Title: Oscillation of Rocking Solid Semi Cylinder

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Abstract

This study investigates the oscillatory motion of a rocking solid semi-cylinder to determine whether it exhibits simple harmonic motion (SHM) and to characterize its damping behavior. Using a theoretical framework derived from classical mechanics, the experiment aims to verify the independence of the oscillation period from amplitude, compare experimental and theoretical natural frequencies, analyze the sinusoidal nature of angular displacement over time, and quantify the system's damping coefficient. The experimental setup involved tracking the motion of a semi-cylinder displaced by small angles using video analysis and computational tools. The results confirm that the oscillation closely follows SHM principles, with a nearly constant period and frequency. The damping coefficient, relaxation time, and quality factor (Q-value) were also determined, indicating that the system exhibits underdamped oscillatory motion. These findings contribute to a deeper understanding of the dynamics of rocking semi-cylinders and their practical applications in mechanical and engineering systems.

1 Introduction

A roly-poly toy, also known as a tumbler, is a round-bottomed toy hat tends to right itself when pushed at an angle. In fact, this behaviour can be observed from any type of object that shares the shape of a hemisphere, moreover, one can also observed a similar motion pattern in a solid semi-cylinder as the reference 2-dimensional plane to the motion of a hemisphere. When a roly-poly is placed upon a horizontal support surface and displaced from its equilibrium state, the solid hemisphere or semi-cylinder will manifest oscillatory excursions due to the effect of gravity, which acts downward along the vertical direction. Therefore, our intuition immediately associates the oscillatory motion to a damped simple harmonic motion.

This experiment studies the rocking oscillation motion of a solid semi-cylinder. The main objective of this experiment is to prove that this oscillation exhibits simple harmonic motion and describe the damping of the oscillation. It is hypothesized that the rocking oscillation of a solid semi-cylinder is an damped simple harmonic motion. The figure below shows the coordinate system and dimensions of a semi-cylinder from xy-plane in its equilibrium state.

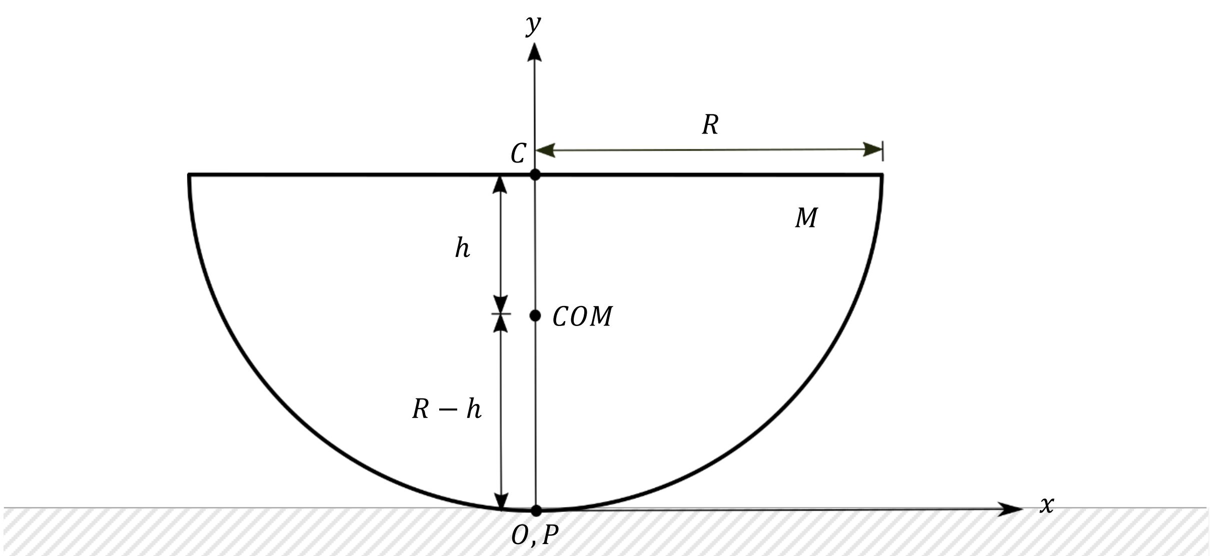


Figure 1. Coordinate System and Dimensions of Semi-Cylinder

The acronyms on Fig. 1 represent its description. COM represents Centre of Mass, C represents the Centre of the Circle, O represents the Origin of the coordinate system, P represents Point of contact, R represents the Radius of the semi-cylinder, M represents the Mass of the semi-cylinder, whereas h represents the distance between C and COM.

