New Tricks For Memorizing Trigonometric Identities

Timothy Jones

February 10, 2025

Abstract

The product to sum (P2S) and sum to product (S2P) trigonometric identities are generally not memorized, but puzzled out using the sum of two angles for the P2S and then P2S for S2P. We show here a faster way to recall these using what might be called heuristic generalizations. We touch on all 27 of the standard identities.

Introduction

DeMoivre's Theorem implies

$$(\cos a/2 + i \sin a/2)^2 = \cos^2 a/2 - \sin^2 a/2 + 2i \cos a/2 \sin a/2$$

$$\cos^2 a/2 - \sin^2 a/2 + 2i \cos a/2 \sin a/2 = \cos a + i \sin a \qquad (*)$$

and we claim this motivates and suggests many of the 27 standard trigonometric identities (listed in section 3) [1], including the allusive product to sum (P2S) and sum to product ones (S2P). You need to generalize function arguments to see the various identities. So, for example, the double angle formula for $\cos$ and $\sin$ can be seen with function argument adjustments:

$$\cos 2a = \cos^2 a - \sin^2 a \qquad (7)$$

$$\sin 2a = 2 \sin a \cos a \qquad (10).$$

Equation references are to the list of identities.

S2P

Some sum to product identities are easy to immediately recall. For example, just add and recall that $\cos 0 = 1$, for

$$\sin a + \sin a = 2\sin\left(\frac{a}{2} + \frac{a}{2}\right)\cos\left(\frac{a}{2} - \frac{a}{2}\right).$$

Working with pencil or pen, just scratch out the second as and replace them with bs for

$$\sin a + \sin b = 2\sin\left(\frac{a}{2} + \frac{b}{2}\right)\cos\left(\frac{a}{2} - \frac{b}{2}\right).$$

That's (24). Take note of the multiple of 2 and average of 2 structure of these identities; train your fingers to edit a start up recall and you'll have the drift of technique.

Just using $\sin$ is odd we have

$$\sin a - \sin b = \sin a + \sin(-b) = 2\sin\left(\frac{a}{2} - \frac{b}{2}\right)\cos\left(\frac{a}{2} + \frac{b}{2}\right),$$

using (24): (25). We can condense these two:

$$\sin a \pm \sin b = 2\sin\frac{a \pm b}{2}\cos\frac{a \mp b}{2}. \tag{1}$$

Can we get to (1) with a recall of $\sin(2a) = 2\sin a\cos a$ or $\sin(a \pm b) = \sin a\cos b \pm \cos a\sin b$?

This heuristic generalization step at some point might not be necessary. I suspect

$$\cos a + \cos b = 2\cos\left(\frac{a}{2} + \frac{b}{2}\right)\cos\left(\frac{a}{2} - \frac{b}{2}\right)$$

and, indeed, that's (26).

The difference in cosines is a slight stretch. We need a couple of minor identities:

$$\sin\left(a \pm \frac{\pi}{2}\right) = \pm\cos a \text{ and } \cos\left(a \pm \frac{\pi}{2}\right) = \mp\sin a.$$

If you move sine backward ($+\pi/2$), you get cosine; forward, negative. If you move cosine backwards ($+\pi/2$), you get negative sine; forward ($-\pi/2$), positive. Make a concave down parabola with your index finger and thumb to see these. Then, using (25),

$$\cos a - \cos b = \sin(a + \pi/2) - \sin(b + \pi/2) = \sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2} + \frac{\pi}{2}\right),$$

giving

$$\cos a - \cos b = -2\sin\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right),$$

for (27).

Whether or not these two equations are helpful is a question:

$$\sin a + \sin a + \sin a - \sin a = 2\sin\frac{a+a}{2}\cos\frac{a-a}{2} + 2\cos\frac{a+a}{2}\sin\frac{a-a}{2}$$

$$\cos a + \cos a + \cos a - \cos a = 2\cos\frac{a+a}{2}\cos\frac{a-a}{2} - 2\sin\frac{a+a}{2}\sin\frac{a-a}{2}.$$

But perhaps these four identities now are known with just two expressions that must be recalled: $2\cos\frac{a+a}{2}$ and $-2\sin\frac{a+a}{2}$. But maybe $\sin(2a) = 2\sin a\cos a$ (heterogeneous products) and $\cos(2a) = \cos^2 a - \sin^2 a$ (homogeneous products) – that pattern guides enough for a easy recall of the negative and multiple of 2.

