The electromagnetic duality in a quaternionic vacuum

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The electromagnetic duality in vacuum is an intriguing property characterizing Maxwell's equations. It was the starting point of numerous developments. One of the most important topic related to that property certainly is the discussion due to Dirac concerning magnetic monopoles. This exploration proposes representations of the duality with elements in M(4, H) involving the three generators of the imaginary part of H, the non-commutative set of quaternions.

Part 1: The matrices representing the duality – first criterion

The context

Due to J.C. Maxwell's work, the behaviour of electromagnetic fields expanding in the empty regions of the universe can be remarkably summarized through four equations [01]:

$$\mathbf{rot}_x\, \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\mathrm{div}_x\, \mathbf{E} = 0$$

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In addition to their conciseness, these equations exhibit a strange property which can literally be observed with the naked eye. Exchanging the pair (**E**, **B**) into the pair (c.**B**, -**E**/c) has no impact on their formalism. This fact is what the literature calls "the electromagnetic duality" in vacuum.

Former works

This amazing property was the starting point of numerous developments; see an introduction, e.g. in: [02]. One of most important topic related to that property certainly is the discussion concerning Dirac's magnetic monopoles.

The main purpose of this document

My aim is to prove that the electromagnetic duality in vacuum (up to now: EDV) can be described with the help of matrices in M(4, H) where H denotes the non-commutative set of quaternions [03].

Characteristics

Recalling that: (i) electromagnetic fields are currently believed to be well represented by antisymmetric 2-forms [04]; (ii) this type of forms can be represented by antisymmetric matrices – e.g.: elements in SU(4), I propose to check if the EDV can be represented by specific matrices denoted $[\chi]$ too:

Equ.(1)

$$[{}^*F(2,0)] = [\chi].[F(2,0)]$$

Furthermore, since a repetition of the substitution delivers the initial electromagnetic field, up to a minus sign, i.e.:

$$(\mathbf{E}, \mathbf{B}) \rightarrow (c.\mathbf{B}, -\mathbf{E}/c) \rightarrow (-\mathbf{E}, -\mathbf{B}) = -(\mathbf{E}, \mathbf{B})$$

... the matrices representing the EDV should also be such that:

Equ.(2)

$$[\chi]^2 = -\mathrm{Id}_4$$

Main results

Each of both equations is associated with a set of four relations when matrices are treated with a visual representation such that:

$$[\chi] = \begin{bmatrix} \text{corner} & < \textbf{North wing}| \\ |\textbf{West wing} > & [\text{Heart}] \end{bmatrix}$$

With this vision, it can be proved that the best candidate for a representation of the first equation has the generic formalism:

Equ.(3)

$$[\chi] = \begin{bmatrix} \Xi & < \frac{\mathbf{S}}{c}| \\ |\frac{\mathbf{S}}{c^3} > & [\alpha.\mathrm{Id}_3 + \frac{\beta}{c^2}.\mathrm{T}(\mathbf{S},\mathbf{S}) - \frac{\gamma}{c}.\Phi(\mathbf{S})] \end{bmatrix}$$

With:

- Ξ as a term proportional to an energy per unit of volume
- c as the speed of light in vacuum,
- μ_0 as the magnetic permittivity in vacuum
- S as the Poynting's vector:

$$\mathbf{S} = \frac{1}{\mu_9}.\mathbf{E} \wedge \mathbf{B}$$

- The Pythagorean table:

$$T(\mathbf{S},\mathbf{S}) = \begin{bmatrix} S^1S^1 & S^2S^1 & S^3S^1 \\ S^1S^2 & S^2S^2 & S^3S^2 \\ S^1S^3 & S^2S^3 & S^3S^3 \end{bmatrix}$$

- The axial rotation matrix:

$$\Phi(\mathbf{S}) = \begin{bmatrix} 0 & -S^3 & S^2 \\ S^3 & 0 & -S^1 \\ -S^2 & S^1 & 0 \end{bmatrix}$$

