Group Theory: Problems and Solutions (Part 2)

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Abstract. There is nothing new about group theory in this paper. It presents group theory problems at undergraduate level and their solutions. In presenting the solutions, we avoid using advanced theorems from group theory but we try to discuss the solutions using elementary facts in group theory.

Let $P \in Syl_p(G)$ and assume $N \trianglelefteq G$. Use conjugacy part of Sylow's Theorem to prove that $P \cap N$ is a Sylow p-subgroup of N.

Let $Q \in Syl_p(N)$. Since $P \in Syl_p(G)$ and Q is a p-subgroup of G, there exists $g \in G$ such that

$$Q \le gPg^{-1}. \tag{1}$$

Clearly, $Q \le N$ and $N \le gNg^{-1}$ by the normality of N in G; so

$$Q \le gNg^{-1}. \tag{2}$$

From (1) and (2),

$$Q \le \; gPg^{-1} \cap \; gNg^{-1} \le \; g(P \cap N)g^{-1}. \tag{3}$$

Since $Q \in Syl_p(N)$ and $P \cap N$ is a p-subgroup of N, there exists $n \in N$ such that

$$P \cap N \le nQn^{-1}. \tag{4}$$

To show $|P \cap N| = |Q|$. From (4), $|P \cap N| \le |nQn^{-1}|$ and $|nQn^{-1}| = |Q|$; so

$$|P \cap N| \le |Q|. \tag{5}$$

From (3), $|Q| \le |g(P \cap N)g^{-1}|$ and $|g(P \cap N)g^{-1}| = |P \cap N|$; so

$$|Q| \le |P \cap N|. \tag{6}$$

From (5) and (6), $|P \cap N| = |Q|$ and thus $P \cap N \in Syl_p(N)$.

Reference:
D. S. Dummit, R. M. Foote, *Abstract Algebra*, John Wiley and Sons, Inc., 1990.

Let P be a normal Sylow p-subgroup of G and let H be any subgroup of G. Prove that $P \cap H$ is the unique Sylow p-subgroup of H.

Let $Q \in Syl_p(H)$. Since $Q \in Syl_p(H)$ and $P \cap H$ is a p-subgroup of H, there exists $h \in H$ such that $P \cap H \le hQh^{-1}$ and thus

$$|P \cap H| \le |hQh^{-1}|. \tag{7}$$

From (7),

$$|P \cap H| \leq |Q|. \tag{8}$$

Since $P \in Syl_p(G)$ and Q is a p-subgroup of G, there exists $g \in G$ such that $Q \leq gPg^{-1}$. Since $Q \leq gPg^{-1}$ and $gPg^{-1} \leq P$, $Q \leq P$. Moreover, since $Q \leq P$ and $Q \leq H$, $Q \leq P \cap H$ and thus

$$|Q| \leq |P \cap H|. \tag{9}$$

From (8) and (9), $|P \cap H| = |Q|$ and thus $P \cap H \in Syl_p(H)$. Since $P \cap H \trianglelefteq H$, $P \cap H$ is the unique Sylow p-subgroup of H.

Reference:

D. S. Dummit, R. M. Foote, *Abstract Algebra*, John Wiley and Sons, Inc., 1990.

Let $P \in Syl_p(G)$ and let $H \leq G$. Prove that $gPg^{-1} \cap H$ is a Sylow p-subgroup of H for some $g \in G$.

Let $Q \in Syl_p(H)$. Then there exists $g \in G$ such that $Q \leq gPg^{-1}$. Since $Q = Q \cap H$ and $Q \cap H \leq gPg^{-1} \cap H$, $Q \leq gPg^{-1} \cap H$ and thus

$$|Q| \leq |gPg^{-1} \cap H|. \tag{10}$$

Since $Q \in Syl_p(H)$ and $gPg^{-1} \cap H$ is a p-subgroup of H, there exists $h \in H$ such that $gPg^{-1} \cap H \leq hQh^{-1}$ and thus

$$|gPg^{-1} \cap H| \leq |hQh^{-1}| = |Q|. \tag{11}$$

From (10) and (11), $|Q| = |gPg^{-1} \cap H|$ and hence $gPg^{-1} \cap H$ is a Sylow p-subgroup of H.

Reference:

D. S. Dummit, R. M. Foote, *Abstract Algebra*, John Wiley and Sons, Inc., 1990.

Suppose G is a group of order 385. Prove that G has exactly one subgroup of order 77.

