T \leq T_{cd}$, $n^2 < 0$, which indicates strong light scattering by $\gamma(\lambda)$ condenson or damped $\gamma(c)$ condenson. However, when $T_\lambda < T \leq T_{\lambda d}$, the $\gamma(\lambda)$ condenson coexists with the damped $\beta(c)$ condensons, which does not scatter light strongly. As a result, the reflection of light at the critical point is stronger than that at the lambda point.

5.2 Specific Heat Across All Temperatures

One of the most important physical quantities for superfluid ^4He is a sharp, narrow, and singular peak in the specific heat at the so-called lambda point and critical point, T_λ and T_c [6]. Based on the renormalization group (RG)[7, 8] approach, the specific heat can be calculated closely near the transition point. However, far from T_c , the RG approach does not agree well with experimental data. Moreover, the RG approach does not provide a physical mechanism to explain why the system fluctuates so strongly near the lambda and critical points. As for Tisza's[9] and Landau's two-fluid model[4], it does not predict any singularity in the specific heat. Based on the condenson model, we find that below T_λ the specific heat is mainly contributed by $\beta(\lambda)$ condensons, while above T_λ , it is mainly contributed by $\gamma(\lambda)$ condensons. Similarly, below T_c , the specific heat is mainly contributed by $\beta(c)$ condensons, while above T_c , it is mainly contributed by $\gamma(c)$ condensons. This means that the physical mechanisms underlying the specific heat on either side of the singularity are different.

The specific heat of the condensons at constant volume is given by the following equation:

$$C_v = k_B \int_0^\infty \left(\frac{\hbar\omega}{2k_B T} \right)^2 \text{csch}^2 \left(\frac{\hbar\omega}{2k_B T} \right) g(k) dk \quad (24)$$

Where k_B is the Boltzmann constant. Using Equation (20) and temperature as the energy unit, the specific heat of λ condensons can be calculated with three temperature intervals:

$$\begin{aligned} C_{v\beta}(T) &= \frac{C_\lambda}{v^2 T^2} \int_0^\infty S(b\omega_\beta) dk, & T \in [0, T_\lambda) \\ C_{v\gamma}(T) &= \frac{C_\lambda}{v_\gamma^2 T^2} \int_\infty^0 S(b\omega_\gamma) dk, & T \in (T_\lambda, T_{\lambda d}] \\ C_{v\alpha}(T) &= \frac{C_\lambda}{v^2 T^2} \int_0^\infty S(b\omega_\alpha) dk, & T \in [T_{\lambda d}, \infty) \\ S(\omega) &\equiv \frac{\omega^2(T, k) - \omega_d^2}{\omega^2(T, k) - \omega_\lambda^2} \omega^4(T, k) \text{csch}^2(\omega(T, k)/2T). \end{aligned} \quad (25)$$

where $C_\lambda \equiv R/(4\pi^2 \rho_{s\lambda})$, $R = k_B N_A$, where N_A is the Avogadro constant. $\rho_{s\lambda} \equiv N_\lambda/V$ is the density of superfluid, which should vary with temperature but is not predicted by our model and must be obtained experimentally. Here, it is treated as a constant because the experimental data for specific heat includes contributions from both the superfluid and non-superfluid components, so this constant represents the portion contributed by the superfluid component. For the damped c condensons, one can substitute the parameters for T_c into Equation (25) to obtain a different set of C_v values. There are six types of condensons that contribute to the specific heat: $\alpha(\lambda)$, $\beta(\lambda)$, $\gamma(\lambda)$, $\alpha(c)$, $\beta(c)$ and $\gamma(c)$ condenson. Since each condenson is independent of one another, the specific heat for any given temperature is the superposition of the

two where the temperatures overlap as shown in Fig. 3. According to the Dulong-Petit law, $C_v = 3R/2$ when $T > T_c$.

