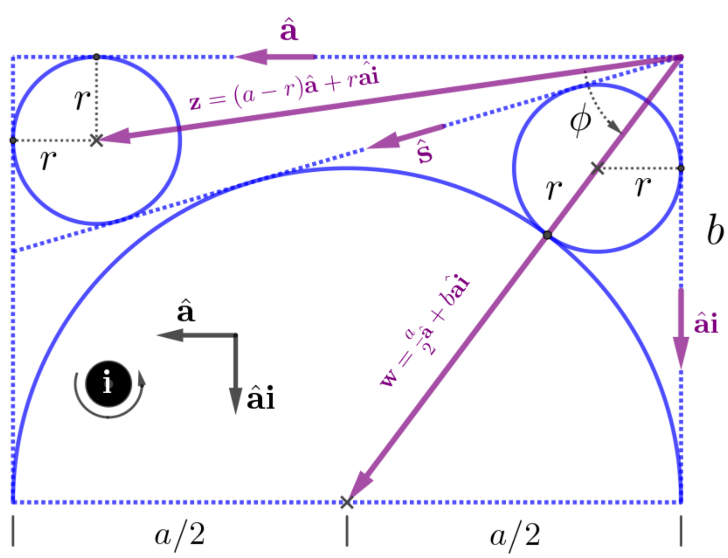
A Geometric Algebra (GA) Solution to a Multiple-Tangency Problem

February 17, 2025

James Smith
LinkedIn group "Pre-University Geometric Algebra"
https://www.linkedin.com/groups/8278281/

Abstract

Using GA's capacities for formulating reflections, we solve an interesting multiple-tangency problem. The solution is obtained in two ways, the easiest of which transforms the relevant reflections into a single rotation. The solution is validated via a GeoGebra worksheet.



The two small circles are congruent. What is the ratio $^b/_a$?

1 Statement of the Problem

Fig. 1 Shows the problem statement ([1], [2]).

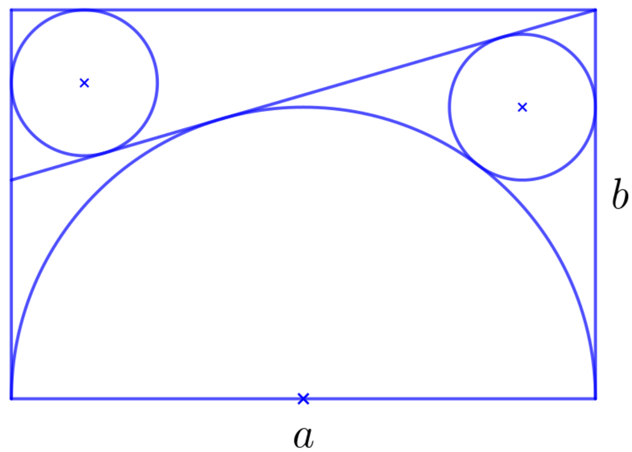


Figure 1: Problem statement: The two small circles are congruent. What is the ratio $^b/_a$?

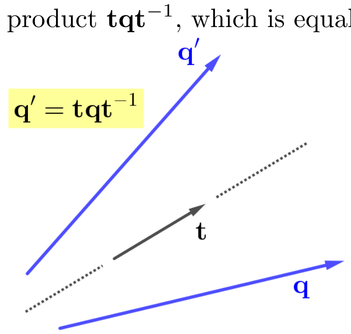
2 Ideas that We Will Use

See also Macdonald [3].

1. Two multivectors A and B are equal if and only if all of their respective parts of equal grade are equal to each other. For example, in the case of two vectors $\mathbf{u}$ and $\mathbf{v}$, if $\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\wedge\mathbf{v}=\alpha+\beta\mathbf{B}$, where α and β are scalars, and $\beta\mathbf{B}$ is a bivector, then $\mathbf{u}\cdot\mathbf{v}=\alpha$, and $\mathbf{u}\wedge\mathbf{v}=\beta\mathbf{B}$.

