Physical interpretation of the zero spatial curvature universe models

Fernando Salmon Iza*

fernandosalmoniza@gmail.com

Abstract

The standard cosmological model ΛCDM cannot respond to some important new results of modern cosmology. Challenges arise such as the Microwave Background Uniformity, the Hubble Tension, the El Gordo collision or impossible galaxies (z > 10) that the standard cosmological model does not solve. On the other hand, other models are proposed as alternatives. Among the models proposed in recent years, the Rh=ct universe is one of the most studied and it seems that it does solve these challenges favorably. Therefore, it is the obligation of every scientist to adequately substantiate the model and study the physical meaning it has through its equations. This model is based on the following equations, the restriction R_h = ct, where R_h is the gravitational horizon and the condition of zero active mass, (ρ+3p) = 0. We have carefully studied the foundation of these two equations and have obtained that the model responds to the condition of a universe of zero spatial curvature, being the only model compatible with it. Thus, to investigate the physical meaning of the models of universe with zero spatial curvature is to investigate the physical meaning of this model. To do so, we have obtained an equation that relates spatial curvature to the density of matter, resulting in zero curvature only being obtained if the density of matter is zero. Thus, we deduce that the physical meaning of models with strictly zero spatial curvature is an empty universe in all cases, highlighting the importance of spatial curvature.

Keywords: **G**ravitation, R_h=ct universe, large scale structure of Universe, general relativity

1.- The R_h = ct universe as a consequence of a spatially flat universe

1.1.- The "R_h = ct constraint" in a spatially flat universe

We consider an isotropic, homogeneous and spatially spherical universe, which responds to the FLRW metric and which therefore expands. This universe has a certain energy density ρ at each instant of time. We are going to refer our calculations to an observer located in the center of it. We call R_h its gravitational horizon [1] and $M_{(r)}$ the mass, which comes from its energy density ρ, contained in a sphere of radius "r" centered at the observer's point.

We are going to calculate the kinetic energy and potential energy produced by the expansion of the universe for that observer considering that sphere. The increase in its kinetic energy, "ΔK", during the expansion process is given by [2]:

$$\Delta K = (4\pi r^2 \rho \Delta r)\,(\Delta r/\Delta t)^2/2$$

and the corresponding increase in its potential energy "ΔU", is given by [2]:

$$\Delta U = -(4\pi r^2 \rho \Delta r)\, G M_{(r)}/r$$

According to [2], in a universe dominated by matter, the curvature parameter of the Friedmann equation, *k,* is proportional to the sum of the kinetic energies, K, and potential energies, U, brought into play by the expansion, and this parameter is zero in a spatially flat universe:

$$k \sim (K + U)$$

$$k = 0$$

and using differential calculus:

$$\int_0^r dK + \int_0^r dU = 0$$

$$1 = dK/(-dU) = (dr/dt)^2\, r/2G\, M_{(r)}$$

Let's do the calculation for $r = R_h$:

$$R_h = 2G\, M_{(Rh)}/c^2$$

$$1 = (dR_h/dt)^2 R_h / R_h c^2$$

$$dR_h/dt = c$$

$$R_h = ct + \text{Const.}$$

Let's calculate the value of the Const.:

For a time, t = o, $R_h = 0$, then Const. = 0

Thus, the following equation is obtained:

$$\mathbf{R_h = ct}$$

We have obtained the constraint that characterizes the universe of linear expansion $R_h = ct$ as a consequence of the zero value of the curvature parameter of the Friedmann equation, that is, of a spatially flat universe.

1.2.-The "zero active mass condition ($\rho + 3p$) = 0", in a spatially flat universe

Given the Friedmann equations of the FLRW metric:

$$H^2 = \left(\frac{a'}{a}\right)^2 = \frac{8\pi G\rho}{3c^2} - \frac{kc^2}{a^2}$$

$$\left(\frac{a''}{a}\right) = -\frac{4\pi G}{3c^2}(\rho+3p)$$

The zero active mass condition $(\rho+3p) = 0$ is equivalent to the condition $\left(\frac{a''}{a}\right) = 0$. Therefore, we are going to show [3] that for universes with FLRW metric and zero spatial curvature, k = 0, $a'' = 0$ is fulfilled.