Now, a force is acted upon the semi-cylinder that it displaces an angle of θ from its axis. This causes its centre of mass and the point of contact to the surface to shift in position. Now, there are additional forces acting on the semi-cylinder in order to return to its equilibrium state.

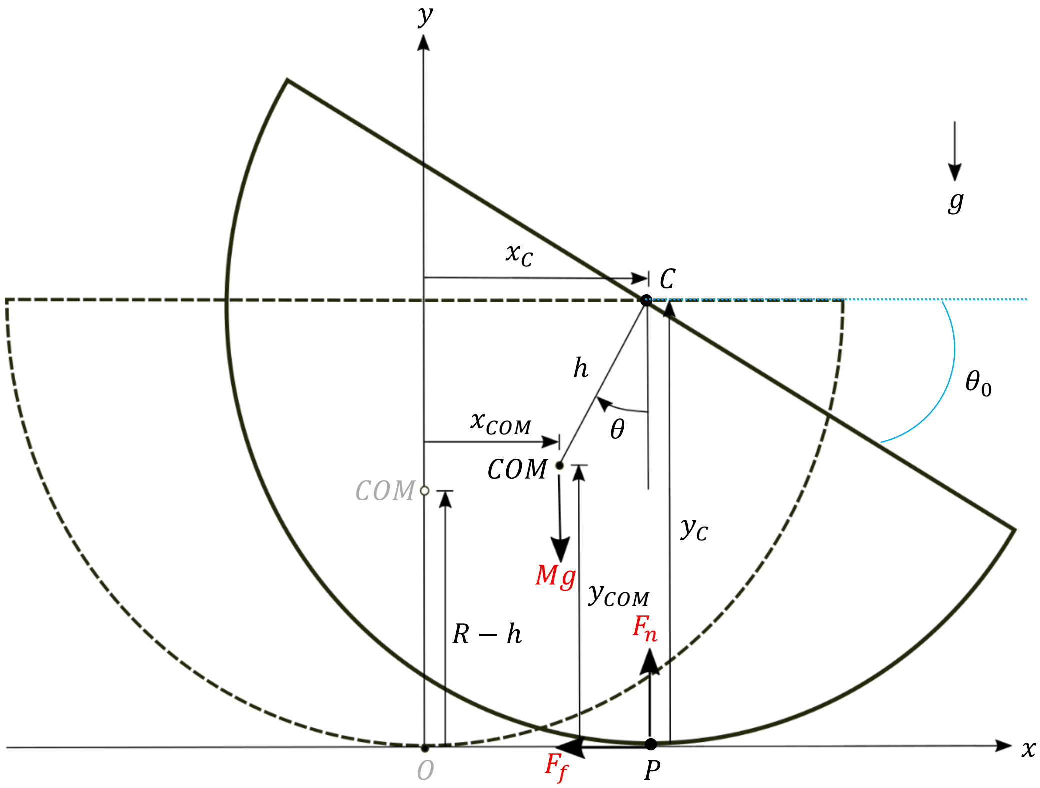


Figure 2. Free-Body Diagram of Semi-Cylinder

The angular displacement θ is measured negative as following the using of counter-clockwise direction as the positive direction due to the effect of torque. The initial angular displacement or amplitude, θ_0 , is measured at the instance of the external force is applied.

