Geometrical Meaning of Covariant and Contravariant Vector

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Abstract

This article presented the accurate understanding of the geometrical meaning of covariant and contravariant vectors, critically analyzing three prevalent misconceptions about this concept found in numerous sources.

Unfortunately, after examining numerous papers and resources on the geometric meaning of covariant and contravariant vector or the differences between them, I found three prevalent interpretations, all of which are incorrect. Not a single source provides a correct geometric definition consistent with the mathematical definition of these concepts or the algebraic relationships connecting them.

I will first present the correct definition of the geometric meaning of covariant vector identified with subscripts and contravariant vector identified with superscripts. For simplicity, I will explain the concept in two dimensions, which can then be generalized to any number of dimensions.

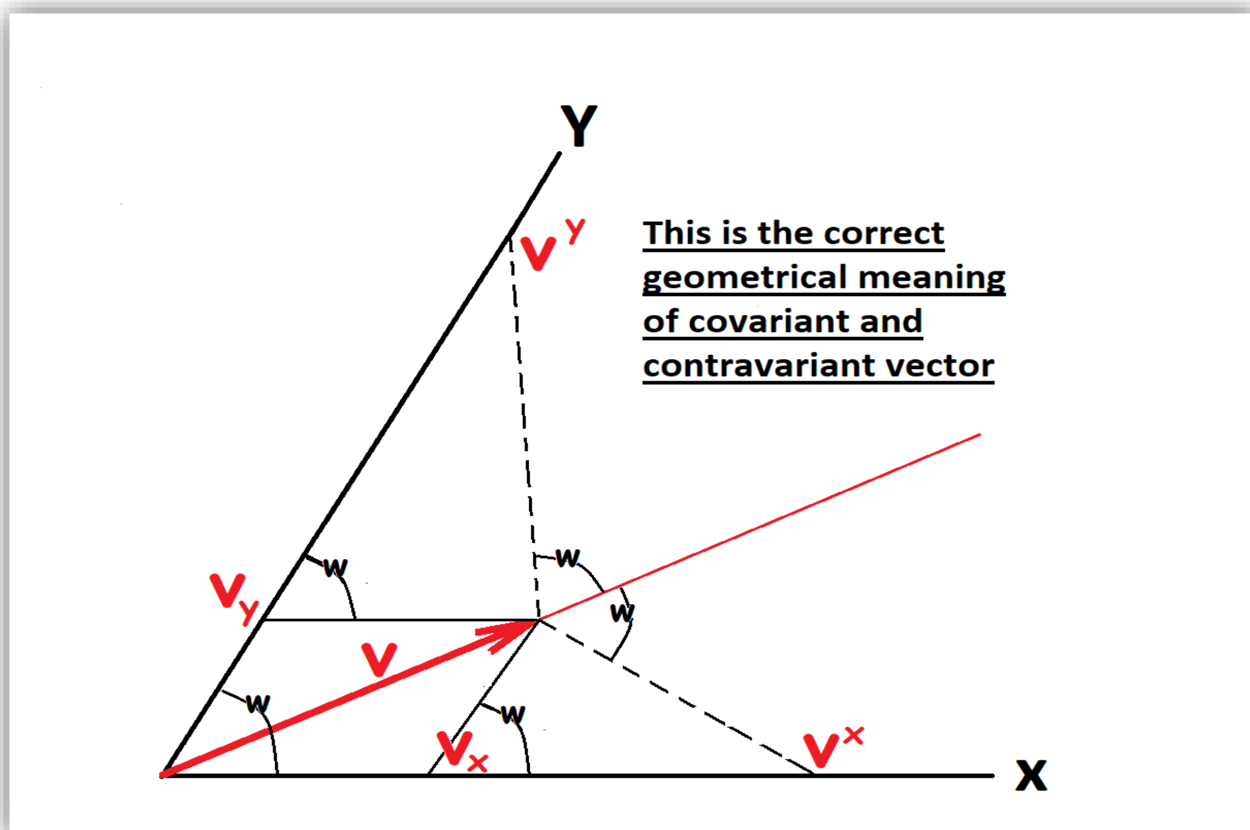

Fig (1)

As shown in fig (1), a vector as an objective reality is a single entity, but the method of analyzing it into components divides it into two categories. The components resulting from the first method of analysis are called the covariant components of the vector, while the components from the second method are called the contravariant components. It is essential to note that both sets of components refer to the same physical reality. In the first method, illustrated with solid lines, we project a straight line from the arrow's tip (representing the vector) to intersect the x-axis at an angle equal to the angle between the x and y axes. This determines the value of the covariant component of the vector in the x-axis direction. Similarly, the value in the y-axis direction is determined.

In the case of contravariant components, the projection method, indicated by dashed lines, is used. To determine the contravariant component of the vector along the x-axis, a straight line is drawn from the arrow's tip to the x-axis, ensuring that the angle between the vector and this line equals the angle between the x and y axes. The value in the y direction is calculated similarly.

