ULAM NUMBERS HAVE ZERO LOGARITHMIC DENSITY

THEOPHILUS AGAMA

Abstract. Determining the natural *density* of Ulam numbers remains an open question. We denote the sequence of all Ulam numbers by U. In this paper, we show for the *logarithmic* density of Ulam numbers

$$\mathcal{D}_{\log}(U) := \lim_{n\longrightarrow\infty} \frac{1}{\log x} \sum_{\substack{n\leq x\\ n\in U}} \frac{1}{n} = 0.$$

1. Introduction

The notion of Ulam numbers was first introduced by the Polish mathematician Stanislaw Ulam, in 1964 [2]. Let us denote, as is standard, the sequence of Ulam numbers by (U_n), then each term in the sequence of Ulam numbers has the unique representation as the sum of two prior distinct Ulam numbers, and it is the smallest such number. More precisely, Ulam numbers is a sequence of **distinct** numbers of the form $1, 2, 3, 4, 6, \ldots, U_i, U_{i+1}, \ldots$, where each term in the sequence is distinct and has the unique representation $U_i = U_j + U_k$ for $i-1 \geq j > k$ and U_i is the smallest such number. The main problem of the sequence of Ulam numbers is very much related to their natural density. This problem is now known as the Ulam density problem, which can be stated as

Question 1. Do the Ulam numbers have positive density?

Ulam is said to have conjectured that the density of these numbers is zero. In this paper, we answer an analogous version of this question. In particular, we show

Theorem 1.1. *Let U denote the sequence of all Ulam numbers. Then*

$$\mathcal{D}_{\log}(U) := \lim_{x\longrightarrow\infty} \frac{1}{\log x} \sum_{\substack{n\leq x\\ n\in U}} \frac{1}{n} = 0.$$

In particular, Ulam numbers have a zero logarithmic density.

In the sequel, we review some elementary properties of Ulam numbers. These properties and their constructions are well known [1].

Lemma 1.2. *There are infinitely many Ulam numbers $(U_m)_{m\geq 1}$.*

Proof. Suppose that the first n Ulam numbers have already been determined, namely $1, 2, 3, 4, \ldots, U_{n-1}, U_n$. Then the representation $U_n + U_{n-1}$ is unique and the number so represented in this form could be the next Ulam number. If not, then this number is not the smallest such number, and since there are other numbers

Date: January 10, 2025.
2010 *Mathematics Subject Classification.* Primary 11P81; Secondary 11P82.
Key words and phrases. additive chain; determiners; regulators; length; generators; partition.

with such unique representations, we choose the smallest from among them bigger than U_n and assigns to U_{n+1} as the next Ulam number. This construction can then be repeated indefinitely, thereby generating an infinite sequence of Ulam numbers. This completes the proof. $\square$

Lemma 1.3. *No Ulam number U_m for $m > 3$ can be the sum of its prior consecutive Ulam numbers.*

Proof. Suppose in contrast that $U_{n-1} + U_n = U_{n+1}$. Then necessarily the representation $U_n + U_{n-2}$ must be unique. Suppose it is not unique, then there exist some $U_i < U_{n-2}$ and $U_j > U_n$ such that

$$\begin{aligned} U_n + U_{n-2} &= U_i + U_j \\ &> U_{n+1} \\ &= U_n + U_{n-1} \end{aligned}$$

and it follows that $U_{n-2} > U_{n-1}$, which is absurd. Now we observe that

$$U_n \leq U_n + U_{n-2} < U_{n+1}$$

contradicting the fact that U_{n+1} is the next Ulam number. $\square$

2. The regulators and determiners of an addition chain

In this section, we recall the notion of an addition chain and introduce the notion of the generators of the chain and their accompanying *determiners* and *regulators*.

