An exploration in recreational mathematics:

Simplified Closed-Form Solution of Kepler's Equation with MacLaurin Expansion

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Abstract

Kepler's equation, a cornerstone of celestial mechanics, relates the mean anomaly (mean angular position) of a celestial body in an elliptical orbit to its eccentric anomaly. While elegant in its formulation (x = y - e sin y), it lacks a general closed-form solution in terms of elementary functions. This article explores the significance of Kepler's equation, briefly traces its historical context, and presents a simplified approach to obtaining a closed-form solution using Mathematica. By expressing the sine function within Kepler's equation as a MacLaurin series, we derive an iterative algorithm that converges rapidly to the eccentric anomaly, providing a practical and computationally efficient method for solving this fundamental equation in celestial mechanics.

1. Introduction

Kepler's laws of planetary motion, formulated by Johannes Kepler in the early 17th century, revolutionized our understanding of celestial mechanics. These laws, derived from meticulous observations of Mars by Tycho Brahe, elegantly describe the motion of planets around the Sun. Among these laws, Kepler's second law, the law of equal areas, states that a planet sweeps out equal areas in equal times. This law directly leads to the formulation of Kepler's equation, a transcendental equation that relates the mean anomaly (M) of a celestial body to its eccentric anomaly (E).

2. Significance of Kepler's Equation

Kepler's equation plays a pivotal role in various fields, including:

- **Celestial Mechanics:** It is fundamental for predicting the positions of planets, asteroids, and comets in the solar system.
- **Spacecraft Navigation:** Accurate determination of spacecraft trajectories relies heavily on solving Kepler's equation.
- **Satellite Orbit Prediction:** Predicting the position and orbital parameters of artificial satellites requires precise solutions to Kepler's equation.

3. Historical Context

Kepler's equation itself was not explicitly derived by Kepler. It emerged later as a consequence of his laws and the geometrical properties of elliptical orbits. Newton's law of universal gravitation provided a deeper understanding of the underlying physics, but it did not alter the fundamental role of Kepler's equation in describing orbital motion.

Kepler's Second Law of Planetary Motion, often referred to as the Law of Equal Areas, states that a planet sweeps out equal areas in equal times as it orbits the Sun. This groundbreaking discovery emerged from years of meticulous analysis of astronomical observations made by Tycho Brahe.

Kepler, a mathematician and astronomer, meticulously examined Brahe's extensive data on the motion of Mars. He meticulously plotted the positions of Mars at various points in its orbit and meticulously calculated the areas swept out by the line connecting Mars to the Sun over equal time intervals. Through this painstaking process, a remarkable pattern emerged: the areas swept out were consistently equal, regardless of the planet's distance from the Sun.

This observation challenged the prevailing geocentric model of the universe, which assumed uniform circular motion. Kepler's insight, however, revealed a profound truth about planetary motion: a planet's orbital speed varies depending on its distance from the Sun. When closer to the Sun, the planet moves faster to cover the same area in the same time, and vice versa. This crucial realization laid the foundation for a more accurate and comprehensive understanding of celestial mechanics.

Kepler's Second Law, a direct consequence of the conservation of angular momentum, has far-reaching implications beyond planetary motion. It applies to any object moving under the influence of a central force, such as a satellite orbiting Earth or an electron orbiting an atomic nucleus. This fundamental principle continues to play a vital role in various fields of physics and astronomy, shaping our understanding of the universe and its intricate workings.

4. Existing Solutions

Kepler's equation, given by: [1]

$$x = y - e \sin y \qquad (1)$$

where:

- x is the mean anomaly
- y is the eccentric anomaly
- e is the eccentricity of the orbit ($0 \le e < 1$)

lacks a general closed-form solution in terms of elementary functions (such as polynomials, trigonometric functions, exponentials, and logarithms).

Numerous methods have been developed to solve Kepler's equation (cf, [1],[2],[3]), including:

- **Iterative Methods:**
    - **Newton-Raphson method:** A widely used iterative method that converges rapidly for most eccentricities.
    - **Lagrange reversion theorem:** Provides an infinite series solution.

- **Series Expansions:**
  - **Fourier series:** Expresses the eccentric anomaly as a series of trigonometric functions of the mean anomaly.
- **Graphical Methods:**
  - Provide approximate solutions by visualizing the intersection of curves.

5. Simplified Closed-Form Solution with MacLaurin Expansion

This section outlines a simplified approach to obtaining a closed-form solution for Kepler's equation using Mathematica. This approach of MacLaurin series expansion has been described initially in Kohout & Layton, cf. ref. [4].

Algorithm:

1. **MacLaurin Series Expansion for sin(y):** Express sin(y) as a MacLaurin series:

$$\sin(y) = y - y^3/3! + y^5/5! - y^7/7! + \dots \qquad (2)$$

2. **Substitute into Kepler's Equation:** Substitute the MacLaurin series expansion for sin(y) into Kepler's equation:

$$x = y - e(y - y^3/3! + y^5/5! - y^7/7! + \dots) \qquad (3)$$

3. **Rearrange and Solve for y:** Rearrange the equation to isolate y. This may involve:
   - Collecting terms of the same order of y.
   - Using algebraic manipulation techniques (e.g., factoring, completing the square).
   - Employing iterative methods within Mathematica to solve the resulting polynomial equation.

