A story of a function

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Abstract

We dedicate this text to a study of a function with interesting properties: besides being, by construction, well suited for a certain type of approximations, the function happens to be lacunary, i.e. without analytic continuation outside the complex unit disk. It however satisfies a functional equation which can be (with some restriction) solved everywhere except the origin. Unfortunately, this solution cannot be understood as its natural continuation beyond the disk.

MSC: 30B10, 30B40

We dedicate this text to a study of a function with interesting properties. It was "discovered" in a specific approximation approach when asking for a simple coefficient formula. Briefly: in the derivative-matching expansions of the form [1]

$$f(z) \approx f(0) + \sum_{n=1}^{\infty} a_n g(z^n), \quad g(0) = 0, \quad g'(0) \neq 0, \tag{1}$$

the coefficients a_n are determined as $a_n = \sum_{d|n} f_d h_{\frac{n}{d}}$, where $d|n$ means "d divides n". Numbers $\{f_d\}$ represent the power-expansion coefficients of f and the sequence $\{h_j\}$ is the inverse of $\{g_i\}$ with respect to the Dirichlet convolution, $\{g_i\}$ being the power expansion coefficients of g. From various possibilities for g one may search for a one giving a simple expression for a_n. Avoiding the trivial choice $h = \mathbf{1} = \{\delta_{1,i}\}_i$, $g(z) = z$ (i.e. the Taylor series), one can consider $h_n|_{n=1,2,3,...} = 1, 1, 0, 0, 0, 0, \ldots$ leading to $(k, n \in \mathbb{N}_0)$

$$g_n = \begin{cases} (-1)^k & \text{for } n = 2^k \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad a_n = \begin{cases} f_n & \text{if } n \text{ is odd} \\ f_n + f_{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}. \tag{2}$$

One then has

$$\text{yo}\,(z) \equiv g\,(z) = \sum_{n=0}^{\infty} (-1)^n z^{2^n}. \tag{3}$$

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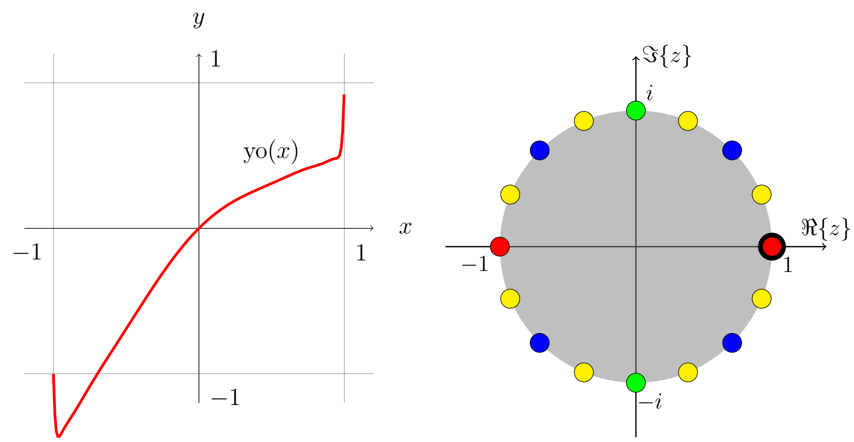


(a) (b)

Figure 1:
a) Graph of $\sum_{n=0}^{6}(-1)^n x^{2^n} \approx \text{yo}(x)$.
b) Repeated square roots of the unity (black annulus): the first generation is represented by the red circles, then second, third and fourth generations by green, blue and yellow circles respectively. The set of all consecutive square roots of unity is dense on a unit circle.

1 Unit circle

The definition (3) converges for all complex $|z| < 1$. On the real segment $\langle 0, 1)$ it is a series with alternating signs whose terms decrease in norm and tend to zero. With the domain of convergence being always a disk around the expansion point, (3) necessarily converges for all $|z| < 1$. For $z \geq 1$ the series diverge because the individuals summands do not tend to zero. And yet, the behavior at the unit circle is interesting: The graph of the function on the real axis (Figure 1-a) seems to tend to a finite value when approaching one.

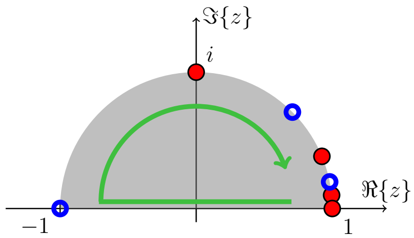
Before a more detailed investigation of the behavior on the unit circle let us state one observation: yo is a lacunary function [2] with the unit circle being its natural boundary. Indeed, the spacing between the powers of z with nonzero coefficients grows rapidly enough for the function to fulfill the conditions of the *Ostrowski–Hadamard gap theorem* (OHGP) implying that the function cannot be analytically continued to and beyond the unit circle.

To investigate the behavior on the unit circle one can notice that the definition (3) implies the validity of the functional equation

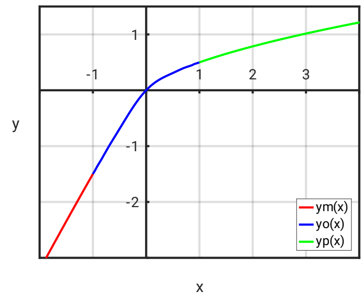
$$\text{yo}(z) = z - \text{yo}(z^2), \tag{4}$$

with two self-consistency points $z = 0$ and $z = 1$. The first gives $\text{yo}(0) = 0$ (in accordance with (3)). The second point leads to

$$\text{yo}(1) = 1 - \text{yo}(1) \Rightarrow \text{yo}(1) = 1/2,$$



(a)



(b)

Figure 2: a) We study the continuity of yo at the unit circle by successively taking the square root starting at $z = 1$. Choosing always the root in the upper-half of the complex plane (all circles), we evaluate yo at every second point (full circles). Doing a round trip, the points come back and approach unity.
b) Functions ym(red), yo (blue) and yp (green).

which extends the definition of the function. Next one proceeds recursively

$$\text{yo}(-1) = -1 - \text{yo}(1) = -3/2, \qquad \text{yo}(i) = i - \text{yo}(-1) = 3/2 + i,$$
$$\text{yo}(-i) = -i - \text{yo}(-1) = 3/2 - i, \qquad \text{yo}(e^{i\pi/4}) = e^{i\pi/4} - \text{yo}(i) = \dots$$

In this way one computes function values at arbitrary square roots of the unity $z = e^{im\pi/2^n}$, $m, n \in N_0$, which represent a dense subset of the unit circle ("square-root points" or SR points), see Figure 1-b. This extension is of course only formal and non-analytic.

