Cyclic representation of the Dirac equation

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A new representation of the Dirac equation is obtained based on the chiral algebra of biquaternions we developed. Our formulation of this fundamental equation combines in a certain way the direct and reverse operators of 4-gradients acting on the right and left chiral components of the particle wave function, as well as the cyclic transformation operator acting on the entire wave function of the particle. The resulting representation allows us to see in the Dirac equation the connection between linear and cyclic times, while the chiral algebra used in this approach provides new methods for studying the relativistic physics of spin. An analog of the Lorentz transformation, a new rotation transformation is derived, that creates the proper rotation of a massive particle. It is shown that the cyclic transformation included in our representation of the Dirac equation is expressed via complex-valued Hadamard matrix, which indicates the connection of the Dirac equation with known algorithms of noise-resistant information transmission.

Keywords: Dirac equation, biquaternions, nullquaternions, isotropic basis, chiral algebra, skew symmetry, chirality, cyclic representation, cyclic conjugation, Hadamard matrices, spin, rotation transformation.

Contents

Introduction.

The Dirac equation, defining the relativistic wave function, occupies a central place in quantum field theory [1]. This equation describes the wave functions of elementary particles of half-integer spin. The Dirac equation is usually derived by factorizing the Klein-Gordon equation. The meaning of the latter equation is that the square of the mass plays the role of an eigenvalue of the square of the energy-momentum operator.

Previously, other authors obtained biquaternion representations of the Dirac equation [5][6]. In [8] a form of this equation in spinors composed of split quaternions was derived. In the current article, we present a new biquaternion formulation of the Dirac equation, which is fundamentally different from the representations obtained in the mentioned above works.

Our article is divided into two main parts. The first part, mathematical, is called "Chiral Algebra". It is devoted to the construction of new methods of biquaternion algebra using isotropic basis of biquaternion space based on nullquaternions. Here we introduce previously unknown methods of biquaternion multiplication and conjugation, and also define types of biquaternions with special projectivity properties. In the second, physical, part of the article, "Dirac Equation", on the basis of chiral algebra, we obtain a new representation of the Dirac equation. Our formulation of this fundamental equation combines the direct and reverse linear gradient operators acting on the right and left chiral parts of the particle's wave function, with the cyclic transformation operator acting on the entire particle's wave function.

Part 1. Chiral algebra.

1.1. Biquaternions.

Biquaternions were discovered by W. Hamilton following his discovery of quaternions, as a complex-valued extension of the latter [9]. L. Silberstein clarified the central role played by biquaternions in relativistic theory, or the theory of united space-time [10]. He also introduced the most convenient and intuitive scalar-vector representation of biquaternions. In the scalar-vector representation biquaternions have the form [2][7]:

$$\mathcal{B} = (s, \mathbf{u}), \qquad s \in \mathbb{C}, \qquad \mathbf{u} \in \mathbb{C}^{\mathbf{3}} \tag{1}$$

As a rule, we will denote biquaternions in capital letters of the Latin alphabet, while scalars and vectors by small letters. As follows from the definition ((1), a biquaternion $\mathcal{B}$ is a pair consisting of a complex number s, called a scalar, and a complex-valued three-dimensional vectora $\mathbf{u}$. s and $\mathbf{u}$ are the scalar and vector parts of the biquaternion $\mathcal{B}$ respectively. The sum of two biquaternions is calculated componentwise, separately for the scalar and для vector parts. Common, or *outer*, product of two biquaternions $\mathcal{B}_1 = (s_1, \mathbf{u}_1)$ and $\mathcal{B}_2 = (s_2, \mathbf{u}_2)$ is calculated according to the formula:

$$\mathcal{B}_1\mathcal{B}_2 = \mathcal{B}_1 \odot \mathcal{B}_2 = (s_1 s_2 + \mathbf{u}_1 \cdot \mathbf{u_2},\ s_1\mathbf{u}_2 + s_2\mathbf{u}_1 + i\mathbf{u}_1 \times \mathbf{u}_2), \tag{2}$$

where $\mathbf{u}_1 \cdot \mathbf{u}_2, \mathbf{u}_1 \times \mathbf{u}_2$ are the scalar and vector products of vectors $\mathbf{u}_1$and, $\mathbf{u}_2$ respectively, i is the imaginary unit. Unlike other types of biquaternion products, which will be discussed below, for the ordinary, or outer, product we will use both equivalent notations $\mathcal{B}_1\mathcal{B}_2$ and $\mathcal{B}_1 \odot \mathcal{B}_2$. The outer product of biquaternions, same way as other types of their products introduced below, is non-commutative – it depends on the order of the multipliers.

An arbitrary complex vector $\mathbf{u} \in \mathbb{C}^{\mathbf{3}}$ is a special case of a biquaternions whose scalar part is zero:

$$\mathbf{u} = \mathbf{A} + i\mathbf{B}, \quad \mathbf{A}, \mathbf{B} \in \mathbb{R}^3 \tag{3}$$

Complex conjugate of the biquaternion $\mathcal{B} = (s, \mathbf{u})$ has the form:

$$\mathcal{B}^* = (s^*, \mathbf{u}^*) \tag{4}$$

Complex conjugate of biquaternions corresponds to Hermitian conjugate of matrix algebras (see приложение **Ошибка! Источник ссылки не найден.**).

Vector conjugation[1] of the biquaternion $\mathcal{B} = (s, \mathbf{u})$ has the form:

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[1] The conjugation referred here as "vector" is often simply called conjugation or "biquaternionic conjugation". We use the name "vector conjugation" to clearly distinguish this type from other types of biquaternion conjugation.

$$\overline{\mathcal{B}} = (s, -\mathbf{u}) \quad (5)$$

Simultaneous application of complex and vector conjugates gives *double conjugation* of the biquaternion*:*

$$\overline{\mathcal{B}}^* = (s^*, -\mathbf{u}^*) \quad (6)$$

Two biquaternions are called *equivalent,* if they are equal to each other up to a scalar (complex number) factor:

$$\mathcal{B}_1 \approx \mathcal{B}_2\colon\ \mathcal{B}_1 = \lambda\,\mathcal{B}_2\,, \lambda \in \mathbb{C}, \lambda \neq 0 \quad (7)$$

Square modulus of the biquaternion $\mathcal{B} = (s, \mathbf{u})$ is a complex number defined according to the formula:

$$|\mathcal{B}|^2 = \mathcal{B}\overline{\mathcal{B}} = s^2 - \mathbf{u}^2\,, \quad |\mathcal{B}|^2 \in \mathbb{C} \quad (8)$$

1.2. Isotropic basis.

In this section, we introduce isotropic basis[2] for the space of biquaternions. This basis is based on biquaternions that have zero square modulus. In physics, such quantities usually describe light and are called isotropic, which determines the name of the basis.

Let's take a closer look at two possible types of biquaternions Q that have zero square modulus (8): $|Q| = 0$. In our terminology, such biquaternions are called *nullquaternions* [2]. The first of the two possible types of nullquaternions is nullvectors[3] – three-dimensional complex vectors whose square is zero. Each nullvector $\mathbf{q}$ is decomposed into sum of two mutually orthogonal real vectors $\mathbf{A}$ and $i\mathbf{B}$ of the same length (Fig. 1):

$$\begin{aligned} &\mathbf{q} = \mathbf{A} + i\mathbf{B}, \quad \mathbf{A}, \mathbf{B} \in \mathbb{R}^3, \ \ \mathrm{A} = \mathrm{B}, \ \mathrm{A} \perp \mathrm{B} \\ &\mathrm{q} \in \mathbb{C}^3, \ \ \mathrm{q}^2 = 0 \end{aligned} \quad (9)$$

Vector $\mathbf{q}^*$, which is complex conjugate to the nullvector $\mathbf{q}$ **,** is also a nullvector. Vector $\lambda\mathbf{q}$ is also nullvector, where λ is an arbitrary nonzero complex number.

The second type of nullquaternions is *uniform* nullquaternions N , each of which can be obtained from the corresponding real vector of unit length $\mathbf{n}$ as follows:

$$N = \lambda(1, \mathbf{n}), \ \mathbf{n} \in \mathbb{R}^3\,, \ \mathbf{n^2} = 1\,, \ \lambda \in \mathbb{C}; \qquad N\overline{N} = 0 \quad (10)$$

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[2] Another name for the isotropic basis *is the light basis* .
[3] Nullvectors are also called isotropic vectors.

The vector conjugation of a uniform nullquaternion N again yields a uniform nullquaternion $\overline{N} = \lambda(1, -\mathbf{n})$.

