Is there an infinite number in which the sum of the divisors is the square number?

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Abstract

The numbers whose sum of divisors is a perfect square may initially appear to follow no specific pattern. However, through this research, I have identified a particular rule related to prime numbers. Furthermore, I establish that the existence of infinitely many numbers whose sum of divisors is a perfect square is a necessary and sufficient condition for the existence of an irreducible polynomial with integer coefficients that generates infinitely many prime numbers. Additionally, I explore its connection to Bunyakovsky's conjecture.

Introduction

While studying perfect numbers — numbers that are equal to the sum of their proper divisors —I extended my research to the sum of all divisors, rather than just the proper ones. In doing so, I observed that a significant number of integers have a sum of divisors that forms a perfect square. This led me to establish certain patterns and conduct further investigation, ultimately revealing a connection to Bunyakovsky's conjecture. If this paper contains no errors, resolving Bunyakovsky's conjecture through this approach would hold even greater mathematical significance than previously recognized.

Keywords : Divisors , Perfect Numbers , Bunyakovsky's Conjecture , Divisor Function

Contents

Background Knowledge

Introduction of the Key Function: The Divisor Sum Function $\sigma(n)$

$$\sigma(n) = \sum_{d|n} d \ \ (\sigma : N \rightarrow N)$$

The function described here is the **sum of divisors function** $\sigma(n)$, which represents the sum of all divisors of a number n. We will refer to numbers whose sum of divisors is a perfect square as **DS numbers**. Thus, the sequence of DS numbers will be called the **DS sequence**, and the set of DS numbers will be referred to as the **DS set**. The **DS set** will be listed in increasing order, excluding duplicates. Therefore, the n-th element of the **DS set** is the n-th DS number and corresponds to the n-th term of the **DS sequence**.

Note that 1 is not considered a DS number.

Upper density: This refers to the upper bound of the proportion of a subset within the set of natural numbers, indicating the density of that subset in the natural numbers.

P is set of prime number

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The numbers whose sum of divisors is a perfect square are listed as follows:

$$3, 22, 66, 70, 81, 94, 214, 282, \ldots$$

It seems that there is no clear pattern. However, let's consider 22, 94, and 214. In doing so, we can easily discover a pattern.

$$22 = 2 \times 11 \ , \ 94 = 2 \times 47 \ , \ 214 = 2 \times 107$$

All three numbers are twice a prime number.

$$66 = 6 \times 11 \ , \ 282 = 6 \times 47$$

Moreover, 66 and 282 are six times 11 and 47, respectively. Therefore, it is a natural conjecture that six times 107 will also be a DS number.

$$6 \times 107 = 642 \text{ , } \sigma(642) = 1 + 2 + 3 + 6 + 107 + 214 + 321 + 642 = 1296 = 36^2$$

We are satisfied. Therefore, although not all DS numbers can be expressed as twice a prime nu mber, we can observe that twice certain primes (11, 47, 107, ...) are DS numbers. If we find a pattern among the primes 11, 47, 107, ..., we may be able to discover a rule for the DS numbe rs.

$$11, 47, 107, \cdots$$

The difference between 11 and 47 is 36, and the difference between 47 and 107 is 60. 36 is the square of 6, and 60 is the product of 6 and 10. Therefore, we have a sequence that increases by a constant amount, denoted as $(6 \times (6 + 4n))$ $(n = 0, 1, 2, \cdots)$, and let's call this sequence $\{a_n\}$. According to this pattern, the next term in the sequence is 107+84=191.
If our conjecture holds, then twice 191 should also be a DS number.

$$\sigma(382) = 1 + 2 + 191 + 382 = 576 = 23^2$$

Twice 191, which is 382, is also a DS number. However, twice 299, the next term 191+108 = 299, is not a DS number.

$$\sigma(598) = 1 + 2 + 13 + 23 + 26 + 46 + 299 + 598 = 1008 = 2^4 \times 3^2 \times 7$$

However, twice 431, the next term 299+132=431, is a DS number.

$$\sigma(862) = 1 + 2 + 431 + 862 = 1296 = 36^2$$

The decisive difference between 11, 47, 107, 191, 431 and 299 is that the former are prime numbers, while 299 is a composite number, being the product of 23 and 13. Therefore, we can make the following conjecture:

Conjecture 1 If a number is included in $\{a_n\}$ and is a prime, then twice that number is a DS n umber.