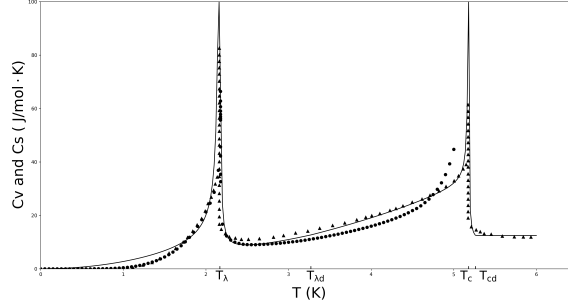


Fig. 3 Temperature dependence of the specific heat of ^4He . Circles[10] represent experimental data of the specific heat at saturated vapor pressure; triangles[6] represent data at constant volume; the solid line is the calculated result from this work. $\rho_{s\lambda}$, ρ_{sc} and the parameters of the spectra used in the calculations are as follows: for the λ point, $\rho_{s\lambda} = 0.00736 \text{ g/cm}^3$ (as a reference, the experimental value[10] of the density of superfluid ^4He at 2.17 K is 0.00729 g/cm^3), $\hbar\omega_\lambda$, $\hbar\omega_p$, v and v_γ are the same as those in Fig. 1; for the critical point, $\rho_{sc} = 0.00364 \text{ g/cm}^3$ (no experimental data is available for reference), $\hbar\omega_c = 100 \text{ K}$, $\hbar\omega_p = 18 \text{ K}$, $v = 61.5 \text{ K}\text{\AA} = 805.2 \text{ m/s}$, $v_\gamma = 90 \text{ K}\text{\AA} = 1178.3 \text{ m/s}$.

5.3 Landau's Phonon-Roton Dispersion Relation

When calculating C_v values, we used six dispersion relations from two types of condensons: λ -condenson and c -condenson. None of these dispersion relations match the P-R curve predicted by Landau[11]. Since then, numerous complex theoretical models, such as Feynman's model[12, 13], have been proposed to calculate it, but they show significant discrepancies with experimental results. Moreover, hundreds of papers have been referenced in Glyde[14] and Godfrin's[15] survey of superfluid ^4He papers. Despite these efforts, the physical picture of the roton[11] remains ambiguous. Since the P-R curve is obtained from neutron inelastic scattering experiments, we will analyze the experimental processes and data using our model.

The kinematics of neutron scattering are illustrated in Fig. 4. Initially, an incident neutron has initial energy E_0 and momentum k_0 . Before exiting, it undergoes two scatterings. At (a) in Fig. 4, it upscatters with an unstable $\gamma(\lambda)$ condenson. The $\gamma(\lambda)$ condenson has excitation energy $\Delta E_\gamma(iQ) = \hbar\omega_\gamma(Q_0 - Q) - \hbar\omega_\lambda$, where $\hbar\omega_\lambda$ is the ground state energy of the γ condenson, and Q_0 is determined by $\hbar\omega_\gamma(Q_0) = \hbar\omega_\lambda$. After (a), the incident neutron energy becomes $E_0 + \Delta E_\gamma(iQ)$, with no change in its momentum due to the imaginary momentum of the γ condenson. Following this, at (b) in Fig. 4, the neutron downscatters with a stable $\beta(\lambda)$ condenson. The $\beta(\lambda)$ condenson gains momentum Q with energy $\hbar\omega_\beta(Q)$. Therefore, the final transferred energy of the outgoing neutron $\Delta\varepsilon(Q) = E_0 - [E_0 + \Delta E_\gamma(iQ) - \hbar\omega_\beta(Q)] = \hbar\omega_\beta(Q) - [\hbar\omega_\gamma(Q_0 - Q) - \hbar\omega_\lambda]$. The final momentum of the outgoing neutron is $k_0 - (k_0 - Q) = Q$. When we substitute parameters from the captions of Fig. 1 and Fig. 3, we obtain Fig. 5. The

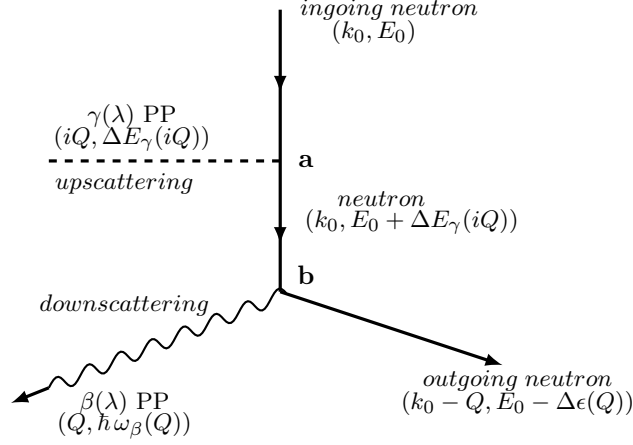


Fig. 4 Schematic diagram of inelastic scattering processes between a neutron and $\gamma(\lambda)$, $\beta(\lambda)$ condensons.

curve $\Delta\epsilon(Q) \sim Q$ is precisely the P-R curve. Hence, we can conclude that there is no such thing as a roton and the P-R curve cannot even be called a dispersion relation because it does not correspond to the spectrum of a single quasiparticle. The P-R curve is merely a spurious effect caused by neutron inelastic scattering.