The OHGP tells us that in no point of the unit circle yo is analytic. We can prove this independently thanks to the functional equation (4) and get some more insight. For this purpose it is convenient to apply the functional equation twice so as to get a value difference in two points $\text{yo}(z^4) = z^2 - z + \text{yo}(z)$. We choose a specific path when taking square roots, starting from 1 we go to -1 and then we remain on the upper complex semi-plane, always taking as the square root the point with half-argument, see Figure 2-a. We have after N steps

$$\text{yo}\,(1) = \left[e^{i\pi} - e^{i\pi/2}\right] + \text{yo}(e^{i\pi/2}) = \left[e^{i\pi} - e^{i\pi/2}\right] + \left[e^{i\pi/4} - e^{i\pi/8}\right] + \text{yo}(e^{i\pi/8})$$
$$= \dots = \left\{\sum_{n=0}^{N}\left[e^{i\pi/2^{2n}} - e^{i\pi/2^{2n+1}}\right]\right\} + \text{yo}(e^{i\pi/2^{2N+1}}).$$

So

$$\text{yo}\,(1) - \text{yo}(e^{i\pi/2^{2N+1}}) = \sum_{n=0}^{N}\left[e^{i\pi/2^{2n}} - e^{i\pi/2^{2n+1}}\right] \equiv \sum_{n=0}^{N} \mathit{\Delta}_n \equiv D_N. \qquad (5)$$

The function arguments on the left-hand side of (5) become arbitrary close when N rises, $\lim_{N\to\infty} e^{i\pi/2^{2N+1}} = 1$. But what about the function values? Let us compute the real part of Δ_n

$$\Re(\Delta_n) = \Re\left(e^{i\pi/2^{2n}}\right) - \Re\left(e^{i\pi/2^{2n+1}}\right) = \cos\frac{\pi}{2^{2n}} - \cos\frac{\pi}{2^{2n+1}}.$$

For all $n \geq 0$ we have $0 < \frac{\pi}{2^{2n}} \leq \pi$, which is an interval where the cosine is monotonic and decreasing. Therefore

$$\cos\frac{\pi}{2^{2n+1}} > \cos\frac{\pi}{2^{2n}} \Rightarrow \Re(\Delta_n) < 0.$$

If each term in (5) has a strictly negative real part then their sum is strictly negative too, implying that $D \equiv D_\infty$ is nonzero. Using the root test we can in addition prove that $\sum_{n=0}^{N} \Delta_n$ converges (absolutely)

$$\begin{aligned}\limsup_{n\to\infty} \sqrt[n]{|\Delta_n|} &= \limsup_{n\to\infty} \sqrt[n]{\left|e^{i\pi/2^{2n+1}}\left(e^{i\pi/2^{2n+1}} - 1\right)\right|} \\ &= \lim_{n\to\infty} \sqrt[n]{\left|e^{i\pi/2^{2n+1}}\right|}\sqrt[n]{\left|\left(e^{i\pi/2^{2n+1}} - 1\right)\right|} = 1 \times \frac{1}{4} < 1,\end{aligned}$$

where the second root tends to $1/4$ since for small positive angles φ one has $\left|e^{i\varphi} - 1\right| \sim \varphi$. The approximate value is $D \approx -(1.2321 + 0.5458i)$. We can conclude that yo is not continuous at one because in its arbitrary small neighborhood the value of yo is significantly different $\mathrm{yo}(1) = \mathrm{yo}\left(e^{i\varepsilon_1}\right) + D$, where $\varepsilon_1 > 0$ is infinitesimal but D is not. This discontinuity is then propagated to all SR points. Indeed, let $e^{i(\pi+\varepsilon_2)}$ be the square root of $e^{i\varepsilon_1}$ which lies in the proximity of -1 in the lower complex half-plane, $\varepsilon_2 = \varepsilon_1/2$. We have

$$\begin{aligned}\mathrm{yo}\left[e^{i(\pi+\varepsilon_2)}\right] &= e^{i(\pi+\varepsilon_2)} - \mathrm{yo}\left(e^{i\varepsilon_1}\right) = -1 - \mathrm{yo}(1) + D \\ &= -1 - [1 - \mathrm{yo}(-1)] + D = -2 + \mathrm{yo}(-1) + D,\end{aligned}$$

where we were manipulating terms intuitively by setting $e^{i(\pi+\varepsilon_2)} = -1$. The result implies $\mathrm{yo}(-1) - \mathrm{yo}\left[e^{i(\pi+\varepsilon_2)}\right] = 2 - D$, i.e. the discontinuity is also situated at $z = -1$. The procedure can be repeated recursively from $z = -1$ to all daughter SR points, meaning that the function is discontinuous on a dense subset. Strictly speaking, for this to be rigorously proven, one should show that the appearing numerical terms (such as $2 - D$) do not exactly cancel in some points. Nevertheless one can do two (and more) loops in the upper half plane: Stopping in an infinitesimal neighborhood of $z = 1$ after the first round trip, i.e. at $z = e^{i\varepsilon_1}$, we start the second round trip by taking this time the square root in the proximity of -1, i.e. $z = e^{i(\pi+\varepsilon_2)} = \sqrt{e^{i\varepsilon_1}}$. Then the second round trip continues in the upper half-plane, as show previously in Figure 2-a. It is evident that the result after approaching $z = 1$ for the second time will be just getting one additional D, $\mathrm{yo}(1) = \mathrm{yo}\left(e^{i\varepsilon_3}\right) + 2D$ for some appropriate and infinitesimal ε_3. This discontinuity will also propagate to all remaining SR points, producing an additional difference of D. If, in the previous case, the numerical factors somehow cancel at some points, then they do not cancel this time, i.e. the function is on the unit circle indeed discontinuous everywhere. Moreover, by adding more round trips one adds more Ds, meaning that the norm of the function takes arbitrarily large values in any nonzero neighborhood of each point of the unit circle.

Another interesting topic is the analysis of the derivatives. It suffices to differentiate (4):

$$\mathrm{yo}'(z) = 1 - 2z\mathrm{yo}'(z^2), \quad \mathrm{yo}''(z) = -2\mathrm{yo}'(z^2) - 4z^2\mathrm{yo}''(z^2), \quad \mathrm{yo}'''(z) = \ldots. \tag{6}$$

Again, there are two self-consistency points $z = 0,\ 1$. Although seemingly trivial, the case $z = 0$ allows us to determine derivatives at zero meaning we can reconstruct the power series (3), i.e. it tells us that the functional equation (4) is an equivalent way of defining yo inside the circle. At $z = 1$ we solve (6) and get

$$\mathrm{yo}'(1) + 2\,\mathrm{yo}'(1) = 1 \Rightarrow \mathrm{yo}'(1) = \frac{1}{3}, \quad \mathrm{yo}''(1) = -\frac{2}{15}, \quad \mathrm{yo}'''(1) = \frac{8}{45}, \ \ldots \tag{7}$$

Arbitrary high derivatives can be determined in this way. Thanks to (6) the derivatives can be propagated to an arbitrary SR point. Thus, for example,

$$\mathrm{yo}'(-1) = 1 + 2\,\mathrm{yo}'(1) = 1 + 2 \times (1/3) = 5/3. \tag{8}$$

One should be aware that these derivatives are purely formal, they do not exist in the sense in which the derivative is defined.