- and the coefficients:

$$\gamma = -\frac{<\mathbf{E},\mathbf{B}>_{\mathrm{Id}_3}}{\mathrm{E}^2}$$

$$\beta = \frac{\mu_0}{c.B^2}$$

$$\alpha = -\gamma.\frac{<\mathbf{E},\mathbf{B}>_{\mathrm{Id}_3}}{\mu_0.c}$$

$$a = \frac{<\mathbf{E},\mathbf{B}>_{\mathrm{Id}_3}}{\mu_0.c}$$

$$b = 2.(\gamma^2 + 1/c^2).\frac{\mathrm{E}^2}{\mu_0.\mathrm{c}}$$

This representation is not perfect because it delivers only:

$$[\chi].[F(2,0)] = [{}^*F] + a.\begin{bmatrix} 0 & <\mathbf{0}| \\ |\mathbf{0}> & \mathrm{b}.\Phi(\mathbf{B}) - \frac{\gamma}{\mathrm{c}}.\{\mathrm{T}(\mathbf{S},\mathbf{B}) + \mathrm{T}(\mathbf{B},\mathbf{S})\} \end{bmatrix}$$

But it deserves at least two remarks:

1. The coefficient b vanishes each time $\gamma.c^2$ is one of the three generators of the imaginary part of H.

$$\gamma.c^2 \in \{I, J, K\} \Rightarrow b = 0$$

2. The sum $T^t(\mathbf{S},\mathbf{B}) + T(\mathbf{S},\mathbf{B})$ (the " t " denotes the transposed of) vanishes when **S** and **B** have components in the anticommutative part of H.

These remarks suggest translating the mathematical discussion into a vector space such that the electrical field and the magnetic field have their components in H; in that case, $[\chi]$, [F(2,0)] and [*F(2,0)] are elements in M(4,H). Furthermore, a transposition of the formula is written:

$$-[F(2,0)].[\chi]^t = -[{}^*F(2,0)] + a.\begin{bmatrix} 0 & <\mathbf{0}| \\ |\mathbf{0}> & -\mathrm{b}.\Phi(\mathbf{B}) - \frac{\gamma}{\mathrm{c}}.\{T^t(\mathbf{S},\mathbf{B}) + T^t(\mathbf{B},\mathbf{S})\} \end{bmatrix}$$

As consequence:

$$[\chi].[F(2,0)] - [F(2,0)].[\chi]^t = [0]$$

And:

$$[\chi].[F(2,0)] + [F(2,0)].[\chi]^t = 2.[{}^*F(2,0)] + a.\begin{bmatrix} 0 & <\mathbf{0}| \\ |\mathbf{0}> & b.\Phi(\mathbf{B}) \end{bmatrix}$$

Due to the first remark, the second matrix on the right side vanishes when $\gamma.c^2$ is one of the three generators of the imaginary part of H. In these conditions, it is legitim to write the expected relation:

$$[\chi].[F(2,0)] = [{}^*F(2,0)]$$

… provided:

- the components of **S** and **B** are in H, and they are anticommutative, i.e.: $S^m.B^n + B^n.S^m = 0$.
- the coefficients of $[\chi]$ are such that:

$$-\frac{<\mathbf{E},\mathbf{B}>_{\mathrm{Id}_3}}{E^2}.\ c^2 = \pm I,\ \pm J \text{ or } \pm K$$

$$\beta = \frac{\mu_0}{c.B^2} \Rightarrow \frac{\beta}{c^2} = \frac{1}{c^3}.\frac{1}{2.W_B}$$

$$\text{with } W = ½.\ \frac{B^2}{\mu_0} = ½.\ \varepsilon_0.\ E^2$$

$$\alpha = -\gamma.\frac{<\mathbf{E},\mathbf{B}>_{\mathrm{Id}_3}}{\mu_0.c} = \frac{1}{\mu_0.c^5}.\ \gamma^2.\ c^4.\ E^2 = -\frac{E^2}{\mu_0.c^5} = -\frac{2}{c^3}.\ W_E$$

