Lecture Notes on a quadratic inequality for gaps between consecutive primes and its extension to the proof of the Binary Goldbach conjecture

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Abstract

This research paper is in the form of lesson notes. In it an identity is established connecting to consecutive primes. Bertrand's postulate is used together with the identity to establish a quadratic inequality that can be used to establish minimum intervals containing at least three primes in between its limits. A generalization of the quadratic inequality is introduced to establish the minimum interval contaimining at least one pair of primes for Goldbach partitition. The concepts of Goldbach partition deviation and Goldbach partition interval are introduced by which it is shown that the minimum number of Goldbach partitions of a composite even number is 1.

Learning objectives

The learner should be able to

1. Derive an identity connecting two consecutive prime numbers
2. Use Bertrand's postulate in a rearranged form to obtain a quadratic inequality for solving the prime gap problem
3. Use the derived quadratic inequality to obtain intervals containing three primes
4. Define a Goldbach partition deviation and a Goldbach partition interval.
5. Extend the derived quadratic inequality to derive an interval containing at least one pair of primes for Goldbach partition
6. Derive inequalities by which the minimum number of Goldbach partitions of a composite even number is equal to 1
7. Establish a relationship between two Goldbach partition counting functions and show that the relationship implies that the minimum number of Goldbach partitions of a composite even number is equal to 1.
8. Estimate the number of Goldbach paritions using a suspected number of Goldbach partitions

Introduction

These lecture notes are derivatives from pages 19 to 26 of paper reference [1]. The expositions in this paper is meant to provide some insight on some cocepts in the paper that may not clear to its readers.

Deriving an identity connecting to consectutive primes

Consider two numbers represented by two algebraic terms a and b. We can establish an identity connecting a and b through the steps below.

$$ab + (\frac{a-b}{2})^2 = \frac{4ab + a^2 - 2ab + b^2}{4} = \frac{a^2 + 2ab + b^2}{4} = (\frac{a+b}{2})^2 \quad (1)$$

Therefore

$$ab + (\frac{a-b}{2})^2 = (\frac{a+b}{2})^2 \quad (2)$$

Now consider two consecutive primes p_i and p_{i+1} If we now set $a = \sqrt{p}_{i+1}$ and $b = \sqrt{p}_i$ then

$$\sqrt{p_i p_{i+1}} + (\frac{\sqrt{p}_{i+1} - \sqrt{p}_i}{2})^2 = (\frac{\sqrt{p}_{i+i} + p_i}{2})^2 \quad (3)$$

For the purpose of achieving a quadratic inequality, the above identity will be rearranged to a more covenient form. That is:

$$\sqrt{p}_{i+1} + \sqrt{p}_i = 2\sqrt{((\frac{\sqrt{p}_{i+1} - \sqrt{p}_i}{2})^2 + \sqrt{p_i p_{i+1}})} \quad (4)$$

It also means that

$$\sqrt{p}_{i+1} + \sqrt{p}_i = 2\sqrt{((\frac{\sqrt{p}_{i+1} - \sqrt{p}_i}{2\sqrt[4]{p_i p_{i+1}}})^2 + 1)} \quad (5)$$

This also means

$$\sqrt{p}_{i+1} + \sqrt{p}_i = 2\sqrt{(p_i p_{i+1})^{\frac{1}{2}}(\frac{p_i - p_{i+1}}{2})^2((\frac{2}{p_i - p_{i+2}})^2 + \frac{1}{p_i p_{i+1}})} \quad (6)$$

That is

$$\sqrt{p}_{i+1}+\sqrt{p}_i=2(p_ip_{i+1})^{\frac{1}{4}}\frac{\sqrt{p}_{i+1}-\sqrt{p}_i}{2}\sqrt{(\frac{2}{\sqrt{p}_{i+1}-\sqrt{p}_i})^2+\frac{1}{(p_ip_{i+1})^{\frac{1}{2}}}} \tag{7}$$

Using Bertrand's postulate in a rearranged form to obtain a quadratic inequality for solving the prime gap problem

