An Unconventional Method for Evaluating Approximations of Irrational Numbers

A Preprint

Alexander C. Sarich
United States of America
acsarich@gmail.com

January 8, 2025

Abstract

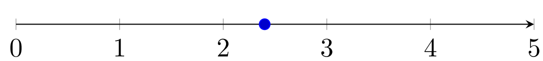
Provided is a method for evaluating irrational numbers in parts; where, any irrational number can be represented by a numerical-approximation taken to some exact decimal place and its remainder. Analysis provided is to be considered incomplete but imaginative. Argumentation provided is intended to simplify ideas pertaining to Irrational numbers, infinite series, and distance sets as a mechanism of solution finding.

1 Introduction

For the purposes herein, as they might pertain directly to the $\pi + e$ proof for whether this value is rational or irrational. The following proof is developed for the evaluation of any number based on an approximation of that number and its true value defined by its limit. As the reader is walked through this unconventional approach, one will see the use of interval notation. The developed notation is used to help differentiate values from standard practices in Set Theory. To help demonstrate the use of this system, the first section will provide information on how an interval from zero to x can be considered, and how this can be written and read without error or loss of generality. Then, to conclude, the method will be applied to a proof for whether $\pi + e$ is an irrational or rational value.

2 New Notation for Defining a Real Number

Let $x > 0$ be some value on the real number line.



When $x > 0$ we will say that this number is synonymous to the length distance between 0 and x. The best way that I might say this is by using the Euclidean Distance Notation, such that

$$x = d(0, x).$$

So whether x is a whole number, rational number or irrational number, We will henceforth be defining it through this notation. Where terms $x > 0$ and $d(0, x)$ are interchangeable. Now, if we have two real numbers, $x, y > 0$, we can perform simple operations like

$$x + y = d(0, x) + d(0, y) = d(0, x + y),$$

$$x * y = d(0, x) * d(0, y) = d(0, x * y),$$

$$\frac{x}{y} = \frac{d(0,x)}{d(0,y)} = d(0, \frac{x}{y}).$$

Next, let us take the case of $x = \pi$. Since π is a transcendental number, obtaining the numerical sequence of digits for π requires an infinite series summation. An example of this would be the Gregory-Leibniz Series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$$

Such that,

$$\pi = 4 * (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...).$$

Thus, we can use this to find any digit of π. Assuming of course somebody interested in finding the hundredth digit themselves is willing to put in the effort or computing power. However, since π is irrational, no matter how long one were to calculate this series, they will always end up with a value less than π.

Another way of saying this is that without being able to write an infinity of numbers, there is no writing π in terms of digits. For example, suppose we have the approximation of π for the first three, five, and nine digits. All of these approximations will be less than the value of π,

$$3.14 < 3.1415 < 3.14159262 < \pi.$$

When a calculation is performed for π, such as $y = \frac{\pi}{2}$, any numerical value found for y will be an approximation. Then whether we approximate π to 3.14, 3.1415, or 3.14159262, we get a solution where

$$\frac{3.14}{2} = 1.57 < \frac{3.1415}{2} = 1.57075 < \frac{3.14159262}{2} = 1.57079631 < \frac{\pi}{2} = d(0, \frac{\pi}{2}).$$

Now, let us write a new notation which will provide us with any arbitrary numerical solution for a value $x > 0$. We will be using interval notation as a substitution in this manner of the form

$$[0, x) \leq d(0, x).$$

Where the parentheses bracket after x means we have some numerical-approximation which approaches x or is x. Using this notation, set $[0, x) = y$ such that y is a rational number greater than zero. Then if x is some rational number such as 2, $\frac{1}{3}$, &c, then $y = [0, x) = d(0, x) = x$. If x is some irrational number such as π, e, $\sqrt{2}$, &c, then $y = [0, x) < d(0, x) = x$. Restating this,

$$\text{Let } y = [0, x) \text{ be a rational number, such that}$$
$$\begin{cases} \text{If } x = d(0, x) \text{ is a rational number, then } [0, x) = d(0, x) \\ \text{If } x = d(0, x) \text{ is an irrational number, then } [0, x) < d(0, x) \end{cases}$$

Regarding the case of $x = \pi$, we have that $[0, x)$ can be any numerical approximation of π. Meaning that whether we use 3.14 or 3.14159 or any other value, we are only ever approaching the true value of π without having the value be set to its infinite sequence of digits. This providing the other difficulty in expressing this notation. Where we have that

$$3.13 < d(0, \pi).$$

But,

$$3.13 \neq [0, \pi).$$

Since 3.13 is not a numerical approximation of π, we can neither use it as a substitution for $[0, \pi)$ nor can we treat it as a possible solution for $[0, \pi)$. Next, in following this type of cleanup for current notation use, another aspect of this notation that does require explanation is that even though we can have $[0, \pi)$ be any numerical-approximation for π, when using $[0, \pi)$ or any other irrational number approximation of the form $[0, x)$, we need to require that once an approximation has been defined, the value does not change. Where if we set $[0, \pi) = 3.14$ and calculate $[0, \pi) + 2 = z$ then $z = 5.14$. There is no changing the value of $[0, \pi)$ once it is defined to give us $z = 3.1415$. Although this fixing of the numerical-approximation may be unnecessary for the following $\pi + e$ example, it may be necessary for future developments.