This is the meaning of the covariant and contravariant components of a vector. The covariant components are so named because, when transformed from one coordinate system to another, they change in the same way as the differentials of dimensions dx, dy …etc. In contrast, components calculated using the other method transform inversely relative to these quantities.

This definition aligns with the fundamental algebraic relationship connecting the covariant and contravariant vector components:

$$\mathbf{v_x v^x = v_y v^y = |v|^2} \qquad \textbf{(1)}$$

Now, let us address the incorrect definitions of the geometric meaning of covariant and contravariant vector, starting with the most inaccurate [1]. This definition attributes the difference between the two concepts to the physical quantity represented by the vector, focusing on measurement units or dimensions. This interpretation is far removed from reality. Changes in measurement units do not affect the relationships between physical quantities. Furthermore, when we classify a quantity as a vector, it suffices to indicate how to handle it mathematically, regardless of whether the vector represents one physical quantity or another.

The second incorrect definition [2], though better than the first, claims that covariant and contravariant vector components of a vector result from analyzing it in two different coordinate systems: the primary system chosen and a secondary, auxiliary system associated with it. This approach is illustrated in Fig (2).

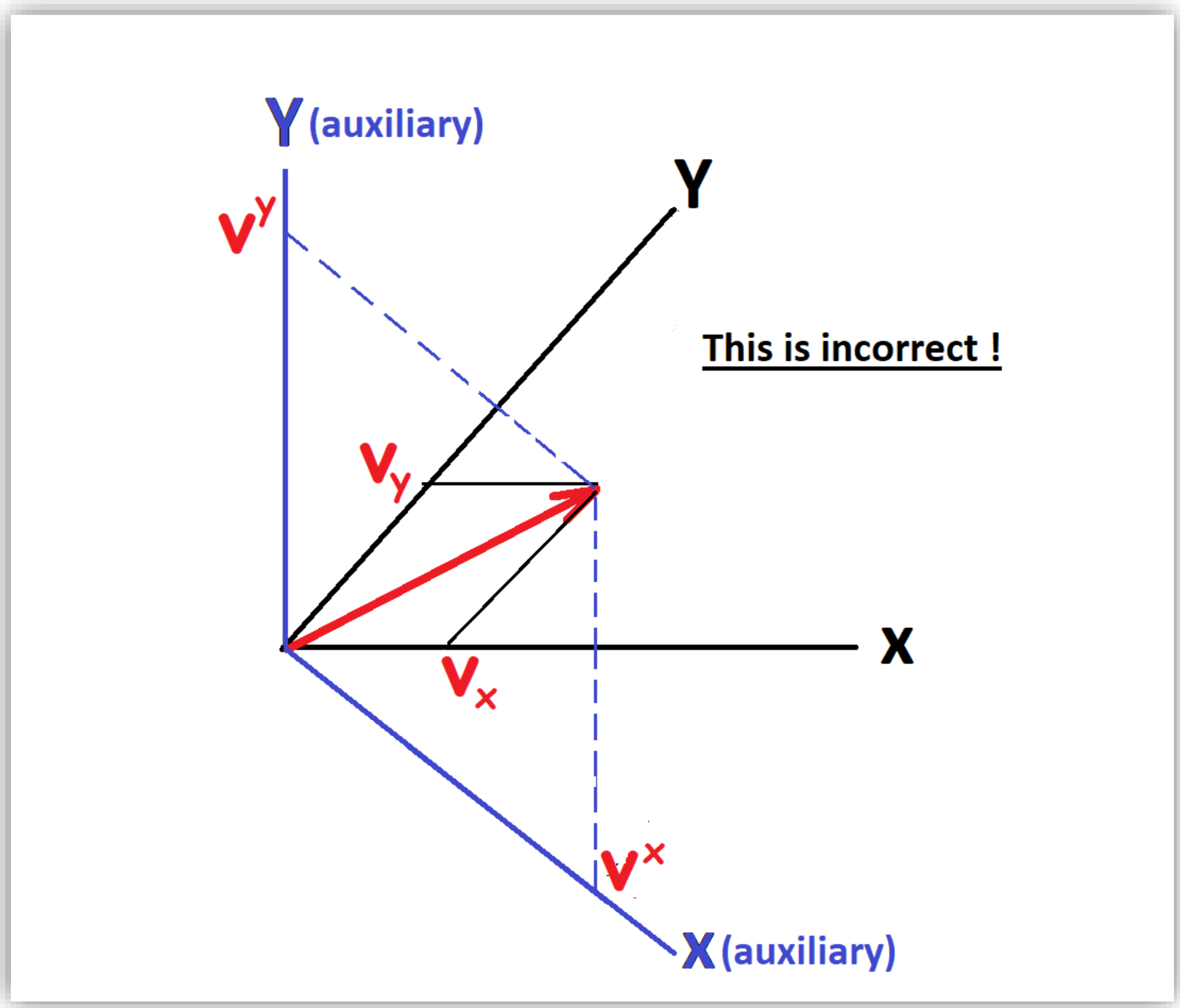

Fig (2)

It becomes evident that this method does not satisfy the fundamental algebraic relationship between covariant and contravariant vector components equation(1), proving its invalidity. Additionally, this approach complicates vector analysis by arbitrarily introducing auxiliary coordinate systems. Since auxiliary coordinate systems can always be defined in multiple ways, analyzing vectors based on them does not create a distinct meaning for covariant versus contravariant components. Instead, it makes the definition of contravariant components identical to that of covariant components but in a different coordinate system which means that the first is merely a transformation of the second.