Bertrand's postulate postulate requres $p_{i+1} < 2p_i$. Therefore. Therefore substituting $p_{i+1} = 2p_i$

$$\frac{\sqrt{p}_{i+1}-\sqrt{p}_i}{2}\sqrt{(\frac{2}{\sqrt{p}_{i+1}-\sqrt{p}_i})^2+\frac{1}{(p_ip_{i+1})^{\frac{1}{2}}}}<\frac{\sqrt{p}_i(\sqrt{2}-1)}{2}\sqrt{(\frac{2}{\sqrt{2p_i}-\sqrt{p}_i})^2+\frac{1}{\sqrt{2}p_i}} \tag{8}$$

Therefore

$$\frac{\sqrt{p}_{i+1}-\sqrt{p}_i}{2}\sqrt{(\frac{2}{\sqrt{p}_{i+1}-\sqrt{p}_i})^2+\frac{1}{(p_ip_{i+1})^{\frac{1}{2}}}}<\sqrt{1+\frac{(\sqrt{2}-1)^2}{4\sqrt{2}}}=1.015 \tag{9}$$

Now because

$$\sqrt{p}_{i+1}+\sqrt{p}_i>2\sqrt{p_{i+1}p_i} \tag{10}$$

$$1<\frac{\sqrt{p}_{i+1}-\sqrt{p}_i}{2}\sqrt{(\frac{2}{\sqrt{p}_{i+1}-\sqrt{p}_i})^2+\frac{1}{(p_ip_{i+1})^{\frac{1}{2}}}}<1.015 \tag{11}$$

The function

$$f(p_i)=1.05^{\frac{1}{p_i}}\wedge p_i>3 \tag{12}$$

lies within the interval $(1, 1.05)$ Therefore intervals containing three primes are determined by solving the quadratic inequality below.

$$\sqrt{p}_{i+1}+\sqrt{p}_i<2\times 1.05^{\frac{1}{p_i}}\sqrt[4]{p_{i+1}p_i} \tag{13}$$

Using the quadratic inequality (13) to obtain intervals containing three primes

Example 1 Find the integer inteval centered around $p_i = 7$ containing at least three primes

Solution

$$\sqrt{x}+\sqrt{7}<\sqrt[4]{7x}\times 1.05^{\frac{1}{7}}$$

calculator step

$$4\leq x\leq 11$$

In this interval the three primes are 5, 7 and 11.

Example 2 Use the inequality above to find at least 3 primes centering around 23.

Solution

$$\sqrt{x}+\sqrt{2}3<\sqrt[4]{23x}\times 1.05^{\frac{1}{7}}$$

Calculator step. The integer interval is:

$$17\leq x\leq 29$$

In the above interval the primes are 17, 19, 23 and 29.

Goldbach partition deviation and interval

If $2m$ is a composite even number, we will define a Goldbach partition deviation as the ratio of m to the number of Goldbach partitions of 2m. If $R(2m)$ is the number of Goldbach partitions of 2m and d_g is Goldbach partition Goldbach partition deviation then

$$d_g=\frac{m}{R(2m)} \tag{14}$$

Thus by the above definition all composite even numbers wih having 1 Goldbach partition $d_g=m$ A Goldbach partition interval is an interval containing at least 1 Goldbach partition and its limits are defined as

$$m-d_g<i_g<m+d_g \tag{15}$$

The number 100 has 6 Goldbach partitions. This means $d_g=8$. An interval containing primes for one Goldbach partition of 100 is $42,58$. The Goldbach partition prime pairs in this are $(47,53)$. Now we can construct an equation that determines this interval given

$$\sqrt{5}0+\sqrt{x}<2\times 1.05^{\frac{1}{50}}\sqrt[4]{50x} \tag{16}$$

and we note that

$$41.9 < x < 59.7$$

In this interval the Goldbach partition pairs are $(41, 59)$ and $(47, 53)$. The above the length of the above interval is $2d_g$

Now Consider the composite even number 12.
The composite even number will have 1 Goldbach partition if $d_g = m = 6$ This would mean that the interval containing primes making up one Goldbach partition would be $(0, 12$. The quadratic inequality:

$$\sqrt{6} + \sqrt{x} < 2 \times 1.05^{\frac{1}{6}}$$

The interval from the solution of the above is $(3.6 < x < 10)$. The Goldbach partition primes pair in this interval is $(5, 7)$.