Now, when we look at the case of $[0, \pi)$, we can see that the difference between $[0, \pi)$ and $d(0, \pi)$ will always differ by some value, ϵ being the digit-sequence of π starting at the terminating decimal place of $[0, \pi)$. An example of this is the following, let

$$[0, \pi) = 3.14 < d(0, \pi), \text{ where}$$

$$[0, \pi) + \epsilon = d(0, \pi) \text{ then}$$

$$\epsilon = 0.001519...$$

Where, since $[0, \pi)$ is a rational number, and $d(0, \pi) = \pi$ is an irrational number, this then requires that ϵ be the irrational number difference, $\epsilon = d(0, \pi) - [0, \pi)$. Then to account for this ϵ difference between $[0, \pi)$ and $d(0, \pi)$, let us introduce new notation. Consider the following and the introduction of a new type of notation of the form $[0, x]$,

$$\text{Let } \pi = d(0, \pi), \text{ such that } \tfrac{\pi}{2} = \tfrac{d(0,\pi)}{2} = \tfrac{d(0,\pi)}{d(0,2)} = d(0, \tfrac{\pi}{2}).$$

Then for some ϵ being an irrational number, we have that

$$[0, \tfrac{\pi}{2}) + \epsilon = d(0, \tfrac{\pi}{2})$$

From here we add a condition of,

$$[0, \tfrac{\pi}{2}) + [0, \tfrac{\pi}{2}] = d(0, \pi), \text{ such that}$$

$$[0, \tfrac{\pi}{2}) < d(0, \tfrac{\pi}{2}) < [0, \tfrac{\pi}{2}].$$

Then if we have that

$$[0, \tfrac{\pi}{2}) < [0, \tfrac{\pi}{2}],$$

$$[0, \tfrac{\pi}{2}) + \epsilon = d(0, \tfrac{\pi}{2}), \text{ and}$$

$$[0, \tfrac{\pi}{2}) + [0, \tfrac{\pi}{2}] = d(0, \pi).$$

Then, since $[0, \frac{\pi}{2})$ is a rational number and both $d(0, \frac{\pi}{2})$ and $d(0, \pi)$ are irrational numbers, we require that $[0, \frac{\pi}{2}]$ be the irrational number difference between $[0, \frac{\pi}{2})$ and $d(0, \pi)$. Which leads to the following conclusion that

$$[0, \tfrac{\pi}{2}) + \epsilon \leq [0, \tfrac{\pi}{2}].$$

Let us now look at the two separate scenarios in which we consider the scenario where $[0, \frac{\pi}{2}) + \epsilon$ is equal to $[0, \frac{\pi}{2}]$, and the scenario where $[0, \frac{\pi}{2}) + \epsilon$ is less than $[0, \frac{\pi}{2}]$. To show that $[0, \frac{\pi}{2}) + \epsilon = [0, \frac{\pi}{2}]$, we can see that if

$$[0, \tfrac{\pi}{2}) + \epsilon = d(0, \tfrac{\pi}{2}), \text{ and}$$

$$[0, \tfrac{\pi}{2}) + [0, \tfrac{\pi}{2}] = d(0, \pi), \text{ such that}$$

$$d(0, \pi) = d(0, \tfrac{\pi}{2}) + d(0, \tfrac{\pi}{2}) = d(0, \tfrac{\pi}{2}) + [0, \tfrac{\pi}{2}) + \epsilon, \text{ then}$$

$$[0, \tfrac{\pi}{2}) + [0, \tfrac{\pi}{2}] = d(0, \tfrac{\pi}{2}) + [0, \tfrac{\pi}{2}) + \epsilon.$$

Whereby subtracting out $[0, \frac{\pi}{2})$, we find that

$$[0, \tfrac{\pi}{2}] = d(0, \tfrac{\pi}{2}) + \epsilon, \text{ but if}$$

$$d(0, \tfrac{\pi}{2}) = [0, \tfrac{\pi}{2}) + \epsilon = [0, \tfrac{\pi}{2}], \text{ then we find that}$$

$$d(0, \tfrac{\pi}{2}) = d(0, \tfrac{\pi}{2}) + \epsilon, \text{ or}$$

$$[0, \tfrac{\pi}{2}) + \epsilon = [0, \tfrac{\pi}{2}) + \epsilon + \epsilon.$$

Which would give us, $\epsilon = \epsilon + \epsilon$. Then this only works if $\epsilon = 0$. But since $[0, \frac{\pi}{2})$ is a rational number and $d(0, \frac{\pi}{2})$ is an irrational number, ϵ must be an irrational number greater than zero for $\frac{\pi}{2} = d(0, \frac{\pi}{2}) = [0, \frac{\pi}{2}) + \epsilon$ to be true. Which gives us that $[0, \frac{\pi}{2}) + \epsilon \leq [0, \frac{\pi}{2}]$ is false when $[0, \frac{\pi}{2}] = [0, \frac{\pi}{2}) + \epsilon$. Then, if we know that

$$[0, \tfrac{\pi}{2}] \neq [0, \tfrac{\pi}{2}) + \epsilon.$$

Let us consider the case when $[0, \frac{\pi}{2}]$ is greater than $[0, \frac{\pi}{2}) + \epsilon$. Such that we get the inequality,

$$[0, \tfrac{\pi}{2}) + \epsilon < [0, \tfrac{\pi}{2}].$$

Restating some of the conditions as follows, where

$$[0, \tfrac{\pi}{2}) + [0, \tfrac{\pi}{2}] = d(0, \pi), \text{ and}$$

$$[0, \tfrac{\pi}{2}) + \epsilon = d(0, \tfrac{\pi}{2}), \text{ such that}$$

$$[0, \tfrac{\pi}{2}) \text{ is a rational number and numerical approximation of } d(0, \tfrac{\pi}{2}) = \tfrac{\pi}{2}.$$