By Sylow's theorems, G has a normal Sylow 7-subgroup H and a normal Sylow 11-subgroup K. So HK is a subgroup of G and $|HK| = \frac{|H||K|}{|H \cap K|} = 77$. Let A and B be subgroups of G of order 77. Notice that $|A \cap B|$ divides $|A|$ and hence $|A \cap B| \in \{1, 7, 11, 77\}$. Suppose $|A \cap B| = 1$. Then $|AB| = \frac{|A||B|}{|A \cap B|} = \frac{77 \cdot 77}{1} = 5929$, a contradiction since AB is a subset of G. Suppose $|A \cap B| = 7$. Then $|AB| = \frac{|A||B|}{|A \cap B|} = \frac{77 \cdot 77}{7} = 847$, a contradiction since AB is a subset of G. Suppose $|A \cap B| = 11$. Then $|AB| = \frac{|A||B|}{|A \cap B|} = \frac{77 \cdot 77}{11} = 539$, a contradiction since AB is a subset of G.

Thus $|A\cap B| = 77$. Since $A\cap B \le A$ and $|A\cap B| = |A|$, $A\cap B = A$. Since $A\cap B \le B$ and $|A\cap B| = |B|$, $A\cap B = B$. To conclude that $A = A\cap B = B$.

A group of order $7\cdot 5\cdot 23^2$ has a normal Sylow 5-subgroup.

Let $|G| = 7\cdot 5\cdot 23^2$. So G has a normal Sylow 23-subgroup N. Let $\bar{G} = G/N$. Its order is 35 and $\bar{G}$ has a normal Sylow 5-subgroup $\bar{H}$. Let $H = \{x\in G \mid xN \in \bar{H}\}$. Then H is a subgroup of G and $\bar{H}\cong H/N$; thus

$$5 = |\bar{H}| = \frac{|H|}{|N|} = \frac{|H|}{23^2}.$$

So $|H| = 5\cdot 23^2$ and H has a Sylow 5-subgroup Q. Let φ be an automorphism of H. Then $\varphi(Q)$ is a Sylow 5-subgroup of H. H has exactly one Sylow 5-subgroup and thus $\varphi(Q) = Q$. Since $\bar{H}$ is normal in $\bar{G}$, H is normal in G. Since Q is a characteristic subgroup of H and H is normal in G, Q is normal in G.

Show that a group G of order 108 has a normal subgroup of order 9 or 27.

By Sylow's theorem, $n_3(G)\in\{1,4\}$. If $n_3(G) = 1$, then there is one and only one Sylow 3-subgroup of G and it is normal in G. Suppose $n_3(G) = 4$. Let S and T be two distinct Sylow 3-subgroups of G. Since $ST\subseteq G$,

$$|ST| = \frac{|S||T|}{|S\cap T|} = \frac{729}{|S\cap T|} \le 108.$$

Thus $|S\cap T| = \{9, 27\}$. If $|S\cap T| = 27$, then $S = S\cap T = T$, a contradiction. Thus $|S\cap T| = 9$. Since $|S\cap T| = 3^2$ and $|S| = 3^3$, $S\cap T$ is a normal subgroup of S and hence $S\le \mathbb{N}_G(S\cap T)$. Similarly, $T\le \mathbb{N}_G(S\cap T)$. So $ST\subseteq \mathbb{N}_G(S\cap T)$. Since $|ST| = \frac{|S||T|}{|S\cap T|} = \frac{729}{9} = 81$,

$$81 \le |\mathbb{N}_G(S\cap T)|. \tag{1}$$

Moreover, $|S|$ divides $|\mathbb{N}_G(S\cap T)|$ and so

$$27 \text{ divides } |\mathbb{N}_G(S\cap T)|. \tag{2}$$

By Lagrange's theorem,

$$|\mathbb{N}_G(S\cap T)| \text{ divides } 108. \tag{3}$$

By (1), (2), (3), $|\mathbb{N}_G(S\cap T)| = 108$. Thus $\mathbb{N}_G(S\cap T) = G$ and so $S\cap T$ is a normal subgroup of G.

Let α be an automorphism of a finite group G which fixes only the unit of G ($\alpha(a) = a \Rightarrow a = 1$). Assume $\alpha^2 = 1$. Then G is abelian of odd order.