Extending the derived quadratic inequality to derive an interval containing at least one pair of primes for Goldbach partition

The interval containing one pair primes for Goldbach partition of a composite even number, 2m can be determining through solving the quadratic inequality below.

$$\sqrt{m} + \sqrt{x} < 2 \times 1.05^{\frac{1}{m}} \sqrt[4]{mx} \tag{17}$$

Laws governing the number of Goldbach partition

From the solution of the quadratic inequality, the length of the interval containing three primes is given

$$(\sqrt{p}_i(1.05^{\frac{2}{p_i}} + \sqrt{(\sqrt{p}_i(1.05^{\frac{2}{p_i}})^2 + p_i})^2 - \sqrt{p}_i(1.05^{\frac{2}{p_i}} - (\sqrt{(\sqrt{p}_i(1.05^{\frac{2}{p_i}})^2 + p_i})^2 \tag{18}$$

The maximum length of interval containing a pair of Goldbach partition primes

$$(\sqrt{m}(1.05^{\frac{2}{m}} + \sqrt{(\sqrt{m}(1.05^{\frac{2}{m}})^2 + m})^2 - \sqrt{m}(1.05^{\frac{2}{m}} - (\sqrt{(\sqrt{m}(1.05^{\frac{2}{m}})^2 + m})^2 \tag{19}$$

It is observed that if

$$(\sqrt{m}(1.05^{\frac{2}{m}}+\sqrt{(\sqrt{m}(1.05^{\frac{2}{m}})^2+m})^2-\sqrt{m}(1.05^{\frac{2}{m}}-(\sqrt{(\sqrt{m}(1.05^{\frac{2}{m}})^2+m})^2 \leq 2m \tag{20}$$

Then 2m has at least one Goldbach partition. It should be noted that 2m is the largest possible Goldbach partition interval, while m is the largest possible Goldbach partition deviation. It is also observed that If

$$(\sqrt{m}(1.05^{\frac{2}{m}}+\sqrt{(\sqrt{m}(1.05^{\frac{2}{m}})^2+m})^2-(\sqrt{m}(1.05^{\frac{2}{m}}-\sqrt{(\sqrt{m}(1.05^{\frac{2}{m}})^2+m})^2 \leq m \tag{21}$$

then 2m has at least 2 Goldbach partitions. Thus by the two equations above, composite even numbers less than 14 have at least one Goldbach partition. and those greater or equal to 14 have at least two Goldbach partitions. Thus the number of Goldbach partitions function $R(2m)$ is governed by the inequality

$$R(2m) \geq \frac{m}{(\sqrt{m}(1.05^{\frac{2}{m}}+\sqrt{(\sqrt{m}(1.05^{\frac{2}{m}})^2+m})^2-(\sqrt{m}(1.05^{\frac{2}{m}}-\sqrt{(\sqrt{m}(1.05^{\frac{2}{m}})^2+m})^2} \tag{22}$$

Let S_G represent the sum of Goldbach Partition primes Now

$$R(2m) = \frac{S_G}{2m} \tag{23}$$

Therefore

$$\frac{S_G}{2m} \geq \frac{m}{(\sqrt{m}(1.05^{\frac{2}{m}}+\sqrt{(\sqrt{m}(1.05^{\frac{2}{m}})^2+m})^2-(\sqrt{m}(1.05^{\frac{2}{m}}-\sqrt{(\sqrt{m}(1.05^{\frac{2}{m}})^2+m})^2} \tag{24}$$

This means that

$$S_G \geq \frac{2m^2}{(\sqrt{m}(1.05^{\frac{2}{m}}+\sqrt{(\sqrt{m}(1.05^{\frac{2}{m}})^2+m})^2-(\sqrt{m}(1.05^{\frac{2}{m}}-\sqrt{(\sqrt{m}(1.05^{\frac{2}{m}})^2+m})^2} \tag{25}$$

Substituting (23) into (24) we establish that

$$S_G \geq m \tag{26}$$

Therefore

$$R(2m) \geq \frac{1}{2} \tag{27}$$

This confirms the Goldbach conjecture to be true.

