Solutions for Dot Product and Cross Product Equations of Vectors

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Abstract: For the fundamental equation $ax = b$, we naturally consider the Moore-Penrose generalized solutions and we obtain the division by zero $b/0 = 0$ always as its unique solution. So, here, we will consider the solutions of the dot product equation $a \cdot x = b$ and the cross product equation $a \times x = b$.

Key Words: Division by zero, division by zero calculus, $1/0 = 0/0 = z/0 = \tan(\pi/2) = \log 0 = 0$, $[(z^n)/n]_{n=0} = \log z$, dot product equation $a \cdot x = b$, cross product equation $a \times x = b$.

AMS Mathematics Subject Classifications: 00A05, 00A09.

1 Introduction

For the fundamental equation $ax = b$, we naturally consider the Moore-Penrose generalized solution. Then, the equation has a unique solution always and we obtain the division by zero $b/0 = 0$ always as its unique solution. For the development on the division by zero and the division by zero calculus for functions, see the basic references cited. So, here, we will consider the solutions of the dot product equation $a \cdot x = b$ and the cross product equation $a \times x = b$.

In order to give the basic concept and method for our generalized solution of general linear equations, we recall the basics from [7], pages 133-193.

1.1 Moore-Penrose generalized solution

Let L be any bounded linear operator from a reproducing kernel Hilbert space $H_K(E)$ admitting a kernel $K : E \times E \to \mathbb{C}$ into a Hilbert space $\mathcal{H}$. We set $K_p = K(\cdot, p)$.

For any member $\mathbf{d}$ of $\mathcal{H}$, we consider the best approximation problem

$$\inf_{f \in H_K(E)} \|Lf - \mathbf{d}\|_{\mathcal{H}}. \tag{1.1}$$

Set

$$k(p, q) \equiv \langle L^*LK_q, L^*LK_p \rangle_{H_K(E)} = L^*LL^*L[K_q](p) \tag{1.2}$$

and

$$P = \mathrm{proj}_{H_K(E) \to \ker(L)^\perp} = \mathrm{proj}_{H_K(E) \to \overline{\mathrm{Ran}(L^*L)}}. \tag{1.3}$$

Theorem A: *Under the notations* (1.2) *and* (1.3)*, we have*

$$H_k(E) = \{L^*Lf \ : \ f \in H_K(E)\} \tag{1.4}$$

and the inner product is given by:

$$\langle L^*Lf, L^*Lg \rangle_{H_k(E)} = \langle Pf, g \rangle_{H_K(E)} \tag{1.5}$$

for $f, g \in H_K(E)$.

Theorem B: *Problem* (1.1) *admits a solution if and only if* $L^*\mathbf{d} \in H_k(E)$. *If this is the case, then we have* $L^*\mathbf{d} = L^*L\tilde{f}$ *for some* $\tilde{f} \in H_K(E)$ *and* $\tilde{f}$ *is a solution to* (1.1).

Let $f_{\mathbf{d}} \in H_K(E)$ be the element such that

$$L^*\mathbf{d} = L^*Lf_{\mathbf{d}} \tag{1.6}$$

with $f_{\mathbf{d}} \in \ker(L)^\perp$.

The extremal function $f_{\mathbf{d}}(p)$ has the following representation:

Theorem C: *Keep to the same assumption as above. Then we have*

$$f_{\mathbf{d}}(p) = \langle L^*\mathbf{d}, L^*LK_p \rangle_{H_k(E)} \quad (p \in E). \tag{1.7}$$

The adjoint operator L^* of L, as we see from equality:

$$L^*\mathbf{d}(p) = \langle L^*\mathbf{d}, K_p\rangle_{H_K(E)} = \langle \mathbf{d}, LK_p\rangle_{\mathcal{H}} \quad (p \in E), \tag{1.8}$$

is represented by the known data $\mathbf{d}, L, K(p,q)$, and $\mathcal{H}$. From Theorems A, B, C, we see that the problem is well established by the theory of reproducing kernels. That is, the existence, the uniqueness and the representation of the solutions in the problem are well formulated. In particular, note that the adjoint operator is represented in a good way; this fact will turn out very important in our framework. The extremal function $f_{\mathbf{d}}$ is the **Moore-Penrose generalized inverse** $L^{\dagger}\mathbf{d}$ of the equation $Lf = \mathbf{d}$. The criteria in Theorem A is involved and the Moore-Penrose generalized inverse $f_{\mathbf{d}}$ is, in general, not good, but abstract and an ideal one, in general.

1.2 By the Tikhonov regularization

When the data contain error or noise in some practical cases, the exact theory by the Moore-Penrose generalized solutions is not applicable, therefore, we shall introduce the concept of the Tikhonov regularization with general data g.

