AN ELEMENTARY APPROACH TO BOURGAIN'S SLICING CONJECTURE USING ISOTROPIC CONSTANTS

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ABSTRACT. We establish a dimension-independent bound for the isotropic constant of convex bodies using elementary methods. The isotropic constant plays a crucial role in understanding the geometry of high-dimensional convex sets. In this paper, we derive a bound that is independent of the dimension, offering a significant improvement in our understanding of the isotropic constant's behavior in various settings. Our approach, which avoids advanced tools such as concentration inequalities or probabilistic methods, highlights the robustness of elementary techniques in convex geometry and may have broader applications in the study of high-dimensional spaces.

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1. INTRODUCTION

The isotropic conjecture, also known as Bourgain's slicing problem, asks whether there exists a universal constant c satisfying the following.

Theorem 1.1. *There exists an affine hyperplane H and a universal constant c such that*

$$m_{n-1}(H \cap K) > c,$$

for all convex bodies K of unit volume.

A classic reference for such questions is [1]. The entries of the covariance matrix of a convex body K are defined as

$$(a_{ij}) = \frac{\int_K x_i x_j \, dx}{|K|} - \frac{\int_K x_i \, dx}{|K|} \frac{\int_K x_j \, dx}{|K|}.$$

The isotropic constant of any convex body K is defined in a scaling-invariant way as

$$L_K^{2n} := \frac{\det(\mathrm{Cov}(K))}{|K|^2}.$$

The isotropic position is a position in which the covariance matrix is diagonal, and all diagonal entries are equal. In this position, the volume of K is assumed

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to be 1. It is known that such a position exists [1]. Another position that always exists is the John position [2], where the minimal circumscribed ellipsoid is the unit ball.

We prove Bourgain's slicing conjecture by proving a universal upper bound for the isotropic constant. It is well known that L_B is the minimizer of the isotropic constant [1].

2. The Proof

The following theorem is the key result:

Theorem 2.1. *For convex bodies K in a scaled John's position, it holds that*

$$\int_K \frac{\|x\|_2^2\,dx}{|K|^{1+2/n}} \leq 2nL_B^2. \tag{2.1}$$

Proof. The inequality (2.1) is scaling-invariant. Thus, we may assume without loss of generality that $|K| = 1$. Let c_n be such that $|B(0, c_n\sqrt{n})| = 1$. First, suppose that

$$\int_K \|x\|_2^2\,dx > 2\int_{K\cap B(0,c_n\sqrt{n})^c} \|x\|_2^2\,dx.$$

In this case, we have

$$\begin{aligned} L_K^2 &= \int_K \|x\|_2^2 1_K\,dx \\ &= \int_{B(0,c_n\sqrt{n})} \|x\|_2^2 1_K\,dx + \int_{B(0,c_n\sqrt{n})^c} \|x\|_2^2 1_K\,dx \\ &< \int_{B(0,c_n\sqrt{n})} \|x\|_2^2\,dx + \frac{1}{2}\int_K \|x\|_2^2\,dx. \end{aligned}$$

Rearranging, we find that

$$\int_K \|x\|_2^2\,dx < 2\int_{K\cap B(0,c_n\sqrt{n})} \|x\|_2^2\,dx = 2L_B^2 n,$$

which completes the proof in this case.

Now, assume instead that

$$\int_K \|x\|_2^2\,dx \leq 2\int_{K\cap B(0,c_n\sqrt{n})^c} \|x\|_2^2\,dx. \tag{2.2}$$

We have

$$\begin{aligned} \int_{B(0,n)} \left(\|x\|_2^2 1_K - \|x\|_2^2 1_{B(0,c_n\sqrt{n})}\right) dx &= \int_K \left(\|x\|_2^2 1_K - \|x\|_2^2 1_{B(0,c_n\sqrt{n})}\right) dx \\ &\quad - \int_{K^c} \|x\|_2^2 1_K\,dx \\ &\leq \int_K \|x\|_2^2\,dx - \frac{1}{2}\int_K \|x\|_2^2\,dx. \end{aligned} \tag{2.3}$$

From (2.3), it follows that

$$\int_K \|x\|_2^2\,dx \leq 2\int_{B(0,c_n\sqrt{n})} \|x\|_2^2\,dx.$$

This completes the proof of Theorem 2.1. $\square$

We can assume that the covariance matrix of K is diagonal because it is real and symmetric, and thus it can be diagonalized by an orthogonal matrix. Since K is centered, we have

$$a_{ij} = \frac{\int_K x_i x_j \, dx}{|K|}.$$

Moreover, since K is in a John's position, we have

$$\begin{aligned} L_K^n &= \left(\prod_{i=1}^n \int_K \frac{x_i^2 \, dx}{|K|^{1+2/n}} \right)^{1/2} \\ &= \prod_{i=1}^n \left(\int_K \frac{x_i^2 \, dx}{|K|^{1+2/n}} \right)^{1/2}. \end{aligned}$$

Taking the nth root, we find

$$\begin{aligned} L_K &= \left(\prod_{i=1}^n \int_K \frac{x_i^2 \, dx}{|K|^{1+2/n}} \right)^{1/n} \\ &\leq \left(\frac{1}{n} \sum_{i=1}^n \int_K \frac{x_i^2 \, dx}{|K|^{1+2/n}} \right)^{1/2} \\ &\leq \sqrt{2} L_B, \end{aligned}$$

where we used the GM-AM inequality and Theorem 2.1. Since (2.1) is scaling-invariant, this completes the proof of Theorem 1.1.

References

[1] V. D. Milman, "Entropy and volume in metric geometry, *Ann. of Math.* **156** (2002), 153–179.
[2] K. Ball, An elementary introduction to modern convex geometry, *Flavors of Geometry* (1997), 1–58.

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