From Classical Mechanics, the formula for the distance between C and COM, h, and the Moment of Inertia at COM, I_{COM}, of a solid semi-cylinder can be derived to be:

$$h = \frac{4}{3\pi}R \qquad \text{(1)}$$

$$I_{COM} = \frac{1}{2}MR^2 - Mh^2 \qquad \text{(2)}$$

If the motion is deduced to be simple harmonic, the linearized differential equation of motion has to follow:

$$\ddot{\theta} + \omega_0^2\theta = 0 \qquad \text{(3)}$$

Therefore, using Newton's Second Law, the derivation of the formula for the theoretical undamped natural frequency ω_0 of oscillation for the rocking semi-cylinder is obtained to be:

$$\omega_0 = \sqrt{\frac{8g}{R(9\pi - 16)}} \qquad (4)$$

From this formula, the theoretical period, T, can be evaluated as well, since the period will be constant and independent of amplitude for an undamped system. Therefore, this experiment consists of 4 elements to verify that the oscillation is indeed simple harmonic, which are:

1. Verifying if, as theory predicts, the period of an undamped harmonic oscillator, $\boldsymbol{T}$ is amplitude independent by investigating the period over a time interval of 5 oscillations.
2. Comparing the experimental natural oscillating frequency, $\boldsymbol{\omega_o}$ with the theoretical value.
3. Plotting the angular displacement against time graph of the oscillation and verifying the sinusoidal variation of the angular displacement.
4. Investigate the damping coefficient, $\boldsymbol{\mu}$, of the system by measuring the maximum amplitude of the angular displacement after 5 consecutive periods and from that compare the results.

2 Experiment Setup

Material

No.	Item Description	Quantity	No.	Item Description	Quantity
1.	Solid Semi-Cylinder Block	1	5.	Meter Rule	1
2.	Protractor	1	6.	Marker Pen	1
3.	A4 Paper	1	7.	Video Recorder	1
4.	Tape	15cm	8.	OSP Tracker Software	1

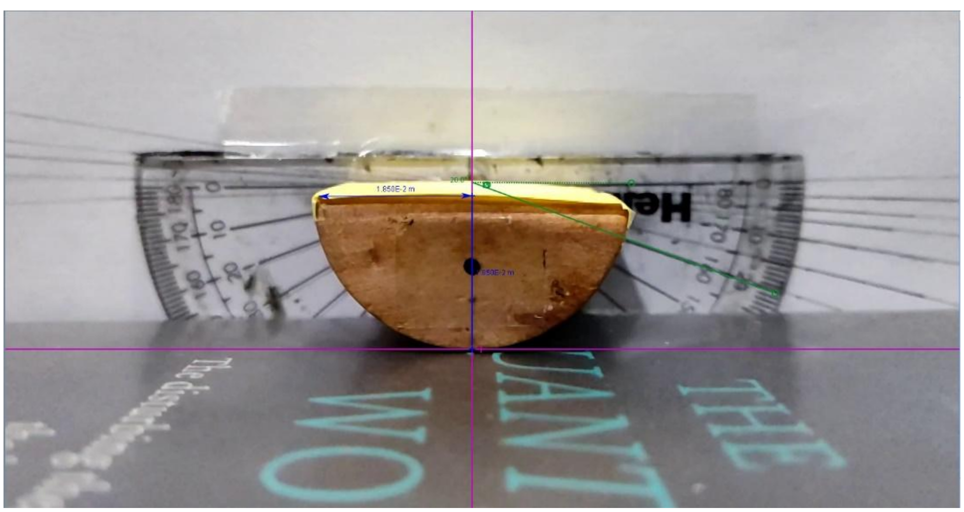


Figure 3. Experiment Setup and Calibration using Tracker Tool

Initial Setup

1. Refer to Fig. 3 above, to ease the process of calibration using Tracker software later, measure the radius of the semi-cylinder block using a ruler beforehand. Stick a piece of tape onto the block and plot a black dot on the centre of mass of the block using a marker pen. Tape a protractor onto a piece of A4 paper and mark several lines of angle such as 10°,15°and 20°. Place the A4 paper as the background.
2. Place the semi-cylinder block on a flat surface. Make sure the centre of the block is positioned with the origin axis of the protractor. Tape a piece of paper on the horizontal side of the block, but this step is optionally as it only eases the process to locate the angles.

