Computational Complexity of Quantum Error Correction

Masataka Ohta
School of Computing, Institute of Science Tokyo, Tokyo, Japan
mohta@necom830.hpcl.titech.ac.jp or necom830@yahoo.co.jp

Abstract

Errors on computations depend on their arguments and are, in general, different arguments by arguments. Though Shor used a simple error model that a qubit is disturbed/decohered locally by its environment state only, he overlooked a fact that an initial environment state around a qubit is output from a QEC (Quantum Error Correction) encoder and is affected by and entangled with all the argument qubits used to compute the qubit, which makes the errors depend on the argument qubits. As linear superposition for quantum parallelism keeps the errors different, usual QEC schemes just do not work. It is actually demonstrated that Shor code fails to correct a single qubit error if an input qubit to an encoder is entangled with an external qubit. Quantum block codes are not helpful to correct errors on qubits of a block entangled with qubits outside of the block. Though it may be possible to construct an improved QEC circuit for *N* quantum parallel computations to correct *N* different errors, detections and corrections of *N* different errors require O(*N*) information, which makes hardware complexity of the circuit O(*N*), which is no better than classical *N* parallel computations with *N* parallel hardware, which means quantum supremacy with QEC is denied.

Keywords: Quantum Entanglement, Quantum Error Correction, Quantum Supremacy

I. Introduction

As unitary quantum operators are linear, instead of evaluating a unitary operator serially with *N* different input arguments one by one for *N* times, the operator with linear combination or quantum superposition of *N* different input arguments may be evaluated once to obtain a superpositioned evaluation result, which is, so called, quantum parallelism.

However, results of quantum computations suffer from errors and are not so correct against which various QEC (Quantum Error Correction) schemes were developed.

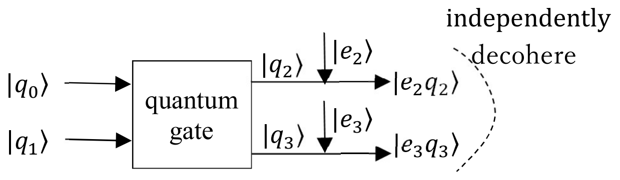
With quantum serial computations to obtain outputs of a quantum circuit with different input arguments, errors may be corrected by QEC. QEC is "quantum analog of error correcting codes" [1] and is not very different from CEC (Classical Error Correction).

With CEC, there are finite number of correctable error types including a type with no error. If an encoded code suffers from a correctable error type, during decoding, syndrome bits are computed to identify the error type. CEC scheme is chosen to make the probability of the codes suffer from an uncorrectable error type negligible. An (n, k, d) block code encodes k bits into n bit codes with minimum distance between the codes d, which can correct $m = \lfloor (d-1)/2 \rfloor$ bit errors [2]. The block code works even if errors on m bits in a n bit block are strongly correlated. Though relative phase errors between two polarization modes are not negligible for high-speed and long-distance optical communications and known to cause PMD (Polarization Mode Dispersion), because PMD characteristics are stable in time, the characteristics are measured and equalized by modern digital coherent detection technologies [3].

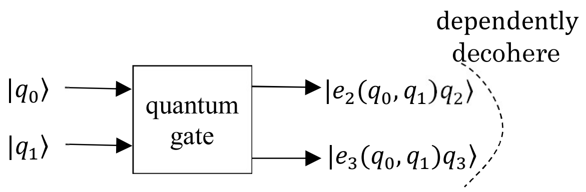
With QEC, an encoded code suffers from errors (called "decoherence" in [1]) to be an erroneous state, which is a superposition of states suffering from various correctable and uncorrectable error types. During decoding, syndrome qubits are computed and observed to converge the erroneous state into a state suffering from a correctable or, with negligibly small probability, an uncorrectable error type. The correctable error type is identified by the observed result. Block quantum codes are also available [4]. Though correlated errors between nearby qubits are discussed in [5], if qubits are well isolated from environment and from each other, the number of qubits with correlated errors is bounded below a certain constant, for which a block quantum code with large enough distance can be applied. Both bit and relative phase errors can be corrected by full-fledged QEC schemes such as that of [1].