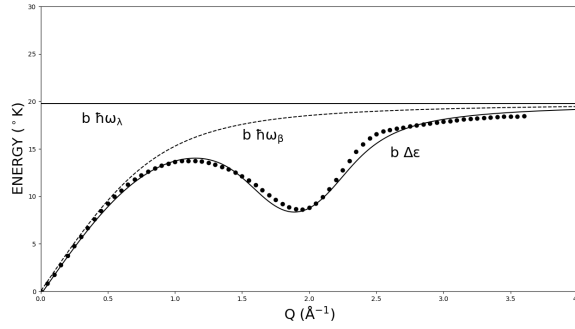


Fig. 5 Shows the curves of $b\hbar\omega_\lambda$, $b\hbar\omega_\beta$ and the P-R curve $b\Delta\epsilon(Q)$ along with experimental data [10, 16] obtained from inelastic neutron scattering at a temperature of $T_{\text{exp}}=1.1$ K. The parameters used here are the same as those in Fig. 1. The wavevector Q_0 is determined by the condition $\hbar\omega_\gamma(Q_0) = \hbar\omega_\lambda$, yielding a value of $Q_0 \approx 1.91\text{\AA}^{-1}$. The reduced temperature b is defined as the ratio of the experimental temperature to the lambda temperature, i.e., $b = T_{\text{exp}}/T_\lambda = 1.1/2.17$.

5.4 Quantization of Circulation and Magnetic Flux of Rotating Helium II

When a cylindrical container of He II is rotated about its axis at a constant angular velocity exceeding a small critical value Ω_{c1} , many Onsager–Feynman vortex lines [17]

are formed in the container. They are arranged parallel to the axis of rotation and are approximately uniformly distributed over the cross section of the container. When the angular velocity exceeds another critical value Ω_{c2} , the Onsager–Feynman vortex lines and the superfluidity of He II disappear, and the system turns into He I[18]. According to the currently prevailing view[19], the solid-like behavior of He II during overall rotation is attributed to the appearance of Onsager–Feynman vortex lines. The velocity $\mathbf{v}_s$ of the superfluid component satisfies the equation $\nabla \times \mathbf{v}_s = n\Gamma_0 \sum_i \delta(\mathbf{r} - \mathbf{r}_i)$, where $n = 0, \pm 1, \pm 2, \dots$, $\Gamma_0 \equiv h/m$ is the quantum of circulation, and $\mathbf{r}_i$ is the two-dimensional position vector of a vortex line in an arbitrary plane perpendicular to the rotation axis.

The following provides an alternative explanation based on our condensation model. Before the cylindrical container begins to rotate, the superfluid and non-superfluid components of ^4He are uniformly mixed, forming an isotropic liquid in thermodynamic equilibrium, and no Onsager–Feynman vortex lines appear. However, when the cylindrical container rotates at a constant angular velocity ω_0

$$\Omega_{c1} < \omega_0 < \Omega_{c2},$$

consider two ^4He atoms located at positions $\mathbf{r}_i$ and $\mathbf{r}_j$ on the free surface of the liquid. Due to the rotation of the container, they acquire additional velocities and kinetic energies,

$$\mathbf{v}_i = \boldsymbol{\omega}_0 \times \mathbf{r}_i, \quad \mathbf{v}_j = \boldsymbol{\omega}_0 \times \mathbf{r}_j,$$

$$E_i = \frac{1}{2}m\omega_0^2 r_i^2, \quad E_j = \frac{1}{2}m\omega_0^2 r_j^2.$$

To eliminate this effect induced by rotation, the difference in kinetic energy between these two atoms is converted into a static potential energy difference along the rotation axis, thereby avoiding mutual friction caused by their velocity difference. Let their heights be h_i and h_j , respectively; then

$$mgh_i = \frac{1}{2}m\omega_0^2 r_i^2, \quad mgh_j = \frac{1}{2}m\omega_0^2 r_j^2.$$

This implies that the two atoms lie on the same static rotating paraboloid,

$$h = \omega_0^2 r^2 / 2g$$

Inside the liquid ^4He , the transformation of the free surface from a plane to a rotating paraboloid leads to a static pressure that increases with r along the same horizontal plane. Under these conditions, the liquid ^4He is no longer isotropic; it transitions from a thermodynamic equilibrium state to a stable state. Since our analysis does not depend on whether these atoms are in the superfluid state, liquid ^4He —whether He I, He II, or a mixture of both—exhibits solid-like behavior during overall rotation, just like He I. These conclusions are consistent with experimental observations [19, 20].