Then if $[0, \frac{\pi}{2})$ is a rational number and $d(0, \pi) = \pi$ is an irrational number, then ϵ and $[0, \frac{\pi}{2}]$ will be irrational numbers. Also, since $[0, \frac{\pi}{2}) + \epsilon < [0, \frac{\pi}{2}]$ is the same inequality $[0, \frac{\pi}{2}) < [0, \frac{\pi}{2}] - \epsilon$, we can define a new variable $\delta > \epsilon$, such that

$$[0, \tfrac{\pi}{2}) + \epsilon < [0, \tfrac{\pi}{2}) + \delta = [0, \tfrac{\pi}{2}].$$

From here we have the following equations,

$$[0, \tfrac{\pi}{2}) + \epsilon = d(0, \tfrac{\pi}{2}),$$

$$[0, \tfrac{\pi}{2}) + [0, \tfrac{\pi}{2}] = d(0, \pi),$$

$$d(0, \tfrac{\pi}{2}) + d(0, \tfrac{\pi}{2}) = d(0, \pi).$$

Substituting the first equation into the third gives the equation

$$[0, \tfrac{\pi}{2}) + \epsilon + [0, \tfrac{\pi}{2}) + \epsilon = d(0, \pi)$$

Next, substituting the second equation in gives an equation of the form,

$$[0, \tfrac{\pi}{2}) + \epsilon + [0, \tfrac{\pi}{2}) + \epsilon = [0, \tfrac{\pi}{2}) + [0, \tfrac{\pi}{2}], \text{ and since}$$

$$[0, \tfrac{\pi}{2}] = [0, \tfrac{\pi}{2}) + \delta, \text{ then}$$

$$[0, \tfrac{\pi}{2}) + \epsilon + [0, \tfrac{\pi}{2}) + \epsilon = [0, \tfrac{\pi}{2}) + [0, \tfrac{\pi}{2}) + \delta.$$

Then, subtracting out $[0, \frac{\pi}{2})$ from both sides leads to a solution such that

$$\epsilon + \epsilon = \delta.$$

Where, going back to the inequality

$$[0, \tfrac{\pi}{2}) + \epsilon < [0, \tfrac{\pi}{2}) + \delta, \text{ and applying}$$

$$\epsilon + \epsilon = \delta, \text{ then we get}$$

$$[0, \tfrac{\pi}{2}) + \epsilon < [0, \tfrac{\pi}{2}) + \epsilon + \epsilon.$$

By subtracting out $[0, \frac{\pi}{2})$ from both sides we find that

$$\epsilon < \epsilon + \epsilon.$$

Where, having already shown that ϵ must be some irrational value greater than zero, then $[0, \frac{\pi}{2}]$ is some irrational number greater than $[0, \frac{\pi}{2}) + \epsilon$. Now, rewriting and restating some of these equations, we can see that in order to have

$$[0, \tfrac{\pi}{2}) + \epsilon = d(0, \tfrac{\pi}{2}),$$

We must have ϵ be an irrational number since $[0, \frac{\pi}{2})$ is a rational number, and if we can write

$$d(0, \tfrac{\pi}{2}) + d(0, \tfrac{\pi}{2}) = d(0, \pi), \text{ as}$$

$$[0, \tfrac{\pi}{2}) + \epsilon + [0, \tfrac{\pi}{2}) + \epsilon = d(0, \pi),$$

Then if we subtract out the numerical-approximation values, $[0, \frac{\pi}{2})$, from the previous equation, we get

$$\epsilon + \epsilon = d(0, \pi) - [0, \tfrac{\pi}{2}) - [0, \tfrac{\pi}{2})$$

Since subtracting two rational numbers from the irrational number, $d(0, \pi)$ will still give an irrational number, then $\epsilon + \epsilon$ must be an irrational number. Similarly, by looking at the equation,

$$[0, \tfrac{\pi}{2}) + [0, \tfrac{\pi}{2}] = d(0, \pi), \text{ such that}$$

$$[0, \tfrac{\pi}{2}) + [0, \tfrac{\pi}{2}) + \delta = d(0, \pi),$$

If we subtract the two rational numbers from the irrational number, $d(0, \pi)$, once again we end up with an irrational number which will be δ. Which as we have seen is also equal to $\epsilon + \epsilon$. Which if this whole process were to be repeated for any arbitrary irrational number greater than zero, what we would find is that at the core of this method, by taking any irrational number and adding itself once or multiplying by two, we still get an irrational number. What we can now do with this is apply it to the situation of two distinct irrational numbers being added to each other.

