Find the Functional Dependency Relationship Between Functions

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Abstract

An algorithmic method is proposed to solve functional dependency relationships between functions. To do so, a simple theorem is stated and three examples are provided, including the solution to demonstrate the effectiveness of the method.

I am a mechanical industrial engineer, but I am still interested in mathematics. Two months ago, I reread a book I had studied and found a problem in which the author established the functional dependency relationship between functions by simple empirical observation but not through mathematical analysis. I tried to submit this work to `https://arxiv.org/`, but I lack endorsements to review my work because I am not dedicated to research or teaching.

Theorem 1. *Method to Find the Functional Dependency Relationship in the Case of Dependent Functions $m = n$*

Remark. *Let the family $F = \{f_1, f_2, ..., f_m\}$ of functions mapping $\mathbb{R}^n$ to $\mathbb{R}$ be defined as:*

$$F = \left\{ \begin{array}{l} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{array} \right\}$$

For the case $n = m$, we consider an open set $A \subset \mathbb{R}^n$ and the family F of class-one functions A to $\mathbb{R}$. Then:

$$F = \left\{ \begin{array}{l} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{array} \right\}$$

It is known that the necessary and sufficient condition for the family F to be functionally dependent in A is that the determinant of its Jacobian matrix is identically zero within A. Since $m = n$, the matrix is square and the Jacobian determinant can be computed.

Then, if the functional relation for the family F to be functionally dependent is:

$$F = c_1 f_1^{a_1} + c_2 f_2^{a_2} + \cdots + c_n f_n^{a_n} = 0 \tag{1}$$

where the values $c_1 = c_2 = \cdots = c_n \neq 0$ are numerical coefficients and the values $a_1 = a_2 = \cdots = a_n \neq 0$ are numerical exponents of the applications. We can start obtaining the calculation of equation (1) from the following expression:

$$dF = b_1 df_1 + b_2 df_2 + \cdots + b_n df_n = 0 \tag{2}$$

where $b_1, b_2, \ldots, b_n$ are terms (numerical coefficients that can be multiplied, some of them, by any of the applications $f_1, f_2, \ldots, f_n$) after solving the following system of equations.

The system of equations derived from equation (2) can be rewritten as follows:

$$\left\{ \begin{array}{l} b_1 \frac{\partial f_1}{\partial x_1} + b_2 \frac{\partial f_1}{\partial x_2} + \cdots + b_n \frac{\partial f_1}{\partial x_n} = 0 \\ b_1 \frac{\partial f_2}{\partial x_1} + b_2 \frac{\partial f_2}{\partial x_2} + \cdots + b_n \frac{\partial f_2}{\partial x_n} = 0 \\ \vdots \\ b_1 \frac{\partial f_n}{\partial x_1} + b_2 \frac{\partial f_n}{\partial x_2} + \cdots + b_n \frac{\partial f_n}{\partial x_n} = 0 \end{array} \right.$$

Technically, equation (2) could also be rewritten as:

$$c_1 a_1 f_1^{a_1 - 1} df_1 + c_2 a_2 f_2^{a_2 - 1} df_2 + \cdots + c_n a_n f_n^{a_n - 1} df_n = 0 \tag{2a}$$

From the identification of terms, the following relations hold:

$$\begin{cases} b_1 = c_1 a_1 f_1^{a_1-1} \\ b_2 = c_2 a_2 f_2^{a_2-1} \\ \vdots \\ b_n = c_n a_n f_n^{a_n-1} \end{cases}$$

However, the system of equations derived from Equation (2) is already sufficient to solve the functional dependency relationship among the applications, making Equation (2a) unnecessary.