Let the FLRW metric be in coordinates (t, x^1, x^2, x^3) where "t" is the commoving time and x^i are the spatial coordinates, (c=1).

$$ds^2 = dt^2 - a(t)^2(g_{\mu\nu}\, dx^\mu dx^\nu)$$

In a spatially flat universe, k= 0, the 3D hypersurface corresponding to each section of cosmic spacetime is the Euclidean space R^3.

We are looking for a coordinate transformation that will convert this metric into a conformal metric. We make the following coordinate change:

$$dt = d\tau.\, a(\tau)$$

$$a = a\,(t(\tau))$$

$$a(\tau) = dt/d\tau$$

The conforming metric will be:

$$ds^2 = a(\tau)^2(d\tau^2 - g_{\mu\nu}\, dx^\mu dx^\nu)$$

where the scale factor $a(\tau) = a(t(\tau))$ is now a function of conformal time. Conformal time is not the proper time of any particular observer, but these coordinates have some advantages, such as making it clear that FLRW metrics with k = 0 is a locally conformally flat metric.

According to reference [4] in this metric the term $R_{\tau\tau}$ of the Ricci tensor is given by:

$$R_{rr} = 3((a(\tau)''/a(\tau)) - (a(\tau)'/a(\tau))^2)$$

In a conformally flat metric, the curvature tensors are zero.

Therefore, in our case of a spatially flat universe, k=0. It will be true that:

$$0 = R_{rr} = 3((a(\tau)''/a(\tau)) - (a(\tau)'/a(\tau))^2)$$

We show below that: $a''(t) \sim R_{\tau\tau} = 0$;

$$a = a(t(\tau))$$

$$a(\tau)' = da(\tau)/d\tau = (da(t(\tau))\,/dt).\,(dt/d\tau) = a(t)'.\, a(\tau).$$

$$a(t)' = a(\tau)'/a(\tau)$$

$$a(t)'' = d(a(t)')\,/dt = d(a(\tau)'/a(\tau))\,/dt = (d\,(a(\tau)'/a(\tau))\,/d\tau).\,(d\tau/dt) =$$

$$= ((a(\tau)''.\ a(\tau) - a(\tau)'^2) / a(\tau)^2).\ (1/a(\tau)) = R_{\tau\tau} / (3a(\tau)) = 0$$

Thus, we have shown that for spatially flat universes in the FLRW metric it is true that;

$$a(t)'' = 0$$

Or what is the same according to Friedmann's equations;

$$\mathbf{(\rho + 3p) = 0}$$

1.3.- The R_h = ct universe, the only valid model in the FLRW metric compatible with zero spatial curvature

Let a universe have zero spatial curvature, that implies that its scale factor is:

$a = Dt + C$, being D y C constants

For $t = 0$, $a = 0$, then:

$C = 0$

Furthermore, letting $a = 1$ for $t = t_o$ = age of the universe, we have:

$1 = Dt_o$, $\quad D = 1/t_o$

So:

$a = t/t_o$, $\quad H = a'/a = 1/t$

that is, the resulting universe is the R_h=ct universe.

1.4.- Discussion

Of the alternative models that are proposed to update the standard cosmological model, the linear expansion universe R_h=ct, responds very well to the new challenges that the cosmos reveals to us today. This model is based on the study of the so-called "cosmological horizon, R_h" and on the equations that characterize it, the restriction R_h=ct and the condition of zero active mass $(\rho+3p) = 0$. Deducing that it is also a consequence of a spatially flat space and being the only valid model in the FLRW metric compatible with this assumption, it is the result we have achieved.

