Newton's Laws and the Principle of Least Action

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Abstract

The Principle of Least Action is a fundamental pillar of theoretical physics that unifies diverse phenomena under a variational formulation. This deeply elegant formulation of physics provides an alternative and powerful approach to derive Newton's Laws. This principle states that the path followed by a physical system between two states is the one that minimizes (or makes stationary) a quantity called action, defined as the time integral of the Lagrangian. In this article, we explore how the Principle of Least Action connects with Newton's laws, showing that the latter are a direct consequence of a variational approach. We present an introduction to the action and the Lagrangian, followed by an application of variational calculus to obtain the Euler-Lagrange equations, which translate into Newton's second laws under specific conditions. This approach demonstrates the elegance and generality of the Lagrangian formalism.

1 Introduction

Newton's laws of motion are the foundation of classical mechanics. However, they can be derived from a more general principle: the Principle of Least Action, initially introduced by Pierre-Louis Maupertuis and later developed by Joseph-Louis Lagrange and William Rowan Hamilton. This principle states that the path followed by a physical system between two points in spacetime is the one that minimizes (or makes stationary) a scalar quantity called action.

2 The Principle of Least Action

The action, S, is defined as the time integral of a function called the Lagrangian, L, which depends on the generalized coordinates q_i, their time derivatives $\dot{q}_i$, and time t:

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt,$$

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where $L = T - V$ is the Lagrangian, T is the kinetic energy, and V is the potential energy. The Principle of Least Action states that the actual trajectories of the system make this action stationary, i.e., $\delta S = 0$. This means that the actual trajectories followed by a physical system make the action stationary (it may be a minimum, a maximum, or an inflection point).

3 Derivation of the Equations of Motion

To find the equations of motion, we apply variational calculus. Consider a small variation δq_i in the generalized coordinates. Let there be a small variation in the particle's path, $q(t) \to q(t) + \eta(t)$, where $\eta(t)$ is an arbitrary function satisfying $\eta(t_1) = \eta(t_2) = 0$. Substituting into the action and expanding in a Taylor series, the corresponding change in the action is:

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \eta + \frac{\partial L}{\partial \dot{q}} \dot{\eta} \right) dt.$$

Integrating by parts the term containing $\dot{\eta}$ and applying $\eta(t_1) = \eta(t_2) = 0$, we obtain:

$$\delta S = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \eta \, dt.$$

For $\delta S = 0$ for any $\eta(t)$, the term within the brackets must be zero:

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = 0.$$

This is the Euler-Lagrange equation, describing the system's equations of motion.

4 Relation to Newton's Laws

In a particle system under Cartesian coordinates, the typical Lagrangian is expressed as:

$$L = T - V = \frac{1}{2} m \dot{q}^2 - V(q),$$

where T is the kinetic energy and V the potential energy. Substituting into the Euler-Lagrange equations, we obtain:

$$m\ddot{q} = -\frac{\partial V}{\partial q},$$

which is Newton's second law $F = ma$, where the force F is related to the gradient of the potential energy, $F = -\nabla V$.

5 Conclusion

The Principle of Least Action provides a more general and elegant framework for deriving the laws of motion. This unifying approach to classical mechanics allows deriving Newton's laws from a variational formulation. While Newton's laws focus on forces and accelerations, the variational approach reveals that the dynamics of classical systems are rooted in the optimization of a scalar quantity: the action. This formalism is not only elegant and encapsulates Newton's laws but also extends naturally to other areas of physics such as quantum mechanics and general relativity, demonstrating its unifying power in physics.

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