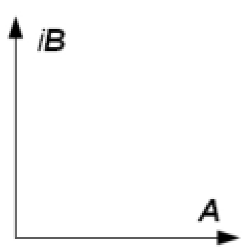
Isotropic basis of a biquaternion space consists of the following four elements, each of which is a nullquaternion:

$$\begin{cases} \mathbf{q} = \frac{1}{2}(\mathbf{A} + i\,\mathbf{B}) \\ \mathbf{q}^* = \frac{1}{2}(\mathbf{A} - i\,\mathbf{B}) \\ N = \frac{1}{2}(1,\ \mathbf{n}) \\ \overline{N} = \frac{1}{2}(1, -\mathbf{n}) \end{cases} \qquad \begin{matrix} \mathbf{q}, \mathbf{q}^*, N, \overline{N} = const \\ \mathbf{A}, \mathbf{B}, \mathbf{n} \in \mathbb{R}^3 \\ \mathbf{A}^2 = \mathbf{B}^2 = \mathbf{n}^2 = 1 \end{matrix} \qquad (11)$$

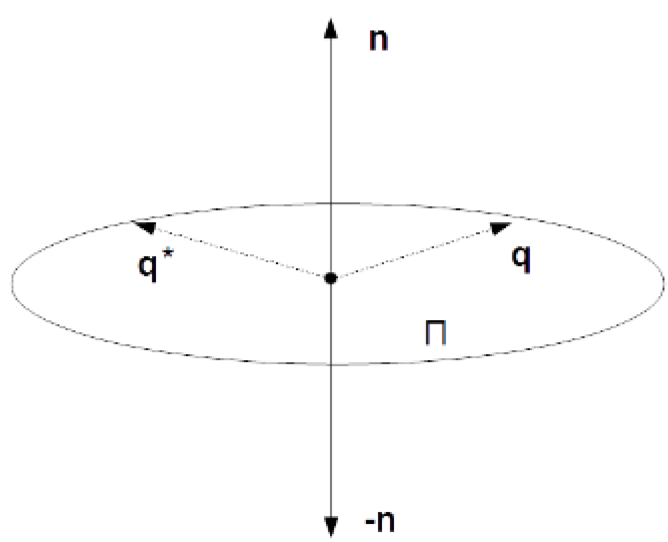
The first two of these elements $\mathbf{q}$ and $\mathbf{q}^*$ are nullvectors, and the remaining two N and $\overline{N}$ are uniform nullquaternions. Nullvectors $\mathbf{q}$ and $\mathbf{q}^*$ lie in the same plane Π, which we will call *the transverse plane*; the real vectors A and B that compose $\mathbf{q}$ and $\mathbf{q}^*$ also lie in the plane Π. The unit *longitudinal* real vector $\mathbf{n}$ is normal to this plane. Isotropic basis is thus defined by a certain constant direction in space (vector $\mathbf{n}$) and a fixed turn angle in the plane Π, which determines angular position of the pair $\mathbf{A}$ and $\mathbf{B}$. The nullvectors $\mathbf{q}$ and $\mathbf{q}^*$ and the uniform nullquaternions N and $\overline{N}$ are bound by the following relations:

$$\mathbf{q}\mathbf{q}^* = N, \quad \mathrm{q}^*\mathrm{q} = \overline{N} \qquad (12)$$

In (12) $\mathbf{q}\mathbf{q}^*$ and $\mathbf{q}^*\mathbf{q}$ are common, or outer, biquaternion products (2). The vector relationship takes place: $\mathbf{A} \times \mathbf{B} = \mathbf{n}$, where $\mathbf{A} \times \mathbf{B}$ denotes the vector product of vectors $\mathbf{A}$ and $\mathbf{B}$. Figure 2 gives a schematic representation of isotropic basis.



1. Nullvector **q** (plane Π).



2. Isotropic basis.

In the Cartesian basis constructed on the real vectors $\mathbf{A}, \mathbf{B}, \mathbf{n}$, the vectors $\mathbf{n}, \mathbf{q}, \mathbf{q}^*$ have the followinge complex coordinates:

$$\mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{q} = \frac{1}{2}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \quad \mathbf{q}^* = \frac{1}{2}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \tag{13}$$

An arbitrary biquaternion $\mathcal{B}$ is decomposed within isotropic basis using the complex-numeric coordinates $\alpha, \beta, \xi, and\ \eta$:

$$\mathcal{B} = \alpha\mathbf{q} + \beta\mathbf{q}^* + \xi N + \eta\bar{N}, \quad \alpha, \beta, \xi, \eta \in \mathbb{C} \tag{14}$$

As easily shown, this decomposition is unique.

Decompose the biquaternion $\mathcal{B}$ (14) into two components:

$$\mathcal{B} = \mathbf{u} + \mathcal{P}, \qquad \begin{cases} \mathbf{u} = \alpha\mathbf{q} + \beta\mathbf{q}^* \\ \mathcal{P} = \xi N + \eta\bar{N} \end{cases} \tag{15}$$

The first component $\mathbf{u}$, call it *transverse* component, is a complex vector lying in the plane Π. The second component $\mathcal{P}$, call it *longitudinal* component, is a biquaternion whose vector part is parallel to the vector $\mathbf{n}$. The sum (15) thus gives *a longitudinal-transverse* representation of the biquaternion $\mathcal{B}$.

Let us take a certain point of a biquaternion space considered as space-time. This point is described by Cartesian coordinates including time: t, x, y, z. The relation between Cartesian and isotropic coordinates is expressed as follows:

$$\begin{cases} \alpha = x - iy \\ \beta = x + iy \\ \xi = t + z \\ \eta = t - z \end{cases} \tag{16}$$

Relations like (16) are naturally applicable not only to the coordinates of a point, but also to the components of any other biquaternion. From (16), we can obtain the relation between partial derivatives in isotropic and Cartesian bases:

$$\begin{cases} \partial_\alpha = \frac{1}{2}(\partial_x + i\partial_y) \\ \partial_\beta = \frac{1}{2}(\partial_x - i\partial_y) \\ \partial_\xi = \frac{1}{2}(\partial_t + \partial_z) \\ \partial_\eta = \frac{1}{2}(\partial_t - \partial_z) \end{cases} \tag{17}$$

1.3. Signed biquaternions and projectors.

Group the members of a biquaternion (14) to represent it as follows:

$$\mathcal{B} = \mathcal{B}_+ + \mathcal{B}_-, \qquad \begin{cases} \mathcal{B}_+ = \alpha\mathbf{q} + \eta\bar{N} \\ \mathcal{B}_- = \beta\mathbf{q}^* + \xi N \end{cases} \tag{18}$$

Biquaternions of the form $\mathcal{B}_+$ and $\mathcal{B}_-$ are called *signed* – positive and negative, respectively. We will denote the fact that a certain biquaternion $\mathcal{B}$ is positive in symbolic form as $\mathcal{B} = B_+$, and negative signed as $\mathcal{B} = B_-$.

We now represent the same biquaternion $\mathcal{B}$ (14) in a different way:

$$\mathcal{B} = P^+ + P^- , \qquad \begin{cases} P^- = \alpha\mathbf{q} + \xi N \\ P^+ = \beta\mathbf{q}^* + \eta\bar{N} \end{cases} \tag{19}$$

Biquaternions of the form P^- and P^+ are called *projectors* – negative and positive, respectively. Further on we will explain this name. From the uniqueness of isotropic basis expansion, it follows that each biquaternion is unambiguously decomposed into both the sum of signed biquaternions and the sum of projectors.

Signed biquaternions and projectors are connected to each other by the vector conjugation operation (5):

$$\begin{cases} \overline{B_+} = P^- \\ \overline{B_-} = P^+ \end{cases} \qquad \begin{cases} \overline{P^-} = B_+ \\ \overline{P^+} = B_- \end{cases} \tag{20}$$

The correspondence (20) implies an isomorphism between signed biquaternions and projectors by outer multiplication, provided that the product of two projectors must be taken in the reverse order to the product of signed biquaternions. It should be noted that although both types of biquaternions under consideration, projectors and signed biquaternions, have a sign characteristic, we apply the term "sign" and "signedness" only to the former. As will be shown later, in the wave function representation we use, the chiral states are projectors.