The reason why the exponential coefficients $a_1 = a_2 = \cdots = a_n$ *are ignored is straightforward: these coefficients are absorbed into the terms* $b_1, b_2, \ldots, b_n$*, during differentiation.*

Demonstration

Formally, we need to perform the following operation, taking the product of the row vector of coefficients by the column vector of the applications to arrive at the functional dependency relationship:

$$F(x_1, x_2, \ldots, x_n) = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix} \begin{bmatrix} f_1^{a_1} \\ f_2^{a_2} \\ \vdots \\ f_n^{a_n} \end{bmatrix} = 0$$

$$F = c_1 f_1^{a_1} + c_2 f_2^{a_2} + \cdots + c_n f_n^{a_n} = 0 \tag{1}$$

where we have easily arrived at equation (1).

Since F is of class one, it admits the first total derivative. Differentiation functions as an inverse process to integration. Therefore, knowing Equation (2),

$$dF = b_1 df_1 + b_2 df_2 + \cdots + b_n df_n = 0 \tag{2}$$

we can obtain equation (1) by integration, although equation (2) must first be constructed by obtaining the terms $b_1, b_2, \ldots, b_n$ instead of determining them directly using the coefficients $c_1, c_2, \ldots, c_n$ from equation (1).

It is known by hypothesis that the family F is functionally dependent if it consists of $m \times n$ family of functions where $m = n$, provided that its Jacobian determinant is identically zero.

The Jacobian matrix of the family F is:

$$J(x_1, x_2, \ldots, x_n) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

And its Jacobian determinant is identically null, a necessary and sufficient condition for the family of applications to be functionally dependent. This has been previously proven in another theorem, so:

$$|J(x_1, x_2, \ldots, x_n)| = \det J = 0$$

If $|J(x_1, x_2, \ldots, x_n)| = \Delta = 0 \iff F$ is a family of functionally dependent functions.

Because the family F is of class one, it admits the first derivative. Therefore, the total derivative of the family can be obtained:

$$dF = \begin{cases} df_1(x_1, x_2, \ldots, x_n) \\ df_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ df_n(x_1, x_2, \ldots, x_n) \end{cases}$$

We express the total derivatives of the functions in terms of their partial derivatives:

$$\begin{cases} df_1 = \frac{\partial f_1}{\partial x_1}dx_1 + \frac{\partial f_1}{\partial x_2}dx_2 + \cdots + \frac{\partial f_1}{\partial x_n}dx_n \\ df_2 = \frac{\partial f_2}{\partial x_1}dx_1 + \frac{\partial f_2}{\partial x_2}dx_2 + \cdots + \frac{\partial f_2}{\partial x_n}dx_n \\ \vdots \\ df_n = \frac{\partial f_n}{\partial x_1}dx_1 + \frac{\partial f_n}{\partial x_2}dx_2 + \cdots + \frac{\partial f_n}{\partial x_n}dx_n \end{cases} \tag{3}$$

Including the partial derivatives from equation (3) to equation (2):

$$\begin{cases} b_1\left(\frac{\partial f_1}{\partial x_1}dx_1 + \frac{\partial f_1}{\partial x_2}dx_2 + \cdots + \frac{\partial f_1}{\partial x_n}dx_n\right) + \\ b_2\left(\frac{\partial f_2}{\partial x_1}dx_1 + \frac{\partial f_2}{\partial x_2}dx_2 + \cdots + \frac{\partial f_2}{\partial x_n}dx_n\right) + \\ \vdots + \\ b_n\left(\frac{\partial f_n}{\partial x_1}dx_1 + \frac{\partial f_n}{\partial x_2}dx_2 + \cdots + \frac{\partial f_n}{\partial x_n}dx_n\right) = 0 \end{cases} \tag{4}$$

Regrouping equation (4):

$$\begin{cases} \left(b_1\frac{\partial f_1}{\partial x_1} + b_2\frac{\partial f_2}{\partial x_1} + \cdots + b_n\frac{\partial f_n}{\partial x_1}\right)dx_1 + \\ \left(b_1\frac{\partial f_1}{\partial x_2} + b_2\frac{\partial f_2}{\partial x_2} + \cdots + b_n\frac{\partial f_n}{\partial x_2}\right)dx_2 + \\ \vdots + \\ \left(b_1\frac{\partial f_1}{\partial x_n} + b_2\frac{\partial f_2}{\partial x_n} + \cdots + b_n\frac{\partial f_n}{\partial x_n}\right)dx_n = 0 \end{cases} \tag{5}$$