The situation is different for the ^4He atoms in contact with the surface of the rotating container. From Table 1, we observe that although the non-superfluid components (He I and $\beta(c)$ condensons) decrease as the temperature is lowered, they still exist even below $T_\lambda(2.17\text{K})$. These non-superfluid ^4He atoms experience friction with the container surface (especially the cylindrical side wall). Under the combined effects of centrifugal force, Coriolis force, and the radial pressure gradient, they give rise to Onsager–Feynman vortex lines composed of non-superfluid ^4He atoms. As the density of vortex lines increases, they move toward the central region of the container. In the ^3He system, the birth of individual vortices can be observed as jumps in the NMR signal [21].

It should be further noted that some ^4He atoms within β condensons may, through friction with the container surface, satisfy the condition $\zeta > 1/\sqrt{2}$ and consequently transform into non-superfluid ^4He atoms, thereby contributing to the formation of Onsager–Feynman vortex lines.

Similar to Vinen’s experiment [20], an experiment based on our condenson model for measuring He II circulation and Magnus force is shown in the schematic in Fig. 6. Let $\mathbf{v}_{ni}$ and $\mathbf{v}_{sj}$ denote the velocity of i th and j th ^4He atom of the normal fluid

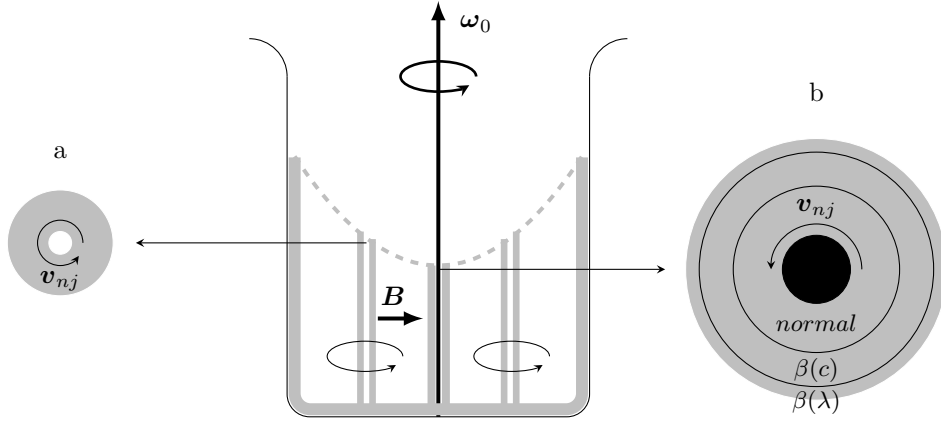


Fig. 6 Schematic diagram of an experiment similar to Vinen’s experiment[20]. A constant magnetic field, $\mathbf{B}$, is applied. (a) Cross-sectional view of the annulus transition region. (b) Cross-sectional view of the cylindrical wire and transition region. The exaggerated dimensions of this cross-section are used to clearly show the details of the composition and structure of the gray region: near the surface of the wire is normal fluid ^4He , and the thickness of a thin layer near the outer surface of the gray region represents the penetration depth of the superfluid component, $\beta(\lambda)$ condensons. The middle layer is mainly composed of damped $\beta(c)$ condensons. The two inner layers are both non-superfluid. Other gray regions in this figure can be understood similarly. The microstructures of the transition region closely resemble the response of a Type-II superconductor to a magnetic field, known as the mixed state. In Vinen’s experiment[20], there are only two components—the normal fluid and the superfluid—and no component separation occurs; therefore, there are no gray regions.