1.4. Types of multiplication of biquaternions.

In addition to the usual or outer method of multiplying biquaternions (2), we will introduce other methods for multiplying them. In this paper, we use four different methods for multiplying biquaternions. Below we show different product types for two biquaternions $\mathcal{B}_1$ and $\mathcal{B}_2$, represented in the longitudinal-transverse representation and in isotropic basis as:

$$\begin{cases} \mathcal{B}_1 = \mathbf{u_1} + \mathcal{P}_1 = \alpha_1\mathbf{q} + \beta_1\mathbf{q}^* + \xi_1 N + \eta_1\bar{N} \\ \mathcal{B}_2 = \mathbf{u_2} + \mathcal{P}_2 = \alpha_2\mathbf{q} + \beta_2\mathbf{q}^* + \xi_2 N + \eta_2\bar{N} \end{cases} \tag{21}$$

According to (18) each of these biquaternions can be decomposed into signed parts:

$$\begin{cases} \mathcal{B}_1 = \mathcal{B}_{1+} + \mathcal{B}_{1-} \\ \mathcal{B}_2 = \mathcal{B}_{2+} + \mathcal{B}_{2-} \end{cases} \tag{22}$$

The first, outer, type of biquaternion multiplication given below corresponds to the first two possible ways of multiplying second-order square matrices – by adding the products of row elements of the first matrix to column elements of the second matrix . The second, inner, type, corresponds to the matrices with subtraction rule instead. (see Appendix **Ошибка! Источник ссылки не найден.**). Outer and inner multiplications are directly involved in the formulation of the cyclic representation of the Dirac equation. The third, diagonal, type of multiplication in the matrix representation uses the addition of the multiplied components not horizontally or vertically, but diagonally. We will use this type of multiplication when we describe the spin of a particle. The fourth, crossing type, in a certain way combines the first two types.

1) Outer product $\odot$

The outer[4], or common, product of biquaternions was defined above in the formula (2). In isotropic basis, the outer product of two biquaternions $\mathcal{B}_1$ and $\mathcal{B}_2$ is expressed as:

$$\mathcal{B}_1 \odot \mathcal{B}_2 = (\xi_1\alpha_2 + \alpha_1\eta_2)\mathbf{q} + (\eta_1\beta_2 + \beta_1\xi_2)\mathbf{q}^* + (\alpha_1\beta_2 + \xi_1\xi_2)N + (\beta_1\alpha_2 + \eta_1\eta_2)\overline{N} \quad (23)$$

2) Inner product $\otimes$

In isotropic basis, the inner product of two biquaternions $\mathcal{B}_1$ and $\mathcal{B}_2$ is expressed as:

$$\mathcal{B}_1 \otimes \mathcal{B}_2 = (\alpha_1\alpha_2 + \xi_1\eta_2)\mathbf{q} + (\beta_1\beta_2 + \eta_1\xi_2)\mathbf{q}^* + (\beta_1\xi_2 + \xi_1\alpha_2)N + (\alpha_1\eta_2 + \eta_1\beta_2)\overline{N} \quad (24)$$

Appendix **Ошибка! Источник ссылки не найден.** gives pairwise products of elements of isotropic basis with respect to outer and inner multiplication. Based on these products, you can get general formulas for the corresponding products (23),(24).

Inner multiplication has a number of surprising and unexpected properties, which are beyond the scope of this article to study in detail. Here is just one example of such properties, such as the fact that the inner product of two complex numbers λ_1 and λ_2 is a vector: $\lambda_1 \otimes \lambda_2 = \lambda_1\lambda_2\mathbf{A}$**.** Also, the inner product of a number on a biquaternion does not hold the usual distributivity for such products: $\lambda \otimes (\alpha\mathbf{q} + \beta\mathbf{q}^* + \xi N + \eta\overline{N}) \neq \lambda\alpha\mathbf{q} + \lambda\beta\mathbf{q}^* + \lambda\xi N + \lambda\eta\overline{N}$.

3) Diagonal product $\times$

In isotropic basis, the diagonal product of two biquaternions $\mathcal{B}_1$ and $\mathcal{B}_2$ is expressed as[5]:

$$\mathcal{B}_1 \times \mathcal{B}_2 = (\alpha_1\alpha_2 + \beta_1\xi_2)\mathbf{q} + (\alpha_1\beta_2 + \beta_1\eta_2)\mathbf{q}^* + (\xi_1\alpha_2 + \eta_1\xi_2)N + (\xi_1\beta_2 + \eta_1\eta_2)\overline{N} \quad (25)$$

4) Crossing product $\diamondsuit$

The crossing product of biquaternions combines outer and inner multiplications in a certain way. The crossing product of two biquaternions $\mathcal{B}_{B1}$ and $\mathcal{B}_{B2}$ has the form:

$$\mathcal{B}_1 \diamondsuit \mathcal{B}_2 = \mathbf{u_1} \odot \mathbf{u_2} + \mathcal{P}_1 \otimes \mathcal{P}_2 \quad (26)$$

$$\mathcal{B}_1 \diamondsuit \mathcal{B}_2 = \eta_1\xi_2\mathbf{q} + \xi_1\eta_2\mathbf{q}^* + \alpha_1\beta_2 N + \beta_1\alpha_2\overline{N} \quad (27)$$

The crossing product of a positive projector on the left side with any biquaternion on the right side always gives a positive signed biquaternion, and the crossing product of a negative projector on the left side with any biquaternion on the right side always gives a negative signed

[4] *Outer* and *inner products* used in this paper have different meanings than in Grassmann algebra.

[5] It is important not to confuse the diagonal product sign × applied to biquaternions with the similar looking cross product sign applied to vectors.

biquaternion. The action of projectors on the right side have similar properties. Write down all four possible variations of the projector products with an arbitrary biquaternion $\mathcal{B}$ on the left and right sides:

$$\forall\mathcal{B}: \begin{cases} P^{+} \diamond \mathcal{B} = B_{+} \\ P^{-} \diamond \mathcal{B} = B_{-} \\ \mathcal{B} \diamond P^{+} = B_{-} \\ \mathcal{B} \diamond P^{-} = B_{+} \end{cases} \qquad (28)$$

The relations (28) determine the name of projectors: biquaternions of this type project an arbitrary biquaternion onto a positive or negative signed biquaternion. Note that there are relations similar (28), in which signed biquaternions have projection properties with respect to the projectors themselves.

1.5. Exchange cconjugates ofя biquaternions.

Above, we considered the classical types of conjugation of biquaternions: complex $\mathcal{B}^{*}$ (4) and vector $\overline{\mathcal{B}}$ (5). In addition to the above conjugations, we can introduce other types of conjugation, which we will consider below. Each of these conjugates represents a certain permutation of the coordinates of a given biquaternion in isotropic basis.

1) Exchange conjugation of the first type $\mathcal{B}^{\star}$ *.*
We define the exchange conjugation of the first type[6] as follows:

$$\mathcal{B} = \alpha\mathbf{q} + \beta\mathbf{q}^{*} + \xi N + \eta\overline{N} \quad \rightarrow \quad \mathcal{B}^{\star} = \alpha\mathbf{q}^{*} + \eta N + \beta\mathbf{q} + \xi\overline{N} \qquad (29)$$

Schematically, this conjugation is represented as:

$$\mathcal{B} = \alpha\mathbf{q} + \beta\mathbf{q}^{*} + \xi N + \eta\overline{N} \quad \rightarrow \quad \mathcal{B}^{\star}$$

2) Exchange conjugation of the second type $\widetilde{\mathcal{B}}$ *.*
We introduce the exchange conjugation of the second type as follows:

$$\mathcal{B} = \alpha\mathbf{q} + \beta\mathbf{q}^{*} + \xi N + \eta\overline{N} \quad \rightarrow \quad \widetilde{\mathcal{B}} = \xi\mathbf{q} + \eta\mathbf{q}^{*} + \alpha N + \beta\overline{N} \qquad (30)$$

or schematically:

$$\mathcal{B} = \alpha\mathbf{q} + \beta\mathbf{q}^{*} + \xi N + \eta\overline{N} \quad \rightarrow \quad \widetilde{\mathcal{B}}$$

In the future, the exchange conjugation of the second type will also be referred simply as the exchange conjugation.

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[6] In our work [3], the first type of exchange conjugation was called "symbolic conjugation", the second type of exchange conjugation was called "exchange conjugation". It is also important not to confuse the symbol of the first type of exchange conjugation ✶ with the symbol of complex conjugation *.

3) Exchange conjugation of the third type $\dddot{\mathcal{B}}$.