2. We will use Macdonald's definitions for the inner and outer product ([3], p. 101) . In the case of vectors, $\mathbf{u}\cdot\mathbf{v}=\langle\mathbf{u}\cdot\mathbf{v}\rangle_0$, and $\mathbf{u}\wedge\mathbf{v}=\langle\mathbf{u}\cdot\mathbf{v}\rangle_2$.

3. The reflection of a vector $\mathbf{q}$ with respect to vector $\mathbf{t}$ can be written as the product $\mathbf{t}\mathbf{q}\mathbf{t}^{-1}$, which is equal to $[\mathbf{t}\mathbf{q}\mathbf{t}]/\|\mathbf{t}\|^2$.



Thus, the reflection of $\mathbf{q}$ with respect to $\hat{\mathbf{t}}$ is $\hat{\mathbf{t}}\mathbf{q}\hat{\mathbf{t}}$.

4. Products of unit vectors as rotation operators. (See Fig. 2.) The product $\hat{\mathbf{p}}\hat{\mathbf{q}}$ of two unit vectors $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ rotates (without dilation) any vector $\mathbf{v}$ that is parallel to the plane that contains $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$.

Or more correctly, any vector $\mathbf{v}$ that is parallel to the bivector $\hat{\mathbf{p}} \wedge \hat{\mathbf{q}}$. Macdonald ([3], p. 98) presents a set-theoretic definition as well.

(a) $\mathbf{v}\,(\hat{\mathbf{p}}\hat{\mathbf{q}})$ evaluates to the rotation of $\mathbf{p}$ by the bivector angle from $\hat{\mathbf{p}}$ to $\hat{\mathbf{q}}$.

(b) $(\hat{\mathbf{q}}\hat{\mathbf{p}})\,\mathbf{v} = \mathbf{v}\,(\hat{\mathbf{p}}\hat{\mathbf{q}})$.

(c) Therefore,

$$\begin{aligned}(\hat{\mathbf{q}}\hat{\mathbf{p}})\,\mathbf{v}\,(\hat{\mathbf{p}}\hat{\mathbf{q}}) &= [(\hat{\mathbf{q}}\hat{\mathbf{p}})\,\mathbf{v}]\,(\hat{\mathbf{p}}\hat{\mathbf{q}}) \\ &= [\mathbf{v}\,(\hat{\mathbf{p}}\hat{\mathbf{q}})]\,(\hat{\mathbf{p}}\hat{\mathbf{q}})\,,\end{aligned}$$

which evaluates to the rotation, twice, of $\mathbf{v}$ through the bivector angle from $\hat{\mathbf{p}}$ to $\hat{\mathbf{q}}$. Or equivalently, through the double of the bivector angle from $\hat{\mathbf{p}}$ to $\hat{\mathbf{q}}$.

(d) The product $\hat{\mathbf{p}}\hat{\mathbf{q}}$ is by definition $\hat{\mathbf{p}} \cdot \hat{\mathbf{q}} + \hat{\mathbf{p}} \wedge \hat{\mathbf{q}}$. It can also be written in the "trigonometric" form $\cos\theta + \mathbf{i}\sin\theta$, where θ is the angle of rotation from $\hat{\mathbf{p}}$ to $\hat{\mathbf{q}}$. (The equivalent exponential form is $e^{\mathbf{i}\theta}$.) Equating the scalar parts of $\hat{\mathbf{p}} \cdot \hat{\mathbf{q}} + \hat{\mathbf{p}} \wedge \hat{\mathbf{q}}$ and $\cos\theta + \mathbf{i}\sin\theta$, we have $\cos\theta = \hat{\mathbf{p}} \cdot \hat{\mathbf{q}}$. Equating the bivector parts, we have $\mathbf{i}\sin\theta = \hat{\mathbf{p}} \wedge \hat{\mathbf{q}}$, from which $\sin\theta = -\mathbf{i}\,(\hat{\mathbf{p}} \wedge \hat{\mathbf{q}})$.