4. **Iterative Refinement:** Use the initial solution obtained from the MacLaurin series expansion as a starting point for a more accurate iterative method, such as Newton-Raphson, to refine the solution for the eccentric anomaly. The Newton-Raphson method, also known as Newton's method, is a powerful iterative algorithm used to find the roots (or zeros) of a real-valued function. It's based on the idea of linear approximation, where the function is approximated by its tangent line at a given point.

   The essence of the Newton-Raphson method lies in its iterative formula:

   x_{n+1} = x_n - f(x_n) / f'(x_n)

   where:

   x_n is the current estimate of the root.

   f(x_n) is the value of the function at x_n.

   f'(x_n) is the derivative of the function at x_n.

   The process begins with an initial guess, x₀, for the root.

   Then, the formula is applied repeatedly to generate a sequence of approximations, x₁, x₂, x₃, and so on. Ideally, this sequence converges to the actual root of the function.

Mathematica Code (Illustrative Example):

```
(* Define Kepler's equation with MacLaurin series approximation *) keplerEq[x_, y_, e_, n_] := x == y -
e*(Sum[(-1)^k*y^(2*k + 1)/(2*k + 1)!, {k, 0, n}]) (* Example: Solve for y with e = 0.5, x = 1.0, and n = 5
*) e = 0.5; x = 1.0; n = 5; sol = Solve[keplerEq[x, y, e, n], y][[1]] (* Refine solution using Newton-
Raphson method *) refinedSol = FindRoot[x - y + e*Sin[y], {y, y /. sol}] (* Print results *)
Print["Eccentric Anomaly (MacLaurin Series): ", y /. sol] Print["Eccentric Anomaly (Refined): ", y /.
refinedSol]
```

6. Discussion: Can Kepler's Equation Account for Out-of-Plane Deviations?

Kepler's equation, a cornerstone of celestial mechanics, elegantly describes the motion of a body in an idealized two-body system with elliptical orbits. However, its applicability is limited when considering real-world scenarios where out-of-plane deviations occur. One such deviation arises from the Magnus effect, a phenomenon where a spinning object experiences a force perpendicular to both its motion and its spin axis.

Limitations of Kepler's Equation:

- Two-Dimensional Assumption: Kepler's equation is inherently two-dimensional, confined to the plane of the elliptical orbit. It cannot directly account for forces that induce motion perpendicular to this plane, like the Magnus effect.
- Idealized Conditions: The derivation of Kepler's equation relies on several simplifying assumptions, such as the absence of external forces beyond the gravitational attraction of the central body. The Magnus effect, by introducing an additional force, violates these assumptions.

Addressing Out-of-Plane Deviations:

To incorporate out-of-plane deviations, such as those arising from the Magnus effect, more sophisticated mathematical frameworks are necessary:

- Perturbation Theory: This approach treats the Magnus force as a small perturbation to the idealized two-body problem. By employing techniques like the variation of parameters, it's possible to estimate the deviations from the Keplerian orbit caused by the Magnus effect.
- Kustaanheimo-Stiefel (KS) Transformation: The KS transformation is a powerful mathematical tool that maps the three-dimensional Kepler problem into a four-dimensional harmonic oscillator. This transformation may be extended further to include perturbing forces, such as the Magnus effect, and offers a more elegant and potentially more accurate way to model the out-of-plane motion. [2]

Challenges and Considerations:

- Complexity: Incorporating the Magnus effect into orbital calculations significantly increases the complexity of the problem. Analytical solutions become more elusive, and numerical methods often become necessary.
- Data Requirements: Accurate modeling of the Magnus effect requires precise knowledge of the object's spin rate, orientation, and the surrounding environment (e.g., atmospheric density, wind conditions).
- Limitations of KS Transformation: While powerful, the KS transformation also has its limitations. It can introduce numerical instabilities, and its application to highly perturbed systems may require careful consideration.

Kepler's equation, while a fundamental tool in celestial mechanics, cannot directly account for out-of-plane deviations caused by forces like the Magnus effect. More advanced techniques, such as perturbation theory and the KS transformation, are required to model these effects accurately. While these approaches present challenges, they offer valuable insights into the complex dynamics of real-world orbital systems.

7. Concluding remark

This article presented a simplified approach to solving Kepler's equation using a MacLaurin series expansion. While this approach may not provide the most efficient solution for all cases, it offers a valuable framework for understanding the mathematical underpinnings of this fundamental equation in celestial mechanics.

Kepler's equation, while a fundamental tool in celestial mechanics, cannot directly account for out-of-plane deviations caused by forces like the Magnus effect. More advanced techniques, such as perturbation theory and the KS transformation, are required to model these effects accurately. While these approaches present challenges, they offer valuable insights into the complex dynamics of real-world orbital systems.

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Discussion with several physicists are acknowledged, including Prof Jean de Climont etc. The above algorithm is for illustration purposes only. The actual Mathematica code and the level of approximation will depend on the desired accuracy and the specific requirements of the application. This article is for informational purposes only and may not be suitable for all applications.

References:

[1] Raposo-Pulido, V. & Pelaez, J. An efficient code to solve the Kepler equation. Elliptic case. *Mon. Not. Royal Astron. / MNRAS* 467, 1702–1713 (2017)

[2] Vrbik, J. Solving Kepler's Problem. *The Mathematica Journal 13,* 2011. Wolfram Media, Inc

[3] Deakin, R.E. *Solutions of Kepler's Equation* (2017). Bonbeach VIC, 3196, Australia

[4] Kohout, J.M. and Layton, L. *Optimized Solution of Kepler's Equation*. NASA TN D-6712, Goddard Space Flight Center