Definition 2.1. Let $n \geq 3$, then by an addition chain of length $k-1$ producing n, we mean the sequence

$$1, 2, \ldots, s_{k-1}, s_k$$

where each term s_j $(j \geq 3)$ in the sequence is the sum of two earlier terms i.e $s_k = s_i + s_j$ $(s_k > 1)$ with $i \leq j < k$, with the corresponding sequence of partition

$$2 = 1 + 1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n$$

where $a_{i+1} = a_i + r_i$ and $a_{i+1} = s_i$ for $2 \leq i \leq k$. We call the partition $a_i + r_i$ the i^{th} **generator** of the chain for $2 \leq i \leq k$. We call a_i the **determiner** and r_i the **regulator** of the i^{th} generator of the chain. We call the sequence (r_i) the regulators of the addition chain and (a_i) the determiners of the chain for $2 \leq i \leq k$. We call the subsequence (s_{j_m}) for $2 \leq j \leq k$ and $1 \leq m \leq t \leq k$ a truncated addition chain producing n.

At any rate, we do not expect the regulators to be a part of the chain, although the determiners must be the terms in the chain.

Lemma 2.2. *Let $1, 2, \ldots, s_{k-1}, s_k$ be an addition chain producing $n \geq 3$ with associated generators*

$$2 = 1 + 1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n.$$

Then the following relation for the regulators

$$\sum_{j=2}^{k} r_j = n - 1$$

hold.

Proof. We notice that $r_k = n - a_k$. It follows that

$$\begin{aligned} r_k + r_{k-1} &= n - a_k + r_{k-1} \\ &= n - (a_{k-1} + r_{k-1}) + r_{k-1} \\ &= n - a_{k-1}. \end{aligned}$$

Again we obtain from the following iteration

$$\begin{aligned} r_k + r_{k-1} + r_{k-2} &= n - a_{k-1} + r_{k-2} \\ &= n - (a_{k-2} + r_{k-2}) + r_{k-2} \\ &= n - a_{k-2}. \end{aligned}$$

By iterating downwards in this manner the relation follows. □

3. **Preliminary results**

We derive an *asymptotic* formula for the *logarithmic* partial sums of terms in an addition chain.

Theorem 3.1. *Let $n \geq 2$ be fixed positive integer and let $1, 2, \ldots, s_{\delta(n)-1}, s_{\delta(n)} = n$ be an addition chain producing n and of length $\delta(n)$, with associated sequence of generators*

$$1 + 1, s_2 = a_2 + r_2, \ldots, s_{\delta(n)-1} = a_{\delta(n)-1} + r_{\delta(n)-1}, s_{\delta(n)} = a_{\delta(n)} + r_{\delta(n)} = n$$

then

$$\sum_{l=1}^{\delta(n)} \log s_l = \delta(n) \log n - O(\delta(n)).$$

Proof. Let $n \geq 2$ be a fixed positive integer and consider an addition chain $1, 2, \ldots, s_{\delta(n)-1}, s_{\delta(n)} = n$ producing n and of length $\delta(n)$, with associated sequence of generators

$$1 + 1, s_2 = a_2 + r_2, \ldots, s_{\delta(n)-1} = a_{\delta(n)-1} + r_{\delta(n)-1}, s_{\delta(n)} = a_{\delta(n)} + r_{\delta(n)} = n$$

and put (a_j) and (r_j) to be the sequence of determiners and regulators, respectively, in the chain. We make the following observations: $s_{\delta(n)-1} = a_{\delta(n)} = a_{\delta(n)-1} + r_{\delta(n)-1} = s_{\delta(n)-2} + r_{\delta(n)-1} = a_{\delta(n)-2} + r_{\delta(n)-2} + r_{\delta(n)-1} = \cdots = 1 + \sum\limits_{j=1}^{\delta(n)-1} r_j = n + 1 - r_{\delta(n)}$, where we have used Lemma 2.2. Similarly, we can write $a_{\delta(n)-1} = 1 + \sum\limits_{j=1}^{\delta(n)-2} = n + 1 - r_{\delta(n)} - r_{\delta(n)-1}$. Thus by induction, we can write $a_l = n + 1 - \sum\limits_{j=l}^{\delta(n)} r_j$ for each $3 \leq l \leq \delta(n)$. We observe that

$$\sum_{l=1}^{\delta(n)} \log s_l = \log 2 + \sum_{l=3}^{\delta(n)} \log a_l + \log n.$$