Exchange conjugation of the third type has the form:

$$\mathcal{B} = \alpha\mathbf{q} + \beta\mathbf{q}^* + \xi N + \eta\overline{N} \quad \to \quad \dddot{\mathcal{B}} = \eta\mathbf{q} + \xi\mathbf{q}^* + \beta N + \alpha\overline{N} \tag{31}$$

or schematically:

$$\mathcal{B} = \alpha\mathbf{q} + \beta\mathbf{q}^* + \xi N + \eta\overline{N} \quad \to \quad \dddot{\mathcal{B}}$$

Let's describe some properties of the three exchange conjugates introduced above. Each of these conjugates is self-inverse: $\mathcal{B}^{\star\star} = \mathcal{B}, \widetilde{\widetilde{\mathcal{B}}} = \mathcal{B}, \dddot{\dddot{\mathcal{B}}} = \mathcal{B}$. Each of the three types of exchange conjugation is symmetric operation: $\mathcal{B}_2 = \mathcal{B}_1^\star \Leftrightarrow \mathcal{B}_1 = \mathcal{B}_2^\star$; $\mathcal{B}_2 = \widetilde{\mathcal{B}}_1 \Leftrightarrow \mathcal{B}_1 = \widetilde{\mathcal{B}}_2$; $\mathcal{B}_2 = \dddot{\mathcal{B}}_1 \Leftrightarrow \mathcal{B}_1 = \dddot{\mathcal{B}}_2$. Operations of different exchange conjugates are commutable: $\left(\widetilde{\mathcal{B}}\right)^\star = \widetilde{\mathcal{B}^\star}$, etc. There is a transitive connection between different types of exchange conjugation: the combines application of two different conjugations gives a conjugation of the third type: $\widetilde{\mathcal{B}^\star} = \dddot{\mathcal{B}}$; $\dddot{\mathcal{B}^\star} = \widetilde{\mathcal{B}}$; $\dddot{\left(\widetilde{\mathcal{B}}\right)} = \mathcal{B}^\star$.

Exchange conjugations make it possible to reverse the multipliers of the biquaternion product with changing product type: for two arbitrary biquaternions $\mathcal{B}_1, \mathcal{B}_2$, the following identities hold:

$$\begin{aligned} \mathcal{B}_1 \otimes \mathcal{B}_2 &= \mathcal{B}_2 \odot \dddot{\mathcal{B}}_1 \\ \widetilde{\mathcal{B}_1} \otimes \mathcal{B}_2 &= \mathcal{B}_2 \odot \mathcal{B}_1^\star \end{aligned} \tag{32}$$

In addition to the types of biquaternion conjugates discussed above, we present below another type of biquaternion conjugation – cyclic conjugation (40), which plays a key role in our formulation of the Dirac biquaternion equation.

Part 2. The Dirac equation

2.1. The Dirac equation in Weyl spinors.

In the representation of Weyl spinors, the Dirac equation has the following form [12]:

$$\begin{cases} \dfrac{\partial \psi_L}{\partial t} = -(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\psi_L - im\,\psi_R \\ \dfrac{\partial \psi_R}{\partial t} = +(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\psi_R - im\psi_L \end{cases} \qquad (33)$$

Here $\boldsymbol{\sigma}$ is a three -dimensional vector composed of the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$:

$$\boldsymbol{\nabla} = (\sigma_x, \sigma_y, \sigma_z) \qquad (34)$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

m – mass of the particle; $\boldsymbol{\nabla} = (\partial_x, \partial_y, \partial_z)$ – a three-dimensional nabla operator; ψ_L and ψ_R –Weyl spinor representing the left- and right-chiral states respectively:

$$\begin{cases} \psi_L = \begin{pmatrix} u \\ v \end{pmatrix}_L \\ \\ \psi_R = \begin{pmatrix} u' \\ v' \end{pmatrix}_R \end{cases} \qquad u, v, u', v' \in \mathbb{C} \qquad (35)$$

The subscript characters L and R for columns in parentheses indicate that these columns are different in their type: the first is left-chiral, and the second is right-chiral. In the section "Spin" below, we will show how these types are expressed in matrices. For future purposes, we write the equations (33) in an expanded form using the complex-numeric components u, v, u', v' and partial derivatives in isotropic basis (17):

$$\begin{cases} \partial_\xi u + \partial_\beta v = -imu' \\ \partial_\eta v + \partial_\alpha u = -imv' \\ \partial_\eta u' - \partial_\beta v' = -imu \\ \partial_\xi v' - \partial_\alpha u' = -imv \end{cases} \qquad (36)$$

2.2. Biquaternion wave function.

Let's represent the 4-coordinate of a space-time point as a biquaternion $Z = (t, \mathbf{r})$, where $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. In isotropic basis, this biquaternion has the form[7]: $Z = \alpha\mathbf{q} + \beta\mathbf{q}^* + \xi N + \eta\bar{N}$. The correspondence between Cartesian and isotropic coordinates is given by the formulas (16),(17). The wave function of a Dirac particle[8] is described by a biquaternion function of space-time coordinates:

$$F = \mathbf{q}f_\alpha + \mathbf{q}^* f_\beta + N f_\xi + \bar{N} f_\eta \,, \tag{37}$$

where $f_\alpha, f_\beta, f_\xi, f_{nn}$ are scalar (complex-number) functions of space-time coordinates: $f_\alpha = f_\alpha(t, \mathbf{r}), f_\alpha \in \mathbb{C}$, etc.

Let us decompose F (37) by the sum of its projection components according to (19):

$$F = P^+ + P^- \tag{38}$$

$$\begin{cases} P^- = \mathbf{q}f_\alpha + N f_\xi \\ P^+ = \mathbf{q}^* f_\beta + \bar{N} f_\eta \end{cases}$$

2.3. Gradients.

Consider the 4-gradient operator expressed in scalar-vector biquaternion form: $D = (\partial_t, \boldsymbol{\nabla})$. The vector-conjugate (inverse) operator has the form: $\bar{D} = (\partial_{t\,t}, -\boldsymbol{\nabla})$. Using the coordinate transformations (16),(17), these two operators can be represented in isotropic basis:

$$\begin{cases} D = 2(\mathbf{q}\partial_\beta + \mathbf{q}^*\partial_\alpha + N\partial_\xi + \bar{N}\partial_\eta) \\ \bar{D} = 2(-\mathbf{q}\partial_\beta - \mathbf{q}^*\partial_\alpha + N\partial_\eta + \bar{N}\partial_\eta) \end{cases} \tag{39}$$

In the operator D partial derivative ∂_α stands at $\mathbf{q}^*$, and the partial derivative ∂_β stands at $\mathbf{q}$, and not vice versa, as would be expected at first glance.

The Dirac equation in the representation of Weyl spinors (33), can be apparently formulated using differential operator of the "spin 4-gradient" $D_s = (\partial_t, \boldsymbol{\sigma} \cdot \boldsymbol{\nabla})$. It is noteworthy that in isotropic basis, this operator coincides up to a factor $\frac{1}{2}$ with the 4-gradient operator D (39): $D_s = \frac{1}{2}D$.

The transition between forward and reverse 4-gradients $D = (\partial_t, \boldsymbol{\nabla})$ and $-\bar{D} = (-\partial_t, \boldsymbol{\nabla})$ is a reflection of time derivative: $\partial_t \leftrightarrow -\partial_t$. Therefore, the operator D is associated with changing the

[7] Note that here and below the isotropic coordinates α, β, ξ, η describe specifically the 4-coordinate of space-time. Above, these same variables could describe the components of an arbitrary biquaternion.
[8] Following the generally accepted convention, we call a Dirac particle a particle that satisfies the Dirac equation.

object of differentiation forward in time, wile the operator $-\overline{D}$ is associated with changing the object back in time.

2.44. Cyclic conjugation.

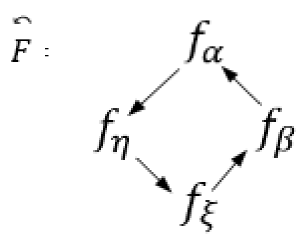
The biquaternion representation of the Dirac equation introduced below uses the cyclic conjugation operation. *The cyclic conjugate* $\overset{\frown}{F}$ of the biquaternion F, is calculated according to the formula:

$$F = \mathbf{q}f_\alpha + \mathbf{q}^* f_\beta + N f_\xi + \overline{N} f_\eta \quad \longrightarrow \quad \overset{\frown}{F} = \mathbf{q}f_\beta + \mathbf{q}^* f_\xi + N f_\eta + \overline{N} f_\eta \tag{40}$$

The cyclic conjugation (40) can be expressed using the following diagram, which shows a cyclic permutation of coordinates for a given biquaternion F, giving the output $\overset{\frown}{F}$:

$$F = N f_\xi + \mathbf{q}^* f_\beta + \mathbf{q} f_\alpha + \overline{N} f_\eta \quad \rightarrow \quad \overset{\frown}{F} \tag{41}$$

or:



In the scheme (41), the basis elements remain in place, and the components of the wave function $f_\alpha, f_\beta, f_\xi, f_n$ are transferred according to the arrows. Applying the cyclic conjugation operation four times returns the biquaternion to its original state, i.e. it implements a full cycle. The cyclic conjugation (40) is expressed via exchanged conjugations (29) (30):

$$\overset{\frown}{F} = (\widetilde{P^-} + P^+)^\star = \widetilde{P^{-\star}} + P^{+\star} = P_1^+ + P_1^-$$

$$\begin{cases} P_1^+ = \widetilde{P^{-\star}} = \ddot{P}^{-} \\ P_1^- = P^{+\star} \end{cases} \tag{42}$$

Applying the cyclic conjugation operation twice to the given wave function F, we obtain its exchange conjugation of the second type:

$$\overset{\frown}{\overset{\frown}{F}} = \widetilde{F} \tag{43}$$

The formula (43) provides the expression for *double cyclic conjugation*, which will be used later in the section "Spin" to find out the physical meaning of the cyclic conjugation operation.