Part A: Measuring the period of oscillation using Tracker Software

1. Exert a force with your finger on the right side of the block till it tilts to angle 10°. Next, start the recording button on the video recorder. Release your finger and let the block to oscillate until it returns to its equilibrium state. Stop the recording.
2. Upload the recordings to Tracker Software. Refer to Fig. 3 above, calibrate the origin O of the block by using the Coordinate Axes Tool. Next, calibrate the dimension of the radius by using the Calibration Stick Tool. Using the Calibration Tape Tool, make sure that the horizontal radius has the same value as the vertical radius. Mark the initial angular displacement angle using the Protractor Tool.
3. Determine the frame number which the finger is released as the start frame and the frame number which the block has returned to its equilibrium state as the end frame. Using the Point Mass Tool, create a point mass by selecting the black dot area. Next, select 'Search' and the displacement against time data will be recorded.
4. From the graph and data chart, determine the time taken for 5 complete oscillations.
5. Repeat Step 1 to Step 4 two more times by using different initial displaced angle, 15°and 20°. Record the findings onto a data table.

Part B: Plotting the sinusoidal graph for the oscillation of solid semi-cylinder

1. From the Tracker Software, export the data collected, angular displacement θ and time elapsed t, to an Excel Spreadsheet. Using Microsoft Excel, plot the sinusoidal graph of angular displacement against time and label the graph.
2. After verifying that the graph is sinusoidal and the motion is simple harmonic, create a MATLAB Programme to produce a more accurate angular displacement over time graph. Compare the experimental graph from Excel with the theoretical graph from MATLAB.
3. Determine the equation of motion for the damped simple harmonic motion.

3 Data Collection to Verify the System is SHO

Part A: Measuring the period of oscillation using Tracker Software

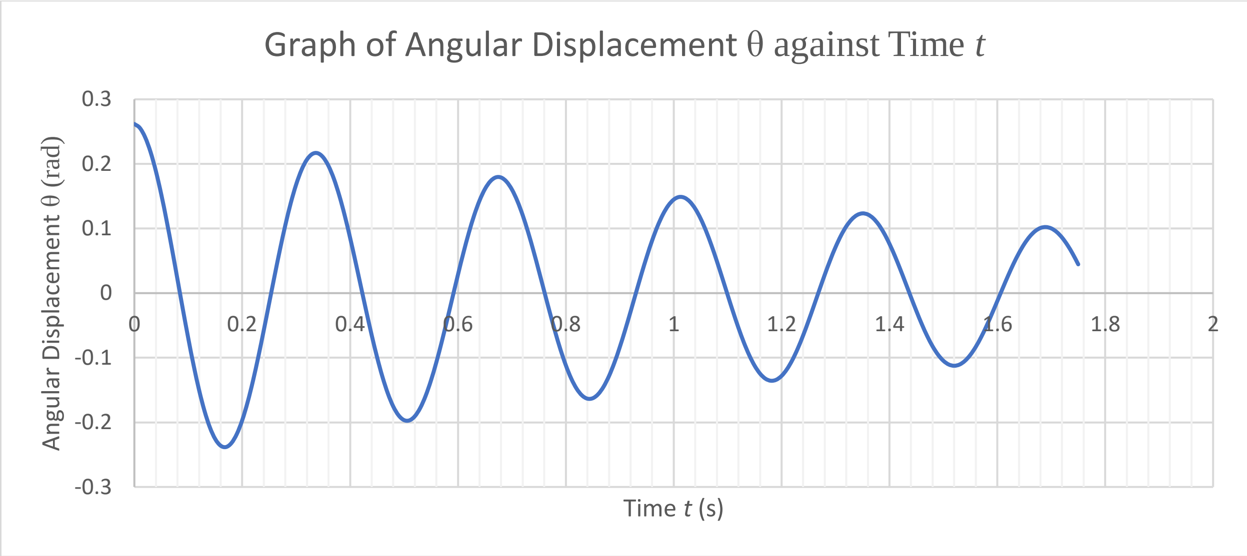
$\theta_0\ (deg)$	n	$\Delta t_1\ (s)$	$\Delta t_2\ (s)$	$\Delta t_3\ (s)$	$\Delta t_{avg}\ (s)$	$T\ (s)$
10°	5	1.691	1.687	1.693	1.690	0.338
15°	5	1.692	1.695	1.685	1.691	0.338
20°	5	1.710	1.698	1.715	1.708	0.342

