Duality between Lines and Points

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Abstract

There are several notions of duality between lines and points. In this note, it is shown that all these can be studied in a unified way. Most interesting properties are independent of specific choices.

It is also shown that either dual mapping can be its own inverse or it can preserve relative order (but not both).

Generalisation to higher dimensions is also discussed. An elementary and very intuitive treatment of relationship between arrangements in $d+1$ dimensions and searching for k-nearest neighbour in d-dimensions is also given.

1 Dual Mapping

A non-vertical line is determined by two parameters— e.g., m and c in $y = mx + c$ or a and b in $\frac{x}{a} + \frac{y}{b} = 1$. And a point in 2-d also requires two parameters (coordinates, e.g., x and y in Cartesian and r and θ in polar).

Various authors suggest different ways to map points to lines and vice-versa

Ja'Ja'[5],Lee and Ching[4] point (r, s) is mapped to line $y = rx + s$ and line $y = mx + c$ is mapped to point $(-m, c)$

O'Rourke[8] Suggests two mappings first maps line $y = mx + c$ to point (m, c) and conversely. The other which he actually uses is maps point (r, s) to line $y = 2rx - c$ and conversely.

Berg et.al.[1] point (r, s) is mapped to line $y = rx - s$ and dual of line $y = mx + c$ is the point $(m, -c)$.

Chazelle, Guibas and Lee[2] point (r, s) is mapped to line $rx + sy + 1 = 0$ and conversely.

Let us look at general dual mapping in which point (r, s) is mapped to line $y = \alpha rx + \beta s$. And a line $y = mx + c$ gets mapped to point $(\mu m, \lambda c)$.

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Observation 1 *[8] There is a one-to-one correspondence between all non-vertical lines and all points in the plane.*

REMARK This property will not be true for more general "dual" lines like $y = (ap + bq)x + (cp + dq)$.

We will like the following property to be true for the mapping:

> If a point p lies on a line L, then dual of line L, say $d(L)$ (is a point) $d(L)$ which should lie on dual of point p, say $d(p)$ (which is a line $d(p)$).

A general line through point (r, s) will be $y - s = m(x - r)$. This line gets mapped to a point $(\mu m, \lambda(s - mr))$. As this point must lie on line $y = \alpha rx + \beta s$, we get

$$
\begin{aligned}
\lambda(s - mr) &= \alpha(\mu m) + \beta s \text{ or} \\
s(\lambda - \beta) &= m(\alpha\mu + r\lambda \text{ or} \\
\lambda &= \beta \text{ and} \\
\alpha\mu &= -\lambda \text{ or} \\
\alpha\mu &= -\beta
\end{aligned}
$$

Thus, the **permissible transforms** will map point (r, s) to line $y = \alpha(rx - \mu s)$ and will map line $y = mx + c$ to point $(\mu m, -\alpha\mu c)$.

Lemma 1 *If $d(\ )$ is a permissible dual transform, then a point p lies on a line L, then dual of line L, $d(L)$ lies on $d(p)$, the dual of point p.*

Corollary 1 *[8] Two lines L_1 and L_2 intersect in a point P, iff, dual $d(P)$ passes though $d(L_1)$ and $d(L_2)$.*

As we have two free parameters we can impose additional "desirable" conditions. Two of popular desirable conditions are

1. If dual of point p is line L, then dual of line L should be point p.
2. If point p lies above line L, then dual of p should lie above dual of L.

Let us first look at first condition. Dual of line $y = (\alpha r)x + (-s\mu\alpha)$ will be point $(\mu(\alpha r), (-\mu\alpha)(-s\mu\alpha))$. For this point to be (r, s) we need $\mu\alpha = 1$.

Lemma 2 *If $\alpha\mu = 1$, then dual d is its own inverse.*

REMARK These transforms have only one free parameter. Point (r, s) gets mapped to line $y = \alpha rx - s)$ and line $y = mx + c$ gets mapped to the point $(\frac{1}{\alpha}m, -c)$.

Let us now look at the second condition. A point (r, s) is above line $y = mx + c$ if $s - rm > c$ or $0 > c + rm - s$ or $c + rm - s < 0$.

