A probabilistic approach to the inductive step of $3n + 1$

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1 Abstract

It was Paul Erdős, one of the brightest mathematicians of our time, who famously stated that 'Mathematics is not yet ripe for such questions', setting the tone for almost all related mathematical inquiry for the next generation. The notorious Collatz Conjecture has stumped a plethora of hobbyist and professional mathematicians alike who have tried their hand. Among many ways in which this problem has been tackled, including [Lag11], [Lag12], and others, one of them seems most suggestive - strong induction. In this paper, we claim to show that the inductive step of the Collatz Conjecture is probabilistically true for large odd seeds. Further, in this paper I put forth a novel reformulation that, if true, would imply the truth of the Collatz Conjecture.

2 The Setup

Let C be the Collatz map, that is:

$$C : \mathbb{Z}^+ \to \mathbb{Z}^+$$
$$C(n) = \begin{cases} \frac{n}{2} & , n \text{ is even} \\ 3n+1 & , n \text{ is odd.} \end{cases}$$

The Collatz conjecture states that, for any positive integer, n, there exists $k \geq 1$ such that $C^k(n) = 1$.

A Collatz sequence is a sequence of positive integers, a_k, such that $a_{k+1} = C(a_k)$.

Let S_k be the k-th odd term in an arbitrary Collatz sequence and b_k the number of even terms in that sequence between S_k and S_{k+1}, then:

$$S_{k+1} = \frac{3S_k + 1}{2^{b_k}}$$

$$S_k = \frac{3S_{k-1} + 1}{2^{b_{k-1}}}$$

So, recursively, we also have,

$$S_{k+1} = \frac{3^k S_1}{2^{B_k}} + \sum_{i=0}^{k-1} \frac{3^{k-i-1}}{2^{B_k - B_i}}, \tag{1}$$

for all $k \geq 1$, where $B_k = \sum_{l=1}^{l=k} b_l$ and $b_0 = 0$. Now, notice that,

$$3S_1 + 1 = 2^{b_1} S_2 \tag{2}$$

and, similarly,

$$3S_2 + 1 = 2^{b_2} S_3 \tag{3}$$

and so on and so on, so that, by recursion we know that, for all $k \geq 1$,

$$2^{B_k} = \frac{S_1}{S_{k+1}} \prod_{l=1}^{k} \left(3 + \frac{1}{S_l}\right) \tag{4}$$

also, define C to be the largest known number that adheres to the Collatz Conjecture.

3 The Proof

Now, assume by way of contradiction that 'for some $S_1 > C$ for all $k \geq 1$ we have that $S_{k+1} \geq S_1$'. Since, for all $k \geq 1$,

$$\frac{S_1}{S_{k+1}} \prod_{l=1}^{k} \left(3 + \frac{1}{S_l}\right) = 2^{B_k} \tag{5}$$

we see that, for all $k \geq 1$,

$$S_1 = \frac{3^{k-1}}{\prod_{l=1}^{k}\left(3+\frac{1}{S_l}\right) - 3^k} \cdot \sum_{i=0}^{k-1} \frac{\prod_{l=1}^{i}\left(3+\frac{1}{S_l}\right)}{3^i} \cdot \frac{S_1}{S_{i+1}} \text{ and further, by assumption, for all } k \geq 1, \tag{6}$$

$$k \leq S_1 \leq \frac{3^{k-1}}{\prod_{l=1}^{k}\left(3+\frac{1}{S_l}\right) - 3^k} \cdot \sum_{i=0}^{k-1} \frac{\prod_{l=1}^{i}\left(3+\frac{1}{S_l}\right)}{3^i} \text{ for all } k \in [1, S_1] \tag{7}$$

$$k \leq \frac{\prod_{l=1}^{k-1}\left(3+\frac{1}{S_l}\right)}{\prod_{l=1}^{k}\left(3+\frac{1}{S_l}\right) - 3^k} \cdot k \text{ clearly,} \tag{8}$$