2 Beyond the unit circle

Quite some body of literature focuses on the question of non-analytic continuation of functions beyond their natural boundary, see e.g. [3]. Looking at the latter (remark 6.9.12) and on other sources, it seems that the lacunary functions are among the most resistant in this regard, no "natural" way of extending them seems to be commonly accepted. In what follows we make an attempt to extend yo using (4), yet we basically confirm the previous statement.

Anticipating our results, we are going to use for the attempted extension a different function label, namely $\mathrm{yp}\,(z)$. We start by considering the real axis outside the unit circle (using x instead z). For $\mathrm{yp}\,(x)$ we assume only that it satisfies the functional equation (4) and the value at one: $\mathrm{yp}\,(1) = 1/2$. Again, we apply the functional equation (4) twice so as to make appear the difference of yp values at two points $\mathrm{yp}\,(z) = z - z^2 + \mathrm{yp}\left(z^4\right)$. We start in the proximity of one, $x = 1 + h$, and we will be interested in the limit $h \to 0^+$. We have

$$\begin{aligned} \mathrm{yp}(1+h) &= \left[(1+h) - (1+h)^2\right] + \mathrm{yp}\left[(1+h)^4\right] \\ &= \left[(1+h) - (1+h)^2\right] + \left[(1+h)^4 - (1+h)^8\right] + \mathrm{yp}\left[(1+h)^{16}\right] \\ &= \ldots = \left\{\sum_{n=0}^{N}\left[(1+h)^{2^{2n}} - (1+h)^{2^{2n+1}}\right]\right\} + \mathrm{yp}\left[(1+h)^{2^{2N+2}}\right]. \end{aligned} \tag{9}$$

Here two limits appear, $h \to 0^+$ and $N \to \infty$. To proceed in a consistent way we relate them choosing $h = \alpha/2^{2N+2}$ where $\alpha > 0$ is a real parameter. Applying the limit we have

$$\mathrm{yp}(1) = \lim_{N\to\infty}\left\{\sum_{n=0}^{N}\left[(1+\frac{\alpha}{2^{2N+2}})^{2^{2n}} - (1+\frac{\alpha}{2^{2N+2}})^{2^{2n+1}}\right]\right\} + \mathrm{yp}\left(e^{\alpha}\right),$$

or, equivalently

$$\begin{aligned}\mathrm{yp}\left(e^{\alpha}\right) &= \frac{1}{2} + \lim_{N\to\infty}\sum_{n=0}^{N}\left[(1+\frac{\alpha}{2^{2N+2}})^{2^{2n+1}} - (1+\frac{\alpha}{2^{2N+2}})^{2^{2n}}\right] \\ &\equiv \frac{1}{2} + \lim_{N\to\infty}\sum_{n=0}^{N}\omega_{N,n} = \frac{1}{2} + \lim_{N\to\infty}\Theta_N\left(\alpha\right). \end{aligned} \tag{10}$$

Our main concern now is to show that the sum converges to a finite value. The proof of the latter is important to our text but technical, and thus we present it in Appendix. There we also derive the boundaries ($x \geq 1$)

$$\mathcal{B}\left[\ln\left(x\right)/2\right] \leq \mathrm{yp}\left(x\right) \leq \mathcal{B}\left[2\ln\left(x\right)\right], \quad \mathcal{B}\left(q\right) = \frac{1}{\ln\left(4\right)}\left[\mathrm{Ei}\left(q\right) - \mathrm{Ei}\left(\frac{q}{2}\right)\right], \tag{11}$$

where Ei is the exponential integral. The limiting functions are shown in Figure 3-a. Furthermore, the Appendix contains the derivation of an elegant alternative expression [4] for an efficient computation of yp

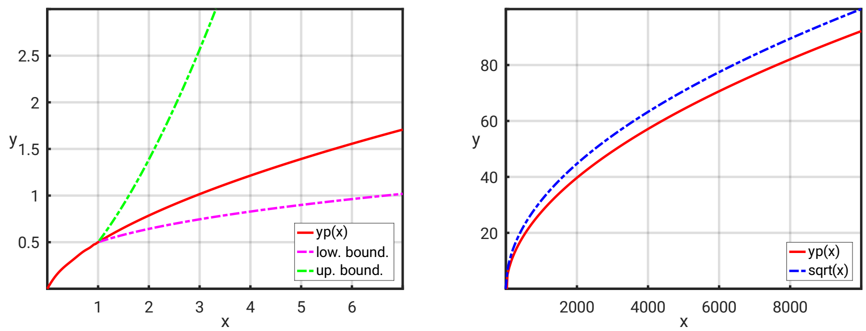
$$\mathrm{yp}\left(e^{\alpha}\right) = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{\alpha^k}{k!\left(2^k+1\right)} = \frac{1}{2} + \Theta\left(\alpha\right),\ \alpha \in \mathbb{R}. \tag{12}$$

There are interesting remarks to make:

- Function $\mathrm{yp}\left(e^{\alpha}\right) = 1/2 + \Theta\left(\alpha\right)$ is, as function of α, an entire function. Indeed, for $\alpha \geq 0$ all terms in the sum (12) are positive, thus partial sums rise. Yet, they are bounded by the exponential $\Theta\left(\alpha\right) \leq \sum_{k=0}^{\infty}\alpha^k/k! = \exp\left(\alpha\right)$. This implies convergence for all $\alpha \geq 0$. Because the convergence domain is a disk centered at the expansion point, the yp series converges for all $\alpha \in \mathbb{C}$.

- We used the functional equation (4) only to determine the value at $x = 1$, we did not use the derivatives (7). Yet, because we derived the expression for yp respecting (4), yp reproduces these values and they are not formal (as for yo), but true derivatives of $\mathrm{yp}(x)$ at $x = 1$.

- Writing $\mathrm{yp}\left(z\right) = 1/2 + \Theta\left(\ln z\right)$ one sees that analytic properties of $\mathrm{yp}\left(z\right)$ are driven by the logarithm in the argument. It implies yp is not defined at $z = 0$ and that it has a cut starting at zero with infinitely many Riemann sheets.



(a) (b)

Figure 3: The function $\mathrm{yp}(x)$ with its bounds (a) given by (11) for $x > 1$ and its behavior compared to $\sqrt{x}$ on a large interval (b).