To prove **Conjecture 1,** let's first find the general term of the sequence $\{a_n\}$. The recurrence relation for $\{a_n\}$ is as follows:

$$a_n = a_{n-1} + 6 \times (6 + 4(n-2)) \quad (a_1 = 11)$$

Solving the recurrence relation gives us the following:

$$a_n = a_{n-1} + 24n - 12$$

The difference between two consecutive terms is as follows:

$$a_{n+1} - a_n = a_n + 24n + 12 - (a_{n-1} + 24n - 12) = a_n - a_{n-1} + 24$$

Let the sequence $\{A_n\}$ be defined as follows:

$$A_n = a_{n+1} - a_n \ (n \geq 1 \ , \ a_1 = 11)$$

Then, $\{A_n\}$ is an arithmetic sequence with a common difference of 24. Therefore, we can conjecture that the general term of $\{a_n\}$ is in the form of a quadratic expression.

Theorem 1 If $\{v_n\}$ is an arbitrary sequence with all positive terms that is monotonically increasing, and $\{v_{n+1} - v_n\}$ is an arithmetic sequence, then the general term of $\{v_n\}$ is a quadratic expression.

-proof-

Since $\{v_{n+1} - v_n\}$ is an arithmetic sequence, we can consider $\{v_{n+1} - v_n\} = \{u_n\}$. Then, for the common difference d, we have:

$$v_{n+1} - v_n = u_1 + (n-1)d \ \ (d \neq 0)$$

$$v_n = v_1 + \sum_{k=1}^{n-1} u_k = v_1 + \sum_{k=1}^{n-1} u_1 + (k-1)d = v_1 + u_1 \sum_{k=1}^{n-1} 1 + d \sum_{k=1}^{n-1} (k-1)$$

$$\sum_{k=1}^{n-1} 1 = n-1 \ , \ \sum_{k=1}^{n-1} (k-1) = \frac{(n-1)(n-2)}{2}$$

$$v_n = v_1 + u_1(n-1) + \frac{d(n-1)(n-2)}{2}$$

This is a quadratic expression in terms of n. ■

Applying **Theorem 1** to $\{a_n\}$, we get the following:

$$a_n = a_1 + (a_2 - a_1)(n-1) + 12(n-1)(n-2)$$
$$a_n = 11 + 36n - 36 + 12(n^2 - 3n + 2)$$
$$a_n = 12n^2 - 1$$

In this way, we have found the general term of $\{a_n\}$.
Now, we can state **Conjecture 1** as follows:

Conjecture 1 For any natural number n≥1 , if p is a prime of the form $12n^2-1$, then $\sigma(2p)$ is a perfect square.

-proof-

The divisors of a prime p are only 1 and p. Therefore, for any prime p, the divisors of p are just 1 and p. The divisors of 2p are 1,2,p, and 2p. Thus, the sum of the divisors of 2p is 3p+3. If p has the form $12n^2-1$, then the sum of the divisors of 2p is as follows:

$$\sigma(2p) = 3(12n^2 - 1) + 3 = 36n^2 = (6n)^2$$

Therefore, if p has the form $12n^2-1$, then 2p is a DS number. ■

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Lemma 2 If two arbitrary natural numbers a and b are coprime, then $\sigma(ab) = \sigma(a)\sigma(b)$.

-proof-

$$\sigma(a) = \sum_{d|a} d \ , \ \sigma(b) = \sum_{k|b} k \Rightarrow \sigma(ab) = \sum_{d|a}\sum_{k|b} dk$$

Because the divisors d and k of a and b, respectively, are all distinct (other than 1), we can separate the sum of the divisors as follows:

$$\sum_{d|a}\sum_{k|b} dk = \sum_{d|a} d \sum_{k|b} k = \sigma(a)\sigma(b)$$
$$\therefore \gcd(a,b) = 1 \Rightarrow \sigma(ab) = \sigma(a)\sigma(b)$$

■

Theorem 2 For a prime p and a natural number t that is not a multiple of p , $\sigma(tp) = \sigma(t)p + \sigma(t)$.

-proof-

For a prime p and a natural number t that is not a multiple of p, p and t are coprime. Therefore, by **Lemma 2**, we have:

$$\sigma(pt) = \sigma(t)\sigma(p) = \sigma(t)(p+1) = \sigma(t)p + \sigma(t)$$

■

Lemma 2-1 For a prime p of the form $\sigma(q)n^2 - 1$, except when p and q are the same, p and q are always coprime (q,n are natural numbers).