Table 1. Table of Data for Part A

Part B: Plotting the sinusoidal graph for the oscillation of solid semi-cylinder

Time, $t\ (s)$	**Angular Displacement, $\theta\ (rad)$**	**Time, $t\ (s)$**	**Angular Displacement, $\theta\ (rad)$**
0.00	0.261799	0.85	-0.16258
0.01	0.255866	0.90	-0.08359
0.05	0.152401	0.95	0.056464
0.10	-0.07020	1.00	0.144808
0.15	-0.22590	1.05	0.115185
0.20	-0.19659	1.10	-0.00287
0.25	-0.01521	1.15	-0.11230
0.30	0.168254	1.20	-0.12803
0.35	0.210278	1.25	-0.04283
0.40	0.085664	1.30	0.071234
0.45	-0.09917	1.35	0.123452
0.50	-0.19649	1.40	0.076349
0.55	-0.13496	1.45	-0.02788
0.60	0.028734	1.50	-0.10468
0.65	0.161115	1.55	-0.09550
0.70	0.160399	1.60	-0.01217
0.75	0.034346	1.65	0.076171
0.80	-0.11174	1.70	0.100193

Table 2. Table of Selected Data for Part B



Graph 1. Graph of Data for Part B

4 Analysis and Discussion

Analysis

Part A: Measuring the period of oscillation using Tracker Software

From the initial setup, the radius of the semi-cylinder R was measured to be 1.85 cm and the mass of the semi-cylinder was measured to be 11.5 g. On the other hand, from Table 1, the θ_0 represents the initial angular displacement or amplitude whereas n denotes the number of complete cycles of oscillations. The total time for 5 complete oscillations, Δt,was recorded up to 3 times in order to obtain the mean value for Δt_{avg}. The period for 1 oscillation T was then calculated by dividing Δt_{avg} with n.Now, the experimental period and natural frequency are compared to its theoretical counterpart in the Table 3 below.

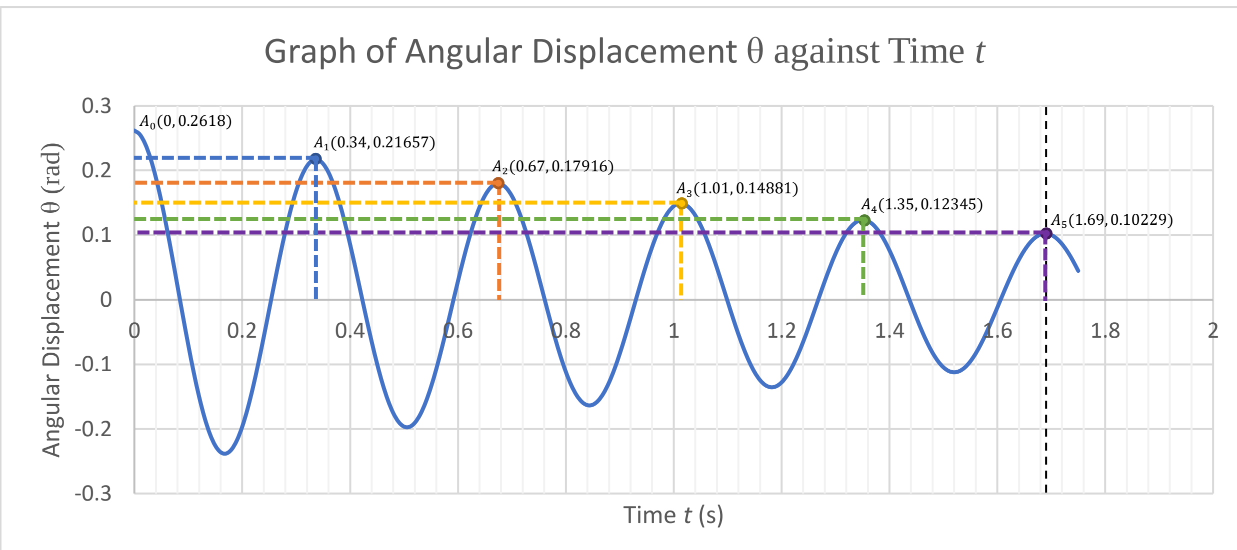
$\boldsymbol{\theta_0}$ $(\boldsymbol{deg})$	$\boldsymbol{T_{exp}}$	$(\boldsymbol{\omega_0})_{\boldsymbol{exp}}$	$(\boldsymbol{\omega_0})_{\boldsymbol{theo}}$	$\boldsymbol{T_{theo}}$	**Relative Error**
10°	0.338 s	18.59 rad/s	18.59 rad/s	0.338 s	0.00374 %
15°	0.338 s	18.59 rad/s	18.59 rad/s	0.338 s	0.06287 %
20°	0.342 s	18.37 rad/s	18.59 rad/s	0.338 s	1.18343 %