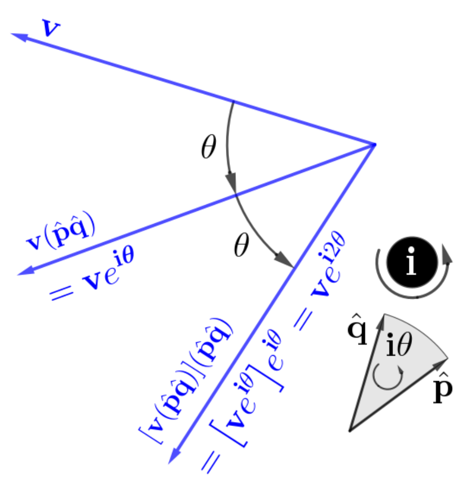


Figure 2: The product $\mathbf{v}\,(\hat{\mathbf{p}}\hat{\mathbf{q}})$ evaluates to the rotation of $\mathbf{p}$ by the bivector angle from $\hat{\mathbf{p}}$ to $\hat{\mathbf{q}}$. The product $[\mathbf{v}\,(\hat{\mathbf{p}}\hat{\mathbf{q}})]\,(\hat{\mathbf{p}}\hat{\mathbf{q}})$ evaluates to the rotation, twice, of $\mathbf{v}$ through the bivector angle from $\hat{\mathbf{p}}$ to $\hat{\mathbf{q}}$. Or equivalently, through the double of the bivector angle from $\hat{\mathbf{p}}$ to $\hat{\mathbf{q}}$.

3 Analysis of the Problem, and Initial Formulation

Because GA is particularly suited to formulating/manipulating rotations and reflection, we analyze and formulate the problem in those terms (Fig. 3). Note, too, that we use the similarity of triangles to derive the relation $(a+2r)\,/\,(a-2r) = \left(\sqrt{a^2+4b^2}\right)/a$, where r is the radius of the two small circles (Fig. 3).

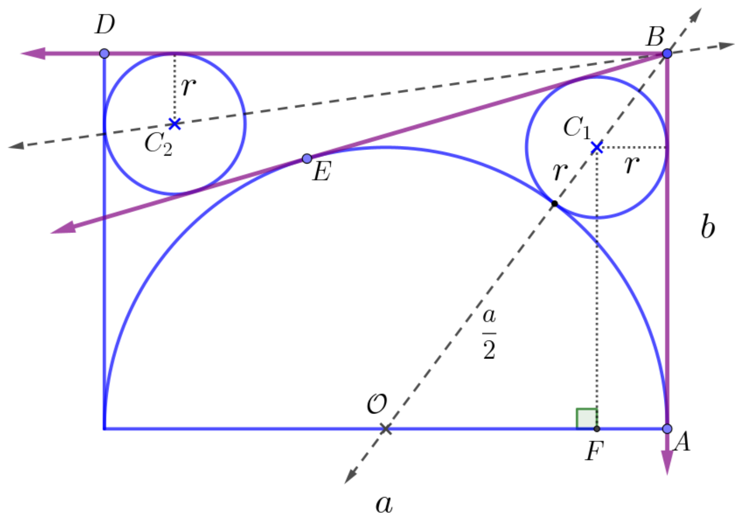


Figure 3: Diagram for exploration, analysis, and initial formulation. We can see that ray $\overrightarrow{BE}$ is the reflection of ray $\overrightarrow{BD}$ with respect to ray $\overrightarrow{BC_2}$, and that $\overrightarrow{BE}$ is also the reflection of ray $\overrightarrow{BA}$ with respect to ray $\overrightarrow{BC_1}$. In addition, from the similarity of triangles $\triangle C_1\mathcal{O}F$ and $\triangle B\mathcal{O}A$, $\dfrac{C_1\mathcal{O}}{\mathcal{O}F} = \dfrac{B\mathcal{O}}{\mathcal{O}A}$, from which $\dfrac{a/2+r}{a/2-r} = \dfrac{\sqrt{(a/2)^2+b^2}}{a/2}$. That relation can be simplified to $\dfrac{a+2r}{a-2r} = \dfrac{\sqrt{a^2+4b^2}}{a}$.