Therefore, for all $k \in [1, S_1]$,

$$\prod_{l=1}^{k}\left(3+\frac{1}{S_l}\right) - 3^k \leq \prod_{l=1}^{k-1}\left(3+\frac{1}{S_l}\right) \tag{9}$$

$$1 - \frac{3^k}{\prod_{l=1}^{k}\left(3+\frac{1}{S_l}\right)} \leq \frac{1}{3+\frac{1}{S_k}} \tag{10}$$

$$\frac{\prod_{l=1}^{k}\left(3+\frac{1}{S_l}\right)}{3^k} < \frac{3}{2} \tag{11}$$

Now, substituting (11) into (6), we see that, for all $k \in [1, S_1]$

$$2\left[\frac{\prod_{l=1}^{k}\left(3+\frac{1}{S_l}\right)}{3^k} - 1\right] < \sum_{i=0}^{k-1} \frac{1}{S_{i+1}} \tag{12}$$

Moreover, by the Cauchy-Schwarz inequality, we see that, for all $k \in [1, S_1]$,

$$3\left[\frac{\prod_{l=1}^{k}\left(3+\frac{1}{S_l}\right)}{3^k}-1\right] = \sum_{i=0}^{k-1}\frac{\prod_{l=1}^{i}\left(3+\frac{1}{S_l}\right)}{3^i}\cdot\frac{1}{S_{i+1}} \tag{13}$$

$$< \sqrt{\sum_{i=0}^{k-1}\left(\frac{\prod_{l=1}^{i}\left(3+\frac{1}{S_l}\right)}{3^i}\right)^2\sum_{i=0}^{k-1}\left(\frac{1}{S_{i+1}}\right)^2} \tag{14}$$

$$\frac{3S_1}{\sqrt{k}}\left[\frac{\prod_{l=1}^{k}\left(3+\frac{1}{S_l}\right)}{3^k}-1\right] < \sqrt{\sum_{i=0}^{k-1}\left(\frac{\prod_{l=1}^{i}\left(3+\frac{1}{S_l}\right)}{3^i}\right)^2} \text{ by assumption,} \tag{15}$$

Further, define $\alpha_k \in (0, \frac{\pi}{2})$ to be the angle in-between,

$$\left\langle 1, \frac{\prod_{l=1}^{1}\left(3+\frac{1}{S_l}\right)}{3^1}, \frac{\prod_{l=1}^{2}\left(3+\frac{1}{S_l}\right)}{3^2}, \ldots, \frac{\prod_{l=1}^{k-1}\left(3+\frac{1}{S_l}\right)}{3^{k-1}}\right\rangle \tag{16}$$

and,

$$\left\langle \frac{1}{S_1}, \frac{1}{S_2}, \ldots, \frac{1}{S_k}\right\rangle \tag{17}$$

for all $k \in [1, S_1]$. Therefore, we know that, since equivalently,

$$\left\langle 1, \frac{\prod_{l=1}^{1}\left(3+\frac{1}{S_l}\right)}{3^1}, \frac{\prod_{l=1}^{2}\left(3+\frac{1}{S_l}\right)}{3^2}, \ldots, \frac{\prod_{l=1}^{k-1}\left(3+\frac{1}{S_l}\right)}{3^{k-1}}\right\rangle \cdot \left\langle \frac{1}{S_1}, \frac{1}{S_2}, \ldots, \frac{1}{S_k}\right\rangle \tag{18}$$

$$= 3\left[\frac{\prod_{l=1}^{k}\left(3+\frac{1}{S_l}\right)}{3^k}-1\right] \tag{19}$$

we see that, for all $k \in [1, S_1]$

$$\sqrt{\sum_{i=0}^{k-1}\left(\frac{\prod_{l=1}^{i}\left(3+\frac{1}{S_l}\right)}{3^i}\right)^2\sum_{i=0}^{k-1}\left(\frac{1}{S_{i+1}}\right)^2}\cos(\alpha_k) \tag{20}$$