For equation (5) to represent a functionally dependent family, the following conditions must hold simultaneously:

1. $dx_1 = dx_2 = \cdots = dx_n \neq 0$
2. $$\begin{cases} b_1\frac{\partial f_1}{\partial x_1} + b_2\frac{\partial f_2}{\partial x_1} + \cdots + b_n\frac{\partial f_n}{\partial x_1} = 0 \\ b_1\frac{\partial f_1}{\partial x_2} + b_2\frac{\partial f_2}{\partial x_2} + \cdots + b_n\frac{\partial f_n}{\partial x_2} = 0 \\ \vdots \\ b_1\frac{\partial f_1}{\partial x_n} + b_2\frac{\partial f_2}{\partial x_n} + \cdots + b_n\frac{\partial f_n}{\partial x_n} = 0 \end{cases} \tag{6}$$

Equation (6) represents a system of n equations with n unknowns. Its expanded matrix, including the null coefficients, is given by:

$$\begin{bmatrix} b_1\frac{\partial f_1}{\partial x_1} & b_2\frac{\partial f_2}{\partial x_1} & \cdots & b_n\frac{\partial f_n}{\partial x_1} \\ b_1\frac{\partial f_1}{\partial x_2} & b_2\frac{\partial f_2}{\partial x_2} & \cdots & b_n\frac{\partial f_n}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ b_1\frac{\partial f_1}{\partial x_n} & b_2\frac{\partial f_2}{\partial x_n} & \cdots & b_n\frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0 \tag{7}$$

The system could be solved by obtaining an upper triangular matrix using the Gauss method and applying backward substitution to find the terms $b_1, b_2, \ldots, b_n$, which contain numerical coefficients or numerical coefficients that multiply some of the applications $f_1, f_2, \ldots, f_n$. It can also be solved directly by manipulating the equations of the system to find the terms.

When finding the terms $b_1, b_2, \ldots, b_n$, we must identify the equalities that arise from the variables $x_1, x_2, \ldots, x_n$ and their relationships with the applications $f_1, f_2, \ldots, f_n$, ensuring that the terms $b_1, b_2, \ldots, b_n$ contain only numerical coefficients multiplying some of the applications instead of the variables.

Finding the terms $b_1, b_2, \ldots, b_n$ for equation (2):

$$dF = b_1 df_1 + b_2 df_2 + \cdots + b_n df_n = 0 \tag{7}$$

The integration process of F is carried out for each of the summands in its variable, leading to equation (1):

$$F = c_1 f_1^{a_1} + c_2 f_2^{a_2} + \cdots + c_n f_n^{a_n} = 0 \tag{8}$$

Thus, the problem of finding the linear dependency relationship between $n \times n$ applications has been solved.

Q.E.D.

Solved Practical Examples of the Previous Theorem

Exercise 1

Let the family $\{f_1, f_2, f_3\}$ of functions from $\mathbb{R}^3$ to $\mathbb{R}$ be defined by:

$$\begin{aligned} f_1(u,v,w) &= x = u^2 + v^2 + w^2, \\ f_2(u,v,w) &= y = u + v + w, \\ f_3(u,v,w) &= z = uv + vw + wu. \end{aligned}$$

1) Show that the family is functionally dependent in all $\mathbb{R}^3$.

2) Find an appropriate method to detail the functional relationship. You will have to find a functional relationship expression of this type:

$$F(x,y,z) = mx^i + ny^j + pz^k = 0$$

Note that if the previous expression took the value $F(x,y,z) \neq 0$, the family would be functionally independent.

SOLUTION.