and superfluid, respectively. Since the the mechanical momentum $m\mathbf{v}_{ni}$ equals the canonical momentum $\mathbf{P}_{ni}$, applying the Bohr-Sommerfeld quantization condition, the

circulation of the normal fluid ^4He around any circuit in the transition region is:

$$\oint_{\ell} \mathbf{v}_{ni} \cdot d\mathbf{r}_i = \frac{1}{m} \oint_{\ell} \mathbf{P}_{ni} \cdot d\mathbf{r}_i = n\Gamma_0,$$

$$n = 0, \pm 1, \pm 2, \dots$$

It implies that the circulation of normal fluid ^4He is quantized in units Γ_0 . In the special case of Fig. 6 (a), due to the limitation imposed by the Heisenberg uncertainty principle, the case where $n = 0$ in nh/m should be excluded. This is why the cross-sectional diagram is hollow. It should be pointed out that, in experiments measuring circulation and Magnus force, whether the cylindrical container continues to rotate at a constant angular velocity has a significant effect on the observed results. When the rotation is stopped, circulations of 1, 2 and 3 Γ_0 can be measured [22]. When the rotation is not stopped, even a circulation of $1\Gamma_0$ has not been clearly observed [20, 23]. This clarity may arise from the influence of the entire transition region around the wire.

Another point worth noting is that the Bohr-Sommerfeld quantization condition arises from the wave nature of particles; therefore, the quantization of the circulation of normal fluid ^4He arises from the wave nature of ^4He rather than its superfluidity.

According to Equation (3), the mechanical momentum a superfluid ^4He atom, $m\mathbf{v}_{sj}$, includes both the canonical momentum $\mathbf{P}_{sj}$ and the electromagnetic momentum $\eta\mathbf{A}_j$. Landau[4] postulated that the $\mathbf{v}_{sj}$ satisfies the condition $\nabla \times \mathbf{v}_{sj} = 0$. Therefore, the circulation of the superfluid is:

$$\oint_{\ell} \mathbf{v}_{nj} \cdot d\mathbf{r}_j = \frac{1}{m} \oint_{\ell} (\mathbf{P}_{nj} - \eta\mathbf{A}_j) \cdot d\mathbf{r}_j$$

Using Landau's postulation and the Bohr-Sommerfeld quantization condition

$$\oint_{\ell} \mathbf{v}_{sj} \cdot d\mathbf{r}_j = \iint_S (\nabla \times \mathbf{v}_{sj}) \cdot d\mathbf{s} = 0$$

$$\oint_{\ell} \mathbf{P}_{sj} \cdot d\mathbf{r}_j = nh, \quad n = 0, \pm 1, \pm 2, \dots$$

And

$$\oint_{\ell} \eta\mathbf{A}_j \cdot d\mathbf{r}_j = \eta \iint_S (\nabla \times \mathbf{A}_j) \cdot d\mathbf{s} = \eta\Phi$$

where Φ is the magnetic flux passing through curl surface S . Combining these equations, we obtain $\Phi = n\Phi_0$, which is quantization of magnetic flux. $\Phi_0 \equiv h/\eta$ is the quantum of flux.

In principle, we could design experiments similar to those used to measure the quantized magnetic flux in type-II superconductors to determine the quantized magnetic flux and η in He II. However, as mentioned in the previous hypothesis, each ^4He atom contained in a $\beta(\lambda)$ condenson is electrically neutral; it only couples to photons

through polarization induced by blackbody radiation. When positive and negative charges move together, no quantized magnetic flux should be observed. Only in the case of type-II superconductivity, where a Cooper pair carries a charge of $-2e$, do we have $|\eta| = 2e/c$ and $\Phi_0 = hc/2e = 2 \times 10^{-7} \text{Gs} \cdot \text{cm}^2$, so that quantized magnetic flux can be observed.

6 Superconductivity and Superfluidity of ^3He

As pointed out in Section 2 of this paper, the fundamental constituents required by our condensation model cannot be strongly degenerate fermions. However, when these fermions interact through the exchange of quanta—for example, electric quanta (regular phonons, as in the BCS theory[24]) or magnetic quanta (such as paramagnons, as in the superfluidity of ^3He [25, 26], and so forth)—they may bind together to form Cooper pairs, which are composite bosons. If these Cooper pairs are taken as the fundamental constituents, then their further coupling to BBR photons satisfies the requirements of our condensation model.