The relationship yo $\leftrightarrow$ yp deserves an additional investigation, but let us before study the asymptotic behavior of yp on the real axis using only rough (non-rigorous) arguments. Let us assume that on the positive real axis the behavior of its derivative can be approximated by an exponent $\mathrm{yp}'(x)|_{x\to+\infty} \sim ax^b$, $a, b \in \mathbb{R}$. If we plug this into the functional equation for the first derivative (6) we get for large x

$$\mathrm{yp}'(x) = 1 - 2\,x\,\mathrm{yp}'(x^2) \quad \overset{x\to+\infty}{\Longrightarrow} \quad ax^b = 1 - 2ax^{2b+1}.$$

The latter is in the limit $x \to +\infty$ exactly satisfied only for $a = 1/2$ and $b = -1/2$. Thus

$$\mathrm{yp}'(x)|_{x\to+\infty} \sim \frac{1}{2\sqrt{x}},$$

meaning that for a large positive x one has[1] $\mathrm{yp}(x)|_{x\to+\infty} \sim \sqrt{x}$ (Figure 3-b). As we will argue later, it is non trivial to apply (6) to the last expression in order to get the behavior of the function at large negative x. One can do so, but by precaution we change the function name again yp $\to$ ym

$$\mathrm{ym}'(x)|_{x\to-\infty} = 1 - 2x\frac{1}{2\sqrt{x^2}} = 2,$$

i.e. ym approaches a straight line. Here ym is defined as $\mathrm{ym}(x) = x - \mathrm{yp}(x^2)$, $x \le -1$. A common picture of ym, yo and yp and is shown in Figure 2-b.

3 Yo and yp

The situation as we have it now can be summarized:

[1]Using $\mathrm{yp}(x) < \sqrt{x} \Rightarrow \mathrm{yp}(x^2) < x = \sqrt{x^2}$, it is not difficult to show that $\sqrt{x} > \mathrm{yp}(x)$ for $x \geq 1$. It suffices to recognize this property as true in a small right neighborhood of one and then recursively transport it to an arbitrary large x. The square root is a significantly better upper boundary than $\mathcal{B}\,[2\ln(x)]$.

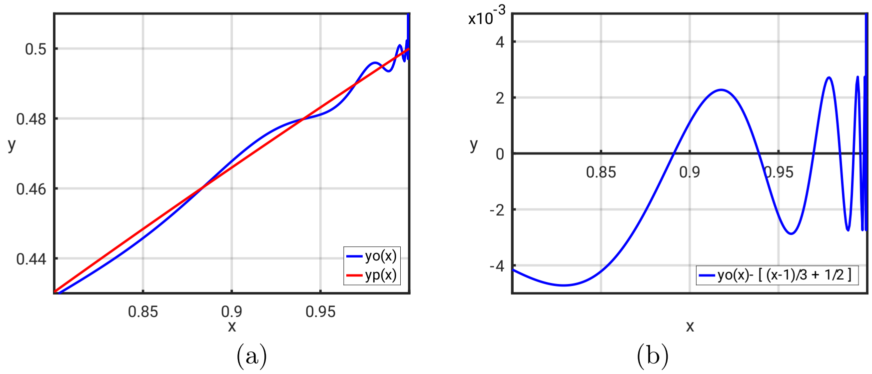


Figure 4: The left neighborhood of one: yo and yp (a) and the difference of yo from the tangent (at $x = 1$) to yp (b).

- The yo function is analytic inside the unit circle and can be expanded in one of the two self-consistency points of (4), namely at $x = 0$. There the derivatives are in agreement with (4). The derivatives one obtains from (4) at $x = 1$ define a different function (yp) which is analytic there and is not the analytic continuation of yo.

- The function yp is analytic in the whole complex plane besides the cut on the negative real axis. Particularly, it is analytic at $x = 1$, implying it is different from yo inside the unit circle (cut excepted). By consequence, the functional equation (4) has on this domain at least two different solutions. The function yp cannot be defined at $x = 0$ since the set of derivatives given by (4) at $x = 0$ is unique. Having these derivatives there means being yo, which yp is not. Yp can be interpreted as an analytic function on a Riemann surface with an infinite number of Riemann sheets generated by the branch point situated at zero.

On $x \in (0, 1)$ yo and yp behave similarly, in Figure 2-b they cannot be distinguished and both can be represented by the right half of the blue segment. However, a more precise numerical inspection shows non-vanishing differences, see Figure 4. The graph suggests that yo has no limit at one, it shows an oscillatory behavior with limited but non-vanishing amplitude.

There is another important difference between yo and yp: Inside the unit circle the function yo fulfills the functional equation (4) without exception, for any z_1 and z_2 such that $z_2 = z_1^2$ one has

$$\mathrm{yo}(z_1) = z_1 - \mathrm{yo}(z_2).$$

This is no longer true for yp, where z_2 cannot be an arbitrary square root, but the root with half-argument:

$$\mathrm{yp}(re^{i\varphi}) = re^{i\varphi} - \mathrm{yp}(r^2 e^{i2\varphi}), \quad r \in \mathbb{R}_{0+}. \tag{13}$$

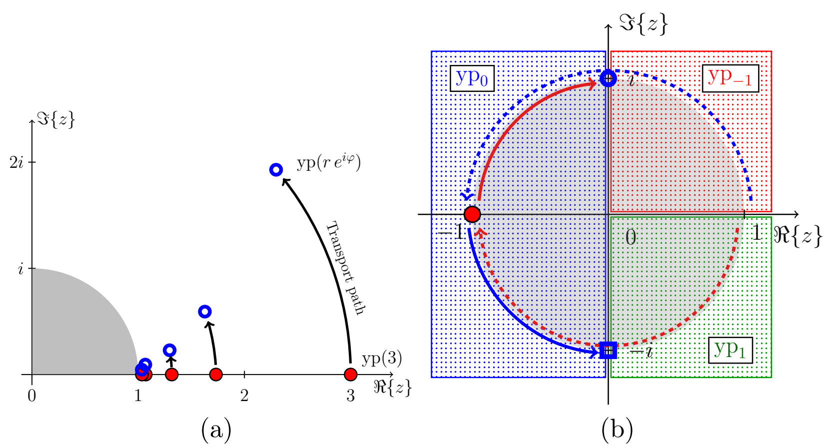


Figure 5: a) The argument of yp is moved away from the real axis, together with all its daughter square roots, along a continuous path to a complex point $re^{i\varphi}$. We require the function value to be transported also in a continuous manner which determines our choice of square roots we use when computing $\mathrm{yp}(re^{i\varphi})$. b) When computing $\mathrm{ym}(z_0)$ we ask for a smooth transition of the value from $z = -1$ along a continuous line. The value of z^2 in the argument of yp (see (14)) may leave the physical sheet for z_0 far from -1. The blue area represents arguments of ym for which the argument of yp is on the physical sheet. If we evaluate ym for some z in the red area then arguments of yp are from the sheet -1, in the green area from the sheet $+1$. Computing e.g. $\mathrm{ym}(i)$ we follow the red line coming to the border of the blue and the red domain. For z transported in this way z^2 follows the dashed red path approaching $-1 = e^{i(-\pi)}$, i.e. $\alpha = -i\pi$ in (12). The evaluation of $\mathrm{ym}(-i)$ corresponds to the blue line which gives dashed blue line for the transport of the yp value, approaching the branch cut from above, $-1 = e^{i(+\pi)}$, i.e. $\alpha = i\pi$ in (12).