Video (right click, new tab): S2P Video

P2S

From (10) comes

$$\sin a\cos a = \frac{1}{2}\left(\sin(a+a) + \sin(a-a)\right), \qquad (2)$$

where we use the simple fact that $\sin 0 = 0$. If you can write and easily believe (2), you can scratch out the second a arguments and replace them with bs:

$$\sin a\cos b = \frac{1}{2}\left(\sin(a+b) + \sin(a-b)\right)$$

and you have a P2S identity (22). Test this with $a = 0$ and $b = 90$ degrees. As

$$\cos a\sin b = \sin b\cos a = \frac{1}{2}\left(\sin(b+a) + \sin(b-a)\right),$$

we get a second P2S:

$$\cos a\sin b = \sin b\cos a = \frac{1}{2}\left(\sin(a+b) - \sin(a-b)\right).$$

That's (23).

One can remember the reduction (R) formulas using

$$1 = \cos^2 a + \sin^2 a = \frac{1+\cos 2a}{2} + \frac{1-\cos 2a}{2}$$

heuristically suggests

$$\cos^2 a = \frac{1+\cos 2a}{2} \text{ and } \sin^2 a = \frac{1-\cos 2a}{2}$$

From these using the same heuristic generalization idea, we have

$$\cos a \cos a = \frac{1}{2}\left(\cos(a-a) + \cos(a+a)\right),$$

using $\cos 0 = 1$. Thence a simple scratch out the second a and replace it with a b for two more P2S identities:

$$\cos a \cos b = \frac{1}{2}\left(\cos(a-b) + \cos(a+b)\right)$$

and

$$\sin a \sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)$$

That's (21) and (20).

Perhaps these P2S identities are well encapsulated with

$$1 = \cos a \cos a + \sin a \sin a$$

$$= \frac{1}{2}(\cos(a-a) + \cos(a+a)) + \frac{1}{2}(\cos(a-a) - \cos(a+a))$$

and

$$\sin(a+a) = \sin a \cos a + \cos a \sin a$$

$$= \frac{1}{2}(\sin(a+a) + \sin(a-a)) + \frac{1}{2}(\sin(a+a) - \sin(a-a))$$

with heuristic generalizations. Video (right click, new tab): P2S Video.

Basic Identities (27)

The 27 identities can be categorized with symbols (number in each category): $\pm$ (6); $2a$ (5); R (3); $a/2$ (5); $P2S$ (4); and $S2P$ (4). We'll show how (*) so motivates these six categories: $trig(a \pm b, 2a, a/2)$; (R)eduction: square to 2a; power to sum (P2S), sum to power (S2P). Here they are.

$$= trig(a \pm b(6), 2a(3 + 2\cos = 5), R(3), a/2(3 + 2\tan = 5)) + P2S(4) + S2P(4)$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b \tag{1}$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b \tag{2}$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \tag{3}$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b \tag{4}$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} \tag{5}$$

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b} \tag{6}$$

$$\cos(2a) = \cos^2 a - \sin^2 a \tag{7}$$

$$\cos(2a) = 2\cos^2 a - 1 \tag{8}$$

$$\cos(2a) = 1 - 2\sin^2 a \tag{9}$$

$$\sin(2a) = 2 \sin a \cos a \tag{10}$$

$$\tan(2a) = \frac{2\tan a}{1 - \tan^2 a} \tag{11}$$

$$\cos^2 a = \frac{1 + \cos 2a}{2} \tag{12}$$

$$\sin^2 a = \frac{1 - \cos 2a}{2} \tag{13}$$

$$\tan^2 a = \frac{1 - \cos 2a}{1 + \cos 2a} \tag{14}$$

$$\cos a/2 = \pm\sqrt{\frac{1 + \cos a}{2}} \tag{15}$$

$$\sin a/2 = \pm\sqrt{\frac{1 - \cos a}{2}} \tag{16}$$

$$\tan a/2 = \pm\sqrt{\frac{1 - \cos a}{1 + \cos a}} \tag{17}$$

$$\tan a/2 = \frac{\sin a}{1 + \cos a} \tag{18}$$

$$\tan a/2 = \frac{1 - \cos a}{\sin a} \tag{19}$$

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right] \tag{20}$$

$$\cos a \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right] \tag{21}$$

$$\sin a \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right] \tag{22}$$

$$\cos a \sin b = \frac{1}{2}\left[\sin(a+b) - \sin(a-b)\right] \tag{23}$$

$$\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2} \tag{24}$$

$$\sin a - \sin b = 2\sin\frac{a-b}{2}\cos\frac{a+b}{2} \tag{25}$$

$$\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2} \tag{26}$$

$$\cos a - \cos b = -2\sin\frac{a+b}{2}\sin\frac{a-b}{2} \tag{27}$$

$\pm$

One can read (*) as

$$\cos(a+a) = \cos a \cos a - \sin a \sin a$$

and using our heuristic generalization we arrive at (1):

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

Similarly, $\sin(2a) = 2\sin a \cos a$ can be re-written as

$$\sin(a+a) = 2\sin a \cos a = \sin a \cos a + \cos a \sin a$$

and with a heuristic generalization this is

$$\sin(a+b) = \sin a \cos b + \cos a \sin b,$$

(3).