1) Trying the Existence of Functional Dependence

Constructing the Jacobian matrix.

Let f be the matrix associated with the family f formed by the given functions:

$$f(u,v,w) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} u^2 + v^2 + w^2 \\ u + v + w \\ uv + vw + wu \end{bmatrix}$$

We can obtain the determinant of its Jacobian matrix. If it is null, there is functional dependence:

$$\det J(u,v,w) = \begin{vmatrix} 2u & 2v & 2w \\ 1 & 1 & 1 \\ v+w & u+w & u+v \end{vmatrix} = \Delta$$

Computing the determinant:

$$\begin{aligned} \Delta &= 2u(u+v) + 2v(v+w) + 2w(u+w) \\ &\quad - 2w(v+w) - 2u(u+w) - 2v(u+v) \\ &= 2u^2 + 2uv + 2v^2 + 2vw + 2wu + 2w^2 \\ &\quad - 2vw - 2w^2 - 2u^2 - 2wu - 2uv - 2v^2 \\ &= 0. \end{aligned}$$

Since the determinant is null, there is functional dependence, implying that the family is functionally dependent throughout $(u,v,w) \in \mathbb{R}^3$.

2) Usual Resolution Method. Trying it is impossible to solve it.

We have assigned each function to a variable such that:

$$f_1 = x,$$
$$f_2 = y,$$
$$f_3 = z.$$

Starting with the equation:

$$F(x, y, z) = mxi + nyj + pzk$$

Developing the equation:

$$F(x, y, z) = m(u^2 + v^2 + w^2)^i + n(u + v + w)^j + p(uv + vw + wu)^k = 0$$

Expanding this equation does not yield a useful result. The equation in this form is analytically unsolvable.

2) Theorem Resolution Method.

To determine the functional dependency relationship between the different functions, we apply Theorem 1.

That is, we must find the equation:

$$F(x, y, z) = mx^i + ny^j + pz^k = 0 \tag{8}$$

which expresses the functional dependency between the functions in the F family using unknown terms.

We need to determine the coefficients m, n, p.

To illustrate the procedure, we introduce the differential equation:

$$adx + bdy + cdz = 0 \tag{9}$$

which is derived from Equation (26) and contains the terms a, b, c.

Note that in this resolution, an alternative nomenclature is used for the variables, functions, and terms to better align with practical problem analysis. The following equivalences are established:

$$\begin{array}{rcl} (f_1, f_2, f_3) & = & (x, y, z) \\ (c_1, c_2, c_3) & = & (m, n, p) \\ (b_1, b_2, b_3) & = & (a, b, c) \end{array}$$

The procedure involves finding the previous terms to integrate the differential equation (9), obtaining the equation of functional dependence among the given functions, arriving at equation (8).

The differential equation for each of the functions has been taken and multiplied by each of the terms.

The variables x, y, z are defined according to the functions as follows:

$$
\begin{aligned}
x &= f_1(u, v, w) = u^2 + v^2 + w^2, \\
y &= f_2(u, v, w) = u + v + w, \\
z &= f_3(u, v, w) = uv + vw + wu.
\end{aligned}
$$

The Jacobian matrix is given by:

$$
J(u, v, w) = \begin{pmatrix} 2u & 2v & 2w \\ 1 & 1 & 1 \\ v+w & u+w & u+v \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} \tag{9}
$$

We develop the expression:

$$
adx + bdy + cdz = 0 \tag{10}
$$

Knowing that the total differentials dx, dy, dz are:

$$
\begin{aligned}
dx &= \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv + \frac{\partial x}{\partial w} dw, \\
dy &= \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv + \frac{\partial y}{\partial w} dw, \\
dz &= \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv + \frac{\partial z}{\partial w} dw.
\end{aligned}
$$

Thus, we derive:

$$
a\left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv + \frac{\partial x}{\partial w} dw\right) + b\left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv + \frac{\partial y}{\partial w} dw\right) + c\left(\frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv + \frac{\partial z}{\partial w} dw\right) = 0
$$