Because fermions possess both spin angular momentum $\mathbf{S}$ and orbital angular momentum $\mathbf{L}$, a Cooper pair may carry a nonzero total angular momentum $\mathbf{J}$, and hence a net magnetic moment $\mathbf{m}_c = \gamma_e \mathbf{J}$, where γ_e is a constant and $\mathbf{J} = \mathbf{S} + \mathbf{L}$. Under the modulation of induced photons, these magnetic moments form spin-wave-like quasiparticles, $\mathbf{m}_c(\mathbf{r}_i) = \mathbf{m}_c e^{i\mathbf{k} \cdot \mathbf{r}_i}$. Let N_c denote the number of Cooper pairs, and let $\mathbf{k}^0$, $\mathbf{l}$, and $\mathbf{j}$ be the unit vectors along the directions of the wave vector $\mathbf{k}$, the orbital angular momentum $\mathbf{L}$, and the total angular momentum $\mathbf{J}$, respectively. The interaction Hamiltonian H_I describing their coupling to BBR is given by

$$H_I = - \sum_i^{N_c} \mathbf{m}_c(\mathbf{r}_i) \cdot \mathbf{B}(\mathbf{r}_i) = - \sum_i^{N_c} \langle \mathbf{m}_c(\mathbf{r}_i) | \mathbf{B}(\mathbf{r}_i) \rangle \quad (26)$$

Where

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = i \sum_{\mathbf{k}\sigma} \left(\frac{2\pi\hbar\omega_k}{V} \right)^{1/2} (\mathbf{k}^0 \times \mathbf{e}_{\mathbf{k}\sigma}) e^{i\mathbf{k} \cdot \mathbf{r}} (a_{\mathbf{k}\sigma} + a_{-\mathbf{k}\sigma}^\dagger)$$

So that

$$H_I = C_0 \sum_{\mathbf{k}\sigma} C_{\mathbf{k}\sigma} (a_{\mathbf{k}\sigma} + a_{-\mathbf{k}\sigma}^\dagger) \quad (27)$$

Where

$$C_0 \equiv N_c m_c \left(\frac{2\pi\hbar}{V} \right)^{1/2}, \quad C_{\mathbf{k}\sigma} \equiv (-i) \mathbf{j} \cdot (\mathbf{k}^0 \times \mathbf{e}_{\mathbf{k}\sigma})$$

After the operators $a_{\mathbf{k}\sigma}$ and $a_{-\mathbf{k}\sigma}^\dagger$ undergo the two scale transformations described in Section 3, H_I becomes

$$H_I = C_0 \sum_{\mathbf{k}\sigma} C_{\mathbf{k}\sigma} \sqrt{\frac{\omega_k}{\omega_A}} (a_{\mathbf{k}\sigma} + a_{-\mathbf{k}\sigma}^\dagger) \quad (28)$$

$$H_{total} = H_4 + H_I \quad (29)$$

We replace

$$a_{\mathbf{k}\sigma} \rightarrow a_{\mathbf{k}\sigma} - i \frac{C_0}{\hbar \omega_k} \left(1 + \frac{\omega_p^2}{(\omega_\lambda - \omega_k)^2} \right)^{1/4} \mathbf{j} \cdot (\mathbf{k}^0 \times \mathbf{e}_{\mathbf{k}\sigma})$$

and

$$b_{\mathbf{k}\sigma} \rightarrow b_{\mathbf{k}\sigma} - \frac{C_0}{\hbar} \frac{\omega_p}{(\omega_\lambda - \omega_k)} \frac{1}{\sqrt{\omega_\lambda \omega_k}} \left(1 + \frac{\omega_p^2}{(\omega_\lambda - \omega_k)^2} \right)^{1/4} \mathbf{j} \cdot (\mathbf{k}^0 \times \mathbf{e}_{\mathbf{k}\sigma})$$

Therefore, the exact eigenvalues $E_{n_{\alpha\mathbf{k}\sigma} n_{\beta\mathbf{k}\sigma}}$ of H_{total} are given by

$$E_{n_{\alpha\mathbf{k}\sigma} n_{\beta\mathbf{k}\sigma}} = \hbar\omega_\alpha(n_{\alpha\mathbf{k}\sigma} + 1/2) + \hbar\omega_\beta(n_{\beta\mathbf{k}\sigma} + 1/2) + E_k \quad (30)$$

Where

$$E_k = -\frac{C_0^2}{\hbar \omega_k} [1 - (\mathbf{j} \cdot \mathbf{k}^0)^2] = -\frac{C_0^2}{\hbar \omega_k} \sin^2 \theta_k \quad (31)$$

Where θ_k is the angle between $\mathbf{k}^0$ and $\mathbf{j}$. From Equations (30) and (31), it can be seen that whenever the total angular momentum of a Cooper pair is nonzero ($\mathbf{J} \neq 0$), the dispersion relation of the condensation becomes direction-dependent with respect to $\mathbf{k}^0$. Such a superfluid is known as an anisotropic superfluid (or anisotropic superconductor).