Let $T : G \to G$ be $T(a) = \alpha(a)a^{-1}$. Suppose $T(a) = T(b)$. Then $\alpha(b^{-1}a) = b^{-1}a$ and hence $b^{-1}a = 1$. So T is injective. Since $T : G \to G$ is injective and G is finite, T is also surjective. Let $g \in G$. Then $g = \alpha(a)a^{-1}$ for some $a \in G$ and $\alpha(g) = \alpha(\alpha(a)a^{-1}) = \alpha(\alpha(a))\alpha(a^{-1}) = \alpha^2(a)\alpha(a^{-1}) = a\alpha(a)^{-1} = (\alpha(a)a^{-1})^{-1} = g^{-1}$. Let $a, b \in G$. So $ab = (b^{-1}a^{-1})^{-1} = \alpha(b^{-1}a^{-1}) = \alpha(b^{-1})\alpha(a^{-1}) = (b^{-1})^{-1}(a^{-1})^{-1} = ba$. Thus G is abelian. Let $a \in G$. Then $T(a) = \alpha(a)a^{-1} = a^{-2}$. Suppose G is a finite group of even order. Then G contains an element $x \neq 1$ such that $x^2 = 1$. So $T(x) = x^{-2} = 1$. Since $x \neq 1$ but $T(x) = T(1)$, T is not injective, a contradiction.

For $x, y \in G$, define

$$[x, y] = x^{-1}y^{-1}xy$$

and the commutator of two subgroups H and K of G is

$$[H, K] = \langle\, [h, k] \mid h \in H, k \in K \,\rangle.$$

Prove that G^i is a characteristic subgroup of G for all i.

For a group G, define

$$G^0 = G, \qquad G^1 = [G, G], \qquad G^{i+1} = [G, G^i].$$

Suppose the statement is true for i. Let σ be an automorphism of G. Then σ^{-1} is also an automorphism of G. Let

$$U = \{x^{-1}y^{-1}xy \mid x \in G, y \in G^i\}.$$

If $u \in U$, then $u = x^{-1}y^{-1}xy$ for some $x \in G, y \in G^i$. Since σ is surjective, $x = \sigma(a)$ for some $a \in G$. By hypothesis, $G^i = \sigma(G^i)$ and thus $y = \sigma(b)$ for some $b \in G^i$. Thus

$$u = x^{-1}y^{-1}xy = \sigma(a^{-1})\,\sigma(b^{-1})\,\sigma(a)\,\sigma(b) = \sigma(a^{-1}b^{-1}ab) \in \sigma(G^{i+1}).$$

Note that $a^{-1}b^{-1}ab \in U \subseteq G^{i+1}$. Both G^{i+1} and $\sigma(G^{i+1})$ are subgroups of G that contain U. Since G^{i+1} is the smallest subgroup of G that contains U, $G^{i+1} \leq \sigma(G^{i+1})$. Moreover, $G^{i+1} \leq \sigma^{-1}(G^{i+1})$ and thus $\sigma(G^{i+1}) \leq G^{i+1}$. To conclude

$$\sigma(G^{i+1}) = G^{i+1}.$$

For a group G, define

$$G^{(0)} = G, \qquad G^{(1)} = [G, G], \qquad G^{(i+1)} = [G^{(i)}, G^{(i)}].$$

Then $G^{(1)}$ is a characteristic subgroup of G.

Let T be an automorphism of G. So T^{-1} is also an automorphism of G. Let $U = \{xyx^{-1}y^{-1} | x, y \in G\}$. If $u \in U$, then $u = xyx^{-1}y^{-1}$ for some $x, y \in G$. Since T^{-1} is surjective, $x = T^{-1}(a)$ and $y = T^{-1}(b)$ for some $a, b \in G$. Thus $u = xyx^{-1}y^{-1} = T^{-1}(a)T^{-1}(b)T^{-1}(a^{-1})T^{-1}(b^{-1}) = T^{-1}(aba^{-1}b^{-1}) \in T^{-1}(G^{(1)})$ since $aba^{-1}b^{-1} \in U$ and $G^{(1)}$ contains U. Hence $T^{-1}(G^{(1)})$ contains U. Both $G^{(1)}$ and $T^{-1}(G^{(1)})$ are subgroups of G that contain U. Since $G^{(1)}$ is the smallest subgroup of G that contains U, $G^{(1)} \leqq T^{-1}(G^{(1)})$. To conclude $T(G^{(1)}) \leqq T(T^{-1}(G^{(1)})) = G^{(1)}$.

Prove that $G^{(i)}$ is a characteristic subgroup of G for all i.

By previous result, $G^{(1)}$ is a characteristic subgroup of G. Suppose $G^{(i)}$ is a characteristic subgroup of G for some integer $i \geqq 1$. Again by the previous result, $G^{(i+1)} = [G^{(i)}, G^{(i)}] = (G^{(i)})^{(1)}$ is a characteristic subgroup of $G^{(i)}$. Since $G^{(i+1)}$ is a characteristic subgroup of $G^{(i)}$ and $G^{(i)}$ is a characteristic subgroup of G, $G^{(i+1)}$ is a characteristic subgroup of G.