2.5. Cyclic representation of the Dirac equation.

The Dirac equation in the cyclic representation[9] has the form:

$$P^{-}\odot\overline{D} + D\otimes P^{+} = im\overset{\curvearrowleft}{F} \qquad (44)$$

In the equation (44), F is the biquaternionic wave function of a Dirac particle (37), decomposed according (37) into the sum of its projection components: $F = P^{+} + P^{-}$;
$\overset{\curvearrowleft}{F}$ is the cyclic conjugate of F (40); m is the mass of a particle, D is the 4-gradient biquaternion operator, $\overline{D}$ – operator vector-conjugated to D (39). The expressions on the left side of the equation (44) are:

$$P^{-}\odot\overline{D} = 2\left(\left(-\partial_{\beta}f_{\xi} + \partial_{\xi}f_{\alpha}\right)\mathbf{q} + (-\partial_{\alpha}f_{\alpha} + \partial_{\eta}f_{\xi})N\right)$$
$$D\otimes P^{+} = 2\left(\left(\partial_{\alpha}f_{\beta} + \partial_{\xi}f_{\eta}\right)\mathbf{q}^{*} + (\partial_{\beta}f_{\eta} + \partial_{\eta}f_{\beta})\overline{N}\right)$$

Let's write the equation (44) component-wise:

$$\begin{cases} \partial_{\xi}f_{\alpha} - \partial_{\beta}f_{\xi} = imf_{\beta} \\ \partial_{\eta}f_{\xi} - \partial_{\alpha}f_{\alpha} = imf_{\eta} \\ \partial_{\eta}f_{\beta} + \partial_{\beta}f_{\eta} = imf_{\alpha} \\ \partial_{\xi}f_{\eta} + \partial_{\alpha}f_{\beta} = imf_{\xi} \end{cases} \qquad (45)$$

The equation (44), or (45), is equivalent to the original Dirac equation in Weyl spinors (33), or (36). The correspondence between the Weyl spinors ψ_L and ψ_R (35) and the biquaternionic wave function, represented in isotropic basis (37), $F = \mathbf{q}f_{\alpha} + \mathbf{q}^{*}f_{\beta} + Nf_{\xi} + \overline{N}f_{\eta}$, is given by:

$$\begin{cases} u = f_{\alpha} \\ v = -f_{\xi} \end{cases} \qquad \begin{cases} u' = -f_{\beta} \\ v' = f_{\eta} \end{cases} \qquad (46)$$

or

$$F = \mathbf{q}u - \mathbf{q}^{*}u' - Nv + \overline{N}v' \qquad (47)$$

From (46),(47), it follows that in the representation we use, <u>chiral states are projectors.</u> The right-chiral state is a positive projector, while the left-chiral state is a negative projector:

$$\begin{cases} P^{-}\sim\psi_L \\ P^{+}\sim\psi_R \end{cases} \qquad (48)$$

Indeed, the positive projector $P^{+} = -\mathbf{q}^{*}u' + \overline{N}v'$ is formed from the same components u', v' that make up the right-chiral Weyl state ψ_R (35). Similarly, the negative projector $P^{-} = \mathbf{q}u - Nv$ is formed from the components u, v that make up the left-chiral state ψ_{L}.

---

[9] The form of the Dirac equation (44) was first presented in our paper [5].

In the section "Spin" below, we will explain the physical meaning of the operation of cyclic conjugation of the wave function $\overset{\curvearrowleft}{F}$.

2.66. Separate presentation.

Equation (44) can be rewritten as two separate equations for the left- and right-chiral states of the described particle:

$$\begin{cases} P^{-} \odot \overline{D} \;= imP^{+\bigstar} \\ D \otimes P^{+} = im\dddot{P}^{-} \end{cases} \qquad (49)$$

Let's write out separately conjugate projectors that are included in the right-hand sides of the equations (49):

$$P^{+\bigstar} = f_{\beta}\mathbf{q} + f_{\eta}N$$
$$\dddot{P}^{-} = f_{\xi}\mathbf{q}^{*} + f_{\alpha}\overline{N}$$

To write the equations (49), we used the expression (42) for cyclic conjugation. Note that in the first of the equations, both sides of the equation $P^{-}\odot\overline{D}$ and $imP^{+\bigstar}$ are negative projectors, i.e. left-chiral states. Similarly, in the second of these equations, both sides are positive projectors, i.e. right-chiral states.

Thus, cyclic representation of the Dirac equation (44) implicitly contains both equations (49), each of which describes one of the two chiral states. By using the cyclic conjugation operation, the equation (44) combines both chiralities.

2.7. Lorentz transformations.

We restrict ourselves to considering the most important special Lorentz transformation of the wave function of a Dirac particle – the boost with velocity $V = th2\theta$ in the direction of the vector $\mathbf{n}$. In the Weyl spinor representation (35), this transformation has the following form [14]:

$$\begin{cases} \psi_L = \begin{pmatrix} u \\ v \end{pmatrix}_L \rightarrow \psi'_L = \begin{pmatrix} e^{-\theta}u \\ e^{\theta}v \end{pmatrix}_L \\ \\ \psi_R = \begin{pmatrix} u' \\ v' \end{pmatrix}_R \rightarrow \psi'_R = \begin{pmatrix} e^{\theta}u' \\ e^{-\theta}v' \end{pmatrix}_R \end{cases} \qquad (50)$$

To avoid confusion, note that in the formula (50), the stroke symbol has two different obvious meanings: for the functions ψ'_L and ψ'_R , this is the Lorentz-transformed quantity, and for the quantities u' and v', these are the components of the right spinor according to the formula (35). As we see below, in terms of chiral algebra, Lorentz transformations are expressed using the operations of outer and inner products.

We sill call the biquaternions L and $\dddot{L}$ the longitudinal and transverse *Lorentz operators*, respectively:

$$\begin{cases} L = Ne^{\theta} + \bar{N}e^{-\theta} \\ \dddot{L} = \mathbf{q}e^{-\theta}\mathbf{q}^{*}e^{\theta} \end{cases} \quad (51)$$

From (32) follows a rule, that allows us to move these operators from the left side to the right side, and vice versa:

$$\forall \mathcal{B}:\ \mathcal{B} \odot L = \dddot{L} \otimes \mathcal{B} \quad (52)$$

The boost of the wave function described above has the form:

$$F \ \rightarrow \ F' = F \odot L = \dddot{L} \otimes F \quad (53)$$

The transformation (53) can also be written by separating the projection components:

$$F \ \rightarrow \ F' = P^{+} \odot L + \dddot{L} \otimes P^{-} \quad (54)$$

According to (50) and (53) and correspondence rule (46), at Lorentz transformations projectors and Weyl spinors behave the same way. Applying the transformations (53) separately to each projection component $P^{\pm}$, it is easy to verify the Lorentz covariance of the Dirac equation in the separated form (49), and thus the covariance of cyclic representation of this equation (44).

In the Cartesian basis, the energy-momentum of a free particle is given by the biquaternion $K = (\epsilon, \mathbf{k}), \mathbf{k} = (k_x, k_y, k_z)$. In isotropic basis, the same value has the form: $K = k_{\alpha}\mathbf{q} + k_{\beta}\mathbf{q}^{*} + k_{\xi}N + k_{n}\bar{N}$, the energy-momentum components in the light and Cartesian bases are connected by the transformations of the form (17).

The Lorentz transformation of energy-momentum K is expressed in the operators L and $T = \dddot{L}$ of the general form, which includes (51) as a special case:

$$K \rightarrow K' = L^{*} \odot K \odot L \quad (55)$$

$$K \rightarrow K' = T^{*} \otimes K \otimes T$$

In (55), both variants of the Lorentz transformation are equivalent to each other. A comparison of the formulas (53) and (55) indicates an intrinsic feature of Lorentz transformations of quantities of various types. Namely, the wave function transformation is one-sided, while the energy-momentum transformation is two-sided in terms of Lorentz operators acting.