However, a problem overlooked by Shor is that, because errors on computations depend on their argument, the errors are, in general, different arguments by arguments.

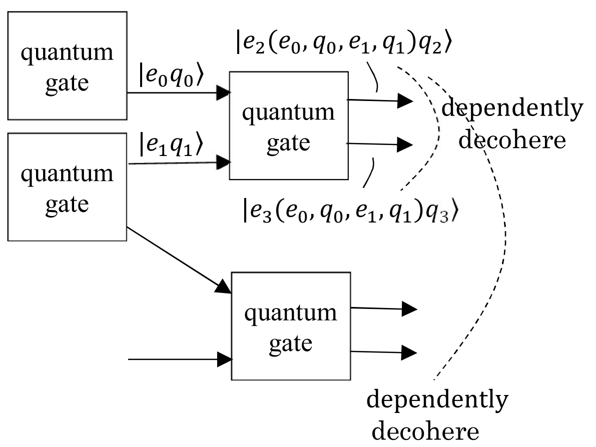
QEC works for quantum serial computations repeated *N* times with *N* different arguments, because computations for QEC are performed each time to correct errors for each set of arguments.
On the other hand, QEC for *N* parallel quantum parallel computation must correct *N* different errors simultaneously, which makes usual QEC assuming only one error type not work. Though it may be possible to develop an improved QEC to be able to correct *N* different errors, O(*N*) syndrome qubits are necessary to identify error types (including cases without error) of *N* errors, which requires O(*N*) hardware complexity, which is no better than classical *N* parallel computation, which denies quantum supremacy with QEC.



(a) qubits and their initial environment states assumed by Shor



(b) qubits and their initial environment states output from a quantum gate



(c) qubits and their environment states output from a multi-stage quantum circuit

Fig. 1 Qubit states $|\boldsymbol{q_i}\rangle$ and their environment states $|\boldsymbol{e_j}\rangle$

Hardware complexity of an error correcting quantum gate with logical qubits [6] is, if such a gate exists, also O(N).

Quantum block codes are not helpful here as errors on qubits of a block, in general, depend on qubits outside of the block. In other words, if an improved QEC to be able to correct N different errors exists, its decoding circuit must be input some ($\log_2 N$ bits) information to differently correct different errors.

The argument so far is almost purely computational and the only quantum mechanical knowledge required is that quantum superposition for quantum parallelism is linear. The linearity keeps N different errors of N quantum serial computations still different even with N quantum parallel computations.

In this paper, in section II, it is explained that even with a simple error model of Shor, quantum entanglement makes errors depend on arguments. In section III, simple examples are given on how Shor code fails to correct a single qubit error when an input qubit of a Shor encoder is entangled with an external qubit. Section IV concludes the paper.

II. Error Model of Shor

Shor assumes a simple error model "Assuming that the decoherence process affects the different qubits in memory independently" [1]. That is, he assumes a qubit state degrades only by interaction with its environment state as follows:

$$|e_0\rangle|0\rangle \rightarrow |a_0\rangle|0\rangle + |a_1\rangle|1\rangle$$
$$|e_0\rangle|1\rangle \rightarrow |a_2\rangle|0\rangle + |a_3\rangle|1\rangle \qquad (1)$$

in [1], where $|e_0\rangle$ is the initial environment state and $|a_0\rangle$, $|a_1\rangle$, $|a_2\rangle$ and $|a_3\rangle$ are the environment states after the decoherence process.

However, his assumption "The important thing to note is that the state of the environment is the same for corresponding vectors from the decoherence of the two quantum states encoding 0 and encoding 1." [1] as if the environment state around a qubit were supplied from an external source independent from computational process to produce the qubit (Fig. 1(a)) is inappropriate.

In both theory and practice, the initial environment state around a qubit is output from a QEC decoder as the final environment state around the qubit from the last quantum gate of the QEC encoder. As such, as shown in Fig. 1(b), the initial environment state ($|e_2\rangle$ or $|e_3\rangle$ of Fig. 1(b)) is affected by input qubits ($|q_0\rangle$ and $|q_1\rangle$) of the quantum gate to be $|e_2(q_0, q_1)\rangle$ or $|e_3(q_0, q_1)\rangle$, which makes errors on q_2 and q_3 depend on q_0 and q_1.