Rearranging terms:

$$
\left(a\frac{\partial x}{\partial u} + b\frac{\partial y}{\partial u} + c\frac{\partial z}{\partial u}\right) du + \left(a\frac{\partial x}{\partial v} + b\frac{\partial y}{\partial v} + c\frac{\partial z}{\partial v}\right) dv + \left(a\frac{\partial x}{\partial w} + b\frac{\partial y}{\partial w} + c\frac{\partial z}{\partial w}\right) dw = 0
$$

Substituting the values of the Jacobian matrix into the equation:

$$
(2au + b + c(v + w))\, du + (2av + b + c(u + w))\, dv + (2aw + b + c(u + v))\, dw = 0
$$

Considering that $du \neq 0$, $dv \neq 0$, $dw \neq 0$, the values inside the brackets must necessarily be zero.

We examine the values in brackets and obtain a system of three equations to solve for the coefficients a, b, and c:

$$2au + b + c(v + w) = 0 \tag{10}$$
$$2av + b + c(u + w) = 0 \tag{11}$$
$$2aw + b + c(u + v) = 0 \tag{12}$$

We perform the following operations:

Equation (10) - Equation (11):

$$2a(u - v) + c(v - u) = 0 \quad \Rightarrow \quad (2a - c)(u - v) = 0 \quad \Rightarrow \quad 2a = c \tag{13}$$

Then,

$$a = \frac{c}{2} \tag{14}$$

Substituting Equation (14) into Equation (10):

$$2\left(\frac{c}{2}\right)v + b + c(u + w) = 0 \quad \Rightarrow \quad b = -c(u + v + w) \quad \Rightarrow \quad b = -cy = -cf_2 \tag{15}$$

Thus,

$$c = 2 \tag{16}$$

To simplify the solution, we take the lowest positive integer values:

$$a = 1, \quad c = 2, \quad b = -2y \tag{17}$$

which provides the solution for the terms in the differential equation (2). Note that both a and c are numerical coefficients, while b is a term.

Substituting these values into Equation (2):

$$adx + bdy + cdz = 0 \quad \Rightarrow \quad dx - 2ydy + 2dz = 0$$

Integrating the previous differential equation, we obtain:

$$x - y^2 + 2z = K = F(x, y, z)$$

Since the integration constant K must be zero from functional dependence to hold, we conclude:

$$x - y^2 + 2z = 0 \quad \Leftrightarrow \quad f_1 - (f_2)^2 + 2f_3 = 0 \tag{18}$$

Verification Equation 18.

Substituting the given functions:

$$(u^2 + v^2 + w^2) - (u + v + w)^2 + 2(uv + vw + wu) = 0$$

Expanding:

$$(u + v + w)^2 = u^2 + v^2 + w^2 + 2uv + 2vw + 2wu$$

Developing the expression:

$$\begin{gathered} u^2 + v^2 + w^2 - (u^2 + v^2 + w^2 + 2uv + uvw + 2wu) + 2(uv + vw + wu) = \\ u^2 + v^2 + w^2 - u^2 - v^2 - w^2 - 2uv - 2vw - 2wu + 2uv + 2vw + 2wu = 0 \end{gathered} \tag{11}$$

Thus, the solution is correct, which confirms that the theorem is valid for determining functional dependency relationships.

Exercise 2

Let the family $\{f_1, f_2, f_3\}$ of applications from $\mathbb{R}^3$ to $\mathbb{R}$ be defined by:

$$\begin{aligned} f_1(u,v,w) &= x = u^2+v^2+w^2, \\ f_2(u,v,w) &= y = u+v+w, \\ f_3(u,v,w) &= z = u^2+v^2+w^2+6uv+6vw+6uw. \end{aligned}$$

1. Show that the family is functionally dependent at all $\mathbb{R}^3$.
2. Find an appropriate method to detail the functional relationship. You will have to find an expression for a functional relationship of the type:

$$F(x,y,z) = mx^i + ny^j + pz^k = 0.$$

Note that if the previous expression were $F(x,y,z) \neq 0$, the family would be functionally independent.