Yet, yp fulfills this equation on a much larger domain. This restriction follows from how yp was constructed. Considering (9)–(10) one sees, that the functional equation was applied in steps, starting close to one. We followed the real axis and a pair of consecutive arguments always respected the restricted equation (13). We can of course apply the procedure also to an h with nonzero imaginary part, but we want the function yp to be continuous when going off the positive real axis. With a sequence of points $\{x_n\}_{n=0}^{\infty}$, $x_n \overset{n\to\infty}{\to} 1$ such that $x_{n+1} = \sqrt{x_n}$, where the usual square root is used, we associate a sequence of complex points $\{z_n\}_{n=0}^{\infty}$, $z_n \overset{n\to\infty}{\to} 1$ meant to compute the value of yp at some $z_0 \in \mathbb{C}$. For that we transport each x_n along a smooth path to become z_n. Along the path connecting x_0 and z_0 we transport also the value of yp, see Figure 5-a. For this transport to be smooth (yp is analytic), we need to keep our convention for taking the square root. This naturally explains the origin of the restriction, yp does not respect the equation (4) for an inappropriate root.

At last one can analyze the behavior of yp with respect to (13) on the whole Riemann surface. We will name the zeroth sheet as "physical" and put the index of the sheet in the subscript next to the function name. The index increases when going around zero in the counter-clock direction. For the physical sheet $\varphi = 0$ the equation (13) stands

$$\mathrm{yp}_0(1) = 1 - \mathrm{yp}_0(1) \quad \text{i.e.} \quad 1/2 = 1 - 1/2.$$

This changes for different sheets. With $\varphi = 2\pi$ one for example has (a numerical approximation is shown)

$$\mathrm{yp}_1(1) = 1 - \mathrm{yp}_2(1) \quad \text{i.e.} \quad (-0.7321 - 0.5458i) = 1 - (1.7321 + 0.5458i)\,.$$

The key point is of course $\mathrm{yp}_1(1) \neq \mathrm{yp}_2(1)$. The functional equation is thus genuinely satisfied only by yp on the physical sheet. Estimates for $\mathrm{yp}_{1,2}(1)$ come from the numerical evaluation of $\Theta\,(2\pi i)$ and $\Theta\,(4\pi i)$ using (12)[2].

4 Ym

We use the expression

$$\mathrm{ym}(z) = z - \mathrm{yp}(z^2) = 2z + \mathrm{yp}(-z), \quad z \approx -1 \tag{14}$$

to define ym in the neighborhood of $z = -1$. This definition is unambiguous because for any z its square z^2 is unique. The expansion point of ym is $x = -1 \in \mathbb{R}$, there its derivatives can be determined, see (8). Because we understand ym as defined by its expansion at $x = -1$ and because we require continuity, we compute the value of ym at some complex point z_0 by transporting its value from $x = -1$ to z_0 along some continuous path, see Figure 5-b. Any point at this path has a unique partner point z^2 which follows a different path starting at $x = 1$ and which is the argument of yp. By its relation to yp, one sees that ym is an analytic function on the whole infinitely-sheeted Riemann surface, the branch point situated at $x = 0$ excepted. One can constrain ym to a single sheet (complex plane) by introducing a cut, its most natural place may be the positive real axis, situated symmetrically to the expansion point. The function ym also inherits from yp its behavior with respect to the functional equation (4). Let φ be a small angle and $r \in \mathbb{R}_{0+}$. One has:

$$\mathrm{yp}(r^2 e^{i2\varphi}) = r^2 e^{i2\varphi} - \mathrm{yp}(r^4 e^{i4\varphi}).$$

Then (14) leads to

$$\mathrm{ym}(-z) = -z - 2z^2 - \mathrm{ym}(-z^2) \quad \text{or} \quad \mathrm{ym}(z) = z - 2z^2 - \mathrm{ym}(-z^2), \tag{15}$$

where z is the root of z^2 with the half-argument (left equation) or the one with the half-plus-π argument (right equation). An analogous approach to what is presented here can be used for any function defined in a way sketched in (6) at some SR point.

[2] Interestingly $\Re\,\mathrm{yp}_2(1) \approx \sqrt{3}$ and $\Im\,\mathrm{yp}_2(1) \approx \sin\left(\frac{1}{\sqrt{3}}\right)$.

5 Discussion

Considering z inside the unite circle, $z \neq 0$, one observes that

$$\mathrm{y}\lambda\left[z, \lambda\left(z\right)\right] \equiv \lambda\left(z\right)\mathrm{yo}(z) + \left[1 - \lambda\left(z\right)\right]\mathrm{yp}(z)$$

is (as a function of its first argument) also a solution to the restricted equation for an arbitrary function $\lambda(z)$. We do not know about other continuous solutions (analytic or not). If existing, they are not defined at zero and one. This follows from our solution-constructing procedure (9)–(10) which provides a unique result (without any specific assumptions, e.g. about analyticity). We presume an analogical construction exists at $z = 0$ giving yo as the unique solution.

For what concerns approximation properties, monomials are easy to express in terms of yo[3]:

$$\mathrm{yo}(x^m) = x^m - \mathrm{yo}(x^{2m}) \quad \Rightarrow \quad x^m = \mathrm{yo}(x^m) + \mathrm{yo}(x^{2m}), \tag{16}$$

which agrees with (2). The approximation (1) is based on matching derivatives at zero. Yet, at least polynomials can be constructed also using yp, $x^m = \mathrm{yp}(x^m) + \mathrm{yp}(x^{2m})$, despite yp not being at zero defined. Using yp, polynomials can be expressed[4] outside the unit circle too.