The space-time coordinate (4-coordinate) $Z = (t, \mathbf{r})$ is transformed like the energy-momentum (55). From the energy-momentum of the particle and the 4-coordinate, we can form the well-known scalar Lorentz invariant:

$$\Phi = \epsilon t - \mathbf{k} \cdot \mathbf{r} = \frac{1}{2}(-k_\beta \alpha - k_\alpha \beta + k_\eta \xi + k_\eta N) \qquad (56)$$

2.8. Plane waves.

The simplest solution of the Dirac equation (44) has the form *of a plane wave*:

$$F = Ae^{i\Phi} + Be^{-i\Phi}, \qquad (57)$$

where Φ is the Lorentz-invariant phase of the wave defined according (56), and A and B are constant biquaternions having the following form for a moving particle:

$$A = a_\alpha \mathbf{q} + a_\beta \mathbf{q}^* + a_\xi N + a_\eta \bar{N}, \qquad B = b_\alpha \mathbf{q} + b_\beta \mathbf{q}^* + b_\xi N + b_\eta \bar{N} \qquad (58)$$

$$a_\alpha = a_1 \qquad b_\alpha = b_1$$
$$a_\beta = a_2 \qquad b_\beta = b_2$$
$$a_\xi = \frac{ma_2 - k_\eta a_1}{k_\alpha} \qquad b_\xi = -\frac{mb_2 + k_\eta b_1}{k_\beta}$$
$$a_\eta = \frac{k_\xi a_2 - ma_1}{k_\alpha} \qquad b_\eta = \frac{k_\xi b_2 + mb_1}{k_\beta}$$

From the Dirac equation follows the dispersion relation:

$$\epsilon^2 - \mathbf{k}^2 = k_\xi k_\eta - k_\alpha k_\beta = m^2 \qquad (59)$$

which implies the subluminal character of a massive Dirac particle. Below we will also consider an analog of the Dirac equation for a superluminal particle. The components of the wave function $Ae^{\,i\Phi}$ and $Be^{-i\Phi}$ are called positive-frequency and negative-frequency, respectively.

For a massive particle, such as an electron or positron, the state of rest can be determined. In this case, $K = m$, or $k_\alpha = k_\beta = 0, k_\xi = k_\eta = m$. The plane wave (57) takes the form:

$$F = A_0 e^{imt} + B_0 e^{-imt}, \qquad (60)$$

Where biquaternionic coefficients A_0 and B_0 are:

$$A_0 = a_1 \mathbf{q} + a_1 \mathbf{q}^* + a_2 N + a_2 \bar{N}, \qquad B_0 = b_1 \mathbf{q} - b_1 \mathbf{q}^* + b_2 N - b_2 \bar{N}$$

2.9. Spin.

Let us ask ourselves how operators of spin projections look in our biquaternionic representation. In the representation of Weyl spinors, these operators, up to a factor $\frac{1}{2}$, are the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ (34). Above we provided the standard expression for left-and right-chiral Weyl spinors in the form of matrix columns. Now we express these states in the form of 2x2 matrices themselves:

$$\begin{cases} \psi_L = \begin{pmatrix} u \\ v \end{pmatrix}_L \equiv \begin{pmatrix} u & 0 \\ v & 0 \end{pmatrix} \\ \\ \psi_R = \begin{pmatrix} u' \\ v' \end{pmatrix}_R \equiv \begin{pmatrix} 0 & u' \\ 0 & v' \end{pmatrix} \end{cases} \tag{61}$$

The wave function of a general particle state is then expressed as:

$$\psi = \psi_L + \psi_R = \begin{pmatrix} u & u' \\ v & v' \end{pmatrix} \tag{62}$$

As we can see, the matrix representation of Weyl spinors no longer requires specifying, which type, left - or right-chiral – this follows from the form of the matrix. At the same time, many of the operations that occur with columns can be carried out by these matrices, provided you use the usual matrix multiplication. Let us write out the form of eigenstates of the spin projection operators [14] for Weyl spinors in their matrix representation:

$$\begin{cases} \psi_{\mathrm{x}\uparrow} = \begin{pmatrix} u & u' \\ u & u' \end{pmatrix}, & \psi_{\mathrm{x}\downarrow} = \begin{pmatrix} u & u' \\ -u & -v' \end{pmatrix} \\ \psi_{y\uparrow} = \begin{pmatrix} u & u' \\ iu & iu' \end{pmatrix}, & \psi_{y\downarrow} = \begin{pmatrix} u & u' \\ -iu & -iu' \end{pmatrix} \\ \psi_{z\uparrow} = \begin{pmatrix} u & u' \\ 0 & 0 \end{pmatrix}, & \psi_{z\downarrow} = \begin{pmatrix} 0 & 0 \\ v & v' \end{pmatrix} \end{cases} \tag{63}$$

In (63), $\psi_{x\uparrow}$ is the state with the *x*-projection of the spin pointing up; $\psi_{x\downarrow}$ is the state with the *x*-projection of the spin pointing down, and so on. Let us establish the following correspondence between the matrices that map the Weyl spinors (62) and the biquaternions:

$$F = f_\alpha \mathbf{q} + f_\beta \mathbf{q}^* + f_\xi N + f_\eta \bar{N} \quad \leftrightarrow \quad \psi = \begin{pmatrix} f_\alpha & -f_\beta \\ -f_\xi & f_\eta \end{pmatrix}, \tag{64}$$

or $u = f_\alpha, v = -f_\xi, u' = -f_\beta, v' = f_\eta$. Above, in (46), when establishing the equivalence of different representations of the Dirac equation, we already used this correspondence with respect to Weyl spinors-columns. Using this correspondence between matrices and biquaternions, we establish an isomorphism by addition and multiplication, provided we use the usual multiplication for matrices and diagonal multiplication for biquaternions (25):

$$\psi_1\psi_2 \leftrightarrow F_1 \times F_2 \tag{65}$$

Using this isomorphism, we can express the spin projection operators in biquaternions corresponding to the Pauli matrices:

$$\begin{cases} \hat{S}_x = -(\mathbf{q}^* + N) \\ \hat{S}_y = i(\mathbf{q}^* - N) \\ \hat{S}_z = \mathbf{q} - N \\ \hat{S}_t = \mathbf{q} + \overline{N} \equiv E \end{cases} \tag{66}$$

Each of the spin operators (66) is a biquaternion and acts on the wave function on the left side. The fourth of these operators $\hat{S}_t$ corresponds to the identity matrix and is the unit for inner multiplication E. Let's express the spin states (63) in biquaternion form:

$$\begin{cases} F_{x\uparrow} = f_\alpha \mathbf{q} + f_\beta \mathbf{q}^* - f_\alpha N - f_\beta \overline{N}, & F_{x\downarrow} = f_\alpha \mathbf{q} + f_\beta \mathbf{q}^* + f_\alpha N + f_\beta \overline{N} \\ F_{y\uparrow} = f_\alpha \mathbf{q} + f_\beta \mathbf{q}^* - i f_\alpha N - i f_\beta \overline{N}, & F_{y\downarrow} = f_\alpha \mathbf{q} + f_\beta \mathbf{q}^* + i f_\alpha N + i f_\beta \overline{N} \\ F_{z\uparrow} = f_\alpha \mathbf{q} + f_\beta \mathbf{q}^*, & F_{z\downarrow} = f_\xi N + f_\eta \overline{N} \end{cases} \tag{67}$$

According to the isomorphism (65), the spin operators (66) use diagonal multiplication. It is easy to verify that the wave functions (67) are indeed eigenstates of these operators with eigenvalues ± 1 (doubled spin):

$$\begin{cases} \hat{S}_x \times F_{x\uparrow} = F_{x\uparrow}, \quad \hat{S}_x \times F_{x\downarrow} = -F_{x\downarrow} \\ \hat{S}_y x F_{y\uparrow} = F_{y\uparrow}, \quad \hat{S}_y x F_{y\downarrow} = -F_{y\downarrow} \\ \hat{S}_y \times F_{z\uparrow} = F_{z\uparrow}, \quad \hat{S}_x \times F_{z\downarrow} = -F_{z\downarrow} \end{cases} \tag{68}$$

The longitudinal direction $\mathbf{n}$ (or z) is distinguished in our approach according to the choice of isotropic basis[10]. We define *the longitudinal spin* as the projection of the spin on the direction z. Applying exchange conjugate (30) to the eigenstates of the longitudinal spin $F_{z\uparrow}$ and $F_{z\downarrow}$, we find that this operation reverses the direction of the longitudinal spin:

$$\tilde{F}_{z\uparrow} = F_{z\downarrow}, \quad \tilde{F}_{z\downarrow} = F_{z\uparrow} \tag{69}$$

We saw above in (43) that the exchange conjugation coincides with the double cyclic conjugation. Thus, (69) can be rewritten as:

$$\overset{\frown\frown}{F}_{z\uparrow} = F_{z\downarrow}, \quad \overset{\frown\frown}{F}_{z\downarrow} = F_{z\uparrow} \tag{70}$$

At the rest state (60) with the spin directed up along z-axis, the wave function (for a positive-frequency component) has the form: $F_{z\uparrow} = a_{a1}\mathbf{q} + a_{a1}\mathbf{q}^*$. By applying cyclic conjugation transformation to this state, we will receive the new state F_0: $F_0 = \overset{\frown}{F}_{z\uparrow} = a_1(\mathbf{q}^* + N)$. Let us determine the value of z-projection of the spin for the state F_0. To do this, we obtain a representation of this state in Weyl spinors (46): $\psi_{0L} = \binom{0-1}{-1}_L, \psi_{0R} = \binom{-1}{0}_{R.}$ The average value of

[10] In our terminology, the longitudinal direction may not coincide with the direction of the particle's momentum, so the projection of the spin onto the longitudinal direction is not helicity.

the spin projection s_z in this state is 0: $< s_z > = \psi_{0L}^* \sigma_z \psi_{0L} + \psi_{0R} \sigma_z \psi_{0R} = 0$. We will call F_0 the positive-frequency *zero state of spin.* Similarly defined is the negative-frequency zero state of spin.