SOLUTION.

1) Check Functional Dependence

Let f be the matrix associated with the given applications:

$$f(u,v,w) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} u^2+v^2+w^2 \\ u+v+w \\ u^2+v^2+w^2+6uv+6vw+6uw \end{pmatrix}$$

We compute the determinant of its Jacobian matrix. If the determinant is null, functional dependency exists:

$$J(u,v,w) = \begin{vmatrix} 2u & 2v & 2w \\ 1 & 1 & 1 \\ 2u+6(v+w) & 2v+6(u+w) & 2w+6(u+v) \end{vmatrix}$$

Computing the determinant:

$$\begin{aligned} \Delta &= 2u[2w+6(v+u)] + 2v[2u+6(v+w)] + 2w[2v+6(u+w)] \\ &\quad - 2w[2u+6(v+w)] - 2v[2w+6(v+u)] - 2u[2v+6(u+w)] \\ &= 2u(4v-4w) + 2v(4w-4u) + 2w(4u-4v) \\ &= 8uv - 8uw + 8vw - 8uv + 8uw - 8vw = 0. \end{aligned}$$

Since the determinant is null, there is functional dependence, and the family is functionally dependent for all $(u,v,w) \in \mathbb{R}^3$.

2) Functional Dependency Relationship

The theorem that develops a systematic method to find the dependency relationship between different functions will be used again.

We must find the relationship:

$$F(x, y, z) = mx^i + ny^j + pz^k = 0 \tag{18}$$

that describes the functional dependency among the functions in the F family, using unknown coefficients.

We just need to determine the coefficients m, n, p.

We will explain the procedure again in a practical way for this new exercise.

Consider the expression:

$$adx + bdy + cdz = 0 \tag{19}$$

This is the differential equation derived from Equation (18), with the terms a, b, and c.

The following equivalences are established again:

$$\begin{aligned}(f_1, f_2, f_3) &= (x, y, z),\\ (c_1, c_2, c_3) &= (m, n, p),\\ (b_1, b_2, b_3) &= (a, b, c)\end{aligned}$$

The procedure involves identifying these terms to integrate differential Equation (19), obtaining the equation of functional dependence among the functions, leading to Equation (18).

The differential equation for each function has been taken and multiplied by the corresponding terms.

The variables x, y, z are defined according to the functions as follows:

$$\begin{aligned}x &= f_1(u, v, w) = u^2 + v^2 + w^2,\\ y &= f_2(u, v, w) = u + v + w,\\ z &= f_3(u, v, w) = u^2 + v2 + w2 + 6uv + 6vw + 6uw\end{aligned}$$

The Jacobian matrix $J(u, v, w)$ is given by:

$$J(u, v, w) = \begin{pmatrix} 2u & 2v & 2w \\ 1 & 1 & 1 \\ 2u + 6(v + w) & 2v + 6(u + w) & 2w + 6(v + u) \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}.$$

We develop the expression:

$$adx + bdy + cdz = 0$$

Knowing that the total differentials dx, dy, dz are:

$$\begin{aligned} dx &= \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw, \\ dy &= \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv + \frac{\partial y}{\partial w}dw, \\ dz &= \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv + \frac{\partial z}{\partial w}dw. \end{aligned}$$

Thus, we obtain the following result:

$$a\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw\right) + b\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv + \frac{\partial y}{\partial w}dw\right) + c\left(\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv + \frac{\partial z}{\partial w}dw\right) = 0$$

Rearranging terms:

$$\left(a\frac{\partial x}{\partial u} + b\frac{\partial y}{\partial u} + c\frac{\partial z}{\partial u}\right)du + \left(a\frac{\partial x}{\partial v} + b\frac{\partial y}{\partial v} + c\frac{\partial z}{\partial v}\right)dv + \left(a\frac{\partial x}{\partial w} + b\frac{\partial y}{\partial w} + c\frac{\partial z}{\partial w}\right)dw = 0$$