Dual of point (r, s) is the line $y = \alpha(rx - \mu s)$ and dual of line $y = mx + c$ is the point $(\mu m, -\alpha\mu c)$. Line $y = \alpha(rx - \mu s)$ is above the point $(\mu m, -\alpha\mu c)$, if

$$(-\alpha\mu c) - \alpha r(\mu m) < -\alpha\mu s$$

or

$$-(\alpha\mu)(c + rm - s) < 0$$

As $c + rm - s < 0$, $-(c + rm - s) > 0$ or $\alpha\mu < 0$.

Lemma 3 *If $\alpha\mu < 0$, then if a point p lies above line L then $d(p)$ lies above $d(L)$. And if*

If $\alpha\mu > 0$, then if a point p lies above line L then $d(p)$ lies below $d(L)$ (i.e. order gets reversed).

Observation 2 *Both Conditions can not be simultaneously satisfied.*

If we define *vertical distance*[6] between a point (r, s) and line $y = mx + c$ to be $|mr + c - s|$, then, from the proof of lemma, in the dual space the vertical distance gets scaled by $|\alpha\mu|$.

Parallel lines $y = mx + c$ and $y = mx + b$ will be mapped to points $(\mu m, -\alpha\mu c)$ and $(\mu m, -\alpha\mu b)$. Or (vertical) distance between these two points is $|\alpha\mu(b - c)|$.

Thus, if wish vertical distance to be preserved, then $\alpha\mu = \pm 1$.

Har-Peled [6, Exercise 31.2] observes that no duality can preserve exactly orthogonal distances between points and lines.

2 Applications

Assume S is a set of points in the plane. A point p is on convex hull, iff there is a non-vertical line L through p s.t., all points of S are below the line L.

For each point of S (including p), there is a line in dual space. Dual of L is a point which lies on line $d(P)$. All dual lines $d(S)$ should be on one side of $d(L)$. Or $d(L)$ lies on (upper or lower, depending on sign of $\alpha\mu$) envelope of lines $d(S)$.

Let us consider intersection of half-planes. Each half plane is either $y \leq m_i x + c_i$, which will be called **top constraint** or $y \geq m_i x + c_i$, which will be called **bottom constraint**.

The "feasible region" for top (respectively, bottom) constraints will be a convex region. Assume point (x_0, y_0) is on the boundary of this convex region. As it satisfies all top constraints it is below lines $y = m_i x + c_i$ (if i is top constraint). And as it is on the boundary, it is on at least one such line. In dual space, point (x_0, y_0) gets mapped to a line (say L_0), and each line $y = m_i x + c_i$ will get mapped to a point (say p_i). Moreover, all points p_i in dual space, will be on one side of the line L_0 and at least one point will be on this line. Thus, line L_0 will be tangent to the convex hull.

An edge e in the convex hull (in dual space) is a line between two dual points, say $d(L_1)$ and $d(L_2)$. This edge e will correspond to a point (say X) in the untransformed domain. As (dual) line $d(X)$ (which contain segment e) passes through points $d(L_1)$ and $d(L_2)$, point X will lie on lines L_p and L_q, i.e., point X will be the point of intersection of these two lines.

Thus, line segments of convex hull define define extreme point of feasible solution space, and both sets occur in the same relative order. Let us assume we have determined the convex hull.

We can similarly, deal with bottom constraints B. As both these hulls are sorted (say on x coordinate), these can be merged in linear time. The merged sets define a vertical slab, in which there are at most two boundary segments, one from lower and one from upper. Hence, we can determine the boundary in linear time.

In linear programming, assume we wish to maximise $cx + dy$. As each corner point (p_i, q_i) of solution space, we find $V_i = cp_i + dq_i$ and find the largest V_i. This again takes linear time, if the solution space is known.

Kernel of a polygon is the region from which entire polygon is visible. We interpret each edge of polygon as a half plane (the side containing the interior). Intersection of these regions will give the kernel.

3 Duality in d-dimensions

Let us generalise dual maps used (see e.g.,[6, 3]). Assume point $P = (p_1, \ldots, p_d)$ is mapped to hyperplane $x_d = \sum_{i=1}^{d-1} a_i p_i x_i + a_d p_d$ for some constants a_i. And hyperplane L with equation $x_d = \sum_{i=1}^{d-1} m_i x_i + c$ is mapped to point $(b_1 m_1, b_2 m_2, \ldots, b_{d-1} m_{d-1}, b_d c)$.