4 Solution Strategy

In Fig. 3, we have already identified a relationship between a, b, and r from similarity of triangles. We will derive a second, independent relationship between a, b, and r from the reflection relationships shown in Fig. 3. After eliminating r from that pair of equations, we will find the ratio $^b/_a$.

5 Final Formulation

To implement our solution strategy, we formulate the problem as shown in Fig. 4.

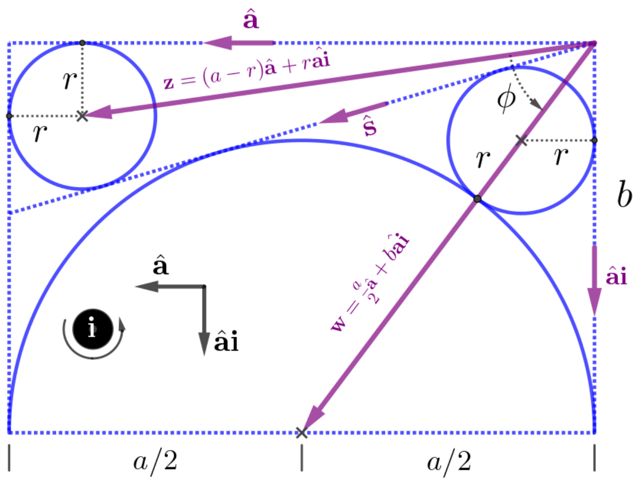


Figure 4: Formulation in GA terms. We define two vectors: $\mathbf{w} = \frac{a}{2}\hat{\mathbf{a}} + b\mathbf{ai}$, and $\mathbf{z} = (a-r)\,\hat{\mathbf{a}} + r\mathbf{ai}$. The vector $\hat{\mathbf{s}}$ is the reflection of $\hat{\mathbf{a}}$ with respect to $\mathbf{z}$, and is also the reflection of $\hat{\mathbf{a}}\mathbf{i}$ with respect to $\mathbf{w}$. Thus, $\hat{\mathbf{z}}\hat{\mathbf{a}}\hat{\mathbf{z}} = \hat{\mathbf{w}}\,(\hat{\mathbf{a}}\mathbf{i})\,\hat{\mathbf{w}}$. Also shown, for convenience, is the angle ϕ, to which we will refer later.

We define two vectors:

$$\begin{aligned} \mathbf{w} &= \frac{a}{2}\hat{\mathbf{a}} + b\mathbf{ai}, \text{ for which } \|\mathbf{w}\|^2 = \frac{a^2}{4} + b^2, \text{ and} \\ \mathbf{z} &= (a-r)\,\hat{\mathbf{a}} + r\mathbf{ai}, \text{ for which } \|\mathbf{z}\|^2 = (a-r)^2 + r^2. \end{aligned} \tag{5.1}$$

We also have two independent equations in a, b, and r:

$$\frac{a+2r}{a-2r} = \frac{\sqrt{a^2+4b^2}}{a} \tag{5.2a}$$

$$\hat{\mathbf{z}}\hat{\mathbf{a}}\hat{\mathbf{z}} = \hat{\mathbf{w}}\,(\hat{\mathbf{a}}\mathbf{i})\,\hat{\mathbf{w}}. \tag{5.2b}$$

6 Solution

6.1 Derivation of a relation between a, b, and r

Starting from Eq. 5.2, we will derive a relation between a, b, and r in two different ways.

6.1.1 First Method: Substitution into Eq. (5.2)

We begin by substituting our expressions for $\mathbf{w}$ and $\mathbf{z}$ (Eqs. 5.1) into Eq. 5.2. Because the product of any three coplanar vectors evaluates to a vector, each side of the equation $\hat{\mathbf{z}}\hat{\mathbf{a}}\hat{\mathbf{z}} = \hat{\mathbf{w}}\left(\hat{\mathbf{a}}\mathbf{i}\right)\hat{\mathbf{w}}$ is a vector. By substituting our expressions for for $\mathbf{w}$ and $\mathbf{z}$ into that equation, we transform each side into a linear combination of $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}\mathbf{i}$. Then, we use the fact that $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}\mathbf{i}$ are perpendicular to each other to derive an equation for r in terms of a and b.