3 How We can use This Approximation Method to Demonstrate That $\pi + e$ is an Irrational Number

Continuing with the notation already developed, let us consider the structure of the two irrational numbers, π and e. We currently have the basic setup and information that allows us to look at an irrational number in terms of some arbitrary numerical-approximation and its irrational number remainder in the form of

$$[0, \tfrac{x}{2}) + [0, \tfrac{x}{2}] = d(0, x), \text{ such that}$$

$$\text{If } x \text{ is an irrational number, then}$$

$$[0, \tfrac{x}{2}) < d(0, \tfrac{x}{2}) < [0, \tfrac{x}{2}].$$

From the previous section, we have demonstrated, using π as an example, that if we take an irrational number and subtract a numerical-approximation of that number, we will end up with an irrational number remainder. For the case of $\frac{\pi}{2}$ we treated ϵ as the irrational number remainder, and for π, we treated δ as the irrational number. With this, we were able to show a direct relationship between ϵ and δ where $\epsilon + \epsilon = \delta$. Now, this is equivalent to saying that for some irrational number ϵ, the value 2ϵ will also be an irrational number. Where, from here, we can use this analysis of basic elements thus far developed to show that $\pi + e$ is an irrational number. To begin, let us consider the infinite-series summations that defines π and e such that,

$$\pi = 4(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - ...)$$

$$e = 1 + \tfrac{1}{1!} + \tfrac{1}{2!} + \tfrac{1}{3!} + \tfrac{1}{4!} + ...$$

We can equivalently write these in terms of the notation $d(0,x)$, where

$$d(0,\pi)=\pi=4(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-...)$$

$$d(0,e)=e=1+\tfrac{1}{1!}+\tfrac{1}{2!}+\tfrac{1}{3!}+\tfrac{1}{4!}+...$$

From here, using the form of $d(0,x)=[0,\frac{x}{2})+[0,\frac{x}{2}]$ such that $[0,\frac{x}{2})$ will be a rational number approximation of the irrational number, $\frac{x}{2}$. Then, for the number e, whether we use 2.71 or 2.7182 or 2.7182818 or any sequence of numbers that approximates e, as long as the value is a rational number, then we can use this as a substitute for e. Since we know that,

$$\pi=3.14159262...,\ \text{and}$$

$$e=2.71828182...$$

If we were to say,

$$\text{Let } 3.14 \text{ be an approximation of } \pi,\ \text{and}$$

$$\text{Let } 2.71 \text{ be an approximation of } e,\ \text{then}$$

$$3.14=[0,\pi),\ \text{and } 2.71=[0,e).$$

Comparatively, suppose we were to say,

$$\text{Let } 3.13 \text{ be an approximation of } \pi,\ \text{and}$$

$$\text{Let } 2.72 \text{ be an approximation of } e,\ \text{then}$$

$$3.13\neq[0,\pi),\ \text{and } 2.72\neq[0,e).$$

Since the use of $[0,x)$ already requires that any numerical-approximation be rational and contain the same sequence of digits of x up to some terminating decimal place. Another way of saying this is that for some whole number n, where n can be 1 or 2 or 3 or 4 and so on, then whatever n is, we will have that be the terminating decimal place for the approximation $[0,x)$. Now let us add another notation element to our system. Let n be added to the bottom right corner of an approximation, where, $[0,x)_n$, gives the position of approximation termination, such that we can get specific values,

$$[0,\pi)_1=3,$$

$$[0,e)_3=2.71,$$

$$[0,\pi)_8=3.1415926,\ \text{\&c.}$$

This does not change any of the previous definitions or statements where we still have that,

$$[0,x)_n+[0,x]=[0,x)+[0,x]=d(0,2x),\ \text{where}$$

$$x \text{ is an irrational number, and } n \text{ is a whole number.}$$

We can use this to our advantage. Where, a useful property this gives us is that if we know that any $[0,x)_n$ is a rational number, then we can turn this into a whole number. An example of this would be if we look at,

$$[0,\pi)_4=3.141,\ \text{then}$$

$$10^3[0,\pi)_4=3141.$$

Which gives us a whole number. Simplifying this for any irrational number, x, we can get any whole number using this approximation method, where if

$$x = d(0, x) \text{ is an irrational number, and}$$

$$[0, x)_n \text{ is a rational number, then}$$

$$10^{n-1}[0, x) \text{ will be a whole number.}$$

Looking at the case of $\pi + e$, we can rewrite this in the following ways, where we let

$$e = d(0, e) = [0, \tfrac{e}{2})_n + [0, \tfrac{e}{2}] = [0, \tfrac{e}{2})_n + [0, \tfrac{e}{2})_n + \delta_{n,e}, \text{ and}$$

$$\pi = d(0, e) = [0, \tfrac{\pi}{2})_n + [0, \tfrac{\pi}{2}] = [0, \tfrac{\pi}{2})_n + [0, \tfrac{\pi}{2})_n + \delta_{n,\pi}, \text{ such that}$$

$$\delta_{n,e} \text{ and } \delta_{n,\pi} \text{ are the irrational number remainders.}$$

Where we add the subscript to δ to differentiate between e and π. Now, this allows us to write our equation as,

$$\pi + e = d(0, \pi) + d(0, e) = [0, \tfrac{\pi}{2})_n + [0, \tfrac{\pi}{2})_n + \delta_{n,\pi} + [0, \tfrac{e}{2})_n + [0, \tfrac{e}{2})_n + \delta_{n,e}$$