$$= 3\left[\frac{\prod_{l=1}^{k}\left(3+\frac{1}{S_l}\right)}{3^k}-1\right] \tag{21}$$

$$\tag{22}$$

So that, by (15),

$$\cos(\alpha_k) < \frac{\sqrt{k}}{S_1\sqrt{\sum_{i=0}^{k-1}\left(\frac{1}{S_{i+1}}\right)^2}} \tag{23}$$

$$= \frac{1}{S_1\sqrt{\frac{\sum_{i=0}^{k-1}\left(\frac{1}{S_{i+1}}\right)^2}{k}}} \tag{24}$$

$$\leq \frac{k}{S_1\sum_{i=0}^{k-1}\frac{1}{S_{i+1}}} \text{ by QM-AM,} \tag{25}$$

$$< \frac{k}{2S_1\left[\frac{\prod_{l=1}^{k}\left(3+\frac{1}{S_l}\right)}{3^k}-1\right]} \text{ by (12)} \tag{26}$$

and since $\alpha_k \in (0, \frac{\pi}{2})$ and $\cos(t)$ is concave along $t \in (0, \frac{\pi}{2})$ we know that,

$$1 - \frac{2}{\pi}\alpha_k < \cos(\alpha_k) \tag{27}$$

Therefore, we see that, for all $k \in [1, S_1]$

$$\alpha_k > \frac{\pi}{2}\left(1 - \frac{1}{\frac{2S_1}{k}\left[\frac{\prod_{l=1}^{k}\left(3+\frac{1}{S_l}\right)}{3^k}-1\right]}\right) \tag{28}$$

Further, since $\cos(t)$ is decreasing along $t \in (0, \frac{\pi}{2})$, we see that, for all $k \in [1, S_1]$,

$$3\left[\frac{\prod_{l=1}^{k}\left(3+\frac{1}{S_l}\right)}{3^k}-1\right] < \sqrt{\sum_{i=0}^{k-1}\left(\frac{\prod_{l=1}^{i}\left(3+\frac{1}{S_l}\right)}{3^i}\right)^2\sum_{i=0}^{k-1}\left(\frac{1}{S_{i+1}}\right)^2}\cos\left(\frac{\pi}{2}\left(1 - \frac{1}{\frac{2S_1}{k}\left[\frac{\prod_{l=1}^{k}\left(3+\frac{1}{S_l}\right)}{3^k}-1\right]}\right)\right) \tag{29}$$

$$< \frac{3k}{2S_1}\cos\left(\frac{\pi}{2}\left(1 - \frac{1}{\frac{2S_1}{k}\left[\frac{\prod_{l=1}^{k}\left(3+\frac{1}{S_l}\right)}{3^k}-1\right]}\right)\right) \text{ by (11) and our assumption,} \tag{30}$$

$$\leq \frac{3}{2}\cos\left(\frac{\pi}{2}\left(1 - \frac{1}{2\left[\frac{\prod_{l=1}^{k}\left(3+\frac{1}{S_l}\right)}{3^k}-1\right]}\right)\right) \text{ since } \cos(t) \text{ is decreasing in } t \in [0, \pi/2] \tag{31}$$