-proof-

Since q is a natural number, it is at least 1. If q=1, then it is obvious that q and p are coprime. Thus, we only need to consider the case where q is greater than 1.

$$q \geq 2 \Rightarrow \sigma(q) \geq 1 + q$$
$$p = \sigma(q)n^2 - 1 \geq n^2(1+q) - 1 \geq q$$
$$\therefore p > q \ (\because p \neq q)$$

When q is greater than 1 and not equal to p, we know that p is greater than q.
Now, suppose that p and q are not coprime; that is, they share a common divisor greater than 1. Since p is a prime number, any common divisor other than 1 must be p itself. Thus,

p must be a divisor of q. Therefore, p must be smaller than q (since they are not equal). This contradicts our earlier observation that p is greater than q. Thus, our assumption is false, Therefore, there can be no common divisor of p and q other than 1, and p and q are always coprime when they are not equal. ■

Theorem 2-1 For a prime p of the form $\sigma(q)n^2-1$, pq is a DS number. (q,n are natural numbers, and p≠q).

-proof-

By **Lemma 2-1**, p and q are always coprime. Therefore, by **Lemma 2**, we have:

$$\sigma(pq)=\sigma(p)\sigma(q)$$

And since p is a prime of the form $\sigma(q)n^2-1$, by **Theorem 2**, we have:

$$\sigma(pq) = \sigma(q)(\sigma(q)n^2-1)+\sigma(q) = (\sigma(q)n)^2$$

Therefore, the sum of the divisors of pq is a perfect square, so pq is a DS number. ■

Above, we showed that when p≠q and p is a prime of the form $\sigma(q)n^2-1$, pq is a DS number. Now, we examine the case when p and q are equal.

Theorem 2-2 If p=q and p is a prime, then pq is not a DS number.

-proof-

$$\sigma(pq)=\sigma(p^2)=1+p+p^2$$

Assume that a perfect square of this form exists.

$$1+p+p^2=k^2 \ \ (k\in N)$$

$$p=\frac{-1\pm\sqrt{1-4(1-k^2)}}{2}$$

$$\sqrt{1-4(1-k^2)} = z \ (z \in Z) \ \because p \in N$$

$$4k^2 - 3 = z^2$$

$$(2k+z)(2k-z) = 3$$

$$3 = 1 \times 3 \ , \ \wedge \ 2k+z > 2k-z \ \therefore 2k+z = 3 \ , \ 2k-z = 1$$

$$\therefore k = z = 1$$

$$p = \frac{-1 \pm 1}{2} = 0$$

pq can be a DS number only when p=0 and k=1, but 0 is not a prime.
Therefore, such a perfect square does not exist, and there is no prime p for which pq is a DS number when p=q. ■

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Above, we proved that for a prime p of the form $\sigma(q)n^2 - 1$, if p≠q then pq is a DS number. Therefore, the question "Do infinitely many DS numbers exist?" can be rephrased as
"Do infinitely many primes of the form $\sigma(q)n^2 - 1$ exist?"
However, there are a few important points to consider.

Theorem 3 If q is a DS number, then the only prime of the form $\sigma(q)n^2 - 1$ is 3.

-proof-

If q is a DS number, then $\sigma(q)$ is a perfect square. Then,

$$\sigma(q)n^2 - 1 = (\sqrt{\sigma(q)}\,n + 1)(\sqrt{\sigma(q)}\,n - 1) \ \ (\sqrt{\sigma(q)} \in N)$$

p must be a prime. Therefore, the smaller of the two terms must be 1.

$$\sqrt{\sigma(q)}\,n - 1 = 1 \ , \ n = \frac{2}{\sqrt{\sigma(q)}}$$

Since n is a natural number, q must be 1. Therefore, when n=2 and q=1, the value of p is 3.
Therefore, if q is a DS number, the only prime of the form $\sigma(q)n^2 - 1$ is 3. ■

In other words, "When q is not a DS number, there are infinitely many primes of the form $\sigma(q)n^2-1$" is equivalent to "There are infinitely many DS numbers."

Theorem 3-1 If there are infinitely many primes of the form $\sigma(q)n^2-1$ when q is not a DS number, then there are infinitely many DS numbers. the converse does not hold.

