Continued Fraction Representations of Universal Numbers and Approximation

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Abstract

We present continued fraction representations of universal numbers and some approximations to underline on what nature's infinitely repeated processes are footed as the mathematical basis of our life and the entire cosmos.

Continued Fraction Representations

The golden ratio number φ

$$\varphi = \frac{\sqrt{5}-1}{2} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}.$$

$$= 0.618033989\ldots$$

$$\varphi^3 = \frac{\sqrt{20}-4}{2} = \cfrac{1}{4+\cfrac{1}{4+\cfrac{1}{4+\cdots}}}.$$

$$= 0.236067977\ \ldots$$

$$D_{KK} = 5 + \cfrac{1}{4+\cfrac{1}{4+\cfrac{1}{4+\cdots}}}.$$

$$= 5 + \underbrace{0.236067977\ \ldots}_{\text{fractal part of } D_{\mathrm{KK}}}$$

$$\varphi^5 = \frac{\sqrt{125}-11}{2} = \cfrac{1}{11+\cfrac{1}{11+\cfrac{1}{11+\cdots.}}}$$

$$= 0.0901699\ldots$$

$$\varphi^{-5} = 11 + \varphi^5 = 11 + \cfrac{1}{11+\cfrac{1}{11+\cfrac{1}{11+\cdots.}}}$$

$$= 11.0901699\ldots$$

(fractal representation of the dimension in *Witten*'s *M*-theory) [1]

However, one can trace back the fifth power of φ by the relation

$$\varphi^5 = 5\varphi - 3$$

and use again the continued fraction of φ. This may be important when valuate phase transitions governed by the fifth power of φ or *Fibonacci* anyon-based quantum computation [2].

More generally, the infinitely continued fraction representing odd integer powers of the golden mean is given by

$$\varphi^n = \cfrac{1}{b+\cfrac{1}{b+\cfrac{1}{b+\cdots.}}}$$

where

$$b = \varphi^{-n} - \varphi^n$$

For big numbers b, the quotient of consecutive numbers approximates to

$$\frac{b(n)}{b(n+1)} \approx \varphi^2.$$

Even integer powers of the golden mean can be approximated using $b = \varphi^{-n} + \varphi^n$. The largest deviation results from the smallest number of $b = 2$ yielding $\sqrt{2} - 1 = 0.41421356\ldots$ (see below) instead of $\varphi^2 = 0.38196601$.

However, when choosing $b = \varphi^{-2} - \varphi^2 = \sqrt{5} = 2.236067978\ldots$, one can exactly reproduce $\varphi^2 = 0.38196601\ldots$

$$\varphi^2 = \cfrac{1}{\sqrt{5} + \cfrac{1}{\sqrt{5} + \cfrac{1}{\sqrt{5} + \cdots}}}.$$

and further

$$\varphi^4 = \cfrac{1}{3\sqrt{5} + \cfrac{1}{3\sqrt{5} + \cfrac{1}{3\sqrt{5} + \cdots}}}.$$

$$\varphi^6 = \cfrac{1}{8\sqrt{5} + \cfrac{1}{8\sqrt{5} + \cfrac{1}{8\sqrt{5} + \cdots}}}.$$

$$\varphi^8 = \cfrac{1}{21\sqrt{5} + \cfrac{1}{21\sqrt{5} + \cfrac{1}{21\sqrt{5} + \cdots}}}.$$

Table 1 summarizes the results for the continued fractions of powers of the golden mean. One can identify the factors given as numbers of the *Lucas* series *L* = {2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, …}

Table 1. Continued Fraction Results for Powers of the Golden Mean φ^n and for Silver Mean

n	$\varphi^{-n} - \varphi^n$	$\varphi^{-n} + \varphi^n$	φ^n	φ^{-n}
silver mean	2		0.41421356237310	2.41421356237310
1	1		0.61803398874989	1.61803398874989
2	$\sqrt{5}$	3	0.38196601125010	2.61803398874989
3	4		0.23606797749979	4.23606797749979
4	$3\cdot\sqrt{5}$	7	0.14589803375031	6.85410196624968
5	11		0.09016994374947	11.09016994374947
6	$8\cdot\sqrt{5}$	18	0.05572809000084	17.94427190999914
7	29		0.03444185374863	29.03444185374863
8	$21\cdot\sqrt{5}$	47	0.02128623625220	46.97871376374776
9	76		0.01315561749642	76.01315561749642
10	$55\cdot\sqrt{5}$	123	0.00813061875578	122.991869381244110

Rogers-Ramanujan continued fraction [3]

$$r(\tau) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}}$$

where $q = e(\tau) = e^{2\pi i\tau}$. For $\tau = 0$ one gets the golden mean φ, and for $\tau = i = \sqrt{-1}$ the result is [2]

$$r(i) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{1+\sqrt{5}}{2} = \sqrt{1+\varphi^{-2}} - \varphi^{-1} \approx \pi\varphi^5 + (\frac{\pi\varphi^5}{10})^2$$

