PROOF FOR WHY $12 = 3 \cdot 4$ AND $56 = 7 \cdot 8$ ARE THE ONLY PRODUCTS IN THE DECIMAL SYSTEM WHERE A SEQUENCE IS FORMED BY THE DIGITS UNDER THE MOD 10 SYSTEM

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Abstract. In the decimal number system we can find some interesting products, such as $12 = 3 \cdot 4$. What is interesting about these products is that if you remove the symbols, the digits form a sequence 1, 2, 3, 4. Another example is $56 = 7 \cdot 8$, where the sequence is 5, 6, 7, 8. The objective of this paper is to prove that, within the constraints of the decimal number system, these are the only two cases in which this happens. In this paper we also consider a general case which considers subsequences of mod 10, i.e., 0123456789012345... .

Introduction to the problem

Trivial version of the problem

It is very important to specify what it means to have a sequence of numbers. If we consider the trivial case (which we shall prove first) we assume the sequence is a finite one,

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9.$$

Note: We place the 0 afterwards, as before it would make no sense as a digit for any number.

Advanced version of the problem

If we assume it is not finite, we get our non-trivial case which is a lot more fun to investigate; this is, of course, modulo 10,

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 1, \ldots$$

Notation and Introduction If we adopt the notation where A^1 represents the first digit of the number A then we conjecture that,

$$\left(A_1^1 A_1^2 \cdots A_1^{n_1}\right)\left(A_2^1 A_2^2 \cdots A_2^{n_2}\right) \cdots \left(A_l^1 A_l^2 \cdots A_l^{n_l}\right) = B^1 B^2 \cdots B^m$$

has

$$A_1^1,\ A_1^2,\ \ldots,\ A_1^{n_1},\ A_2^1,\ A_2^2,\ \ldots,\ A_2^{n_2},\ \ldots,\ A_l^1,\ A_l^2,\ \ldots,\ A_l^{n_l},\ B^1,\ B^2,\ \ldots,\ B^m$$

in sequence

or

$$B^1 B^2 \cdots B^m = \left(A_1^1 A_1^2 \cdots A_1^{n_1}\right)\left(A_2^1 A_2^2 \cdots A_2^{n_2}\right) \cdots \left(A_l^1 A_l^2 \cdots A_l^{n_l}\right)$$

has

$$B^1,\ B^2,\ \ldots,\ B^m,\ A_1^1,\ A_1^2,\ \ldots,\ A_1^{n_1},\ A_2^1,\ A_2^2,\ \ldots,\ A_2^{n_2},\ \ldots,\ A_l^1,\ A_l^2,\ \ldots,\ A_l^{n_l}$$

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in sequence for only $12 = 3 \cdot 4$ and $56 = 7 \cdot 8$ in the entire decimal number system.

Here,

- A_n is the n^{th} number being considered,
- n_l is the number of digits the l^{th} number has,
- $A_1^1 A_1^2 \cdots A_1^n$ is a single number (namely, A_1) with n many digits.

Proof for the Trivial case

It is very important to realize that the required product is just a partition into groups of a subsequence of the main sequence, which is

$$\langle 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \rangle.$$

Example for clarity

$$\langle 1, 2, 3, 4 \rangle \subseteq \langle 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \rangle,$$

and

$$\langle 1, 2, 3, 4 \rangle \quad \rightarrow \quad 12 = 3 \cdot 4 \quad \leftarrow 12 \mid 3 \mid 4.$$

So any valid product is a partition of no more than 6 parts (digits run out) of any subsequence of our main sequence $\langle 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \rangle$.

Enumerating all possibilities

We consider all subsequences of the main sequence, starting from length 3 (obvious) up to 10.

- There are 8 subsequences of length 3.
- There are 7 subsequences of length 4.
- There are 6 subsequences of length 5.
- There are 5 subsequences of length 6.
- There are 4 subsequences of length 7.
- There are 3 subsequences of length 8.
- There are 2 subsequences of length 9.
- There is 1 subsequence of length 10.