More specifically, we will first obtain an equation for r by equating $\hat{\mathbf{a}}$ terms, after which we will verify that equation by deriving it by equating $\hat{\mathbf{a}}\mathbf{i}$ terms.

First, we transform Eq. (5.2) slightly:

$$\hat{\mathbf{z}}\hat{\mathbf{a}}\hat{\mathbf{z}} = \hat{\mathbf{w}}\left(\hat{\mathbf{a}}\mathbf{i}\right)\hat{\mathbf{w}}$$

$$\frac{\mathbf{z}\hat{\mathbf{a}}\mathbf{z}}{\|\mathbf{z}\|^2} = \frac{\mathbf{w}\left(\hat{\mathbf{a}}\mathbf{i}\right)\mathbf{w}}{\|\mathbf{w}\|^2}.$$

Now, making our substitutions for $\mathbf{z}$, $\mathbf{w}$, $\|\mathbf{z}\|^2$, and $\|\mathbf{z}\|^2$,

$$\frac{\left[(a-r)\,\hat{\mathbf{a}} + r\mathbf{a}\mathbf{i}\right]\hat{\mathbf{a}}\left[(a-r)\,\hat{\mathbf{a}} + r\mathbf{a}\mathbf{i}\right]}{(a-r)^2 + r^2} = \frac{\left[\frac{a}{2}\hat{\mathbf{a}} + b\mathbf{a}\mathbf{i}\right]\left(\hat{\mathbf{a}}\mathbf{i}\right)\left[\frac{a}{2}\hat{\mathbf{a}} + b\mathbf{a}\mathbf{i}\right]}{\frac{a^2}{4} + b^2}$$

After cross-multiplying, expanding, and simplifying, we arrive at

$$\left[\frac{a\,(a-2r)}{(a-r)^2 + r^2}\right]\hat{\mathbf{a}} + \left[\frac{2r\,(a-r)}{(a-r)^2 + r^2}\right]\hat{\mathbf{a}}\hat{\mathbf{i}} = \left[\frac{4ab}{a^2 + 4b^2}\right]\hat{\mathbf{a}} + \left[\frac{4b^2 - a^2}{a^2 + 4b^2}\right]\hat{\mathbf{a}}\hat{\mathbf{i}}. \tag{6.1}$$

Formally, we take the inner product of both sides with $\hat{\mathbf{a}}$, taking advantage of the fact that $\hat{\mathbf{a}}\cdot\left(\hat{\mathbf{a}}\hat{\mathbf{i}}\right) = 0$.

Now, we equate the $\hat{\mathbf{a}}$ coefficients of the two sides, thereby obtaining

$$\left[\frac{a\,(a-2r)}{(a-r)^2 + r^2}\right] = \left[\frac{4ab}{a^2 + 4b^2}\right].$$

From this equation, we obtain the following quadratic in r:

$$8br^2 + 2\,(a-2b)^2\,r - a\,(a-2b)^2 = 0,$$

whose two roots are $r = \frac{a}{2} - b$, and $r = \frac{a}{2} - \frac{a^2}{4b}$. From Fig. 4, we can see that $r = \frac{a}{2} - b$ is a negative number. Therefore,

An additional argument for the uniqueness of the solution $r = \frac{a}{2} - \frac{a^2}{4b}$ is that as we will now see, this root is the only one that makes the $\hat{\mathbf{a}}\mathbf{i}$ coefficients equal, as well as the $\hat{\mathbf{a}}$ coefficients.