Substituting the values from the Jacobian matrix:

$$(2au + b + c(2u + 6v + 6w))\,du + (2av + b + c(2v + 6u + 6w))\,dv + (2aw + b + c(2w + 6u + 6v))\,dw = 0.$$

Since $du \neq 0, dv \neq 0, dw \neq 0$, the expressions inside the brackets must necessarily be zero, leading to the system of equations:

$$2au + b + c(2u + 6v + 6w) = 0 \tag{20}$$
$$2av + b + c(2v + 6u + 6w) = 0 \tag{21}$$
$$2aw + b + c(2w + 6u + 6v) = 0. \tag{22}$$

Performing the subtraction of Equation (20) from Equation (21):

$$\begin{aligned} 2a(u - v) + c\left[2u + 6v + 6w - (2v + 6u + 6w)\right] &= 0 \\ 2a(u - v) + c(4v - 4u) &= 0 \\ 2a(u - v) - 4c(u - v) &= 0 \end{aligned}$$

We do not take u, v into account, then:

$$(2a - 4c)(u - v) = 0 \Rightarrow \begin{cases} a = 2c, \\ u = v. \end{cases} \quad \text{(at least one must hold, or both simultaneously).}$$

We substitute the previously obtained value of a into Equation (20):

$$4cv + b + c(8v + 6w) = 0 \Rightarrow b + 12cv + 6cw = 0 \Rightarrow b = -6c(w + 2v).$$

Since we know that:

$$y = u + v + w,$$

We can express:

$$w + 2v = y - u - v + 2v = y - u + v.$$

From the previous calculation, we obtained:

$$u = v.$$

Thus, we simplify:

$$w + 2v = y.$$

Therefore:

$$b = -6cy.$$

Setting $c = 1$, we obtain:

$$a = 2, \quad b = -6y.$$

Substituting these values into the differential equation (19):

$$2dx - 6ydy + dz = 0.$$

Integrating the above equation:

$$2x - 3y^2 + z = K.$$

Since $K = 0$ due to the linear dependency relationship, we conclude:

$$2x - 3y^2 + z = 0 \iff 2f_1 - 3f_2^2 + f_3 = 0.$$

Unlike the previous example, this relationship has not been explicitly verified.

Exercise 3

Let the family $\{f_1, f_2, f_3\}$ of applications from $\mathbb{R}^3$ to $\mathbb{R}$ be defined by:

$$\begin{aligned} f_1(u,v,w) &= x = u, \\ f_2(u,v,w) &= y = v + w, \\ f_3(u,v,w) &= z = -u^3 + 2v^2 + 2w^2 + 4vw. \end{aligned}$$

1. Show that the family is functionally dependent at all $\mathbb{R}^3$.
2. Find an appropriate method to detail the functional relationship. You will have to find an expression for a functional relationship of the type:

$$F(x,y,z) = mx^i + ny^j + pz^k = 0$$

Note that if the previous expression were $F(x,y,z) \neq 0$, the family would be functionally independent.

SOLUTION.

1) Check Functional Dependence

Consider the matrix associated with the family f formed by the given applications:

$$f(u,v,w) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} u \\ v+w \\ -u^3 + 2v^2 + 2w^2 + 4vw \end{pmatrix}$$

We compute the determinant of its Jacobian matrix. If it is zero, there is a functional dependence:

$$J(u,v,w) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -3u^2 & 4v+4w & 4v+4w \end{vmatrix}$$

Computing the determinant:

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -3u^2 & 4v+4w & 4v+4w \end{vmatrix} \\ &= 4v + 4w + 0 + 0 - 0 - 4v - 4w = 0 \end{aligned}$$

Since the determinant is zero, there is functional dependence and the family is functionally dependent to all $(u,v,w) \in \mathbb{R}^3$.