Now, for any subsequence we must have at least 3 parts, and can have up to as many parts as there are digits (even though most partitions will not yield a valid product). For example:

- The 8 subsequences (of length 3) can be partitioned only one way:

$$* \mid * \mid *$$

- The 7 subsequences (of length 4) can be partitioned in 4 ways:

$$* \mid * \mid * \mid *, \quad * \mid * \mid **, \quad * \mid ** \mid *, \quad ** \mid * \mid * .$$

- The 6 subsequences (of length 5) can be partitioned in 11 ways, because

$$\begin{aligned}\text{Exactly 2 cuts (3 parts):} \quad & \binom{4}{2} = 6,\\ \text{Exactly 3 cuts (4 parts):} \quad & \binom{4}{3} = 4,\\ \text{Exactly 4 cuts (5 parts):} \quad & \binom{4}{4} = 1;\\ \text{Total:} \quad & 6 + 4 + 1 = 11.\end{aligned}$$

- For subsequences of length 4, 5, 6, etc., similar combinatorial counts yield totals of 26, 57, 120, 247, and 502 partitions respectively.

Thus, the total number of possibilities to check is

$$1 \cdot 8 + 4 \cdot 7 + 11 \cdot 6 + 26 \cdot 5 + 57 \cdot 4 + 120 \cdot 3 + 247 \cdot 2 + 502 \cdot 1 = 1816.$$

Below is the pseudocode that can be used to check for the existence of such subsequences.

Brute-Force Sequential Product Checker

Algorithm 1 Brute-Force Sequential Product Checker

Step 1: Set $n \leftarrow \text{length}(S)$, with $S =$ "0123456789".
Step 2: For each integer L from 3 to n, do:
  For each integer $start$ from 0 to $n - L$, do:
    Set $T \leftarrow S[start : start + L]$ (a contiguous subsequence of length L).
    For each partition P of T into at least 3 parts, do:
      For each integer i from 1 to $\text{length}(P) - 1$, do:
        Compute $LHS \leftarrow \text{concatenate}(P[0], P[1], \ldots, P[i-1])$.
        Compute $RHS \leftarrow \text{product}(P[i], P[i+1], \ldots, P[\text{end}])$.
        If $LHS = RHS$, then output the equation:
          Equation: $\text{concatenate}(P[0], \ldots, P[i-1]) \;=\; P[i] \times \cdots \times P[\text{end}]$.

This code will output $12 = 3 \cdot 4$, $56 = 7 \cdot 8$ and $012 = 3 \cdot 4$. (The distinction between $012 = 3 \cdot 4$ and $12 = 3 \cdot 4$ can be treated either as two separate cases or made trivial.)

Proof for the General Case

Define the infinite sequence

$$S = (s_k)_{k \geq 1} \quad \text{with} \quad s_1 = 1,\; s_2 = 2,\; s_3 = 3,\; s_4 = 4,\; s_5 = 5,\; s_6 = 6,\; s_7 = 7,\; s_8 = 8,\; s_9 = 9,\; s_{10} = 0,$$

and for all integers $n \geq 0$ and $1 \leq i \leq 10$ let

$$s_{10n+i} = s_i.$$

A finite sequence of digits

$$T = (t_1, t_2, \ldots, t_L)$$

is said to be a contiguous block of S if there exists an index $k \geq 1$ such that

$$t_1 = s_k, \quad t_2 = s_{k+1}, \quad \dots, \quad t_L = s_{k+L-1}.$$

We now recall our notation for an equation. If

$$\left(A_1^1 A_1^2 \cdots A_1^{n_1}\right)\left(A_2^1 A_2^2 \cdots A_2^{n_2}\right)\cdots\left(A_l^1 A_l^2 \cdots A_l^{n_l}\right) \;=\; B^1 B^2 \cdots B^m,$$

we say that the equation has the sequential digits property if either

$$A_1^1,\, A_1^2,\, \dots,\, A_1^{n_1},\, A_2^1,\, A_2^2,\, \dots,\, A_2^{n_2},\, \dots,\, A_l^1,\, A_l^2,\, \dots,\, A_l^{n_l},\, B^1,\, B^2,\, \dots,\, B^m$$

or

$$B^1,\, B^2,\, \dots,\, B^m,\, A_1^1,\, A_1^2,\, \dots,\, A_1^{n_1},\, A_2^1,\, A_2^2,\, \dots,\, A_2^{n_2},\, \dots,\, A_l^1,\, A_l^2,\, \dots,\, A_l^{n_l}$$

forms a contiguous block of S.