Edelsbrunner[3, p4] in first map takes all $a_i = 1$ and in the second [3, p13] $a_i = 2$ for $1 \leq i \leq d-1$ and takes $a_d = -1$. In another map[3, p17], point P is mapped to hyper-plane $\sum_{i=1}^{d} p_i x_i = 1$.

If point P lies on hyperplane L then $p_d = \sum m_i p_i + c$. For transformed point $(b_1 m_1, b_2 m_2, \ldots, b_{d-1} m_{d-1}, b_d c)$ to lie on transformed plane, we must have

$$(b_d c) = \sum a_i (b_i m_i) p_i + a_d p_d = \sum a_i (b_i m_i) p_i + a_d \left(\sum m_i p_i + c \right) = \sum m_i p_i (a_i b_i + a_d) + a_d c$$

Thus, $b_d c = a_d c$ or $b_d = a_d$ and $a_i b_i = -a_d$ or $b_i = -a_i / a_d$.

Or the general transforms are

Point $P = (p_1, \ldots, p_d)$ is mapped to hyperplane $x_d = \sum_{i=1}^{d-1} a_i p_i x_i + a_d p_d$ for some constants a_i.
Plane And hyperplane L with equation $x_d = \sum_{i=1}^{d-1} m_i x_i + c$ is mapped to point

$$\left(-a_d \frac{m_1}{a_1}, -a_d \frac{m_2}{a_2}, \ldots, -a_d \frac{m_{d-1}}{a_{d-1}}, a_d c \right)$$

Dual of point $\left(-a_d \frac{m_1}{a_1}, -a_d \frac{m_2}{a_2}, \ldots, -a_d \frac{m_{d-1}}{a_{d-1}}, a_d c \right)$ is the plane

$$x_d = \sum a_i \left(-a_d \frac{m_i}{a_i} \right) x_i + a_d (a_d c) = -\sum a_d m_i x_i + a_d^2 c$$

For this to be the original plane (dual to be its own inverse) $a_d = -1$ Thus, for this case, the transforms are:

Point $P = (p_1, \ldots, p_d)$ is mapped to hyperplane $x_d = \sum_{i=1}^{d-1} a_i p_i x_i - p_d$ for some constants a_i.
Plane And hyperplane L with equation $x_d = \sum_{i=1}^{d-1} m_i x_i + c$ is mapped to point

$$\left(\frac{m_1}{a_1}, \frac{m_2}{a_2}, \ldots, \frac{m_{d-1}}{a_{d-1}}, -c \right)$$

Let us now look at the second condition. A point P is above the plane L if $p_d \geq \sum m_i p_i + c$ or

$$p_d - \sum m_i p_i - c \geq 0$$

Dual of P is the hyper plane $x_d = \sum_{i=1}^{d-1} a_i p_i x_i + a_d p_d$.

And dual of hyper plane L is the point $\left(-a_d \frac{m_1}{a_1}, -a_d \frac{m_2}{a_2}, \ldots, -a_d \frac{m_{d-1}}{a_{d-1}}, a_d c\right)$

Dual of point (hyperplane) is above the dual of plane (point) if the dual point (of original hyper plane) is below dual plane (of original point) thus:

$$ca_d \leq \sum a_i p_i \left(-a_d \frac{m_i}{a_i}\right) + a_d p_d = a_d \left(p_d - \sum p_i m_i\right)$$

Or

$$0 \leq a_d \left(p_d - \sum p_i m_i - c\right)$$

As the term in the brackets is positive, it implies a_d is also positive, i.e., $a_d > 0$. Thus, again, both conditions can not be true.

Parallel hyperplanes $x_d = \sum m_i x_i + c$ and $x_d = \sum m_i x_i + b$ will get mapped to points

$$\left(-a_d \frac{m_1}{a_1}, -a_d \frac{m_2}{a_2}, \ldots, -a_d \frac{m_{d-1}}{a_{d-1}}, a_d c\right)$$

and

$$\left(-a_d \frac{m_1}{a_1}, -a_d \frac{m_2}{a_2}, \ldots, -a_d \frac{m_{d-1}}{a_{d-1}}, a_d b\right)$$

Thus, the vertical distance $|c - b|$ gets scaled to $|a_d(c - b)|$ to preserve vertical distance $a_d = \pm 1$ (as before).