Moreover, as shown in Fig. 1(c), for the last quantum gate, considering not only input qubit states ($|q_0\rangle$ and $|q_1\rangle$) but also initial environment states around the input qubits ($|e_0\rangle$ and $|e_1\rangle$) as the final environment states around output qubits of the previous quantum gates, the final quantum states of the output environment states of the last quantum gate is affected by and entangled with $|e_0 q_0\rangle$ and $|e_1 q_1\rangle$ to be $|e_2(e_0, q_0, e_1, q_1)q_2\rangle$ and $|e_3(e_0, q_0, e_1, q_1)q_3\rangle$.

Recursive applications of such arguments make an environment state around a qubit depends on and entangled with all the input qubit states of all the quantum gates used to compute the qubit, which, in turn, makes a state of the qubit after decoherence depends on all the input qubit states of all the quantum gates used to compute the qubit. In other words, errors on computations depend on their arguments and are, in general, different arguments by arguments.

III. A Simple Example of How Shor Code Fails

In this section, with a simple quantum circuit composed of a CNOT gate followed by a Shor code encoder and a Shor code decoder (Fig. 2) , it is shown that Shor code decoder fails to decode the original state if an environment state $|e_2(q_0, q_1)\rangle$ around an input qubit $|q_2\rangle$ of a Shor code encoder is entangled with an external qubit of $|q_0\rangle$.

In the previous section, it is discussed that $|e_3\rangle$ depends on $|e_2(q_0, q_1)\rangle$ and $|q_2\rangle$ to be denoted as $|e_3(e_2(q_0, q_1), q_2)\rangle$. But, as $|e_2(q_0, q_1)\rangle$ and $|q_2\rangle$, then, depend on $|q_0\rangle$ and $|q_1\rangle$, a notation of $|e_3(q_0, q_1)\rangle$ is used for simplicity.

It is assumed that only one output qubit $|q_3\rangle$ of the Shor code decoder suffers from an error of a bit flip error with probability ε only if its initial environment state is $|e_3(1,0)\rangle$. That is, following the notation of (1), only $|q_3\rangle$ suffers from an error to be $|q_3'\rangle$ as:

$$\begin{gathered} |e_3(0,0)\rangle|0\rangle \rightarrow |a_0\rangle|0\rangle \\ |e_3(0,0)\rangle|1\rangle \rightarrow |a_3\rangle|1\rangle \\ (||a_0\rangle| = ||a_3\rangle| = 1) \end{gathered} \tag{2}$$

and

$$\begin{gathered} |e_3(1,0)\rangle|0\rangle \rightarrow |a_0\rangle|0\rangle + |a_1\rangle|1\rangle \\ |e_3(1,0)\rangle|1\rangle \rightarrow |a_2\rangle|0\rangle + |a_3\rangle|1\rangle \\ (||a_0\rangle| = ||a_3\rangle| = \sqrt{1-\varepsilon}, \\ ||a_1\rangle| = ||a_2\rangle| = \varepsilon) \end{gathered} \tag{3}$$

Then, as shown in Table 1 (A column labelled with "M" shows observation results), when q_0 and q_2 are not entangled, that is, $|q_0q_1\rangle = |00\rangle$ or $|q_0q_1\rangle = |10\rangle$, a bit flip error, if any, on q_3 is properly corrected. That is, $|q_0q_2\rangle = |q_0'q_2''\rangle$.

However, with a superpositioned state of them, that is, if $|q_0q_1\rangle = (|00\rangle + |10\rangle)/\sqrt{2}$, q_2 is entangled with q_0 as $|q_0q_2\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ and different errors on q_3 will make

$$\begin{gathered} |q_0q_3'q_4'q_5'\rangle = (|0000\rangle + |0100\rangle + \\ \sqrt{1-\varepsilon}(|1000\rangle - |1100\rangle) + \\ \sqrt{\varepsilon}(|1111\rangle - |1011\rangle)/2 \end{gathered} \tag{4}$$

which is an entangled state involving q_0 with coefficients depending on q_0. As such, observations on q_3' and q_4' at a vertical dashed line in Fig. 2 have non-local effect involving q_0, which causes QEC fail. As shown by boxes enclosed by fat border lines in Table 1, the original state $|q_0q_2\rangle$ $((|00\rangle + |11\rangle)/\sqrt{2})$ and the resulting state $|q_0'q_2''\rangle$ $((|00\rangle + \sqrt{1-\varepsilon}|11\rangle)/\sqrt{2-\varepsilon}$ or $|11\rangle)$ are significantly different.