2.- Calculation of the spatial curvature of the universe. An equation that relates it to the energy density

2.1. - The cosmic spacetime

We are going to study a uniform and isotropic spacetime from a physical point of view, this is equivalent from a geometric point of view to being invariant under translations and rotations.

According to Professor Fulvio Meliá in reference [1], we define “cosmic spacetime” as the set of points (t, r, Ø Φ) that satisfy the FLRW metric, that is, that satisfy the equation:

$$ds^2 = c^2dt^2 - a(t)^2(\frac{dr^2}{1-kr^2} + r^2d\Omega^2)$$

We define each of the "3D hypersurfaces" of cosmic spacetime as the set of points that have the same temporal coordinate. Thus, cosmic spacetime will have a different hypersurface for each time t. As we have defined them, these hypersurfaces do not intersect, that is, they have no common points and the set of all of them constitutes cosmic spacetime.

It is in these 3D hypersurfaces where we are going to calculate the spatial curvature that constitute our objective.

2.2.- Calculation of the spatial curvature of each of the 3D hypersurfaces of cosmic spacetime

First, we are going to calculate the curvature scalar of a 3D hypersurface of our homogeneous and isotropic cosmic spacetime with a matter density ρ_m.

2.2.1- Birkhoff–Jebsen theorem

We make a brief comment on this theorem of mathematics applied to the theory of generalized relativity [5]. First, we summarize Professor Fulvio Melia in reference [2] to explain it.

“If we have a spherical universe of mass-energy density ρ and radius r and within it a concentric sphere of radius r_s smaller than r, it is true that the acceleration due to gravity at any point on the surface of the sphere of relative radius r_s to an observer at its origin, depends solely on the mass-energy relation contained within this sphere”.

Thus, according to this, to calculate the curvature of the gravitational field of a point located at a distance "r_s" from the geometric center that we are considering in our continuous universe, it is only necessary to consider its interaction with the points that are at a radius smaller than "r_s", therefore, the mass "m" to be considered will only be that contained in the sphere of radius "r_s".

In general relativity Birkhoff´s theorem states that any spherically symmetric solution of the vacuum field equations must be statically and asymptotically flat. This means, that the outer solution (that is, the spacetime outside a gravitational, non-rotating, spherical body) must be the Schwarzschild metric.

2.2.2- Calculating the spatial curvature constant

Let's consider our 3D hypersurface and a sphere of radius r inside, the Birkhoff–Jebsen theorem assures us that if we want to calculate the curvature at a point on its surface, we must consider only the interaction with the gravitational mass found inside, the gravitational mass inside for the sphere external point that we are considering behaves as a point mass of equal magnitude to that of the mass of the sphere and located at its central point. In this case we are already in the Schwarzschild model, and we can use its equations to calculate the corresponding curvature.

For all this, we can treat the problem of calculating the curvature scalar of each of the 3D hypersurfaces of our cosmic spacetime as a problem to be solved by the Schwarzschild model and calculate the curvature scalar from that model. In this model, spacetime is reduced to a 2D surface and so Gaussian curvatures are easily calculated; the curvature scalar in this case is twice the Gaussian curvature.

According to Appendix, we have found an equation that relates the Gaussian curvature K_{gauss} of the spacetime of the Schwarzschild model, with the cosmological parameters mass M and universal gravitation constant G. We are going to use this equation to solve our problem. This equation is the following:

$$K_{gauss}= -GM/c^2r^3$$

Since in our case it is a sphere, its mass will be given by

$$M = 4\pi r^3\rho/3$$

$$K_{gauss} = -4\pi G\ \rho/3c^2$$

having reduced the calculation to a two-dimensional problem, the curvature scalar R will be given by twice the Gaussian curvature and, in our case, it will also have the opposite sign. Thus:

$$R/\rho = 8\pi G/3c^{2,} = 0{,}62.10^{-26}$$

This curvature obtained here R. which is the same at each point of each one of 3D hypersurfaces and proportional to the energy density, $\rho(Kg/m^3)$, we will demonstrate in the discussion what the "spatial curvature K" is.