Now, considering the case of $\pi + e$, let us assume that the solution to $\pi + e$ is a rational number. This means that we can write,

$$\pi + e = \tfrac{p}{q}, \text{ such that}$$

$$p \text{ and } q \text{ are whole numbers.}$$

This gives the setup where since we know that $d(0, \pi)$ and $d(0, e)$ are irrational numbers and assume that $d(0, \pi + e)$ is a rational number, then

$$d(0, \pi) + d(0, e) = d(0, \pi + e) = [0, \tfrac{\pi+e}{2})_n + [0, \tfrac{\pi+e}{2})_n + \delta_{n,\pi+e}, \text{ where}$$

$$\delta_{n,\pi+e} \text{ is a rational number remainder.}$$

This means that at some n-th decimal place of $\pi + e$ we find that $\pi + e = 5.85987...$ begins to repeat. Since we assumed that $\pi + e$ is rational, that means we can say that at some $m = n$, we let m be the first decimal digit where the sequence of digits begins to repeat demonstrating that $\pi + e$ is a rational number. Now, we have that if

$$d(0, \pi) + d(0, e) = [0, \tfrac{\pi}{2})_m + [0, \tfrac{\pi}{2}] + [0, \tfrac{e}{2})_m + [0, \tfrac{e}{2}], \text{ then}$$

$$[0, \tfrac{\pi}{2})_m \text{ and } [0, \tfrac{e}{2})_m \text{ are rational numbers.}$$

Where we have that

$$[0, \tfrac{\pi}{2}] = [0, \tfrac{\pi}{2})_m + \delta_{m,\pi},$$

$$[0, \tfrac{e}{2}] = [0, \tfrac{e}{2})_m + \delta_{m,e}, \text{ such that}$$

$$\delta_{m,\pi} \text{ and } \delta_{m,e} \text{ are irrational numbers.}$$

Using the initial assumption that $\pi + e$ is a rational number, we can setup our equation as follows, where

$$\pi + e = d(0, \pi + e) = d(0, \tfrac{p}{q}) = \tfrac{p}{q}, \text{and since}$$

$$d(0, \pi) + d(0, e) = d(0, \pi + e), \text{ and}$$

$$[0, \tfrac{\pi}{2})_m \text{ and } [0, \tfrac{e}{2})_m \text{ are rational numbers, then}$$

$$d(0, \pi + e) = [0, \tfrac{\pi}{2})_m + [0, \tfrac{\pi}{2})_m + \delta_{n,\pi} + [0, \tfrac{e}{2})_m + [0, \tfrac{e}{2})_m + \delta_{m,e}, \text{ gives us}$$

$$d(0, \pi + e) - [0, \tfrac{\pi}{2})_m - [0, \tfrac{\pi}{2})_m - [0, \tfrac{e}{2})_m - [0, \tfrac{e}{2})_m = \delta_{m,\pi} + \delta_{m,e}.$$

Then from the left side of the last equation, we know that $d(0, \pi + e)$, $[0, \frac{\pi}{2})_m$, and $[0, \frac{e}{2})_m$ are rational numbers, so any finite addition and subtraction will still give us a rational number. Which means that,

$$\delta_{m,\pi} + \delta_{m,e} \text{ is a rational number.}$$

Now, recalling from the previous section that for some irrational number, x, we can write

$$d(0, x) = [0, \tfrac{x}{2}) + [0, \tfrac{x}{2}) + \delta = [0, \tfrac{x}{2}) + \epsilon + [0, \tfrac{x}{2}) + \epsilon, \text{ with}$$

$$\delta = \epsilon + \epsilon = 2\epsilon.$$

Then instead of using ϵ we can break up the values of $d(0, \pi)$ and $d(0, e)$ further by setting,

$$\delta_{m,\pi} = \tfrac{\delta_{m,\pi}}{2} + \tfrac{\delta_{m,\pi}}{2},$$

$$\delta_{m,e} = \tfrac{\delta_{m,e}}{2} + \tfrac{\delta_{m,e}}{2}.$$

Which leads to a restating of $d(0, \pi + e)$ in terms of

$$d(0, \pi + e) - [0, \tfrac{\pi}{2})_m - [0, \tfrac{\pi}{2})_m - [0, \tfrac{e}{2})_m - [0, \tfrac{e}{2})_m = \tfrac{\delta_{m,\pi}}{2} + \tfrac{\delta_{m,\pi}}{2} + \tfrac{\delta_{m,e}}{2} + \tfrac{\delta_{m,e}}{2}.$$

Where we have that,

$$\delta_{m,\pi} = \tfrac{\delta_{m,\pi}}{2} + \tfrac{\delta_{m,\pi}}{2},$$

$$\delta_{m,e} = \tfrac{\delta_{m,e}}{2} + \tfrac{\delta_{m,e}}{2}.$$

Continuing in this manner of algebra, since

$$d(0, \pi) = [0, \tfrac{\pi}{2})_m + [0, \tfrac{\pi}{2}],$$

$$d(0, e) = [0, \tfrac{e}{2})_m + [0, \tfrac{e}{2}], \text{ then}$$

$$d(0, \pi) - [0, \tfrac{\pi}{2})_m = [0, \tfrac{\pi}{2}],$$

$$d(0, e) - [0, \tfrac{e}{2})_m = [0, \tfrac{e}{2}].$$

Which gives an irrational number for both $d(0, \pi) - [0, \frac{\pi}{2})_m$ and $d(0, e) - [0, \frac{e}{2})_m$. Since m is the terminating decimal place, when we subtract one of these rational number values from π or e we are still left with the sequence of digits starting at $m + 1$ that will be infinite and non-repeating. A means to demonstrate this is as follows, where we let $n = 3$ such that if