-proof-

Let proposition P be "There are infinitely many primes of the form $\sigma(q)n^2-1$ when q is not a DS number." Let proposition Q be "There are infinitely many DS numbers."
Thus To show that P is a sufficient condition for Q, we need to prove that if P holds, then Q also holds.

$$P \Rightarrow Q$$

suppose that P is true, that is, assume there are infinitely many primes of the form $\sigma(q)n^2-1$.
By **Theorem 2-1**, if p is of the form $\sigma(q)n^2-1$ and p≠q, then pq is a DS number.
By the assumption, there are infinitely many p determined by n and q, and there are also infinitely many q that generate p. Therefore, for primes of the form $\sigma(q)n^2-1$, there are infinitely many pq, and since all of these pq are DS numbers, there are infinitely many DS numbers. that is, Q holds.

$$Q \Rightarrow P$$

(Assuming that Q is true) That is, suppose that DS numbers exist infinitely.
It suffices to show that for every DS number p of the form $\sigma(q)n^2-1$, it is of the form pq .

$$3,22,66,70,81,94,214,282,\cdots$$

However, it can be seen that for prime numbers p of the form $\sigma(q)n^2-1$, such as 70 or 81, they are not of the form pq. Therefore, in general, not every DS number is of the form pq with p = $\sigma(q)n^2-1$ Therefore, if P then Q, but the converse does not hold. ■

Therefore, if there exist infinitely many prime numbers p of the form $\sigma(q)n^2-1$, then DS numbers exist infinitely.

Theorem 3-2 $\sigma: N \rightarrow N$ is neither surjective nor injective.

-proof-

$$\sigma(14) = 24 = \sigma(15)$$

Therefore, it does not satisfy injectivity.

Since there does not exist a α that satisfies $\sigma(\alpha) = 2$, it does not satisfy surjectivity. ■

Theorem 3-3 The image of $\sigma : N \rightarrow N$, denoted by S, is a set that is vastly sparser than the domain, the set of natural numbers. [1]

-proof-

$$n > 1 \Rightarrow \sigma(n) \geq 1 + n \quad (n \in N)$$
$$\therefore \sigma(n) \leq x \Rightarrow n \leq x$$
$$\therefore \{n : \sigma(n) \leq x\} \subseteq \{1, 2, \ldots, x\}$$

Therefore, the choice of n is limited to up to x.

By the Fundamental Theorem of Arithmetic, every natural number can be uniquely expressed as a product of prime numbers. Thus

$$n = \prod_{i=1}^{k} p_i^{e_i} \ (p_1, p_2, \ \ldots \ , p_k \text{ are distinct prime numbers }, \ e_i \geq 1) \text{ then}$$

By **Lemma 2**, $\sigma(n) = \sigma(\prod_{i=1}^{k} p_i^{e_i}) = \prod_{i=1}^{k} \sigma(p_i^{e_i})$

Since $\sigma(p_i^{e_i})$ is the sum of a geometric series,

$$\sigma(p_i^{e_i}) = \frac{p_i^{e_i+1} - 1}{p_i - 1}$$

For each p_i, $\sigma(p_i^{e_i})$ is monotonically increasing with respect to e_i, and since it is

$$p_i^{e_i+1} - 1 < p_i^{e_i+1}$$

$$\sigma(p_i^{e_i}) = \frac{p_i^{e_i+1}-1}{p_i-1} < \frac{p_i^{e_i+1}}{p-1} \le 2p_i^{e_i}$$

When we denote it as $\sigma(n) \le x$.

$$\sigma(p_i^{e_i}) = \frac{p_i^{e_i+1}-1}{p_i-1} \le x$$

Since there exists an appropriate positive constant A, it can be deduced that it is $p_i^{e_i+1} \ll x$.

$$e_i+1 \ll \frac{\log x}{\log p_i} \Rightarrow e_i \ll \frac{\log x}{\log p_i}$$

Therefore, for a fixed prime p, the number of possible values for e can be estimated, and it is as follows.

$$N(p,x) = \#\{e \in N \cup \{0\} : \sigma(p^e) \le x\} \ll \frac{\log x}{\log p} + O(1)$$

An upper bound for the number of values R(x) within the range S can be determined.
The elements m of S are determined by the selection of a finite number of primes and their corresponding exponents. Therefore,

$$R(x) = \#\{m \in S : m \le x\} \le \sum_{k \ge 0} \sum_{\substack{p_1 < p_2 < \cdots < p_k \\ p_i \le P(x)}} \prod_{i=1}^{k} N(p_i, x)$$

P(x) is a sufficiently large upper bound that guarantees x is greater than or equal to m.