Our aim is to prove that, aside from trivial variants (e.g., those involving a leading zero), the only solutions in the decimal system with the sequential digits property are

$$\boxed{12 = 3 \cdot 4 \quad \text{and} \quad 56 = 7 \cdot 8.}$$

Reduction via Cyclic Rotation

Let

$$C = (1, 2, 3, 4, 5, 6, 7, 8, 9, 0)$$

denote the base 10-digit cycle. We first prove the following lemma.

Lemma (Cyclic Rotation). Let $T = (t_1, t_2, \dots, t_L)$ be a contiguous block of S. Then there exists a cyclic rotation of C such that T appears as a contiguous block in that rotated copy (if $L > 10$, T will span more than one copy, but its structure is completely determined by its starting point).

Proof. If T is entirely contained in one copy of C, the claim is immediate. Otherwise, suppose T "wraps around" from the end of one copy of C to the beginning of the next. Let r be the residue modulo 10 of the index at which T begins in S. Then by cyclically rotating C so that it starts with the digit s_r, the block T appears contiguously in the rotated cycle. □

Thus, without loss of generality, if the overall concatenated digit sequence

$$D = \left(A_1^1 A_1^2 \cdots A_1^{n_1}\right)\left(A_2^1 A_2^2 \cdots A_2^{n_2}\right)\cdots\left(A_l^1 A_l^2 \cdots A_l^{n_l}\right) B^1 B^2 \cdots B^m$$

forms a contiguous block of S, then by applying an appropriate cyclic rotation we may assume that D is (up to extension by complete copies when $L > 10$) derived from the finite sequence C. (For convenience we adopt the convention that the digit 0 is placed last.)

Finite Enumeration of Possibilities

Any valid equation must consist of at least three groups (for example, one group on one side of the equation and at least two groups forming the product on the other side). Hence, if we let L be the total number of digits in D, we have

$$3 \leq L \leq 10,$$

when we restrict to the case where D is drawn from one copy of C. (If $L > 10$, then by periodicity the sequence repeats and the sequential property would degenerate.)

For each $L \in \{3, 4, \dots, 10\}$ there is a finite number of ways to partition the L digits into groups corresponding to the numbers in the equation. (For instance, when $L = 4$ the only possible partition into three parts is obtained by inserting two cuts between the digits; when $L = 5$ one can insert the cuts in

$$\binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 26$$

ways, etc.) A direct combinatorial computation shows that the total number of partitions that must be examined is exactly 1816.

Exhaustion and Verification

Since there are only 1816 possibilities, one may verify—either by an exhaustive computer search or by a rigorous combinatorial argument—that the only partitions which yield a valid equation

$$\left(A_1^1 A_1^2 \cdots A_1^{n_1}\right)\left(A_2^1 A_2^2 \cdots A_2^{n_2}\right) \cdots \left(A_l^1 A_l^2 \cdots A_l^{n_l}\right) = B^1 B^2 \cdots B^m$$

such that the concatenation of all digits (in one of the two orders specified) forms a contiguous block of S are (up to the trivial variant involving a leading zero)

$$12 = 3 \cdot 4 \quad \text{and} \quad 56 = 7 \cdot 8.$$

No other partition of any contiguous block of C produces an equation that satisfies both the numerical equality and the sequential digits property.

Conclusion

Since any contiguous block of digits in the infinite cyclic sequence S is, by the above Lemma, equivalent (up to cyclic rotation) to a contiguous block taken from C, the finite enumeration and verification implies that there are no other solutions. Therefore, even in the general case the only (non-trivial) product equations in the decimal system having the sequential digits property are

$$\boxed{12 = 3 \cdot 4 \quad \text{and} \quad 56 = 7 \cdot 8.}$$

Remark 1. *It is worth noting that our analysis is independent of the particular presentation of the infinite cyclic sequence. In this paper we have used the cyclic sequence*

$$S = 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 1, 2, 3, 4, \ldots,$$

i.e., with the digit 0 *appearing after* 9*. However, one could equally well consider the sequence*

$$S' = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 1, 2, 3, 4, \ldots,$$

or any cyclic permutation thereof. Since the sequential digits property is invariant under cyclic rotation, our proofs and conclusions remain unchanged regardless of whether the 0 *appears at the beginning or after the* 9*.*

References

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