We now analyze the latter sum of the right-hand side involving the *determiners* of the addition chain. We can write

$$\sum_{l=3}^{\delta(n)} \log a_l = \sum_{l=3}^{\delta(n)} \log((n+1) - \sum_{i=l}^{\delta(n)} r_i)$$

which can be recast as

$$\sum_{l=3}^{\delta(n)}\log a_l=\sum_{l=3}^{\delta(n)}\log(n+1)-\sum_{l=3}^{\delta(n)}\sum_{v=1}^{\infty}\frac{1}{v(n+1)^v}\bigg(\sum_{i=l}^{\delta(n)}r_i\bigg)^v$$

with $\sum\limits_{i=l}^{\delta(n)}r_i<n-1$ for each $3\leq l\leq \delta(n)$ by Lemma 2.2. It is clear that

$$\sum_{v=1}^{\infty}\frac{1}{v(n+1)^v}\bigg(\sum_{i=l}^{\delta(n)}r_i\bigg)^v\ll 1$$

for each $3\leq l\leq \delta(n)$ since $\sum\limits_{i=l}^{\delta(n)}r_i<n-1$ for each $3\leq l\leq \delta(n)$ by Lemma 2.2. It follows that

$$\sum_{l=1}^{\delta(n)}\log s_l=\log 2+(\delta(n)-2)\log(n+1)+\log n-O(\delta(n)).$$

This completes the proof of the claimed formula. □

We derive an *asymptotic* formula for the *harmonic* partial sums of terms in an addition chain.

Theorem 3.2. *Let $n\geq 2$ be fixed positive integer and let $1,2,\ldots,s_{\delta(n)-1},s_{\delta(n)}=n$ be an addition chain producing n and of length $\delta(n)$, with associated sequence of generators*

$$1+1,s_2=a_2+r_2,\ldots,s_{\delta(n)-1}=a_{\delta(n)-1}+r_{\delta(n)-1},s_{\delta(n)}=a_{\delta(n)}+r_{\delta(n)}=n$$

then

$$\sum_{l=1}^{\delta(n)}\frac{1}{s_l}=\frac{3}{2}+O(\frac{\delta(n)}{n}).$$

Proof. Let $n\geq 2$ be fixed positive integer and consider an addition chain $1,2,\ldots,s_{\delta(n)-1},s_{\delta(n)}=n$ producing n and of length $\delta(n)$, with associated sequence of generators

$$1+1,s_2=a_2+r_2,\ldots,s_{\delta(n)-1}=a_{\delta(n)-1}+r_{\delta(n)-1},s_{\delta(n)}=a_{\delta(n)}+r_{\delta(n)}=n$$

and put (a_j) and (r_j) to be the sequence of determiners and regulators, respectively, in the chain. We make the following observations: $s_{\delta(n)-1}=a_{\delta(n)}=a_{\delta(n)-1}+r_{\delta(n)-1}=s_{\delta(n)-2}+r_{\delta(n)-1}=a_{\delta(n)-2}+r_{\delta(n)-2}+r_{\delta(n)-1}=\cdots=1+\sum\limits_{j=1}^{\delta(n)-1}r_j=n+1-r_{\delta(n)}$, where we have used Lemma 2.2. Similarly, we can write $a_{\delta(n)-1}=1+\sum\limits_{j=1}^{\delta(n)-2}=n+1-r_{\delta(n)}-r_{\delta(n)-1}$. Thus by induction, we can write $a_l=n+1-\sum\limits_{j=l}^{\delta(n)}r_j$ for each $3\leq l\leq \delta(n)$. We observe that

$$\sum_{l=1}^{\delta(n)}\frac{1}{s_l}=\frac{3}{2}+\sum_{l=3}^{\delta(n)}\frac{1}{a_l}+\frac{1}{n}.$$