Also, one might ask whether the expansion (1) is equivalent to the Taylor series in the sense that (1) can be built by replacing each power x^m in the Taylor series by (16). The replacement itself is of course allowed, what is questionable is the re-arrangement of the resulting series by powers of x in the argument of yo. Since such a re-arrangement concerns an infinite number of terms one has to provide a rigorous justification for it. We consider this question as open in general, for specific situations the proof can be performed. Consider for example a function with positive expansion coefficients on the positive real axis (such as $\exp(x)$, Figure 6 left)

$$f\left(x\right) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\left[a_n \mathrm{yo}(x^n) + a_n \mathrm{yo}(x^{2n})\right], \quad a_n, x \in \mathbb{R}_{0+}.$$

Evaluated at some point $x_0 \in \mathbb{R}_{0+}$ strictly within the convergence radius, power series are known to be absolutely convergent. We also have $\mathrm{yo}(x_0^{(2)n}) > 0$, thus the positive number $a_n x_0^n \geq 0$ is a sum of two positive numbers $a_n \mathrm{yo}(x_0^n) \geq 0$ and $a_n \mathrm{yo}(x_0^{2n}) \geq 0$. Necessarily $0 \leq a_n \mathrm{yo}(x_0^{(2)n}) \leq a_n x_0^n$ meaning that the two sequences $\sum_{n=0}^{\infty} a_n \mathrm{yo}(x^n)$ and $\sum_{n=0}^{\infty} a_n \mathrm{yo}(x^{2n})$ are each absolutely convergent and so is their sum. We then re-arrange the sum, ordering it by powers appearing in the argument of yo

$$f(x) = \sum_{n=0}^{\infty}\left[\left(a_{2n} + a_n\right)\mathrm{yo}(x^{2n}) + a_{2n+1}\mathrm{yo}(x^{2n+1})\right].$$

[3]We use yo in examples but what is presented applies also to yp or yλ.

[4]One notices, that for polynomials we have an exact finite expression, not an approximation.

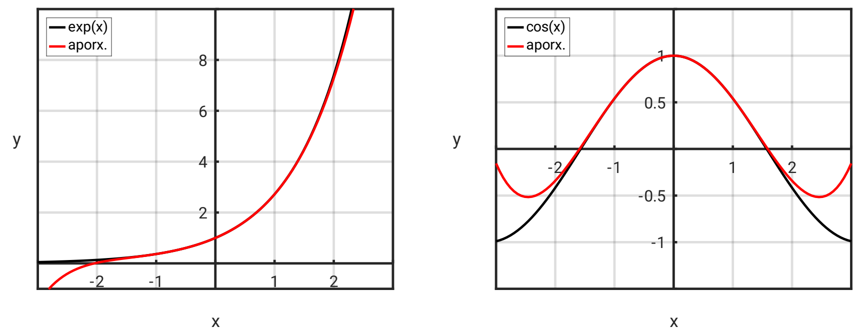


Figure 6: Approximation of $\exp(x)$ and $\cos(x)$ by $f(0) + \sum_{n=1}^{10} a_n \mathrm{ya}(x^n)$.

If true for both, yo and yp, such an equivalence can play a role of the middlemen in the replacement yo $\to$ yp: $\mathrm{yo}(x^m) + \mathrm{yo}(x^{2m}) = x^m = \mathrm{yp}(x^m) + \mathrm{yp}(x^{2m})$ so that one gets an yp-based expansion that goes beyond the unit circle.

A cosmetic defect of using yp appears on the negative real axis where its values (from the upper or the lower edge of the cut) are complex. We therefore propose to build approximations using ya:

$$\mathrm{ya}(x)\,|_{x\le -1} = \mathrm{ym}(x), \quad \mathrm{ya}(x)\,|_{|x|<1} = \mathrm{yo}(x), \quad \mathrm{ya}(x)\,|_{x\ge 1} = \mathrm{yp}(x),$$

$\mathrm{ya}(x)$ already shown in Figure 2-b. Approximations of two example functions using $\mathrm{ya}(x)$ are shown in Figure 6.

6 Summary, outlook

In this text we have introduced a new function yo and analyzed it. We have encountered several interesting properties: Yo is well suited for approximations of a new type (1) whose construction is, rather unexpectedly, related to the Dirichlet convolution. Yo turns out to be a member of a specific function class, namely it is lacunary and no natural way of extending it beyond the unit circle is known. Yet, realizing that yo obeys a functional equation which one is able to solve also outside the unit disc, one may wonder whether a natural continuation is possible. Unfortunately it is not: the solution should be interpreted as an independent function with the related functional equation restricted. Nevertheless one may say that the “events” took an interesting turn.

We believe that the set of functions with these interesting properties may be enlarged by studying other simple approximation forms. For example choosing $h = 1, 0, 1, 0, 0, 0, \ldots$ leads to $g(z) = \sum_{n=0}^{\infty} (-1)^n z^{3^n}$, which is another lacunary function with presumably similar properties to yo. This opens a large space for further investigations.

Appendix: Proof of convergence

We aim to prove the convergence of Θ_N when $N \to \infty$, where

$$\Theta_N(\alpha) = \sum_{n=0}^{N} \left[(1 + \frac{\alpha}{2^{2N+2}})^{2^{2n+1}} - (1 + \frac{\alpha}{2^{2N+2}})^{2^{2n}} \right] \equiv \sum_{n=0}^{N} \omega_{N,n}. \qquad (17)$$

Upper boundary (UB)

First, we note that if n is interpreted as a smooth parameter and N is fixed then $\omega_{N,n}$ increases with increasing n. In this case one can set an upper boundary by replacing the sum with an integral where the upper integration limit is by one higher than the upper limit of the sum, see Figure 7-a. We have ($n \to x$)

$$\begin{aligned}
\Theta_N(\alpha) \leq & \int_0^{N+1} \left[(1 + \frac{\alpha}{2^{2N+2}})^{2^{2x+1}} - (1 + \frac{\alpha}{2^{2N+2}})^{2^{2x}} \right] dx \\
= & \frac{1}{\ln(4)} \left\{ \mathrm{Ei}\left[2^{2x+1} \ln\left(1 + \frac{\alpha}{2^{2N+2}}\right)\right] - \mathrm{Ei}\left[2^{2x} \ln\left(1 + \frac{\alpha}{2^{2N+2}}\right)\right] \right\}_{x=0}^{x=N+1},
\end{aligned}$$

here Ei is the exponential integral. On the upper and lower integration limit we get

$$\begin{aligned}
U_N^{\mathrm{UB}}(\alpha) &= \frac{1}{\ln(4)} \left\{ \mathrm{Ei}\left[2^{2N+3} \ln\left(1 + \frac{\alpha}{2^{2N+2}}\right)\right] - \mathrm{Ei}\left[2^{2N+2} \ln\left(1 + \frac{\alpha}{2^{2N+2}}\right)\right] \right\}, \\
L_N^{\mathrm{UB}}(\alpha) &= \frac{1}{\ln(4)} \left\{ \mathrm{Ei}\left[2 \ln\left(1 + \frac{\alpha}{2^{2N+2}}\right)\right] - \mathrm{Ei}\left[\ln\left(1 + \frac{\alpha}{2^{2N+2}}\right)\right] \right\} \\
&= \frac{1}{\ln(4)} \left\{ L_N^{\mathrm{UB},1}(\alpha) - L_N^{\mathrm{UB},2}(\alpha) \right\}.
\end{aligned}$$