And since it is $N(p_i,x) \ll \frac{\log x}{\log p_i}$,

$$R(x) \ll \sum_{k \ge 0} \sum_{p_1 < \cdots < p_k} \prod_{i=1}^{k} \frac{\log x}{\log p_i}$$

And as the prime p increases, $(\log p)^{-1}$ becomes smaller, so

$$\sum_{p \le P(x)} \frac{\log x}{\log p} \ll \log x \times \sum_{p \le P(x)} (\log p)^{-1} \ll \log x \times O(\frac{P(x)}{\log P(x)})$$

an upper bound can be obtained. By using the more refined Erdős–Pomerance method, the following upper bound can be derived, and the existence of an absolute constant A can be proven.

$$\log R(x) \ll A\sqrt{\log x \log(\log x))} \Rightarrow R(x) \ll e^{A\sqrt{\log x \log(\log x))}}$$
$$e^{A\sqrt{\log x \log(\log x))}} = o(x^{\omega}) \ (\forall\, \omega > 0)$$

Since x is not zero, dividing by x and taking $\omega = 1$ gives the following.

$$\frac{R(x)}{x} \ll \frac{o(x)}{x} \therefore \lim_{x \to \infty} \frac{R(x)}{x} = 0$$
$$\lim_{x \to \infty} \frac{\#\{m \in S : m \le x\}}{x} = 0$$

Therefore, the upper density of the image S of the function $\sigma(n)$ is 0.
On the other hand, the upper density of the set of natural numbers is as follows.

$$\#\{n \in N : n \le x\} = x \therefore \lim_{x \to \infty} \frac{\#\{n \in N : n \le x\}}{x} = \frac{x}{x} = 1$$

Therefore, the image S of the divisor function is vastly sparser compared to the domain, the set of natural numbers. ■

Theorem 3-4 The image S of the divisor function is an infinite set.

-proof-

It suffices to show that the divisor function $\sigma(n)$ takes infinitely many values.
If we denote it by $n = 2^k$ $(k \ge 0)$, then $\sigma(2^k)$ is equal to the sum of a geometric series.
Thus $\sigma(2^k) = 2^{k+1} - 1$ As k increases, $2^{k+1} - 1$ also increases. Thus For distinct values of k, $\sigma(2^k)$ is distinct. Therefore, for k=0,1,2,3,... it can be seen that infinitely many distinct values exist, all of which are included in the image S. Therefore, S is an infinite set. ■

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By **Theorem 3-3**, when considering $\sigma(q)n^2 - 1 = f(n)$, not all natural numbers are the coefficients of leading term. Therefore, for q that is not a DS number, if the images of $\sigma(q)n^2 - 1 = f(n)$ $(f: N \rightarrow N)$ contain infinitely many primes, then there are infinitely many primes of the form $\sigma(q)n^2 - 1$. Thus If the image of $\sigma(q)n^2 - 1 = f(n)$ contains infinitely many primes, then DS numbers exist infinitely.

Theorem 4 "For a non-DS number q and a natural number n, if the image of $\sigma(q)n^2 - 1 = f(n)$ $(f: N \rightarrow N)$ contains infinitely many primes" is equivalent to "There are infinitely many primes of the form $\sigma(q)n^2 - 1$".

-proof-

Proposition A: The image of the function f, $f(N) = \{f(n) : n \in N\} = \{\sigma(q)n^2 - 1 : n \in N\}$ contains infinitely many primes.
Proposition B: There are infinitely many primes of the form $\sigma(q)n^2 - 1$.

$$A \Rightarrow B$$

If the image of f contains infinitely many primes, then the following set is an infinite set.

$$\{p \in P : p = f(n), n \in N\}$$
$$p = f(n) = \sigma(q)n^2 - 1$$

Since the above set is an infinite set, there are infinitely many distinct elements.
Thus There are infinitely many $p = f(n) = \sigma(q)n^2 - 1$

$$B \Rightarrow A$$

Conversely, if there are infinitely many $p = f(n) = \sigma(q)n^2 - 1$, then the following set is an infinite set.

$$\{p \in P : p = f(n), n \in N\}$$

And that set is equal to $f(N) \cap P$. Therefore, the image of f contains infinitely many primes.