Table 3. Table of Data Analysed from Part A

In spite of the T_{exp} was rounded to 3 decimal places, the relative error still persists by using the actual value. However, due to the fact that the relative error is extremely small, it can simply be ignored to show that the period of the oscillation is constant and it was determined to be $0.338\ s$, given the θ_0 was very small so that the damping effect can be ignored. Thus, this result satisfies the equation of natural undamped simple harmonic motion $\ddot{x} + \omega_0^2 x = 0$, or in this case, refer to Eqn. 3, $\ddot{\theta} + \omega_0^2 \theta = 0$. Hence, the natural angular frequency ω_0 is also a constant and can be determined by the equation $\omega = \frac{2\pi}{T}$ to be $18.59\, rad/s$. Using Equation (4), the theoretical undamped natural angular frequency can also be calculated to be $\omega_0 = \sqrt{\frac{8(9.81\, m/s)}{(0.0185\, m)(9\pi - 16)}} \approx 18.59\, rad/s$. Under the condition of θ_0 is very small, the damping effect is hardly noticeable on T and ω_0, However, the damping effect still persists as evidently shown from Graph 1, the data has been enlarged to show that the damping exists, but in reality the damping coefficient is small and only has a significant effect on the amplitude of the motion, these will be discussed below.

Part B: Plotting the sinusoidal graph for the oscillation of solid semi-cylinder

From Graph 1, it is evident that the oscillation of rocking solid semi-cylinder is sinusoidal and amplitude of each cycle of oscillation is decreasing over time, which is the characteristics of a damped simple harmonic motion. For simplicity purposes, the total time of the oscillation for 5 complete cycles will be taken up to 1.69 s in the calculations for the following section.



Graph 2. Graph for Part B, with Amplitudes Highlighted after each cycle

Graph 2 above shown the motion of oscillation with the initial angular displacement θ_0 or the amplitude set to 15° and convert to 0.2618 radian. The amplitude decay can be expressed as a function of time with the equation below that will be further discussed in Discussion section.

$$A = A_0 e^{-\left[\frac{\mu}{2M\left(\frac{3R^2}{2\,h} - 2R\right)}\right]t} \qquad (5)$$

Cycle	Time, t (s)	Amplitude (rad)
0	0.00	0.26180
1	0.34	0.21657
2	0.67	0.17916
3	1.01	0.14881
4	1.35	0.12345
5	1.69	0.10229

Table 4. Table of Amplitude after each Cycle

From Equation (5), A_0 is the initial amplitude whereas μ is damping coefficient. In order to determine the value for μ, logarithmic decay δ is utilised as it is the rate of the decay of amplitude.