IV. Conclusion

It is pointed out that Shor overlooked an environment state around a qubit is entangled with other qubits used to compute the qubit, which makes errors on the qubit depends on the qubits.

Though Shor stated in [1] "The assumption that the qubits decohere independently is crucial" because "This assumption corresponds to independence of errors between different bits in classical information theory", if entanglements are properly considered, the assumption of independent decoherence does not mean locality or independence of errors between different qubits.

References

[1] P. W. Shor, "Scheme for reducing decoherence in quantum computer memory," Phys. Rev. A, 52:R2493, Oct. 1995.

[2] R. W. Hamming, "Error detecting and error correcting codes," The Bell System Technical Journal, vol. 29, no. 2, pp. 147-160, Apr.1950, doi: 10.1002/j.1538-7305.1950.tb00463.x.

[3] S. Boehm, K. Schumacher, D. Goelz, and P. Meissner, "PMD Compensation with Coherent Reception and Digital Signal Processing," in Conference on Lasers and Electro-Optics/Quantum Electronics and Laser Science Conference and Photonic Applications Systems Technologies, OSA Technical Digest, paper JTuA132, 2007.

[4] A. M. Steane, "Error Correcting Codes in Quantum Theory," Phys. Rev. Lett. 77:793, Jul. 1996.

[5] Wilen, C.D., Abdullah, S., Kurinsky, N.A. et al., "Correlated charge noise and relaxation errors in superconducting qubits," Nature 594, pp. 369-373, 2021.

[6] R. W. Heeres, P. Reinhold, N. Ofek, L. Frunzio, L. Jiang, M. H. Devoret, R. J. Schoelkopf, "Implementing a universal gate set on a logical qubit encoded in an oscillator," Nat Commun. 2017 Jul 21;8(1):94.