Now, denote $R_k = \frac{\prod_{l=1}^{k}\left(3+\frac{1}{S_l}\right)}{3^k} - 1 \in (0, \frac{1}{2})$ for all $k \in [1, S_1]$, and let $R_0 = 0$. Therefore, we see that, for all $k \in [1, S_1]$, noting $R_{k+1} > R_k$ for all $k \in [1, S_1 - 1]$,

$$3R_k < \frac{3}{2}\cos\left(\frac{\pi}{2}\left(1 - \frac{1}{2R_k}\right)\right) \tag{32}$$

$$2R_k < \sin\left(\frac{\pi}{4R_k}\right) \tag{33}$$

$$\tag{34}$$

Now, let $T_i = \left\{ \text{ all x} \in [R_i, \frac{1}{2}] : 2x < \sin\left(\frac{\pi}{4x}\right)\right\}$, and let $(X_1, X_2, \ldots, X_{S_1}) \overset{\text{i.i.d.}}{\sim} \mathscr{U}_{[0,\frac{1}{2}]}$. Now, sort X_i to become $X_{(i)}$ such that, $X_{(i+1)} > X_{(i)}$ for all $i \in [1, S_1 - 1]$, and suppose $R_i = X_{(i)}$ for all $i \in [1, S_1]$. Therefore,

$$\mathbb{P}\left(\bigcup_{i=1}^{S_1}\{R_i \notin T_0\}\right) = 1 - \prod_{i=1}^{S_1}\mathbb{P}(R_i \in T_0) \tag{35}$$

$$= 1 - \prod_{i=1}^{S_1}\left(\frac{L_i}{\frac{1}{2} - R_i}\right) \text{ where } L_i = \mu(T_i) \in \left[0, \frac{1}{2} - R_i\right] \text{ for } i \in [1, S_1]\text{, clearly,} \tag{36}$$

$$\approx 1 \text{ for large } S_1 > C\text{, since } \frac{L_i}{\frac{1}{2} - R_i} \leq 1 \text{ for all } i \in [1, S_1] \tag{37}$$

So, as we consider larger $S_1 > C$, assuming R_k is assigned the value of the kth ordered random variable $X_{(k)}$ for $k \in [1, S_1]$, we see, clearly, the probability that 'for some $S_1 > C$ for all $k \geq 1$ we have that $S_{k+1} \geq S_1$' is a contradiction approaches 1. We conclude that the inductive step of the Collatz Conjecture is probabilistically true for large odd seeds. That is to say, we conclude that 'for all $S_1 > C$ for some $k \geq 1$ we have that $S_{k+1} < S_1$' is probabilistically true for large odd seeds, all assuming R_k is assigned the value of the kth ordered random variable $X_{(k)}$ for $k \in [1, S_1]$.

4 A push forward reformulation

In this section, I detail a novel reformulation of the Collatz Conjecture. So, we begin exactly as from (1) to (11). That is, assuming by way of contradiction that 'for some $S_1 > C$ for all $k \geq 1$ we have that $S_{k+1} \geq S_1$', we have that, by substituting $k = S_1$ into (11),

$$\frac{3^{S_1}}{\prod_{l=1}^{S_1}\left(3 + \frac{1}{S_l}\right)} > \frac{2}{3} \tag{38}$$

Now, it is required to show that, in actuality,

$$\frac{3^{S_1}}{\prod_{l=1}^{S_1}\left(3 + \frac{1}{S_l}\right)} \leq \frac{2}{3} \tag{39}$$

For all $S_1 \geq M$ for some $M \geq 1$, and this is our reformulation. As numerically supported, I conjecture that this is the case for all $S_1 > 31$. Moreover, if one were to tackle this conjecture of my own, one would naturally gravitate towards a kind of induction. Unfortunately, the data would suggest that $\frac{3^{S_1}}{\prod_{l=1}^{S_1}\left(3+\frac{1}{S_l}\right)}$ is not always greater than nor always less than $\frac{3^{S_1'}}{\prod_{l=1}^{S_1'}\left(3+\frac{1}{S_l'}\right)}$ where $S_1' = S_1 + 2$, so, another approach is most likely required.

References

[Lag11] Jeffrey C. Lagarias. The 3x+1 problem: An annotated bibliography (1963–1999) (sorted by author), 2011.

[Lag12] Jeffrey C. Lagarias. The 3x+1 problem: An annotated bibliography, ii (2000-2009), 2012.