For $U_N^{\mathrm{UB}}(\alpha)$ the limit $N \to \infty$ is easy to determine

$$2^{2N+3} \ln\left(1 + \frac{\alpha}{2^{2N+2}}\right) = \ln\left[\left(1 + \frac{\alpha}{2^{2N+2}}\right)^{2^{2N+3}} \right] = \ln\left[\left\{ \left(1 + \beta^{-1}\right)^{\beta} \right\}^{2\alpha} \right],$$

where $\beta = 2^{2N+2}/\alpha$. Similarly we have for the second term of $U_N^{\mathrm{UB}}(\alpha)$

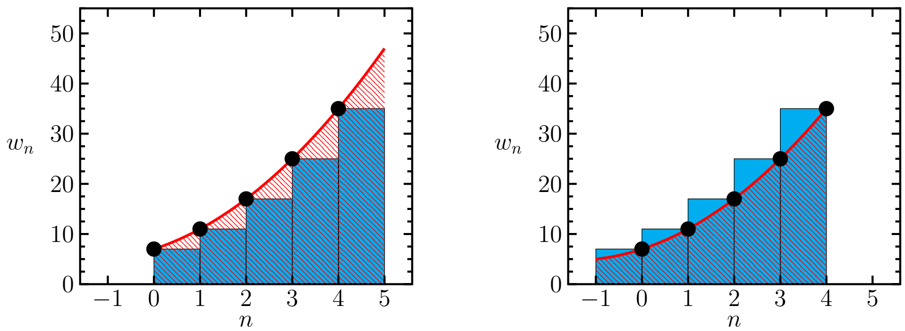
$$2^{2N+2} \ln\left(1 + \frac{\alpha}{2^{2N+2}}\right) = \ln\left[\left\{ \left(1 + \beta^{-1}\right)^{\beta} \right\}^{\alpha} \right].$$

So

$$U^{\mathrm{UB}}(\alpha) \equiv \lim_{N \to \infty} U_N^{\mathrm{UB}}(\alpha) = \frac{\mathrm{Ei}\left[\ln\left(e^{2\alpha}\right)\right] - \mathrm{Ei}\left[\ln\left(e^{\alpha}\right)\right]}{\ln(4)} = \frac{\mathrm{Ei}(2\alpha) - \mathrm{Ei}(\alpha)}{\ln(4)}.$$

For the lower limit one needs to use small argument expansions

$$\mathrm{Ei}(h) = \gamma + \ln(h) + \mathcal{O}(h) \quad \text{and} \quad \ln(1+h) = h + \mathcal{O}\left(h^2\right)$$



(a) (b)

Figure 7: The sum $\sum_{n=0}^{4} w_n$ is in both figures represented by black circles and is exactly given by the area of blue rectangles, constructed either on the right side (a) or on the left side (b) of the points, the histograms are just shifted. If n can be interpreted as a smooth index and w_n rises as function of n (red curve) then an upper boundary is given by the integral (red hatched area) that goes by one further than is the upper summation limit (a) and a lower boundary is given by an integral that starts at the number by one smaller than is the lower summation limit.

(γ is the Euler–Mascheroni constant). One has

$$L_N^{\mathrm{UB},1}(\alpha) = \gamma + \ln\left[2\ln\left(1+\frac{\alpha}{2^{2N+2}}\right)\right] + \mathcal{O}\left[2\ln\left(1+\frac{\alpha}{2^{2N+2}}\right)\right]$$
$$= \gamma + \ln(2\alpha) - \ln\left(2^{2N+2}\right) + \mathcal{O}\left(\frac{\alpha}{2^{2N+2}}\right).$$

Similarly

$$L_N^{\mathrm{UB},2}(\alpha) = \gamma + \ln(\alpha) - \ln\left(2^{2N+2}\right) + \mathcal{O}\left(\frac{\alpha}{2^{2N+2}}\right).$$

Consequently

$$\lim_{N\to\infty} L_N^{\mathrm{UB}}(\alpha) = \lim_{N\to\infty} \frac{L_N^{\mathrm{UB},1}(\alpha) - L_N^{\mathrm{UB},2}(\alpha)}{\ln(4)} = \frac{\ln(2\alpha)-\ln(\alpha)}{\ln(4)} = \frac{1}{2}.$$

To summarize, we have an upper boundary

$$\Theta(\alpha) \equiv \Theta_\infty(\alpha) \le \frac{\mathrm{Ei}(2\alpha) - \mathrm{Ei}(\alpha)}{\ln(4)} - \frac{1}{2},$$

which increases with increasing α and approaches zero for $\alpha \to 0^+$.

Lower boundary (LB)

Similarly, for a sum which rises with the summation index, one can construct a lower boundary by computing an integral, this time the lower integration limit being by one smaller than is the starting value of the summation index, see Figure 7-b

$$\begin{aligned}\varTheta_N(\alpha) \geq & \int_{-1}^{N} \left[(1+\frac{\alpha}{2^{2N+2}})^{2^{2x+1}} - (1+\frac{\alpha}{2^{2N+2}})^{2^{2x}} \right] dx \\ =& \frac{1}{\ln(4)} \left\{ \mathrm{Ei}\left[2^{2x+1}\ln\left(1+\frac{\alpha}{2^{2N+2}}\right)\right] - \mathrm{Ei}\left[2^{2x}\ln\left(1+\frac{\alpha}{2^{2N+2}}\right)\right] \right\}_{x=-1}^{x=N}.\end{aligned}$$

Briefly, we have

$$\begin{aligned}U_N^{\mathrm{LB}}(\alpha) =& \frac{1}{\ln(4)} \left\{ \mathrm{Ei}\left[2^{2N+1}\ln\left(1+\frac{\alpha}{2^{2N+2}}\right)\right] - \mathrm{Ei}\left[2^{2N}\ln\left(1+\frac{\alpha}{2^{2N+2}}\right)\right] \right\}, \\ L_N^{\mathrm{LB}}(\alpha) =& \frac{1}{\ln(4)} \left\{ \mathrm{Ei}\left[\frac{1}{2}\ln\left(1+\frac{\alpha}{2^{2N+2}}\right)\right] - \mathrm{Ei}\left[\frac{1}{4}\ln\left(1+\frac{\alpha}{2^{2N+2}}\right)\right] \right\},\end{aligned}$$

from which follows

$$U^{\mathrm{LB}}(\alpha) = \left[\mathrm{Ei}\left(\frac{\alpha}{2}\right) - \mathrm{Ei}\left(\frac{\alpha}{4}\right)\right] / \ln(4), \quad L^{\mathrm{LB}}(\alpha) = 1/2,$$

where calculations are analogous to those presented for the upper boundary. Thus we conclude

$$\frac{\mathrm{Ei}\left(\frac{\alpha}{2}\right) - \mathrm{Ei}\left(\frac{\alpha}{4}\right)}{\ln(4)} - \frac{1}{2} \leq \varTheta(\alpha) \leq \frac{\mathrm{Ei}(2\alpha) - \mathrm{Ei}(\alpha)}{\ln(4)} - \frac{1}{2}.$$