$$d(0, \pi) = 3.1415926... = [0, \pi)_3 + \delta_{m,\pi} = 3.14 + \delta_{m,\pi}, \text{ where}$$

$$d(0, \pi) - [0, \pi)_3 = 3.141592... - 3.14 = 0.001592... = \delta_{m,\pi}, \text{ such that}$$

$$\delta_{m,\pi} = 0.001592... \text{ is still an irrational number.}$$

Then we can apply this same reasoning for

$$d(0, \tfrac{\pi}{2}) + [0, \tfrac{e}{2})_m, \text{ and}$$

$$d(0, \tfrac{e}{2}) + [0, \tfrac{\pi}{2})_m.$$

Where both solutions will be some irrational number, since they both will always contain the irrational number sequence of $\frac{\pi}{2}$ or $\frac{e}{2}$ after that terminating decimal place, m, for the numerical approximations $[0, \frac{\pi}{2})_m$ and $[0, \frac{e}{2})_m$. Since we have the assumption that $\pi + e$ is a rational number, we setup that

$$d(0, \pi + e) = \frac{p}{q}, \text{ such that}$$

$$p \text{ and } q \text{ are integers.}$$

Then if we have that

$$d(0, \pi + e) \text{ is a rational number, then}$$

$$\frac{d(0,\pi+e)}{2} = \frac{d(0,\pi)+d(0,e)}{2} = \frac{d(0,\pi)}{2} + \frac{d(0,e)}{2} = d(0, \frac{\pi}{2}) + d(0, \frac{e}{2}) = \frac{p}{2q}.$$

Which gives us the following two equations,

$$d(0, \pi) + d(0, e) = \frac{p}{q}, \text{ and}$$

$$d(0, \frac{\pi}{2}) + d(0, \frac{e}{2}) = \frac{p}{2q}.$$

This lets us write our equation for $\pi + e$ as follows, where

$$d(0, \pi) + d(0, e) = [0, \frac{\pi}{2})_m + d(0, \frac{\pi}{2}) + \frac{\delta_{m,\pi}}{2} + [0, \frac{e}{2})_m + d(0, \frac{e}{2}) + \frac{\delta_{m,e}}{2} = \frac{p}{q}.$$

Since we know that for an irrational number, x, that

$$d(0, x) - [0, \frac{x}{2}) = [0, \frac{x}{2}] = [0, \frac{x}{2}) + \delta = [0, \frac{x}{2}) + \frac{\delta}{2} + \frac{\delta}{2}, \text{ where}$$

$$[0, \frac{x}{2}) + \frac{\delta}{2} = d(0, \frac{x}{2}), \text{ then}$$

$$[0, \frac{x}{2}] = d(0, \frac{x}{2}) + \frac{\delta}{2}.$$

Which is how we can write $d(0, \pi) + d(0, e) = \pi + e$ with terms $d(0, \frac{\pi}{2}) = \frac{\pi}{2}$, and $d(0, \frac{e}{2}) = \frac{e}{2}$. Rearranging terms, gives us an equation of the form

$$d(0, \pi) + d(0, e) = d(0, \frac{\pi}{2}) + d(0, \frac{e}{2}) + [0, \frac{\pi}{2})_m + [0, \frac{e}{2})_m + \frac{\delta_{m,\pi}}{2} + \frac{\delta_{m,e}}{2}.$$

Substituting in

$$d(0, \frac{\pi}{2}) + d(0, \frac{e}{2}) = \frac{p}{2q}, \text{ gives that}$$

$$\frac{p}{q} = \frac{p}{2q} + [0, \frac{\pi}{2})_m + [0, \frac{e}{2})_m + \frac{\delta_{m,\pi}}{2} + \frac{\delta_{m,e}}{2}.$$

We can then perform some algebra, and since $[0, \frac{\pi}{2})_m$, and $[0, \frac{e}{2})_m$ are rational numbers we can set

$$[0, \frac{\pi}{2})_m + [0, \frac{e}{2})_m = k, \text{ where}$$

$$k \text{ is a rational number.}$$

Then our equation come out to,

$$\frac{p}{2q} - k = \frac{\delta_{m,\pi}}{2} + \frac{\delta_{m,e}}{2}.$$

From here, we have two rational numbers, $\frac{p}{2q}$, and k, where if we subtract k from $\frac{p}{2q}$ we will still end up with a rational number. This is then equal to the summation of the two irrational numbers, $\frac{\delta_{m,\pi}}{2}$, and $\frac{\delta_{m,e}}{2}$, which must add up together to be the rational number $\frac{p}{2q} - k$. Now, to simplify up our situation a little further, let us multiply both sides by two. Where we get

$$\tfrac{p}{q} - 2k = \delta_{m,\pi} + \delta_{m,e}.$$

Now, since any numerical-approximation of the setup $[0,x)_n$ can be turned into a whole number value by multiplying by 10^{n-1}, we can represent k as a whole number by multiplying by 10^{m-1}, where

$$\tfrac{10^{m-1}p}{q} - 2 * 10^{m-1}k = 10^{m-1}\delta_{m,\pi} + 10^{m-1}\delta_{m,e}.$$