Thus dual cyclic conjugation, which coincides with exchange conjugation of 2nd type, is the operation of turning the longitudinal spin. It follows that cyclic conjugation is spin half-flip operation. In our units, the spin projection varies from +1 to -1, so a half-flip from the state with a projection of +1 or -1 translates the longitudinal spin to zero state F_0 or vice versa – from the zero state F_0 to the state with a projection of ± 1. As we have just seen, z-projection of the spin in zero state is, as it should be, 0. We should keep in mind that the cyclic conjugation operation must be applied separately to the positive- and the negative-frequency zero states. So, applying cyclic conjugate to a state with zero longitudinal spin projection creates a state with a spin projection of ± 1 . But the longitudinal direction can be chosen arbitrarily – when we choose isotropic basis. Therefore, we can conclude that <u>cyclic conjugation is the operator of assigning spin to a massive particle</u>.

2.10. Transformation of rotation.

As you know, general Lorentz transformations include boosts and turns[11]. A boost from the particle's rest state creates a state with a non-zero linear velocity, and a turn rotates this velocity in space. But then there must be analogous transformations: first that moves the particle to the state of its proper rotation and the second that transfers the center of this rotation in space (translation). If we associate the spin of the particle with its proper rotation, then, as we saw in the previous section, the rotation of the particle is created by cyclic conjugation. Cyclic conjugation of the particle wave function is represented by the right-hand side of the Dirac equation (44). This means that for a Dirac particle, the wave function transformation on the left-hand side of the equation also works (although in different manner) as particle rotation generator:

$$F \to F'_{rot} = P^- \odot \bar{D} + D \otimes P^+ \tag{71}$$

We show that (71) has a structure similar to Lorentz transformation (54), but of a slightly different kind. Let's rewrite the expression for the plane wave (57) in the following form:

$$F = Q_1 + Q_2, \tag{72}$$

where $Q_1 = Ae^{i\Phi}$ and $Q_2 = Be^{-i\Phi}$ are the positive and negative frequency components of the wave function F. For the plane wave (57)(72), the following substitutions can be made: $\bar{D} = iK, D = i\bar{K}$ k for the positive-frequency component, and $\bar{D} = -iK, D = -i\bar{K}$ for the negative-frequency component, so that the transformation (71) can be written as:

$$F \to F'_{rot} = Q^- \odot K + \bar{K} \otimes Q^+ \tag{73}$$

$$\begin{cases} Q^- = Q_1^- - Q_2^- \\ Q^+ = Q_1^+ - Q_2^+ \end{cases}$$

[11] Please distinct between turns and rotations,

We call Q^- and Q^+ *subtractive components* of F, and call (73) *rotation transformation*.

Let us put together Lorentz transformation (54) and rotation transform (73) of the wave function:

$$F \rightarrow F'_{lor} = P^+ \odot L + \dddot{L} \otimes P^- \qquad (74)$$
$$F \rightarrow F'_{rot} = Q^- \odot K + \bar{K} \otimes Q^+$$

From (74) we see the similarity of both transformations, with the difference that the Lorentz transformation F'_{lor} includes the usual projection components of the wave function $P^{\pm}$, while rotation transformation F'_{rot} includes its subtractive components $Q^{\pm}$. Also the longitudinal and transverse operators are related to each other in a different way. In the case of the rotation transformation the energy-momentum biquaternion K and its vector conjugate $\bar{K}$ play roles similar to Lorentz operators (L и $\dddot{L}$). The connection between the longitudinal and transverse operators in the first case is carried out through exchange conjugation $L \rightarrow \dddot{L}$, while in the second case –through vector conjugation $K \rightarrow \bar{K}$. Thus, the rotation transformation is similar to Lorentz transformation.

In terms of rotation transformation, the Dirac equation (44) can be re-written as:

$$F \rightarrow F'_{rot} = \overset{\frown}{F} \qquad (75)$$

As a result of rotation transformation applied to the wave function of a Dirac particle, the latter acquires spin, which exhibits by the right-hand side of the equation $\overset{\frown}{F}$. Therefore, this transformation is an analog of the Lorentz transformation, that creates the proper rotation of a particle. We have discussed the necessity of such transformation in the beginning of this section. The question of finding the other, translational, transformation remains open.

2.11. Alternative forms of cyclic representation.

In addition to (44), there are three other analogous representations of the Dirac equation that use their own types of cyclic conjugation. Cyclic conjugations of all four types are closely related to exchange conjugation. A detailed discussion of all these equations and their cyclic conjugates is beyond the scope of this article. Here are all four possible representations of the Dirac equation in projectors, starting with the representation (44) described above:

$$
\begin{aligned}
P^{-}\odot\overline{D} + D\otimes P^{+} &= im\overset{\curvearrowleft}{F} \\
P^{-}\odot\overline{D} - D\otimes P^{+} &= im\overset{\rightsquigarrow}{F} \\
P^{+}\odot\overline{D} + D\otimes P^{-} &= im\overset{\curvearrowright}{F} \\
P^{+}\odot\overline{D} - D\otimes P^{-} &= im\overset{\leftsquigarrow}{F}
\end{aligned}
\tag{76}
$$

In (76) $\overset{\rightsquigarrow}{F}, \overset{\curvearrowright}{F}, \overset{\leftsquigarrow}{F}$ are three other types of cyclic conjugation similar to cyclic conjugation $\overset{\curvearrowleft}{F}$ introduced above in (40),(41). Each of the equations (76) uses its own representation and Weyl spinor correspondence identical or similar to (40). Each of these equations is equivalent to the Dirac equation and therefor they are all equivalent to each other.

Above, we considered cyclic representations of the Dirac equation in projectors (76). The isomorphism of projectors and signed biquaternions (18) was pointed out above. By virtue of this isomorphism, the Dirac equation also has cyclic representations in signed biquaternions, which we do not consider here.

At the end of this section, we give the form of the Dirac equation for *a superluminal particle*:

$$
P^{-}\odot\overline{D} - D\otimes P^{+} = im\overset{\curvearrowleft}{F} \tag{77}
$$

As we can see, the superluminal Dirac equation (77) differs from the subluminal one (44) only by the sign between projection-like components $P^{-}\odot\overline{D}$ and $D\otimes P^{+}$. The dispersion relation for the plane wave (59) is replaced here by $\epsilon^2 - \mathbf{k}^2 = k_\xi k_n - k_\alpha k_\beta = -m^2$, which confirms the superluminal character of the particle described by the equation (77).

2.112. The Dirac equation and the Hadamard matrix.

Using the inverse coordinate transformation *(16)*, we can verify that in Cartesian coordinates, the cyclic conjugation is expressed in terms of a complex Hadamard matrix H_4:

$$
\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & i & 1 & 1 \\ i & -1 & -i & -i \\ -1 & i & -1 & 1 \\ 1 & -i & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = H_4 \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \tag{78}
$$

As is well known, Hadamard matrices work in noise-suppressing algorithms of discrete information transmission. Specifically Walsh functions used for signal encoding are constructed on their basis. The presence of Hadamard matrices in the Dirac equation indicates the informational aspect of this equation.

Similar mechanisms of noise immunity also work in the genetic code of DNA, that is characterized by its multilevel skew symmetry and is described in terms of Petoukhov genetic matrices [11]. Previously, we proposed a mathematical model of the genetic code based on chiral algebra [5]. An explicit similarity between genetic code model and spin theory allowed us to assume that the genetic code contains structures of biological nature similar to spin, which we called *biospin*.

Appendix.