Convergence

Finite upper and lower boundaries do not imply the convergence, one needs to exclude oscillations between them for $N \to \infty$. To show that $\varTheta_N$ increases with increasing N we will focus on the individual terms $\omega_{N,n}$ in (17) and compare $\omega_{N,n}$ with $\omega_{N+1,n+1}$. The sum $\varTheta_{N+1}$ has one term more than $\varTheta_N$, we compare them as represented here

$$\begin{array}{lcccccc} \varTheta_{N+1}: & \omega_{N+1,0} & \omega_{N+1,1} & \omega_{N+1,2} & \dots & \omega_{N+1,N} & \omega_{N+1,N+1} \\ & & \updownarrow & \updownarrow & & \updownarrow & \updownarrow \\ \varTheta_N: & & \omega_{N,0} & \omega_{N,1} & \dots & \omega_{N,N-1} & \omega_{N,N} \end{array} \tag{18}$$

We have

$$\begin{aligned}\omega_{N+1,n+1} - \omega_{N,n} =& \left[(1+\frac{q}{4})^{8t} - (1+\frac{q}{4})^{4t}\right] - \left[(1+q)^{2t} - (1+q)^{t}\right] \\ =& (1+\frac{q}{4})^{4t}\left[(1+\frac{q}{4})^{4t} - 1\right] - (1+q)^{t}\left[(1+q)^{t} - 1\right],\end{aligned}$$

where $q = \alpha/2^{2N+2} \geq 0$ and $t = 2^{2n} \geq 1$. To prove that the expression is positive we need to show that $(1+\frac{q}{4})^{4t} \geq (1+q)^t$. Considering q and t as fixed, we introduce the parametric function $\chi_{q,t}(\alpha) = (1+q/\alpha)^{\alpha t}$ with $\alpha > 1$. The above inequality is then equivalent to $[\chi_{q,t}(\alpha)]_{\alpha=1}^{\alpha=4} \geq 0$. It is now sufficient to demonstrate that $\chi_{q,t}(\alpha)$ rises for $\alpha \geq 1$ for arbitrary $q \geq 0$ and $t \geq 1$. We compute the derivative

$$\frac{d}{d\alpha}\chi_{q,t}(\alpha) = t\left(1+\frac{q}{\alpha}\right)^{\alpha t}\left[\ln\left(1+\frac{q}{\alpha}\right) - \frac{\frac{q}{\alpha}}{1+\frac{q}{\alpha}}\right] = t\left(1+\frac{q}{\alpha}\right)^{\alpha t} R.$$

The only suspicious term is the one in the square brackets. Using the substitution $\beta = 1 + \frac{q}{\alpha}$ we get

$$R = \ln(\beta) - \frac{\beta - 1}{\beta} = \ln(\beta) - \left(1 - \frac{1}{\beta}\right).$$

The function $1 - 1/x$ is a known lower boundary of the logarithm function for $x > 0$, thus R is positive. Moreover, the sum Θ_{N+1} has one additional term, namely $\omega_{N+1,0}$, see (18). All this implies that Θ_N strictly grows with N. A growing and bounded sequence is necessarily convergent.

Alternative proof

At last we want to mention an alternative proof of the convergence presented to us on the online forum [4] which is, indeed, short and elegant. One first re-arranges the sequence in the inverse order and gets ($H_{n<0} = 0$, $H_{n\geq 0} = 1$)

$$\begin{aligned}\Theta(\alpha) &= \lim_{N\to\infty}\sum_{n=0}^{N}\omega_{N,n} = \lim_{N\to\infty}\sum_{m=0}^{N}\omega_{N,N-m} = \lim_{N\to\infty}\sum_{m=0}^{\infty} H_{N-m}\,\omega_{N,N-m} \\ &= \lim_{N\to\infty}\sum_{m=0}^{\infty} H_{N-m}\left[(1+\frac{\alpha}{2^{2N+2}})^{2^{(2N+2)-(2m+1)}} - (1+\frac{\alpha}{2^{2N+2}})^{2^{(2N+2)-(2m+2)}}\right].\end{aligned}$$

Next, after verifying the necessary conditions are met, one applies the Tannery's theorem and interchanges the summation and the limit. This gives

$$\begin{aligned}\Theta(\alpha) &= \sum_{m=0}^{\infty}\left[e^{\frac{\alpha}{2^{2m+1}}} - e^{\frac{\alpha}{2^{2m+2}}}\right] \overset{n=m+1}{=} = \sum_{n=1}^{\infty}\left[e^{\frac{\alpha}{2^{2n-1}}} - e^{\frac{\alpha}{2^{2n}}}\right] \\ &= \sum_{n=1}^{\infty}\left[(+1)\left(e^{\frac{\alpha}{2^{2n-1}}} - 1\right) + (-1)\left(e^{\frac{\alpha}{2^{2n}}} - 1\right)\right] = \sum_{n=1}^{\infty}(-1)^{n+1}\left(e^{\frac{\alpha}{2^{n}}} - 1\right).\end{aligned}$$

Now, expanding the exponential, one checks the required assumptions are obeyed to change the order of the summations

$$\Theta(\alpha) = \sum_{n=1}^{\infty}(-1)^{n+1}\left(\sum_{k=1}^{\infty}\frac{\alpha^k}{k!2^{nk}}\right) = -\sum_{k=1}^{\infty}\frac{\alpha^k}{k!}\sum_{n=1}^{\infty}(-1)^n\frac{1}{2^{nk}},$$

where $\sum_{n=1}^{\infty}\left[-2^{-k}\right]^n$ is a geometric series. This gives an efficient way for computing $\Theta(\alpha)$ and $\mathrm{yp}(x = e^{\alpha})$

$$\Theta(\alpha) = \sum_{k=1}^{\infty}\frac{\alpha^k}{k!\,(2^k+1)}, \quad \mathrm{yp}(e^{\alpha}) = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{\alpha^k}{k!\,(2^k+1)}. \tag{19}$$

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