The example of this being that if we have $[0,\pi)_3 = 3.14$, then $10^2[0,\pi) = 314$, and if we looked at the case of $d(0,\pi) = [0,\pi)_3 + \delta_{3,\pi} = 3.14 + 0.00159...$ then multiplying by 10^2 gives us $10^2 d(0,\pi) = 314.159...$, and $10^2[0,\pi)_3 + \delta_{3,\pi} = 314 + 0.159...$ What we find is that by multiplying 10^{m-1} throughout, we now comprise the ability to state the following such that if we have that p and q are integers, then

$$10^{m-1}p, \text{ and } 2 * 10^{m-1}k \text{ are integers, since}$$

$$k = [0,\pi)_m + [0,e)_m, \text{ and}$$

$$10^{m-1}[0,\pi)_m,\ 10^{m-1}[0,e)_m, \text{ are whole numbers.}$$

Next, let us simplify our terms again, where we set

$$2 * 10^{m-1}k = g,$$

$$10^{m-1}p = r.$$

Then by assuming $\pi + e$ is a rational number, we have defined $n = m$ to be the terminating decimal place by which $\pi + e$ has a sequence which repeats itself forever, making it a rational number. Then by multiplying $\delta_{m,\pi}$, and $\delta_{m,e}$ by 10^{m-1}, we have that

$$0 < 10^{m-1}\delta_{m,\pi} < 1, \text{ and}$$

$$0 < 10^{m-1}\delta_{m,e} < 1.$$

Which if we rewrite our equation, we get

$$\tfrac{r}{q} - g = 10^{m-1}\delta_{m,\pi} + 10^{m-1}\delta_{m,e}, \text{ such that}$$

$$r,\ q,\ g, \text{are whole numbers.}$$

Then, multiplying everything by q, gives the following equation,

$$r - qg = 10^{m-1}q\delta_{m,\pi} + 10^{m-1}q\delta_{m,e} = 10^{m-1}q(\delta_{m,\pi} + \delta_{m,e})$$

Where if r, q, and g are whole numbers, then $r - qg$ will be an integer, such that $10^{m-1}q(\delta_{m,\pi} + \delta_{m,e})$ must be an integer as well. Let us once again make another simplification to our equation. Set

$$10^{m-1}\delta_{m,\pi} = \alpha,$$

$$10^{m-1}\delta_{m,e} = \beta.$$

Then we end up in the following setup, where

$$r,\ q,\ g, \text{ are whole numbers,}$$

$$\alpha,\ \beta, \text{ are irrational numbers,}$$

$$0 < \alpha < 1,$$

$$0 < \beta < 1, \text{ such that}$$

$$r - gq = q\alpha + q\beta.$$

Then $q\alpha$ and $q\beta$ are irrational numbers, where in the previous section, it was shown that multiplying any irrational number by itself will produce an irrational number. Such that multiplying α or β by itself q times with q being a whole number, will still produce an irrational number. Now, we have one final simplification, where we write that,

$$r - gq = v, \text{ with } v \text{ being an integer,}$$

$$q\alpha = t, \text{ with } t \text{ being an irrational number, and}$$

$$q\beta = s, \text{ with } s \text{ being an irrational number.}$$

Which ultimately leads to the final form of our equation being,

$$t + s = v.$$

Which now gives us the setup for when two irrational numbers equal an integer. Another way to show this is that if we divide everything by q we find that

$$\tfrac{t+s}{q} = \alpha + \beta = \tfrac{v}{q}.$$

Where we now have a setup that provides the sum of two irrational numbers equal to a rational number. Which if we were to use this process for any two irrational numbers other than $\pi + e$, we can end up with a similar solution of the form,

$$t + s = v, \text{ such that}$$

$$t, \text{ s are irrational numbers, and}$$

$$v \text{ is an integer.}$$

This gives a solution that if $\alpha = 0$ or $\beta = 0$, then either of the initial Irrational numbers were Rational from the outset, and the solution becomes trivialized. Then for any Rational number, $\frac{p}{q}$, summed with some Irrational number, γ, the solution is always Irrational since,

$$\tfrac{p}{q} + \gamma = d(0, \tfrac{p}{q}) + d(0, \gamma) = [0, \tfrac{p}{q}) + \delta_{\frac{p}{q}} + [0, \gamma) + \delta_\gamma,$$