We now analyze the latter sum of the right-hand side involving the *determiners* of the addition chain. We can write

$$\sum_{l=3}^{\delta(n)}\frac{1}{a_l}=\sum_{l=3}^{\delta(n)}\frac{1}{(n+1)-\sum\limits_{i=l}^{\delta(n)}r_i}$$

which can be recast as

$$\sum_{l=3}^{\delta(n)}\frac{1}{a_l}=\sum_{l=3}^{\delta(n)}\frac{1}{n+1}+\sum_{l=3}^{\delta(n)}\sum_{v=1}^{\infty}\frac{1}{(n+1)^{v+1}}\Bigg(\sum_{i=l}^{\delta(n)}r_i\Bigg)^v$$

with $\sum\limits_{i=l}^{\delta(n)}r_i<n-1$ for each $3\leq l\leq\delta(n)$ by Lemma 2.2. It follows that

$$\sum_{l=1}^{\delta(n)}\frac{1}{s_l}=\frac{3}{2}+\frac{\delta(n)}{n+1}+\sum_{l=3}^{\delta(n)}\sum_{v=1}^{\infty}\frac{1}{(n+1)^{v+1}}\Bigg(\sum_{i=l}^{\delta(n)}r_i\Bigg)^v+O(\frac{1}{n})$$

where $\sum\limits_{i=l}^{\delta(n)}r_i<n-1$ by Lemma 2.2 for each $3\leq l\leq\delta(n)$. It is clear that

$$\sum_{v=1}^{\infty}\frac{1}{(n+1)^{v}}\Bigg(\sum_{i=l}^{\delta(n)}r_i\Bigg)^v\ll 1$$

for each $3\leq l\leq\delta(n)$ since $\sum\limits_{i=l}^{\delta(n)}r_i<n-1$ for each $3\leq l\leq\delta(n)$ by Lemma 2.2. This completes the proof of the claimed formula. □

4. **The notion of Ulam numbers**

In this section, we recall the concept of Ulam numbers and show its profound connection to the concept of an addition chain. The principal goal of this section is to show that any finite sequence of Ulam numbers can be appropriately "inserted" into some addition chain. First, we recall the following definitions.

Definition 4.1. Ulam numbers is a sequence of **distinct** numbers of the form $1,2,3,4,6,\ldots,U_i,U_{i+1},\ldots$, where each term in the sequence is distinct and has the unique representation $U_i=U_j+U_k$ for $i-1\geq j>k$ and each U_i is the smallest such number.

Next, we show that we can "confine" any finite sequence of Ulam numbers (U_n) into a certain addition chain by carefully choosing the regulators of the chain. In fact, this "covering" can be done in a *global* sense so that our addition chains can be extended to contain any finite sequence of Ulam numbers.

Proposition 4.1. *Let $(U_m)_{m=1}^n$ be a finite sequence of Ulam numbers. Then there exist an addition chain (s_k) producing U_n such that*

$$(U_m)_{m=1}^n\subseteq(s_k).$$

Proof. Let $1, 2, 3, 4, \ldots, U_n$ be a finite sequence of Ulam numbers. Then for each term U_m for $m \geq 1$, we choose the regulator $r_j \geq 1$ such that $U_m + r_j \leq U_{m+1}$. If it is the case that $U_m + r_j = U_{m+1}$ then the consecutive sequence U_m, U_{m+1} is also a consecutive sequence in the desired addition chain. If not, then we continue this process by choosing the regulator $r_i \geq 1$ such that $U_m + r_j + r_i = U_{m+1}$. Then in such a case the consecutive Ulam numbers U_m, U_{m+1} are not consecutive numbers in the corresponding addition chain. This construction can be carried out to generate an addition chain producing U_n yet covering the finite sequence of Ulam numbers. This completes the proof of the proposition. □

4.1. **Main result.**

Theorem 4.2. *Let U denote the sequence of all Ulam numbers. Then*

$$\mathcal{D}_{\log}(U) := \lim_{x \longrightarrow \infty} \frac{1}{\log x} \sum_{\substack{n \leq x \\ n \in U}} \frac{1}{n} = 0.$$

In particular, Ulam numbers have a zero logarithmic density.