$$r = \frac{a}{2} - \frac{a^2}{4b}. \tag{6.2}$$

To validate that result, we also solve for r by equating the $\hat{\mathbf{a}}\mathbf{i}$ coefficients of both sides of Eq. (6.1)

$$\frac{2r\,(a-r)}{(a-r)^2 + r^2} = \frac{4b^2 - a^2}{a^2 + 4ab}.$$

After cross-multiplying, expanding, and simplifying, we arrive at

$$16b^2r^2 - 16ab^2r + a^2\left(4b^2 - a^2\right) = 0,$$

which has the roots

$$r = \frac{a}{2} \pm \frac{a^2}{4b}.$$

Because the radius r cannot be larger than $\frac{a}{2}$, the root $\frac{a}{2} + \frac{a^2}{4b}$ is impossible. Therefore, as in Eq. (6.2), $r = \frac{a}{2} - \frac{a^2}{4b}$.

6.1.2 Second Method: Transforming Eq. (5.2) into a Rotation of $\hat{\mathbf{a}}$

This method uses the "rotation operator" properties of products of unit vectors. We begin by transforming Eq. 5.2 :

$$\begin{aligned}
\hat{\mathbf{z}}\hat{\mathbf{a}}\hat{\mathbf{z}} &= \hat{\mathbf{w}}\left(\hat{\mathbf{a}}\mathbf{i}\right)\hat{\mathbf{w}} \\
\hat{\mathbf{w}}\hat{\mathbf{z}}\hat{\mathbf{a}}\hat{\mathbf{z}}\hat{\mathbf{w}} &= \hat{\mathbf{w}}\hat{\mathbf{w}}\left(\hat{\mathbf{a}}\mathbf{i}\right)\hat{\mathbf{w}}\hat{\mathbf{w}} \\
\left[\left(\hat{\mathbf{w}}\hat{\mathbf{z}}\right)\hat{\mathbf{a}}\right]\hat{\mathbf{z}}\hat{\mathbf{w}} &= \hat{\mathbf{a}}\mathbf{i} \\
\left[\hat{\mathbf{a}}\left(\hat{\mathbf{z}}\hat{\mathbf{w}}\right)\right]\hat{\mathbf{z}}\hat{\mathbf{w}} &= \hat{\mathbf{a}}\mathbf{i}.
\end{aligned}$$

Next, we recognize that $\hat{\mathbf{a}}\mathbf{i}$ is the rotation of $\hat{\mathbf{a}}$ by $\pi/2$ in the sense of $\mathbf{i}$, and that $\left[\hat{\mathbf{a}}\left(\hat{\mathbf{z}}\hat{\mathbf{w}}\right)\right]\hat{\mathbf{z}}\hat{\mathbf{w}}$ is the rotation of $\hat{\mathbf{a}}$ by twice the angle of rotation (call it ϕ) from $\hat{\mathbf{z}}$ to $\hat{\mathbf{w}}$ in the sense of $\mathbf{i}$. Thus, $\phi = \pi/4$.

We can express this reasoning in exponential form by writing $\hat{\mathbf{a}}e^{\mathbf{i}2\phi} = \hat{\mathbf{a}}e^{\mathbf{i}\pi/2}$, from which $\phi = \pi/4$.

Now, we equate the expansion of $\hat{\mathbf{z}}\hat{\mathbf{w}}$ to the trigonometric form of that product:

$$\hat{\mathbf{z}} \cdot \hat{\mathbf{w}} + \hat{\mathbf{z}} \cdot \hat{\mathbf{w}} = \cos\phi + \mathbf{i}\sin\phi. \tag{6.3}$$