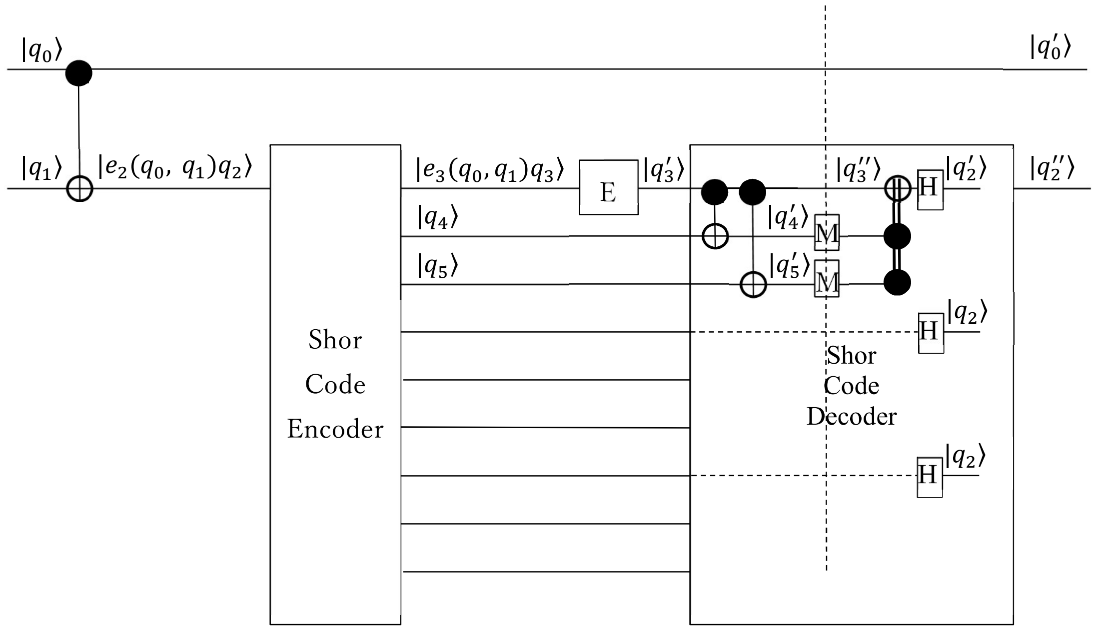


Fig. 2 Simple Circuit with a CNOT gate followed by Shor Code Encoder/Decoder

Table 1. Quantum States of Fig. 2

$\lvert q_0 q_1\rangle$	$\lvert q_0 q_2\rangle$	$\lvert q_0 e_3(q_0, q_1) q_3\rangle$	$\lvert q_0 q_3' q_4 q_5\rangle$	$\lvert q_0 q_3' q_4' q_5'\rangle$	M	$\lvert q_0' q_3''\rangle$	$\lvert q_0' q_2' q_2 q_2\rangle$	$\lvert q_0' q_2''\rangle$
$\lvert 00\rangle$	$\lvert 00\rangle$	$(\lvert 0e_3(0,0)0\rangle + \lvert 0e_3(0,0)1\rangle)/\sqrt{2}$	$(\lvert 0000\rangle + \lvert 0111\rangle)/\sqrt{2}$	$(\lvert 0000\rangle + \lvert 0100\rangle)/\sqrt{2}$	0 0	$(\lvert 00\rangle + \lvert 01\rangle)/\sqrt{2}$	$\lvert 0000\rangle$	$\lvert 00\rangle$
$\lvert 10\rangle$	$\lvert 11\rangle$	$(\lvert 1e_3(1,0)0\rangle - \lvert 1e_3(1,0)1\rangle)/\sqrt{2}$	$(\sqrt{1-\varepsilon}(\lvert 1000\rangle - \lvert 1111\rangle) + \sqrt{\varepsilon}(\lvert 1100\rangle - \lvert 1011\rangle))/\sqrt{2}$	$(\sqrt{1-\varepsilon}(\lvert 1000\rangle - \lvert 1100\rangle) + \sqrt{\varepsilon}(\lvert 1111\rangle - \lvert 1011\rangle))/\sqrt{2}$	0 0	$(\lvert 10\rangle - \lvert 11\rangle)/\sqrt{2}$	$\lvert 1111\rangle$	$\lvert 11\rangle$
					1 1	$(\lvert 10\rangle - \lvert 11\rangle)/\sqrt{2}$	$\lvert 1111\rangle$	$\lvert 11\rangle$
$(\lvert 00\rangle + \lvert 10\rangle)/\sqrt{2}$	$(\lvert 00\rangle + \lvert 11\rangle)/\sqrt{2}$	$(\lvert 0e_3(0,0)0\rangle + \lvert 0e_3(0,0)1\rangle + \lvert 1e_3(1,0)0\rangle - \lvert 1e_3(1,0)1\rangle)/2$	$(\lvert 0000\rangle + \lvert 0111\rangle + \sqrt{1-\varepsilon}(\lvert 1000\rangle - \lvert 1111\rangle) + \sqrt{\varepsilon}(\lvert 1100\rangle - \lvert 1011\rangle))/2$	$(\lvert 0000\rangle + \lvert 0100\rangle + \sqrt{1-\varepsilon}(\lvert 1000\rangle - \lvert 1100\rangle) + \sqrt{\varepsilon}(\lvert 1111\rangle - \lvert 1011\rangle))/2$	0 0	$(\lvert 00\rangle + \lvert 01\rangle + \sqrt{1-\varepsilon}(\lvert 10\rangle - \lvert 11\rangle))/\sqrt{4-2\varepsilon}$	$(\lvert 0000\rangle + \sqrt{1-\varepsilon}\lvert 1111\rangle)/\sqrt{2-\varepsilon}$	$(\lvert 00\rangle + \sqrt{1-\varepsilon}\lvert 11\rangle)/\sqrt{2-\varepsilon}$
					1 1	$(\lvert 10\rangle - \lvert 11\rangle)/\sqrt{2}$	$\lvert 1111\rangle$	$\lvert 11\rangle$