Proof. Set $(U_i)_{i=1}^n$ to be the sequence of the first n Ulam numbers. By Proposition 4.1 there exists an addition chain (s_k) leading to U_n such that $(U_i)_{i=1}^n \subset (s_k)$. In keeping with the notation of the background study, we denote by $\delta(U_n)$ the length of the addition chain (s_k) leading to $U_n \leq x$ for a fixed $x \geq 2$. By using Theorem 3.2, we deduce the inequality

$$\sum_{\substack{j \leq x \\ j \in U}} \frac{1}{j} \leq \sum_{l=1}^{\delta(x)} \frac{1}{s_l} = \frac{3}{2} + O(\frac{\delta(x)}{x}).$$

It follows that

$$\mathcal{D}_{\log}(U) := \lim_{x \longrightarrow \infty} \frac{1}{\log x} \sum_{\substack{j \leq x \\ j \in U}} \frac{1}{j} \leq \lim_{x \longrightarrow \infty} \frac{1}{\log x} \sum_{l=1}^{\delta(x)} \frac{1}{s_l} = 0$$

demonstrating that the *logarithmic density* of Ulam number is zero. □

5. **Further remarks**

The primary objective of this study was to establish a comprehensive framework for understanding Ulam numbers, specifically focusing on their structural properties and their embedding into addition chains. In doing so, we developed a method to cover any finite sequence of Ulam numbers with a suitably constructed addition chain, thereby revealing an essential connection between the two concepts. This result is encapsulated in Proposition 4.1, where we demonstrated that any finite sequence of Ulam numbers can be appropriately inserted into an addition chain, creating a seamless bridge between these two important mathematical structures.

We further extended our investigation by exploring the asymptotic behavior of Ulam numbers, particularly their logarithmic density. The main theorem (Theorem 5.2) provided a critical insight into the sparsity of Ulam numbers, proving that they possess a zero logarithmic density. This result, combined with the method of embedding Ulam numbers into addition chains, establishes a powerful understanding of their growth and distribution, shedding light on the subtle intricacies of their behavior as they grow larger.

The results presented here significantly enhance the existing body of knowledge about Ulam numbers, addition chains, and their interrelation. By providing explicit constructions for embedding Ulam numbers into addition chains, we offer a constructive method for analyzing and generating Ulam numbers that could be useful for various applications in number theory and combinatorics. This work also adds to the broader discussion of the asymptotic properties of sequences, specifically focusing on their logarithmic density and the connection between arithmetic structures and additive properties.

The connection between Ulam numbers and addition chains offers new perspectives for researchers working in related areas, particularly in the study of additive number theory, combinatorial number theory, and computational complexity. Additionally, the result regarding the logarithmic density of Ulam numbers contributes to ongoing discussions surrounding the growth rates of special sequences and their comparison to other well-known sequences, such as the primes and Fibonacci numbers.

This work contributes to our understanding of Ulam numbers and their properties in the context of addition chains, providing a rigorous framework for analyzing their asymptotic behavior and embedding them into well-defined structures. The study of logarithmic density, in particular, has offered insight into the sparsity of Ulam numbers, enriching our understanding of their distribution in the natural numbers. Although many questions remain open, the results of this work offer a solid foundation for further investigation into the deep connections between these mathematical objects.

In conclusion, this study has both clarified the structure of Ulam numbers and presented new methods for embedding them in addition chains, while also contributing to our understanding of their asymptotic properties. The research opens up new avenues for exploration and invites further inquiry into the fundamental structures of number theory, combinatorics, and their intersections.

[1].

References

1. Recaman, Bernardo, *Questions on a sequence of Ulam.* The American Mathematical Monthly, vol. 80:8, Taylor & Francis, 1973, 919–920.
2. Ulam, Stanislaw, *Combinatorial analysis in infinite sets and some physical theories*, SIAM Review, vol. 6:4, SIAM, 1964, 343–355.

Department of Mathematics, African Institute for Mathematical science, Ghana
E-mail address: theophilus@aims.edu.gh/emperordagama@yahoo.com

[1] .