Thus, the “spatial curvature K” at the points of each 3D hypersurface is the same and is proportional to the density of matter

$$\mathbf{K = (8\pi G/3c^2)\ \rho = 0{,}62.10^{-26}\ \rho}$$

<u>Identification of the curvatures found, R = K:</u>

We have found a curvature scalar R, which results from the relativistic gravitational interaction between the points that form the cosmic fluid. This curvature has the same value at each point of each 3D hypersurface corresponding to an instant of time in cosmic space-time. Moreover, this curvature depends only on the universal gravitational constant and the matter energy density. It is therefore very reasonable to identify this curvature with the -spatial curvature K- that determines the value of the parameter "k" in the Friedmann equation. According to the equation found we see that the proportionality factor between K and ρ is very similar to the proportionality factor that Einstein finds between the Einstein tensor and the energy-momentum tensor, that is, between curvature and energy, which further confirms our choice as the "spatial curvature".

Through the Friedmann equation, we can relate K with the curvature parameter "k" that appears in it, [4]:

$H^2 = (a'/a)^2 = (8\pi G\rho/3) - kc^2/a^2$

k = K/[K}. Where k = +1, -1, 0, according to the sign and value of K. If K is positive then k=+1, if K is negative then k=-1 and if K is zero, then k = 0.

Some consequences of the equation:

The first thing we can see is that zero spatial curvature is only possible if the energy density is zero. So, it does not seem that our universe has zero spatial curvature. What we do know is that the curvature term appearing in the Friedmann equation is very small, according to experimental data [6]. $\Omega_k = 0{,}001 \pm 0{,}002$, this term [2], $\Omega_k = kc^2/(Ha)^2$ is a function of k, the expansion parameter a(t), and the Hubble constant H, the small measured value of which has led some scientists to consider the possibility that $\Omega_k = 0$, being therefore k= 0. From what is stated here, our equation denies this hypothesis since k = 0 implies K = 0 and that is only possible if ρ = 0, which is not the case in our universe.

Furthermore, our equation will condition the possible physical existence of one of the most studied universes, the Milne universe. This is a universe with zero energy density ρ = 0 and curvature k = -1. It represents an expanding universe without matter. Our equation will condition its possible physical existence by the following. According to our equation, a universe with zero energy density implies a spatial curvature equal to zero K = 0 and therefore k = 0, therefore the Milne universe, with ρ = 0 and k = -1 would not be possible.

Calculation of the value of spatial curvature:

There are several experimental data available concerning the matter energy density, $\rho_{m,}$ in our universe today, [6], according to these data the value is $\rho_m = 0{,}3.10^{-26}$ kg/m^3. Substituting this value into our equation we can calculate the current spatial curvature of our universe:

$$\mathbf{K = (8\pi G/3c^2)\, \rho = (0{,}62.10^{-26})\, (0{,}3.10^{-26}) = 0{,}19.10^{-52}\ m^{-2}}$$

this is therefore an extremely small value.