2) Functional Dependency Relationship

The theorem that develops a systematic method to determine the dependency relationship between different functions will be used again.

We must find the relationship:

$$F(x, y, z) = mx^i + ny^j + pz^k = 0 \tag{23}$$

which expresses the functional dependency between the functions in the F family, using unknown coefficients.

We only need to determine the terms m, n, p.

We will now explain the procedure in a practical way for this new exercise.

Let the expression be:

$$adx + bdy + cdz = 0 \tag{24}$$

This is the differential equation derived from Equation (23) with terms a, b, and c.

The following equivalences are established:

$$(f_1, f_2, f_3) = (x, y, z)$$
$$(c_1, c_2, c_3) = (m, n, p)$$
$$(b_1, b_2, b_3) = (a, b, c)$$

The procedure involves identifying these terms to integrate differential Equation (24), obtaining the equation of functional dependence among the functions, leading to Equation (23).

The differential equation for each function has been taken and multiplied by the corresponding terms.

The variables x, y, z are defined according to the functions as follows:

$$x = f_1(u, v, w) = u,$$
$$y = f_2(u, v, w) = v + w,$$
$$z = f_3(u, v, w) = -u^3 + 2v^2 + 2w^2 + 4vw$$

The Jacobian matrix $J(u, v, w)$ is given by:

$$J(u, v, w) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -3u^2 & 4v + 4w & 4v + 4w \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}.$$

We develop the expression:

$$adx + bdy + cdz = 0$$

Knowing that the total differentials dx, dy, dz are:

$$
\begin{aligned}
dx &= \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw \\
dy &= \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv + \frac{\partial y}{\partial w}dw \\
dz &= \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv + \frac{\partial z}{\partial w}dw
\end{aligned}
$$

So, we have:

$$a\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw\right) + b\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv + \frac{\partial y}{\partial w}dw\right) + c\left(\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv + \frac{\partial z}{\partial w}dw\right) = 0.$$

Rearranging terms:

$$\left(a\frac{\partial x}{\partial u} + b\frac{\partial y}{\partial u} + c\frac{\partial z}{\partial u}\right)du + \left(a\frac{\partial x}{\partial v} + b\frac{\partial y}{\partial v} + c\frac{\partial z}{\partial v}\right)dv + \left(a\frac{\partial x}{\partial w} + b\frac{\partial y}{\partial w} + c\frac{\partial z}{\partial w}\right)dw = 0.$$

Substituting the values of the Jacobian matrix:

$$\left[a + 0 + c(-3u^2)\right]du + \left[0 + b + c(4v + 4w)\right]dv + \left[0 + b + c(4v + 4w)\right]dw = 0.$$

Since $du \neq 0, dv \neq 0, dz \neq 0$, the values in brackets must necessarily be zero.

Examining the expressions inside the brackets, we obtain a system of three equations to solve for the coefficients a, b, and c:

$$a + c(-3u^2) = 0 \tag{25}$$

$$b + c(4v + 4w) = 0 \tag{26}$$

$$b + c(4v + 4w) = 0 \tag{27}$$

From Equation 25:

$$a = 3cu^2 \Rightarrow a = 3cx^2$$

From Equation 26 or 27 (are the same):

$$b = -4c(v + w) \Rightarrow b = -4cy$$

Taking the value $c = 1$, we obtain:

$$a = 3x^2,$$

$$b = -4y.$$

Substituting these terms into differential equation 24:

$$3x^2 dx - 4y dy + dz = 0.$$

Integrating the above equation, we get:

$$x^3 - 2y^2 + z = K = F.$$

Since $K = 0$ due to the linear dependency relationship, we conclude:

$$x^3 - 2y^2 + z = 0 \iff f_1^3 - 2f_2^2 + f_3 = 0.$$

Unlike the first exercise, this relationship has not been explicitly verified.

References

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