Knowing that $\sin$ is an odd function and $\cos$ is even, one can further infer:

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b \tag{1,2}$$

and

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b. \tag{3,4}$$

We'll get $\tan(a \pm b)$ in the next section.

2a

We already have touched on $\sin 2a$ and $\cos 2a$; those two are our givens (*). Using (*)

$$\tan(2a) = \frac{\sin 2a}{\cos 2a} = \frac{2\sin a \cos a}{\cos^2 a - \sin^2 a} \frac{\frac{1}{\cos^2 a}}{\frac{1}{\cos^2 a}},$$

giving

$$\frac{\tan a + \tan a}{1 - \tan a \tan a}. \tag{11}$$

With a heuristic generalization, we get

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}. \tag{5,6}$$

A handy mnemonic to go from $\cos 2a = \cos^2 a - \sin^2 a$ to

$$\cos 2a = 2\cos^2 a - 1 \tag{8}$$

and

$$\cos 2a = 1 - 2\sin^2 a \tag{9}$$

is to remember the 2 goes in front of the one present and a 1 replaces the one not present. It's another scratch out and replace idea, especially easy with pen and pencil.

R

The R stands for reduction, as in reducing a square to a double angle. These identities are used in integration as substitutions are easy, squares of functions less so.

As was mentioned, one can remember the reduction (R) formulas using

$$1 = \cos^2 a + \sin^2 a = \frac{1 + \cos 2a}{2} + \frac{1 - \cos 2a}{2}$$

heuristically suggesting

$$\cos^2 a = \frac{1 + \cos 2a}{2} \text{ and } \sin^2 a = \frac{1 - \cos 2a}{2} \qquad (12, 13),$$

not to mention quickly giving

$$\tan^2 a = \frac{1 - \cos 2a}{1 + \cos 2a}. \tag{14}$$

a/2

In addition to jogging into memory the reduction (R) formulas using

$$1 = \cos^2 a + \sin^2 a = \frac{1 + \cos 2a}{2} + \frac{1 - \cos 2a}{2},$$

this one equation works for half angle formulas. Start with a change in arguments giving

$$\cos^2 a/2 = \frac{1 + \cos a}{2} \text{ and } \sin^2 a/2 = \frac{1 - \cos a}{2}$$

and then take square roots:

$$\cos a/2 = \pm\sqrt{\frac{1+\cos a}{2}} \text{ and } \sin a/2 = \pm\sqrt{\frac{1-\cos a}{2}}. \qquad (15, 16)$$

The three identities for $\tan a/2$ follow:

$$\tan(a/2) = \sqrt{\frac{1-\cos(a)}{1+\cos(a)}}, \sqrt{\frac{1-\cos(a)}{1-\cos(a)}}, \sqrt{\frac{1+\cos(a)}{1+\cos(a)}}$$

that is

$$\tan(a/2) = \sqrt{\frac{1-\cos(a)}{1+\cos(a)}}, \frac{1-\cos(a)}{\sin(a)}, \frac{\sin(a)}{1+\cos(a)},$$

where multiplications by one are given: (17), (18), and (19).

Remember the technique is to recall an identity and then edit it – generally scratching out and penciling in, SOPI for short. Fast!

Conclusion

I think I speak for many when I say the ease with which complex algebra helps recall trigonometric identities is a boon. One might wonder why complex numbers are not introduced early (first thing) in a trigonometry course given the prevalence of this feeling. Currently, proofs of the sum and difference identities for $\sin$ and $\cos$ use geometric arguments. This despite complex numbers being prevalent in the treatment of solving quadratics. Why not introduce the complex plane right there and then? A picture is more convincing: multiplication spins points around the unit circle; division subtracts; look! and the math matches!

Not to go on a tear but triangle trigonometry and even geometry are from a bygone era. It would be much more fruitful to replace both with periodic functions using complex arithmetic and probability and statistics. Both are much more relavent in our modern world.

References

[1] Blitzer, R. (2010). *Algebra and Trignometry*, 3rd ed., Pearson.