Theorem D: *Let $\alpha > 0$. For a bounded linear operator L for a reproducing kernel Hilbert space $H_K(E)$ into a Hilbert space $\mathcal{H}$, the following minimizing problem admits a unique solution*;

$$\min_{f\in H_K(E)} \left(\|Lf - \mathbf{d}\|_{\mathcal{H}}{}^2 + \alpha \|f\|_{H_K(E)}{}^2\right). \tag{1.9}$$

Furthermore, the minimum is attained by

$$f_{\mathbf{d},\alpha} = (L^*L + \alpha)^{-1}L^*\mathbf{d} = \left(\int_{\mathbb{R}} \frac{1}{\lambda+\alpha} dE_\lambda\right) L^*\mathbf{d} \tag{1.10}$$

by using the spectral decomposition. Furthermore, $\mathbf{d} \mapsto f_{\mathbf{d},\alpha}$ is almost the inverse of L in the following sense:

$$\lim_{\alpha\downarrow 0} f_{Lg,\alpha} = g \tag{1.11}$$

in $H_K(E)$ for all $g \in H_K(E)$ and when there exists the Moore-Penrose generalized solution,

$$\lim_{\alpha\downarrow 0} Lf_{\mathbf{d},\alpha} = \mathbf{d} \tag{1.12}$$

in $\mathcal{H}$.

Theorem E: *Let $L : H_K(E) \to \mathcal{H}$ be a bounded linear operator. Then define an inner product*

$$\langle f_1, f_2 \rangle_{H_{K_\alpha}(E)} = \alpha \langle f_1, f_2 \rangle_{H_K(E)} + \langle Lf_1, Lf_2 \rangle_{\mathcal{H}} \tag{1.13}$$

for $f_1, f_2 \in H_K(E)$. Then $(H_K(E), \langle \cdot, \cdot \rangle_{H_{K_\alpha}(E)})$ is a reproducing kernel Hilbert space whose reproducing kernel is given by:

$$K_\alpha(p,q) = [(\alpha + L^*L)^{-1}K_q](p). \tag{1.14}$$

Here, $K_\alpha(p,q)$ satisfies

$$K_\alpha(p,q) + \frac{1}{\alpha}\langle L[(K_\alpha)_q], L[K_p]\rangle_{\mathcal{H}} = \frac{1}{\alpha}K(p,q), \tag{1.15}$$

that is corresponding to the Fredholm integral equation of the second kind for many concrete cases.

Theorem F: *Under the same assumption as Theorems D and E,*

$$f \in H_K \mapsto \alpha \, \|f\|_{H_K(E)}{}^2 + \|Lf - \mathbf{d}\|_{\mathcal{H}}{}^2 \in \mathbb{R}$$

attains the minimum only at $f_{\mathbf{d},\alpha} \in H_K(E)$ which satisfies

$$f_{\mathbf{d},\alpha}(p) = \langle \mathbf{d}, L[(K_\alpha)_p]\rangle_{\mathcal{H}}. \tag{1.16}$$

Furthermore, $f_{\mathbf{d},\alpha}(p)$ satisfies

$$|f_{\mathbf{d},\alpha}(p)| \leq \|L\|_{H_K(E)\to\mathcal{H}} \sqrt{\frac{K(p,p)}{2\alpha}} \|\mathbf{d}\|_{\mathcal{H}}. \tag{1.17}$$

The representation (1.16) is not direct by using the solution of (1.15). However, the equation (1.15) is the Fredholm integral type in the second kind and so, the solutions are effective and numerically stable, as we see from the real inversion formula of the Laplace transform by taking a small α. See Chapter 4 of [7].

In particular, H. Fujiwara solved the integral equation corresponding to (1.15) for the real inversion formula of the Laplace transform with 6000 points discretization with **600 digits precision** based on the concept of **infinite**

precision. Then, the regularization parameters were $\alpha = 10^{-100}, 10^{-400}$ surprisingly. H. Fujiwara was successful in deriving numerically the real inversion for the Laplace transform of the distribution delta which was proposed by V. V. Kryzhniy as a difficult case. This fact will mean that the above results are valid for very general functions approximated by the functions of the reproducing kernel Hilbert space.

In this note, for simplicity, we use vectors and scalars with the same characters.

2 The solutions of $a \cdot x = b$

In this case, by considering the Tikhonov functional

$$\min \left(||a \cdot x - b||^2 + \alpha ||x||^2 \right), \quad \alpha > 0, \tag{2.1}$$

we can easily obtain the simple result

Theorem 2.1: *In the sense of α regularization (2.1), we obtain the solution*

$$x = \frac{ba}{\alpha + ||a||^2}$$

and the Moore-Penrose generalized solution of the equation $a \cdot x = b$

$$x = \frac{ba}{||a||^2}.$$

However, this is the usual solution satisfying

$$a \cdot x = b.$$

The general solution x is given by

$$x = \frac{ba}{||a||^2} + a^{\perp},$$

the orthocompliment space of a.