There is another nice continued fraction approach to represent φ^{-1}

$$r = \sqrt{1+\sqrt{1+\sqrt{1}+\cdots}} = \sqrt{1+r}$$

$$r^2 - r - 1 = 0 \quad \text{yields} \quad r = \varphi^{-1} = 1.6180339887\ldots$$

Archimedes' constant π as an approximation

$$\pi \approx 3 + \frac{3}{5} \cdot \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cdots}}}$$

$$= 3 + 0.1416407\ldots$$

However, the fractal part of π can be well approximated with the aid of the reciprocal of *Sommerfeld*'s fine structure constant α, where $\alpha^{-1} \approx 137$ [4]

$$\pi - 3 \approx \frac{16}{137-24} = 0.141592\ldots,$$

True circle constant π [5]

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cdots}}}$$

$$= 3 + 0.1415926\ldots$$

Curiously, if one applies $b = \pi - \pi^{-1} = 2.823282767\ldots$, then the continued fraction resulted exactly in *Archimedes*' constant π. This value of b is quite near $2\sqrt{2} = 2.828427\ldots$. It is simple to explain this result solving the quadratic equation

$$x^2 - (\pi - \pi^{-1})x - 1 = 0$$

One gets $x_1 = \pi$ and $x_2 = -\pi^{-1}$.

Euler's number *e*

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \cfrac{5}{6 + \cdots}}}}}}$$

$$= 2.718281828459045....$$

The square root of number 2 (compare this representation with that for φ)

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}.$$

$$= 1.41421356237 \ldots$$

$\sqrt{2} - 1 = 0.4142135 \ldots$ is known as silver mean (see Table 1).

A special infinitely continued fraction based on the golden mean was introduced by the author

$$\tilde{\varphi} = \frac{\sqrt{5+\delta} - 1}{2} = \cfrac{1}{1 - \delta_1 + \cfrac{1}{1 - \delta_1 + \cfrac{1}{1 - \delta_1 + \cdots}}}.$$

The calculation with $\delta_1 = 0.00374774 \approx \frac{1}{266.\bar{6}} \ldots \approx \delta/\varphi \approx \frac{\varphi^5}{4!}$ yielded $\tilde{\varphi} = 0.619071099$. One may speak of a nested golden mean based continued fraction. The value $\tilde{\varphi}$ represents the quantized golden mean in our theory of the gyromagnetic factor of the electron without using the half-spin theory. Very interesting is indeed the number $266.\bar{6}$. The division by integers frequently delivers numbers with repeating decimals, exemplified by $266.\bar{6}/ 24 = 11.\bar{1}$. [6]
The importance of the connection between φ and π has been explained in another contribution of the present author [7].

Due to its importance in cosmology two different continued fraction representations for the *tanh(x)* (tangens hyperbolicus) function were added [8]

$$\tanh(x) = \cfrac{x}{1 + \cfrac{x^2}{3 + \cfrac{x^2}{5 + \cdots}}}.$$

$$\tanh(x) = \cfrac{1}{\cfrac{1}{x} + \cfrac{1}{\cfrac{3}{x} + \cfrac{1}{\cfrac{5}{x} + \cdots}}}.$$

An own small *QBASIC* program code to generate simple infinite continued fraction calculations:

You need as number input the mean parameter and a delta parameter. For instance, to generate the fraction for φ use input 1 and 0, for φ^3 use input 4 and 0, for φ^5 use 11 and 0, for φ^7 use 29 and 0, for φ^9 use 76 and 0, and for $\tilde{\varphi}$ use 1 and -0.00374774 [4].
Phi is calculated normally and printed on the top of the output for your control. If you need higher accuracy, enhance the loop parameter I_{max} = 1000 to a suited value larger as 1000.

```
OPEN "o", #2, "contfr.out"
PRINT #2,
PRINT #2, " Auxillary program to generate infinitely continued fractions"
PRINT #2,
PRINT #2,
Phi# = (SQR(5) - 1) / 2
PRINT #2, USING " Phi = ###.##############"; Phi#
PRINT #2,
INPUT " Input parameter: "; XPH#
INPUT " Input delta:     "; XDEL#
REM XDEL# = -0.00374774
XPH# = 1 + XDEL#
XPHR# = 1 / XPH#
XCONT# = XPH# + XPHR#
FOR I = 1 TO 1000
   XCONT# = XPH# + 1 / XCONT#
NEXT I
PRINT #2, USING " Cont Frac = ###.##############"; 1 / XCONT#
CLOSE #2
END
```

A small *QBASIC* program code to generate *Euler*'s number:

```
OPEN "o", #2, "euler.out"
REM Generate Euler's Number
Add# = 1
Euler# = 1
For I = 1 to 2000
Add1# = Add# * 1/I
Euler# = Euler# + Add1#
Add# = Add1#
Next I
Print #2, USING " Euler number e = ###.#############"; Euler#
PRINT #2,
CLOSE #2
END
```

References

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