1. Isomorphism of biquaternions and matrices.

There exists an isomorphism between biquaternions and second-order square matrices [*13*]. Let's fixate some isotropic basis of biquaternionic space (11). Then each biquaternion $\mathcal{B}$ represented in this basis according to (14) can be put in a one-to-one correspondence with a second-order square matrix M:

$$\mathcal{B} = \alpha\mathbf{q} + \beta\mathbf{q}^* + \xi N + \eta\bar{N} \quad \leftrightarrow \quad M = \begin{pmatrix} \xi & \alpha \\ \beta & \eta \end{pmatrix} \tag{79}$$

According to this correspondence rule, there is an isomorphism by addition and outer multiplication between biquaternions and second-order square matrices. The usual rule of matrix multiplication, denoted here by the symbol ⊙ (same as for biquaternions), has the well-known form:

$$M_1 \odot M_2 = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \odot \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{21}b_{12} & a_{11}b_{21} + a_{21}b_{22} \\ a_{12}b_{11} + a_{22}b_{12} & a_{12}b_{12} + a_{22}b_{22} \end{pmatrix} \tag{80}$$

If we replace biquaternionic outer multiplication ⊙ by inner multiplication ⊗, then another isomorphism is established between biquaternions and matrices:

$$\mathcal{B} = \alpha\mathbf{q} + \beta\mathbf{q}^* + \xi N + \eta\bar{N} \quad \leftrightarrow \quad M = \begin{pmatrix} \xi & \alpha \\ -\beta & -\eta \end{pmatrix} \tag{81}$$

However, for this isomorphism , one needs to change the multiplication rule for matrices to the following:

$$M_1 \otimes M_2 = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} - a_{21}b_{12} & a_{11}b_{21} - a_{21}b_{22} \\ a_{12}b_{11} - a_{22}b_{12} & a_{12}b_{12} - a_{22}b_{22} \end{pmatrix} \tag{82}$$

It follows that outer multiplication of biquaternions corresponds to usual multiplication of matrices (80), and inner multiplication corresponds to the multiplication of matrices (82), in which the products of row elements of the first matrix by column elements of the second matrix are not added, but subtracted from each other.

In this paper, we primarily use a representation of biquaternions based on an isomorphism of the first type. Because of this, outer product of biquaternions has the usual properties of matrix multiplication, including associativity. Inner product of biquaternions, unlike outer product, does not possess associativity. It also turns out that outer product of biquaternions does not depend on chosen basis, while their inner product is basis-dependent. However, you can use an alternative representation of biquaternions, in which, on the contrary, inner product will be associative and basis-independent, while outer product will be non-associative and basis-dependent.

2. Products of elements of isotropic basis.

Table 1 shows pairwise products of the elements of biquaternion isotropic basis for two basic types of multiplication: outer $\odot$ and inner $\otimes$.

Table1. Multiplication table of the elements of isotropic basis.

Outer product $\odot$		Inner product $\otimes$	
$\mathbf{q}\odot\mathbf{q}=0$	$\mathbf{q}^*\odot\mathbf{q}^*=0$	$\mathbf{q}\otimes\mathbf{q}^*=0$	$\mathbf{q}^*\otimes\mathbf{q}=0$
$N\odot\bar{N}=0$	$\bar{N}\odot N=0$	$N\otimes N=0$	$\bar{N}\otimes\bar{N}=0$
$\mathbf{q}\odot\mathbf{q}^*=N$	$\mathbf{q}^*\odot\mathbf{q}=\bar{N}$	$\mathbf{q}\otimes\mathbf{q}=\mathbf{q}$	$\mathbf{q}^*\otimes\mathbf{q}^*=\mathbf{q}^*$
$N\odot N=N$	$\bar{N}\odot\bar{N}=\bar{N}$	$N\otimes\bar{N}=\mathbf{q}^*$	$\bar{N}\otimes N=\mathbf{q}$
$\mathbf{q}\odot N=0$	$\mathbf{q}^*\odot\bar{N}=0$	$\mathbf{q}\otimes N=0$	$\mathbf{q}^*\otimes\bar{N}=0$
$N\odot\mathbf{q}=0$	$\bar{N}\odot\mathbf{q}^*=0$	$\bar{N}\otimes\mathbf{q}=0$	$N\otimes\mathbf{q}^*=0$
$\mathbf{q}\odot\bar{N}=\mathbf{q}$	$\mathbf{q}^*\odot N=\mathbf{q}^*$	$\mathbf{q}\otimes\bar{N}=\bar{N}$	$\mathbf{q}^*\otimes N=N$
$N\odot\mathbf{q}=\mathbf{q}$	$\bar{N}\odot\mathbf{q}^*=\mathbf{q}^*$	$N\otimes\mathbf{q}=N$	$\bar{N}\otimes\mathbf{q}^*=\bar{N}$

In particular, as follows from this table, the elements $N,\bar{N}$ are idempotent with respect to outer multiplication $\odot$ and nilpotent with respect to inner multiplication $\odot$, and the elements $\mathbf{q},\mathbf{q}^*$ are, on the contrary, nilpotent with respect to inner multiplication $\otimes$ and idempotent with respect to outer multiplication. Thus, switching from one type of multiplication to the other converts idempotents and nilpotents into each other.

Discussion and conclusions.

By extending multiplication and conjugation operations and using isotropic basis, we constructed a biquaternion algebra, which we call chiral. In terms of chiral algebra, we obtained a new representation of the Dirac equation (44), that we called cyclic. In this biquaternionic representation, the Dirac equation is written in a single biquaternion string, implicitly combining the equations for the states of both chiralities – right and left. Besides the operations of outer and inner multiplication of biquaternions, the new representation is using a special cyclic conjugation operation. Thus, this representation reveals the inner cyclic nature of a Dirac particle.

From the representation (44), we can conclude that <u>the Dirac equation describes the relationship between linear and cyclic times that characterize the development of the wave function of a Dirac particle.</u> As we have seen above, the usual gradient D is associated with the change of the wave function forward in time, while the opposite gradient $-\bar{D}$ is associated with the change of the wave function backward in time. Note that in our equation, each of forward and reverse gradients is combined with its own type of multiplication, inner or outer, and with its own side of action, right or left: as $\odot\overline{D}$ and $D\otimes$. In general, it can be concluded that three types of time flow converge in the Dirac equation: linear forward, linear inverse, and cyclic. A given particle is evolving as a result of the relationship of these types of time. The presence of all three types of time is possible only for a massive particle. Cyclic representation of the Dirac equation clearly reveals the time asymmetry of this equation.

The Dirac equation and chiral algebra are closely related to each other. In the language of chiral algebra, the Dirac equation is written in a concise and homogeneous form, with easily seen physical meaning. At the same time, this equation also provides concrete meaning to purely mathematical operations of chiral algebra. Thus, the exchange conjugation serves as a spin flip operation, and the cyclic conjugation provides the particle with its spin.

We have derived the rotation transformation – an analog of the Lorentz transformation, that creates the proper rotation of a massive particle. The left-hand side of the Dirac equation gives such a transformation for the case of a plane wave. It is noteworthy that the rotation transformation is formed by the joint action of two linear energy-momentum operators, which resembles the action of a pair of linear momenta in classical mechanics, creating a rotation of a rigid body near a fixed center. We still need to determine a translation transformation related to rotation transformation. They will be connected to each other just as turn and boost transformations are related to each other.

Chiral algebra provides a new mathematical framework for spin theory different from the traditional spinor formalism. In many respects, this apparatus, as well as biquaternions in general, is much simpler and easier to handle than the traditionally used tensor-spinor apparatus. A significant advantage of the biquaternion apparatus is the absence of cluttering up formulas with indexes.

When using chiral algebra to describe spin, one does not need to resort to gamma matrices. Accordingly, the formulas are expressed in a more natural way. Especially important is the fact that all four components of the expansion in isotropic basis are fundamentally different from each other and reflect the essential characteristics of the longitudinal-transverse expansion. This allows us to see transverse and longitudinal nature in the obtained solutions of the equation.

An additional, inner, type of biquaternion multiplication allows to bypass the well-known problem of zero divisors of a biquaternion space, which used to be a major obstacle on constructing a biquaternion algebra with division. This problem has always been a stumbling block in the construction of the theory of biquaternionic analytic functions. As we saw above, the product of the same multipliers in one type of multiplication can be zero, but at the same time in another type of multiplication it is non-zero. Using both of these types of multiplication, we can construct division algebras for biquaternions and then, as we hope, a full-fledged biquaternion analysis. In such a two-side analysis, the Dirac equation will presumably turn out to be a condition for analyticity of the wave function. If there is a particle mass, this analytical function will have a singularity point, where its analyticity is violated, and the function residue will be described by the right-hand side of the Dirac equation.

The expression of cyclic conjugation through a complex Hadamard matrix, that we discovered, indicates the connection of the Dirac equation with noise immunity algorithms involved in the processes of information transfer.

In this paper, we have not investigated in the framework of the new approach such questions as quantum probabilities, currents, Fermi-Dirac statistics, secondary quantization, etc., leaving these important items for future consideration.

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