2.3.- Discussion

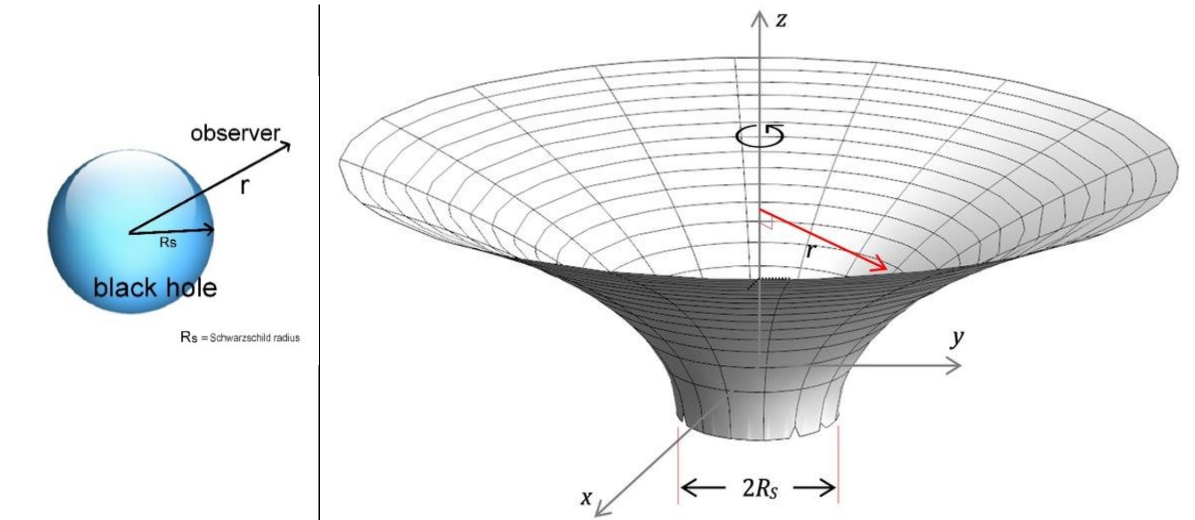
We have found a simple equation that relates, in the FLRW metric, a curvature scalar to the energy density. In the context of this metric, we have identified this curvature scalar with the spatial curvature K of each of the 3D hypersurfaces into which cosmic spacetime is divided. In this found equation, the spatial curvature is proportional to the energy density, with a proportionality constant equal to one third of the proportionality constant existing between the Einstein tensor and the energy-momentum tensor. Knowing the value of the energy density, we have calculated that the value of the current spatial curvature is extremely small. We have also come to the conclusion that a spatial curvature equal to zero is not possible in our universe because, according to our equation, it only occurs if the energy density is equal to zero. Therefore, the Milne universe with $\rho=0$ and $k=-1$ is not physically possible. Our equation is valid in any FLRW metric universe.

3.- Appendix

An equation to calculate the Gaussian curvature of space-time according to the Schwarzschild model

3.1.- Introduction

In Fig. 1, the physical problem [7]. that is posed here is represented, calculating the curvature of space-time at the points surrounding a supposed spherical gravitational mass that we have called a “black hole”. In 1916, Schwarzschild carried out a study of Einstein’s equations for this assumption. The solution studied here proposed by J. Droste is the Flamm paraboloid, represented in Fig. 1. It is a 2D surface of infinite measure and negative Gauss curvature. In addition, we will represent it by means of cylindrical coordinates, which is also a function of the Schwarzschild radius Rs of the gravitational mass that generates it. We will study it algebraically and we will find an equation that allows us to easily calculate the values of the Gauss curvature at each point. As we know, since it is a 2D surface, the value of the curvature scalar will be twice the value of the Gauss curvature.



The physical problem in Euclidean space	J. Droste solution to the Schwarzschild problem Flamm paraboloid

Fig. 1

3.2.-Resolution of the mathematical problem. Gaussian curvature and curvature scalar of spacetime in the J. Droste solution

The Flamm paraboloid, J. Droste's spacetime solution to the problem studied by Schwarzschild [8], is a 2D surface. Its geometry allows us to parameterize the paraboloid as a function of the observer's distance from the point mass "r" and the azimuth angle "ϕ". The problem admits a mathematical treatment of differential geometry of surfaces [9], and with it we are going to calculate the Gaussian Curvature. (R_s = Schwarzschild radius). Since it is a 2D surface, the curvature scalar is obtained by multiplying the Gaussian curvature by two.