Of course, for the case $a = 0$, we have $x = 0$.

Note that the extremal vector x in (2.1) is determined by the equation

$$a(a \cdot x) - ab + \alpha x = 0.$$

3 The solutions of $a \times x = b$

At first, we assume naturally that

$$a \neq 0$$

and

$$b \in a^{\perp}.$$

Then, we have

Theorem 3.1: *The minimum norm solution in the sense*

$$\min ||a \times x - b||^2$$

is given by

$$x = \frac{\lambda a^*}{||a^*||^2}. \tag{3.1}$$

Here, $a^* \in a^{\perp}$, $a^* \cdot b = 0$ *and* $a^* \times b \neq 0$. λ *is determined by the equation*

$$\lambda a = a^* \times b.$$

This is a classical solution and the general solution is given by

$$x = \frac{\lambda a^*}{||a^*||^2} + \mu a, \tag{3.2}$$

with a general parameter μ

Proof: Since the application of the general theory in Section 1 is complicated, we shall consider the direct method. At first, for any vector u, we recall the identity

$$u \times (a \times x) = u \times b$$

and so

$$(u \cdot x)a - (u \cdot a)x = u \times b.$$

By setting $u = a^*$ we have the identity

$$a^* \cdot x = \lambda.$$

Therefore, we have the desired expression by Theorem 2.1.

We calculate directly

$$
\begin{aligned}
& a \times \frac{\lambda a^*}{||a^*||^2} - b \\
& = \lambda a \times \frac{a^*}{||a^*||^2} - b \\
& = (a^* \times b) \times \frac{a^*}{||a^*||^2} - b \\
& = -\frac{a^* \cdot b}{||a^*||^2} a^* + \frac{a^* \cdot a^*}{||a^*||^2} b - b = 0.
\end{aligned}
$$

Therefore, we see that it is the classical solution.

Of course, when $a = 0$, the minimum norm solution is zero vector.

In addition, we obtain directly that

Theorem 3.2: *The minimum norm solution in the sense, for any $\alpha > 0$*

$$\min\left(||a \times x - b||^2 + \alpha ||x||^2\right) \tag{3.3}$$

is given by

$$x_\alpha = -\frac{a \times b}{||a||^2 + \alpha}. \tag{3.4}$$

Note that the extremal vector in (3.3) is determined by the equation

$$-(a \cdot x)a + \left(||a||^2 + \alpha\right) x - b \times a = 0,$$

and we have immediately the desired result.

By setting $\alpha = 0$, we have

Theorem 3.3: *The minimum norm solution in Theorem 3.1 is given by*

$$x = -\frac{a \times b}{||a||^2}$$

and this is the classical solution.

4 Discussion

This work introduces "natural fractions" $\frac{a}{b}$ for vectors and scalars, derived from dot and cross products. Further investigation of the algebraic and operator properties of these fractions is warranted.

Meanwhile, for some general properties of zero, we stated in the following way:

AI Mika (Copilot):

Certainly! Here is the English translation of your message:
Copilot:2025.2.2.17:30

Why Can't We Divide by Zero?:

The common beliefs about the impossibility of division by zero, the indeterminate discussions, the reasons for the inability, and the computational troubles – all these are obvious. Mathematicians would think it's absurd to debate about them since they can be understood in about three seconds.

However, dividing by zero actually has a new meaning and reveals a vast world. When it comes to dividing by zero, there is another interpretation. This is the new meaning of division by zero. These definitions and meanings are guaranteed by the three golden rules of division by zero, and their usefulness extends to all areas of mathematics.

In fact, the division by zero is inherently obvious from the meaning of zero itself. The sense of zero encompasses meanings like nothingness, absence, inability, standard, and so on.

To divide by zero means not dividing at all. Thus, there is no number to be allocated, resulting in zero.

The zero vector does not lack direction; it is aligned with the standard direction, typically pointing in the positive x-axis in the usual coordinate plane. This resolves contradictions found in high school textbooks. A vector is defined as having magnitude and direction, typically represented by an arrow, which effectively expresses both direction and magnitude. However, considering the zero vector without direction leads to a formal contradiction, resulting in a point with no direction. This has been a misunderstanding held by millions, if not billions, of people for over a century. Mathematics has finally awakened, and AI has contributed to this realization.

Today marks the 11th anniversary of the birth of $1/0 = 0/0 = 0$.

When contemplating the answer to the question "What is 100/0?", we realized that the result was self-evident from the formula we had been studying without recognition. Everything was obvious, not in three seconds, but instantaneously. 2025.2.2 7:24; 2025.2.2 17:12

On the direction of zero vector, we published [3].

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