Because $\phi = \pi/4$, $\cos\phi = \sin\phi$. Therefore, making use of the ideas that we reviewed in Section 2, our derivation is as follows:

$$\begin{aligned}
\cos\phi &= \sin\phi \\
\hat{\mathbf{z}} \cdot \hat{\mathbf{w}} &= -\mathbf{i}\left\{\hat{\mathbf{z}} \wedge \hat{\mathbf{w}}\right\} \\
\mathbf{z} \cdot \mathbf{w} &= -\mathbf{i}\left\{\mathbf{z} \wedge \mathbf{w}\right\} \\
\langle \mathbf{z}\mathbf{w}\rangle_0 &= -\mathbf{i}\langle\{\mathbf{z}\mathbf{w}\}\rangle_2 \\
\langle\left[(a-r)\,\hat{\mathbf{a}} + r\hat{\mathbf{a}}\mathbf{i}\right]\left[\frac{a}{2}\hat{\mathbf{a}} + b\hat{\mathbf{a}}\mathbf{i}\right]\rangle_0 &= -\mathbf{i}\langle\left\{\left[(a-r)\,\hat{\mathbf{a}} + r\hat{\mathbf{a}}\mathbf{i}\right]\left[\frac{a}{2}\hat{\mathbf{a}} + b\hat{\mathbf{a}}\mathbf{i}\right]\right\}\rangle_2 \\
\frac{a}{2}(a-r) + br &= -\mathbf{i}\left\{\left[b(a-r) - \frac{a}{2}r\right]\mathbf{i}\right\} \\
\frac{a}{2}(a-r) + br &= b(a-r) - \frac{a}{2}r \\
\therefore r &= \frac{a}{2} - \frac{a^2}{4b}
\end{aligned} \tag{6.4}$$

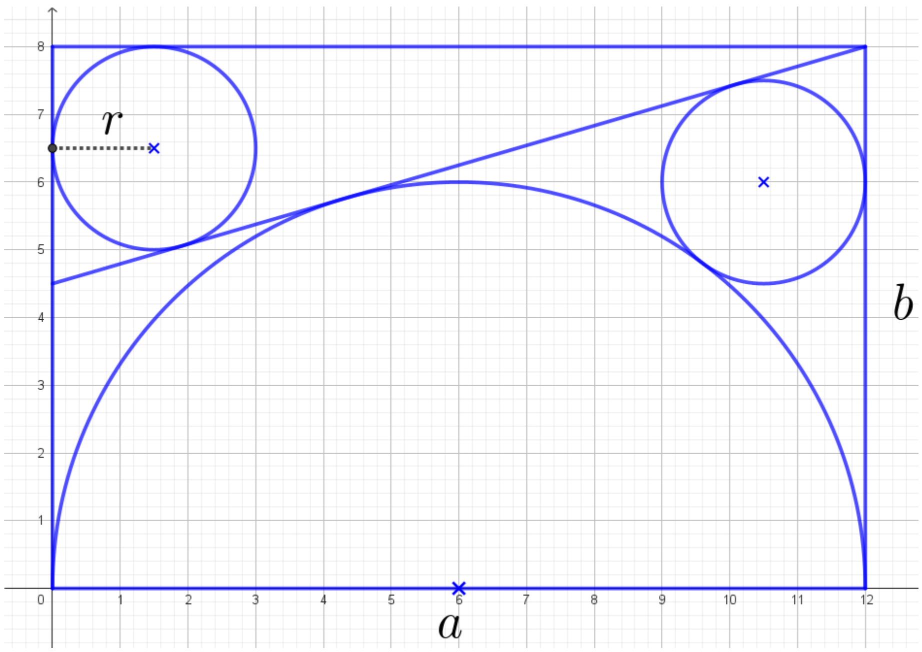


Figure 5: A careful examination of this figure confirms that $b = \frac{2a}{3}$, and $r = \frac{a}{8}$.

7 The result: $b = \frac{2a}{3}$, and $r = \frac{a}{8}$

Substituting the result $r = \frac{a}{2} - \frac{a^2}{4b}$ in Eq. (5.2), we obtain

$$b = \frac{2}{3}a. \tag{7.1}$$

Thus we also find that $r = \frac{a}{8}$. These results are confirmed by Fig. 5.

References

[1] "Two Moons" `mirangu.com/two-moons`

[2] "What's Special in this Geometry Problem?"
`https://www.youtube.com/watch?v=kQUaLNHSYVg`

[3] A. Macdonald, *Linear and Geometric Algebra* (First Edition), CreateSpace Independent Publishing Platform (Lexington, 2012).