The surface

Surface parameters (r, ϕ)

$0 \leq r < \infty, \quad 0 \leq \varphi < 2\pi$

which has this parametric equation:

$x = r \cos\phi$

$y = r \, \mathrm{sen}\phi$

$z = 2(R_s (r - R_s))^{1/2}$

Vector Equation of the Flamm paraboloid

$f(x,y,z) = (r \cos\phi, \; r \, \mathrm{sen}\phi, \; 2(R_s (r - R_s))^{1/2})$

Determination of velocity, acceleration, and normal vectors to the surface

$\eth f/\eth\phi = (-r \, \mathrm{sen}\phi, r \cos\phi, 0)$

$\eth f/\eth r = (\cos\phi, \mathrm{sen}\phi, (r/R_s - 1)^{-1/2})$

$\eth^2 f/\eth\phi^2 = (-r \cos\phi, -r \, \mathrm{sen}\phi, 0)$

$\eth^2 f/\eth r^2 = (0, 0, (-1/2R_s). (r/R_s - 1)^{-3/2})$

$\eth f/\eth\phi\eth r = (-\mathrm{sen}\phi, \cos\phi, 0)$

$$n = \frac{(\eth f/\eth\varphi \text{ x } \eth f/\eth r)}{[\frac{\eth f}{\eth\varphi} \text{x} \frac{\eth f}{\eth r}]}$$

$(\eth f/\eth\phi \text{ x } \eth f/\eth r) = (r \cos\phi/ (r/R_s - 1)^{1/2}, \; r \, \mathrm{sen}\phi/(r/R_s - 1)^{1/2}, \; -r)$

$[\frac{\eth f}{\eth\varphi} \text{X} \frac{\eth f}{\eth r}] = r ((1/ (r/R_s - 1)) + 1)^{1/2}$

Curvature and curvature parameters

Gauss curvature

$K_{gauss} = LN\text{-}M^2/EG\text{-}F^2$

$L = \eth^2 f/\eth\phi^2 . n = -r(r/R_s)^{-1/2}$

$N = \eth^2 f/\eth r^2 . n = (1/2R_s)\,(r/R_s)^{-1/2}\,(r/R_s - 1)^{-1}$

$M = (\eth f/\eth\phi\eth r) . n = 0$

$E = \eth f/\eth\phi . \ \eth f/\eth\phi = r^2$

$G = \eth f/\eth r . \eth f/\eth r = 1 + (1/\,(r/R_s - 1))$

$F = \eth f/\eth\phi . \eth f/\eth r = 0$

An equation of Gauss curvature

$$K_{gauss} = -R_s/2r^3$$

for Schwarzschild radius, $R_s = 2GM/c^2$

4.- Conclusion

Knowing the physical meaning of a model means providing an adequate basis for it. Sometimes this is a difficult task to carry out, other times it is very simple, but it is always essential to advance in science, which is nothing more than knowing the truth and the real scope of its postulates.

Our universe, according to experimental data, is a universe of very small spatial curvature, which has led to the systematic study of models that approximate it to those with strictly zero spatial curvature. In this work we have carried out a theoretical study of the models of strictly zero spatial curvature and we have concluded that these models lead, within the FLRW metric, in all cases to a model of a universe empty of matter and therefore far from the real universe, although in certain cases they respond very well to experimental data, but in the certainty that this will not always be the case.

To reach this conclusion we have studied the Rh=ct universe through the two equations on which it is based, the R_h=ct constraint equation where R_h is the gravitational horizon, and the zero active mass equation (ρ + 3p) = 0, and we have obtained that these two equations result from the zero spatial curvature condition and we have also shown that these equations characterize this model as the only one compatible with the zero spatial curvature condition.

In addition, we have obtained a new equation that relates spatial curvature to matter density, and we have obtained that zero spatial curvature requires a matter density equal to zero, being impossible with a matter density other than zero. Thus, strictly zero spatial curvature universe models have to be empty of matter.

Although our results seem surprising at first, a few years ago, another author, Abhas Mitra, [10], reached similar conclusions comparing the R_h=ct model with the Milne universe model. We have also arrived at the result of an empty universe by other means, which we believe are more rigorous and also more complicated and extensive and which are presented in this work.

For all these reasons we conclude that strictly zero curvature universe models are not going to replace the Standard Cosmological Model, ΛCDM, although they are suitable in some cases where the ΛCDM model fails.

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