$$a = \frac{<\mathbf{E},\mathbf{B}>_{\mathrm{Id}_3}}{\mu_0.c}$$

- the matrix $[\chi]$ is:

$$[\chi] = \begin{bmatrix} \Xi & <\frac{\mathbf{S}}{c}| \\ |\frac{\mathbf{S}}{c^3}> & [-\frac{2}{c^3}.W_E.\mathrm{Id}_3 + \frac{1}{c^3}.\frac{1}{2.W_B}.\mathrm{T}(\mathbf{S},\mathbf{S}) - \frac{\pm I,\ \pm J \text{ or } \pm K}{c^3}.\Phi(\mathbf{S})] \end{bmatrix}$$

Part 2: Matrices representing the duality – second criterion.

The second part explores the necessary conditions for a validation of Equ.(2). The second equation is true when:

- For the corner

$$\Xi^2 + S^2/c^4 = -1$$

- For the wings

$$\Xi + \alpha + \beta.\ S^2/c^2 = 0$$

- For the heart

$$\alpha.\ \gamma + \gamma.\ \alpha = 0$$

$$\alpha^2 - \gamma^2 . S^2/c^2 = -1$$

$$\alpha . \beta/c^2 + \beta/c^2 . \alpha + \beta^2 . S^2/c^4 + \gamma/c^2 + 1 = 0$$

Here, they must be relooked as:

- For the heart

$$W_E . \gamma + \gamma . W_E = 0$$

The electric energy W_E of the EM wave must be (i) J or K if γ = I, (ii) K or I if γ = J and (iii) I or J when γ = K. This is because α and γ must be anticommutative coefficients. Because of this fact:

$$S^2 = 0$$

The Poynting's vector must be isotropic [for a definition see E. Cartan's theory on spinors]. A condition for the vanishing of b was: $\gamma/c^2 + 1 = 0$. The third constraint concerning the heart is then:

$$\alpha . \beta + \beta . \alpha = W_E . \frac{1}{W_B} + \frac{1}{W_B} . W_E = 0$$

This strange constraint can only be realized in a context where the electric energy differs from the magnetic energy, and both are anticommutative quaternions with the same "norm".

- Consequence for the wings

$$\Xi = -\alpha = \frac{2}{c^3} . W_E$$

- Consequence for the corner

$$\Xi^2 = \frac{4}{c^6} . W_E^2 \quad = -1$$

Up to a real factor of proportionality, the electrical energy is a generator for H too.

$$\Xi = \frac{2}{c^3} . W_E \in \{\pm I, \pm J, \pm K\}$$

Conclusion

Hence, the generators of the imaginary part of quaternions give the opportunity to represent the EDV in different exemplars obeying the same model:

$\Xi = -\alpha$	$\frac{1}{c^3} . \frac{1}{2.W_B}$	γ/c
J	-J	I
-J	J	I
K	-K	I
-K	K	I

Etc.

… leaving this exploration with a set of matrices resembling this one:

$$[\chi] = \begin{bmatrix} \mathrm{I} & <\mathbf{S}| \\ |\mathbf{S}> & -\mathrm{I}.\,\mathrm{Id}_3 + \mathrm{K}.\,\mathrm{T}(\mathbf{S},\mathbf{S}) - \mathrm{J}.\,\Phi(\mathbf{S}) \end{bmatrix};\ S^2 = 0$$

This exploration, if it has not already been made by someone else, opens a door into a mysterious terra incognita because it tells more questions than it delivers answers.

- Does a one-to-one correspondence between this type of matrices [χ] and a given type of physical particles or anti-particles (e.g.: quarks and anti-quarks, neutrinos) in the standard model exist?
- Are these representations explaining – at least for a part- the three generations?
- Is this type of elementary particles a set of messengers between two equivalent states of what we perceive as a vacuum? Hence, are these particles carrying the quantized fluctuations characterizing the empty regions of the universe?

Bibliography

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