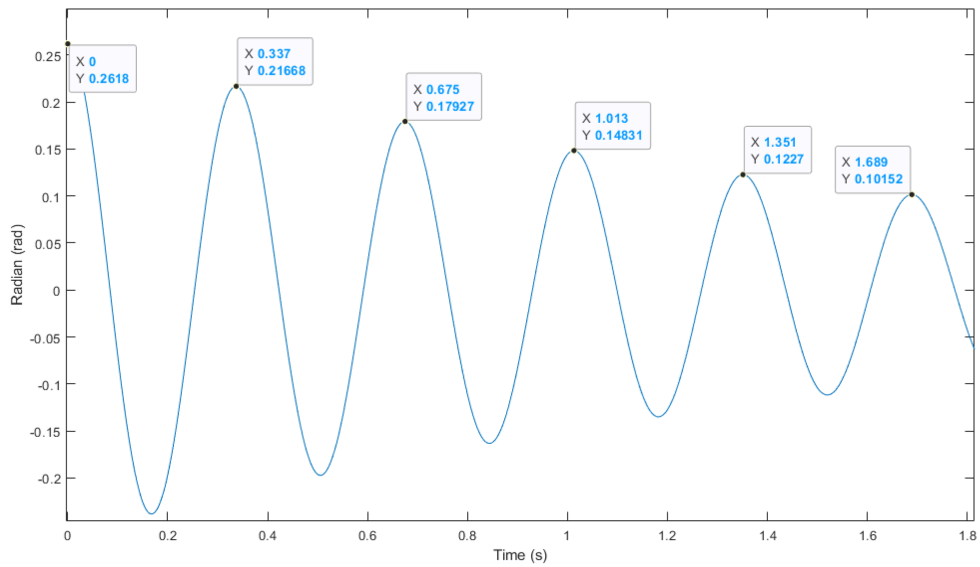
$$\delta = \frac{\mu}{2M(\frac{3}{2}\frac{R^2}{h} - 2R)} T = ln\frac{A_0}{A_1} \quad \text{——— (6)}$$

Given the ratio of amplitude is 1 cycle apart, n=1, and analysing initial amplitude A_0 and the amplitude of the first cycle A_1. After substituting all the known value for each variable, the damping coefficient μ can now be determined from the logarithmic decrement δ to be:

$$\delta = ln\frac{0.26180\ rad}{0.21657\ rad} = 0.189667 \quad ; \quad h = \frac{4}{3\pi}(0.0185\ m) = 7.852 \times 10^{-3} m$$

$$\mu = \frac{2\delta M\left(\frac{3}{2}\frac{R^2}{h} - 2R\right)}{T} = \frac{2(0.189667)(11.5 \times 10^{-3}\ kg)(\frac{3}{2}\frac{(0.0185\ m)^2}{7.852 \times 10^{-3}\ m} - 2(0.0185\ m))}{0.338\ s}$$

Thus, $\mu = 3.66 \times 10^{-4}\ kg\ m/s$. Now, a graph is created using MATLAB with all the known values.



Graph 3. Graph for Part B, with Amplitudes Highlighted after each cycle from MATLAB

The relaxation time is defined as the time taken for the amplitude to decay to e^{-1} of its original value. Therefore, the value for amplitude is: $A = A_0 e^{-1}$ ——— (7)

Refer to Equation (5) and Equation (7) above, the relaxation time t can be determined to be:

$$\left[\frac{\mu}{2M(\frac{3}{2}\frac{R^2}{h} - 2R)}\right] t = 1 \quad => \quad t = \frac{2M(\frac{3}{2}\frac{R^2}{h} - 2R)}{\mu} \qquad \text{——— (8)}$$

Substituting the corresponding value into Equation (8), thus $t = 1.78$ s

$$t = \frac{2(11.5 \times 10^{-3}\ kg)(\frac{3}{2}\frac{(0.0185\ m)^2}{7.852 \times 10^{-3}\ m} - 2(0.0185\ m))}{3.66 \times 10^{-4}\ kg\ m/s.} = 1.78\ s$$

The Q-value of a Damped Simple Harmonic Oscillator describes the number of radians the system vibrated through until the energy decays to e^{-1} of its original value. Given that the energy in a simple harmonic oscillator is proportional to the square of the amplitude, $E \propto A^2$. Therefore, the time function of the energy will be $E = E_0 e^{-\left[\frac{\mu}{M(\frac{3R^2}{2\ h} - 2R)}\right]t}$ where E_0 is the original value of the energy.