$(A \Rightarrow B) \wedge (B \Rightarrow A) \equiv A \Leftrightarrow B$

Therefore, A and B are equivalent. ■

Now, to prove that DS numbers exist infinitely, it suffices to show that the polynomial $f(n)=\sigma(q)n^2-1$, where q is not a DS number, generates infinitely many primes.
The image of $f(n)=\sigma(q)n^2-1$ contains infinitely many primes. ⇔ There are infinitely many primes of the form $\sigma(q)n^2-1$. ⇒ Infinitely many DS numbers exist. (By **Theorem 3-1** and **Theorem 4**)

The statement "For a non-DS number q, $f(n)=\sigma(q)n^2-1$ takes infinitely many prime values" is a special case of the Bunyakovsky conjecture.

5

Bunyakovsky conjecture

The Bunyakovsky conjecture is an unsolved problem in number theory that concerns whether a polynomial with integer coefficients can generate infinitely many primes.
The Bunyakovsky conjecture asserts that for a polynomial $f(x)$ that satisfies the following three conditions, there are infinitely many prime values among the outputs of $f(n)$ for natural numbers n.

1. Integer coefficients and degree. [2]
The polynomial $f(x)$ is an integer-coefficient polynomial with degree at least 1.
And the leading coefficient is positive

2. Irreducibility.
$f(x)$ is an irreducible polynomial over the integers, meaning that $f(x)$ cannot be factored into non-constant polynomials with integer coefficients.

3. The existence of a common divisor.
There no common factor for all the infinitely many values $f(1)$,$f(2)$,$f(3)$, ...
(in particular, the coefficients of $f(x)$ should be relatively prime. it's not necessary for the values $f(n)$ to be pairwise relatively prime.)

Currently, the Bunyakovsky conjecture has been proven to hold only for linear polynomials (Dirichlet's theorem)

The statement "For a non-DS number q, $f(n)=\sigma(q)n^2-1$ takes infinitely many prime values" corresponds to the case where a quadratic polynomial ax^2+bx+c , $b=0$, ax^2+c in the Bunyakovsky conjecture.

Theorem 5 If the Bunyakovsky conjecture is true, then for a non-DS number q, $f(n)=\sigma(q)n^2-1$ takes infinitely many prime values.

-proof-

We just need to check whether $f(n)=\sigma(q)n^2-1$ satisfies the three conditions mentioned above.

1. The leading term is of degree 2, and $\sigma : N \rightarrow N$ thus $\sigma(q)$ have integer coefficients.
2. Since q is not a DS number, $\sigma(q)$ is not a perfect square. Therefore, $f(n)=\sigma(q)n^2-1$ is irreducible over the integers.
3. If there exists a prime r that divides all the values in the image of $f(n)=\sigma(q)n^2-1$, a contradiction arises.

$$f(0) = -1 \ , \ r > 1$$

When n=0 , -1 is not divisible by r. Therefore, there cannot exist a common divisor r of all the values in the image of $f(n)=\sigma(q)n^2-1$.
Since $f(n)=\sigma(q)n^2-1$ satisfies the three conditions of the Bunyakovsky conjecture, if the Bunyakovsky conjecture is true, $f(n)=\sigma(q)n^2-1$ generates infinitely many primes. ■

And since $f(n)=\sigma(q)n^2-1$ is a quadratic polynomial, it follows that even if the Bunyakovsky conjecture is proven only for quadratic polynomials, it can be concluded that $f(n)=\sigma(q)n^2-1$ generates infinitely many primes.
And even if only the case of ax^2+c is proven, it can be concluded that $f(n)=\sigma(q)n^2-1$ generates infinitely many primes.

Therefore, the following logic holds.

The Bunyakovsky conjecture is true, or it is true in the case of quadratic polynomials in the Bunyakovsky conjecture.

$$\Downarrow$$

$f(n)=\sigma(q)n^2-1$ generates infinitely many primes. (By **Theorem 5**)

$$\Updownarrow$$

Infinitely many primes of the form $\sigma(q)n^2-1$ exist. (By **Theorem 4**)

$$\Downarrow$$

Infinitely many DS numbers exist. (By **Theorem 3-1**)

conclusion

If the Bunyakovsky conjecture is true, then infinitely many DS numbers exist. Or
If the Bunyakovsky conjecture is true for quadratic polynomials, then infinitely many DS numbers exist.

References

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