Allows us to multiply by q, with

$$q(\tfrac{p}{q} + \gamma) = [0, p) + \delta_p + [0, q\gamma) + \delta_{q\gamma}; \text{ where,}$$

$$\delta_p = 0, \text{ gives the equation,}$$

$$p + q\gamma = [0, p) + [0, q\gamma) + \delta_{q\gamma}.$$

Then for all $m > 0$, we get $10^{m-1}\delta_\alpha = \alpha = 0$, and solving for $10^{m-1}\delta_{q\gamma} = \beta$; such that, $0 < \beta < 1$, will always produce some Irrational number remainder, requiring that if $t > 0$, and $s > 0$, then,

$$q\alpha + q\beta = q\beta = s \neq t + s = v.$$

Similarly, this solution setup fails if the sum of two Irrational numbers are not equal to some Rational number. Since it has already been shown that for $\frac{\pi}{2} + \frac{\pi}{2}$, that the answer will be equal to the Irrational number value of π, going through the process for $\frac{\pi}{2} + \frac{\pi}{2}$, we will end up with some,

$$10^{m-1}(d(0, \tfrac{\pi}{2}) - [0, \tfrac{\pi}{4})_m - [0, \tfrac{\pi}{4})_m) = 10^{m-1}\delta_{m,\frac{\pi}{2}} = \alpha = \beta, \text{ then if}$$

$$\tfrac{\pi}{2} + \tfrac{\pi}{2} = \pi, \text{ is an irrational number, then any}$$

$$\alpha + \beta = \tfrac{v}{q}, \text{ will be an irrational number, except that}$$

$$\tfrac{v}{q} \text{ is a rational number, then}$$

$$\alpha + \beta \neq \tfrac{v}{q}, \text{ where we cannot have}$$

$$\alpha + \beta \neq \alpha + \beta.$$

Then if we multiply everything by q and set $q\alpha = q\beta = s = t$ then we get the contradiction,

$$t + s \neq v = t + s.$$

Then for the sum of any two Irrational numbers, such as π and e, to equal a Rational number solution, then there exists some Irrational remainder of the form:

$$10^{m-1}\delta_{m,\pi} + 10^{m-1}\delta_{m,e} = 1.$$

Therefore, any two Irrational numbers summing to a number that is Rational requires that,

$$\alpha = 1 - \beta, \text{ and}$$

$$\beta = 1 - \alpha.$$

If $\pi + e$ is equal to a Rational number, then it can be shown that for all unique solutions for π and for e, there exists some Rational number, $\frac{p}{q}$; where,

$$\pi = \tfrac{p}{q} - e, \text{ and}$$

$$e = \tfrac{p}{q} - \pi.$$

Then if $\pi = \frac{p}{q} - e$, we can use the Leibniz series for π and the Euler series for e, remove some $\frac{p}{q}$ value and get an equivalent solution. The two power series are written as,

$$\pi = \sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1}, \text{ and}$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Using the geometric series,

$$\sum_{n=0}^{\infty} \frac{1}{a^n} = \frac{1}{a-1}$$

We can express our equivalence in the form:

$$\pi = \sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1} = \sum_{n=0}^{\infty} \left(\frac{p}{(q+1)^n} - \frac{1}{n!} \right) = \frac{p}{q} - e.$$

In order to simplify the formula, the Leibniz series can be rewritten to give all positive value terms; such that,

$$\sum_{n=0}^{\infty} \left(\frac{4(-1)^{(2n)}}{2(2n)+1} + \frac{4(-1)^{(2n+1)}}{2(2n+1)+1} \right) = \sum_{n=0}^{\infty} \frac{8}{(4n+1)(4n+3)}.$$

Then, whether the power series is represented as,

$$\sum_{n=0}^{\infty} \frac{8}{(4n+1)(4n+3)} = \sum_{n=0}^{\infty} \left(\frac{p}{(q+1)^n} - \frac{1}{n!} \right),$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} = \sum_{n=0}^{\infty} \left(\frac{p}{(q+1)^n} - \frac{8}{(4n+1)(4n+3)} \right), \text{ or}$$

$$\sum_{n=0}^{\infty} \frac{p}{(q+1)^n} = \sum_{n=0}^{\infty} \left(\frac{8}{(4n+1)(4n+3)} + \frac{1}{n!} \right),$$

There exists no sequence within the power series' given such that the geometric series can be shown to equate the sum of π and e; meaning that for all $10^{m-1}\delta_{m,\pi} = \alpha$ and $10^{m-1}\delta_{m,e} = \beta$,

$$\alpha + \beta \neq 1, \text{ and,}$$

$$q\alpha + q\beta = t + s \neq v = t + s, \text{ requiring that}$$

$\pi + e$ is an Irrational number.

References

[1] Boole, George (2007) *An Investigation of the Laws of Thought: on Which Are Founded the Mathematical Theories of Logic and Probabilities*. Kessinger Publishing.

[2] Dedekind, Richard. (1901) *Essays on the Theory of Numbers: I. Continuity and Irrational Numbers, II. The Nature and Meaning of Numbers.* Translated by Prof. Wooster Woodruff Beman, The Open Court Publishing Company.

[3] Leibniz, Gottfried Wilhelm, and Carl Immanuel Gerhardt. (1920). *The Early Mathematical Manuscripts of Leibniz: Translated from the Latin Texts Published by Carl Immanuel Gerhardt with Critical and Historical Notes, by J.M. Child.* The Open Court Publishing Company.

[4] Niven, Ivan (1947) *A Simple Proof That π is Irrational*. Bulletin of the American Mathematical Society, 53(6), p. 509. doi:10.1090/s0002-9904-1947-08821-2.

[5] Roy, Ranjan. (1990) *The Discovery of the Series Formula for π by Leibniz, Gregory and Nilakantha.* Mathematics Magazine, 63(5), 291-306. doi:10.2307/2690896.

[6] Sondow, Jonathan. (2010). *A Geometric Proof that e is Irrational and a New Measure of its Irrationality.* ArXiv. arXiv:0704.1282