The third [3] incorrect approach to defining the geometrical meaning of covariant and contravariant vector is the most widespread and closest to the correct method, as shown in Fig (3):

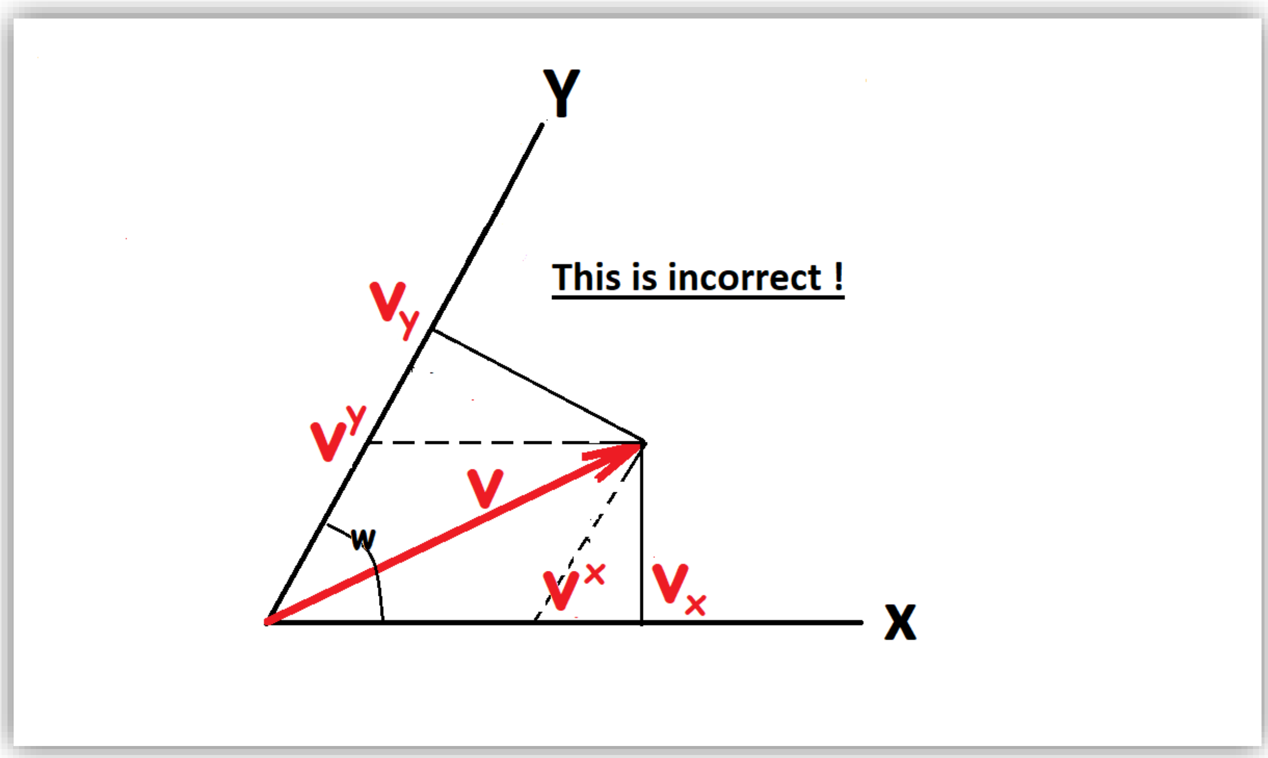

Fig (3)

This approach aligns with the correct method in recognizing that the difference between covariant and contravariant vector lies in the analysis method within the same coordinate system. However, its error lies in how covariant components are analyzed. According to this method, the covariant components are found from the orthogonal projection of the vector onto the coordinate axes. However, it is easy to observe that this interpretation of covariant and contravariant components fails to satisfy the fundamental algebraic relationship of equation (1).

References:

[1] See for example: *Introduction to Differential Geometry & General Relativity*, Stefan Waner.This error is more common in forums and educational videos.

[2] See for example: *General Relativity-Irreducible Minimum*, Alan L Myers.

[3] See for example: *Wikipedia: Covariance and contravariance of vectors*. Searching Google with image on this topic yields numerous results that match Fig (3), this indicates the widespread prevalence of this misconception