When $E = E_0 e^{-1}$, the time taken for the energy decay is,

$$t = \frac{M(\frac{3}{2}\frac{R^2}{h} - 2R)}{\mu} \qquad \text{——— (9)}$$

Therefore, the Q-value is defined as: $Q = \omega' t$ ——— (10)

Since μ is very small, therefore the damping resistance term can be ignored, hence $\omega' \approx \omega_0$.

Thus, by close approximation, the Q-value can be defined as $Q = \omega_0 t$. ——— (11)

Substituting the corresponding value of Equation (9) and ω_0 to Equation (11), it gives:

$$Q = (18.59\ rad/s)\left[\frac{(11.5 \times 10^{-3}\ kg)(\frac{3}{2}\frac{(0.0185\ m)^2}{7.852 \times 10^{-3}\ m} - 2(0.0185\ m))}{3.66 \times 10^{-4}\ kg\ m/s.}\right]$$

$$Q = 16.58\ rad$$

Therefore, the Q-value of 16.58 rad agreed upon the behaviour of the motion of oscillator from Graph 2.

Discussion

The Equation of Motion for a rocking solid semi-cylinder is derived using the Newtonian Method of Analysis. Since the process is extremely long and complicated, thus the full equation of motion for an undamped simple harmonic oscillation obtained is only shown below:

$$M\left(\frac{3}{2}\frac{R^2}{h} - 2Rcos\theta\right)\ddot{\theta} + MRhsin\theta\left(\dot{\theta}\right)^2 + Mgsin\theta = 0 \qquad (11)$$

Since θ very small, using small angle approximation, $sin\theta \approx \theta$, $cos\theta \approx 1$, and the term $\theta\dot{\theta}$ is relatively very small compared to θ, therefore it can be justifiably ignored from the equation. Now the damping coefficient μ is introduced, giving the damped equation of motion:

$$M\left(\frac{3}{2}\frac{R^2}{h} - 2R\right)\ddot{\theta} + \mu\dot{\theta} + Mg\theta = 0 \qquad (12)$$

Now, terms such as stiffness term and other equations can be solved by the ODE process.

$$\theta(t) = A_0 e^{-\left[\frac{\mu}{2M\left(\frac{3}{2}\frac{R^2}{h} - 2R\right)}\right]t} \cos\left(\omega' t\right)$$

$$\omega' = \sqrt{\frac{g}{\frac{3}{2}\frac{R^2}{h} - 2R} - \frac{\mu^2}{4M^2\left(\frac{3}{2}\frac{R^2}{h} - 2R\right)^2}}$$

The Dissipative Force by Friction can be derived using Power function, thus determining the dimensions for the Damping Coefficient μ to be $[kg][\,m][s^{-1}]$.

5 Conclusion

In conclusion, this experiment confirmed that the oscillation of a rocking solid semi-cylinder satisfies the equation of simple harmonic motion shown in Part A and Part B. Therefore, the hypothesis is accepted because ω and T were found to be a constant and were approximately equal to the theoretical predicted value for ω_0, refer to Equation (4), which is 18.59 rad /s with a negligible percentage difference. Thus, T can also be concluded to be constant.

From Part B, the sinusoidal variation of angular displacement against time is verified. It is also determined that the motion is damped with the damping coefficient $\mu = 3.66 \times 10^{-4}\ kg\ m/s$. The time taken for the amplitude to decay to e^{-1} of its original value $t = 1.78\ s$ which is around 5.5 cycles later. The number of radians vibrated through until the energy decays to e^{-1} of its original value $Q = 16.58\ rad$ which is around 2.6 cycles later. It can also therefore be concluded that the damping of the oscillation is underdamped and oscillatory. Both the amplitude and energy decay slightly over time which is shown from both the relaxation time and the Q-value.