Remark As first $d - 1$ coordinates of dual points are identical, other distances are not preserved.

Vertical distance between point $(p_1, \ldots, p_d)$ and hyperplane $x_d = \sum_{i=1}^{d-1} m_i x_i + c$ is $\left|p_d - \sum_{i=1}^{d-1} m_i x_i - c\right|$ and between dual hyperplane $x_d = \sum_{i=1}^{d-1} a_i p_i x_i + a_d p_d$ and dual point $-a_d(m_1/a_1, \ldots, m_{d-1}/a_{d-1}, -c)$ is

$$\left|a_d c + a_d \sum_{i=1}^{d-1} a_i p_i (m_i/a_i) - a_d p_d\right| = |a_d| \left|c + \sum_{i=1}^{d-1} p_i m_i - p_d\right|$$

Thus, again distances will be preserved if $|a_d| = 1$ or $a_d = \pm 1$.

3.1 Application to k-Nearest Neighbours and Arrangements

An elementary and very intuitive treatment of relationship between arrangements in $d + 1$ dimensions and searching for k-nearest neighbour in d-dimensions is also given.

Assume $S = \{P_1, \ldots, P_n\}$ is a set of n points in the R^d, the d-dimensional Euclidean space. Assume $P = (p_1, \ldots, p_d)$ and $Q = (q_1, \ldots, q_d)$ are two points of S.

A query (or test) point $X = (x_1, \ldots, x_d)$ will be closer to P then to Q iff

$$\begin{aligned}
\sum_{i=1}^{d} (x_i - p_i)^2 \quad &< \quad \sum_{i=1}^{d} (x_i - q_i)^2 \text{ or} \\
\sum_{i=1}^{d} \left(\cancel{x_i^2} - 2x_i p_i + p_i^2\right) \quad &< \quad \sum_{i=1}^{d} \left(\cancel{x_i^2} - 2x_i q_i + q_i^2\right) \text{ or} \\
2\sum_{i=1}^{d} x_i q_i - \sum_{i=1}^{d} q_i^2 \quad &< \quad 2\sum_{i=1}^{d} x_i p_i - \sum_{i=1}^{d} p_i^2
\end{aligned}$$

For $R = (r_1, \ldots, r_d)$ define $f(R) = 2\sum_{i=1}^{d} x_i r_i - \sum_{i=1}^{d} r_i^2$.

Then above condition becomes, $f(P) > f(Q)$.

As equation $z = f(R) = 2\sum_{i=1}^{d} x_i r_i - \sum_{i=1}^{d} r_i^2$ is an equation of $d+1$ dimensional hyper-plane, the above condition is equivalent to saying that
hyper-plane $z = 2\sum_{i=1}^{d} x_i p_i - \sum_{i=1}^{d} p_i^2$ is above the hyper-plane $z = 2\sum_{i=1}^{d} x_i p_i - \sum_{i=1}^{d} p_i^2$

Remark Instead of $z = f(R)$ we can also choose any monotonic function (like $z^2 = f(R)$, $\sqrt{z} = f(R)$ etc.). But $z = f(R)$ appears to be the simplest.

Hence, if we draw an arrangement of n-hyper-planes with plane $z = f(P_i)$ the i^{th}point, then the nearest neighbour of query point X will be the topmost hyper-plane above X (the one visible from $(x_1, \ldots x_d, \infty)$). The second nearest neighbour will be the second top most hyper-plane and so on. Hence, to find k^{th}closest point, we need to only consider hyper-planes which have $(k-1)$ hyper-planes above them.

Edelsbrunner[3, p23] considers a map which transforms point P to $x_{d+1} = \sum_{i=1}^{d} 2p_i x_i - \sum_{i=1}^{d} p_i^2$. Here $a_i = 2$ for $i = 1, \ldots, d$ and $c = -\sum_{i=1}^{d} p_i^2$

Thus, this can also be viewed as mapping a point in d-dimensions to a "special plane" (e.g., one constrained to be tangent to a paraboloid or a hyper-sphere) in $d+2$ dimensions.

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