Again it should be noted that

$$R(2m) = \frac{S_G}{2m} = \frac{m}{d_g} \tag{28}$$

This means that

$$S_G = \frac{2m^2}{d_g} \tag{29}$$

since

$$d_g \leq m \tag{30}$$

then $S_G \geq 2m$ and $R(2m) \geq 1$ Again it is noted

$$m \approx \frac{(\sqrt{m}(1.05^{\frac{2}{m}} + \sqrt{(\sqrt{m}(1.05^{\frac{2}{m}})^2 + m})^2 + (\sqrt{m}(1.05^{\frac{2}{m}} - \sqrt{(\sqrt{m}(1.05^{\frac{2}{m}})^2 + m})^2}{2} \tag{31}$$

This result means that the minimum interval one can find Goldbach partition primes of $2m$ is

$$((\sqrt{m}(1.05^{\frac{2}{m}} - \sqrt{(\sqrt{m}(1.05^{\frac{2}{m}})^2 + m})^2, \sqrt{m}(1.05^{\frac{2}{m}} + (\sqrt{(\sqrt{m}(1.05^{\frac{2}{m}})^2 + m})^2) \tag{32}$$

Thus for $2m = 140$, the minimum interval for Goldbach partition of 140 for by the above equation is $(60.3, 81.3)$ or conveniently $(60, 80)$ In this interval the Goldbach partition pairs are $(61, 79)$ and $(67, 73)$

The minimum interval that can be taken to confirm that 4000 has a Goldbach partition is $(1944, 2056)$. In this interval the Goldbach partition pairs are $(1973, 2027)$ and $(1997, 2003)$.

The minimum interval that can be taken to confirm that 128 has a Goldbach partition is $(54, 74)$. In this interval the Goldbach partition pair is $(61, 67)$.

The minimum interval that can be taken to confirm that 32 has a Goldbach partition is $(11, 21)$. In this interval the Goldbach partition pair is $(13, 19)$.

From reference paper [2], the minimum interval of primes of Goldbach partition is

$$(m + \sqrt{m^2 - s_{g_{max}}}, m - \sqrt{m^2 - s_{g_{max}}}) \tag{33}$$

Where $s_{g_{max}}$ largest Goldbach partition semiprime. If

$$a = (\sqrt{m}(1.05^{\frac{2}{m}} - \sqrt{(\sqrt{m}(1.05^{\frac{2}{m}})^2 + m})^2 \tag{34}$$

$$b = (\sqrt{m}(1.05^{\frac{2}{m}} + (\sqrt{(\sqrt{m}(1.05^{\frac{2}{m}})^2 + m})^2 \tag{35}$$

and

$$c = \frac{b - a}{2} \tag{36}$$

Then

$$ab \leq s_{g_{max}} \leq ab + c^2 \tag{37}$$

Thus the maximum Goldbach partition semiprime of 128 is given by $54 \times 74 = 3996 \leq s_{g_{max}} \leq 54 \times 74 + 10^2 = 4096$ The largest Goldbach partition semiprime is actually 4087. If thecomposite even number $2m$ is not semiprime then the Goldbach partition prime pairs with a minimum gap between them is less than or equal to $2c$. That is to say also that there exists Goldbach partition primes within the interval (a, b).

References

[1]Bonaya, Samuel Buya, Confirming Buya's and Bezaleel's proof of the Binary Goldbach conjecture using Bertrand's postulate [Version 2] (May 11, 2024). Available at SSRN: https://ssrn.com/abstract=4828468 or http://dx.doi.org/10.2139/ssrn.4828468

[2] Samuel Bonaya Buya and John Bezaleel Nchima (2024). A Nessary and Sufficient Condition for Proof of the Binary Goldbach Conjecture. Proofs of Binary Goldbach, Andrica and Legendre Conjectures. Notes on the Riemann Hypothesis. International Journal of Pure and Applied Mathematics Research, 4(1), 12-27. doi: 10.51483/IJPAMR.4.1.2024.12-27.