The Cooper pairs in conventional superconductors (BCS theory) are in a state with opposite spins ($S = 0$) and relative orbital angular momentum $l = 0$. This type of pairing is referred to as spin-singlet s -wave pairing. Since the total angular momentum of the Cooper pair is $J = 0$, it follows that $E_k = 0$. Consequently, the energy spectrum of a conventional superconductor is isotropic.

In contrast, in some high- T_c superconductors, the Cooper pairs are in a state with opposite spins ($S = 0$) and relative orbital angular momentum $l = 2$. This type of pairing is referred to as spin-singlet d -wave pairing. According to the present model, their dispersion relation should be anisotropic. Experimental observations are consistent with this conclusion[27].

In the absence of an external magnetic field, superfluid ^3He exhibits two distinct superfluid phases, namely the A phase and the B phase. These two phases share a common feature: the Cooper pairs are in a state with parallel spins ($S = 1$) and relative orbital angular momentum $l = 1$. This type of pairing is referred to as spin-triplet p -wave pairing. For the low-temperature B phase, which is described by the *Balian–Werthamer* (BW) state[28–30], although $S = 1$ and $l = 1$, the relation $\mathbf{S} + \mathbf{L} = \mathbf{J} = 0$ holds. Therefore, the energy spectrum of the BW state is isotropic. For the higher-temperature A phase, which is described by the *Anderson–Brinkman–Morel* (ABM) state[25, 26, 29, 30], although $S = 1$ and $l = 1$, the conditions $\mathbf{S} \perp \mathbf{k}^0$, $\mathbf{J} = \mathbf{L}$ apply. Equation (31) then reduces to

$$E_k = -\frac{C_0^2}{\hbar\omega_k} [1 - (\mathbf{l} \cdot \mathbf{k}^0)^2] = -\frac{C_0^2}{\hbar\omega_k} \sin^2 \theta_k \quad (32)$$

Therefore, the energy spectrum of the ABM state is anisotropic. Experimental studies of superfluid ^3He have likewise confirmed this anisotropy[30, 31].

7 Conclusions

We propose the materialized-photon (MP) hypothesis as a new microscopic mechanism and obtain the exact wave functions of the condensons together with their corresponding dispersion relations. Our analysis shows that the photons coupled to the MPs provide the dynamical origin for the possible superfluidity of the latter.

Based on the condenson model, we derive the phonon–roton (P–R) dispersion curve of superfluid ^4He , the specific-heat curve with singularities over the entire temperature range, and the criterion for condenson superfluidity, namely $\zeta < 1/\sqrt{2}$. The theoretical results are found to be in excellent agreement with available experimental data.

The model further provides a unified framework for understanding a variety of phenomena, including the solid-body-like rotation of He II in a rotating container and the isotropic or anisotropic energy spectra observed in conventional superconductors, high- T_c superconductors, and the BW and ABM phases of superfluid ^3He .

A particularly noteworthy application of the model concerns ^3He near its critical point. Although ^3He is a fermion, it behaves as a non-strongly-degenerate fermion in this regime, so that its statistical correlations may be neglected and it can be treated approximately as a classical single atom. Consequently, both ^3He and ^4He may be regarded as classical atomic systems near their respective critical temperatures. This suggests that the phase transitions of the two systems near their critical points are governed by the same underlying physical mechanism, namely the formation of non-superfluid condensons. Within this picture, the difference in their critical temperatures arises primarily from the difference in their atomic masses. Using the critical temperature of ^4He (5.19 K), one obtains an estimate for the critical temperature of ^3He , $T_c(^3\text{He}) \approx 5.19(m_3/m_4) = 3.89$ K, which is remarkably close to the experimental value of 3.3 K [6].

Although the physical origins of the polariton and the condenson discussed in this paper are different, their Hamiltonians are identical. Therefore, it is logically reasonable to infer that, if the β condenson can exhibit superfluidity, the polariton

may also exhibit superfluidity. In fact, this conclusion has already been experimentally confirmed [32].

In summary, the condensation model provides a unified microscopic framework for describing superfluidity and superconductivity, while yielding quantitative predictions that are consistent with a broad range of experimental observations.

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