Prove that if H is a nontrivial normal subgroup of the solvable group G then there is a nontrivial subgroup A of H with $A \trianglelefteq G$ and A abelian.

Lemma 1
If G is a group and $N \triangleleft G$ such that

$$[a, b] \in N$$

for all $a, b \in G$, then G/N is abelian.

Since H is a subgroup of the solvable group G, H is solvable as well. Since H is solvable, $H^{(k)} = 1$ for some $k \geq 0$. Then there is the smallest nonnegative n for which $H^{(n)} = 1$. Hence $H^{(n-1)} \neq 1$. Since $H^{(n-1)}$ is a characteristic subgroup of H and H is normal in G, $H^{(n-1)}$ is normal in G. Let $a, b \in H^{(n-1)}$. Note that $[a, b]$ is a commutator of elements from $H^{(n-1)}$ and $H^{(n)} = [H^{(n-1)}, H^{(n-1)}]$ is generated by all commutators of elements from $H^{(n-1)}$. Then $[a, b] \in H^{(n)} = 1$. Hence, by Lemma 1, $H^{(n-1)}/1$ is abelian. Moreover, $H^{(n-1)}/1 \cong H^{(n-1)}$. To conclude $H^{(n-1)}$ is abelian.

If $G/Z(G)$ is solvable, then G is solvable.

Since $Z(G)$ itself is an abelian group, it is solvable. Thus G is solvable since $Z(G)$ and $G/Z(G)$ are solvable.

If G is a group, A is a subgroup of G, N is a normal subgroup of G and both A and N are solvable, then AN is also solvable.

Since $A/A \cap N$ is a homomorphic image of A and A is solvable, $A/A \cap N$ is also solvable. Since $AN/N \cong A/A \cap N$, AN/N is a homomorphic image of $A/A \cap N$. Since AN/N is a homomorphic image of $A/A \cap N$ and $A/A \cap N$ is solvable, AN/N is also solvable. Since both N and AN/N are solvable, AN is also solvable.

Let $C(A)$ denote the centralizer of the subset A of a monoid M (or a group G). Then:
(1) $C(C(A)) \supset A$.
(2) If $A \subset B$, then $C(A) \supset C(B)$.
(3) $C(C(C(A))) = C(A)$.
(4) $C(A) = C(\langle A \rangle)$.

(3)
By replacing A with $C(A)$ in (1), $C(C(C(A))) \supset C(A)$. By (1), $A \subset C(C(A))$ and hence by (2), $C(A) \supset C(C(C(A)))$. Since $C(C(C(A))) \supset C(A)$ and $C(A) \supset C(C(C(A)))$, the result follows.

(4)
Let $c \in C(A)$. Then $\{c\} \subset C(A)$ and hence $C(c) \supset C(C(A)) \supset A$ by (1) and (2). Since $C(c)$ is a submonoid of M (or a subgroup of G) containing A and $\langle A \rangle$ is the smallest submonoid of M (or the smallest subgroup of G) containing A, $\langle A \rangle \subset C(c)$. and hence, by (1) and (2), $C(\langle A \rangle) \supset C(C(c)) \supset \{c\}$. Thus $c \in C(\langle A \rangle)$ and so $C(A) \subset C(\langle A \rangle)$. To show the other containment, since $A \subset \langle A \rangle$, $C(A) \supset C(\langle A \rangle)$ by (2). To conclude $C(A) = C(\langle A \rangle)$.

Note that $C(c)$ means $C(\{c\})$.

Let A, B be cyclic groups of order m and n, respectively. Prove that if $A \times B$ is cyclic, then m and n are relatively prime.

Let $G = A \times B$. Assume G is cyclic, $|A| = m$, $|B| = n$, $d = (m, n)$. A has a subgroup X of order d and B has a subgroup Y of order d. Let $U = \{(x, 1) \mid x \in X\}$, $V = \{(1, y) \mid y \in Y\}$. $|U| = |X| = d = |Y| = |V|$. Since G is cyclic, G can have just one subgroup of order d, and thus $U = V$. Let $x \in X$. Then $(x, 1) \in U = V$. So $(x, 1) = (1, y)$ for some $y \in Y$. Thus $x = 1$. To conclude $1 = |X| = d$ as wanted.

Find the maximum possible order for an element of S_8.

We can write any element of S_8 as a product of disjoint cycles, and the order of a product of disjoint cycles is the least common multiple of the orders of the cycles. How many ways can we have disjoint cycles? It just depends on how long the cycles are. So we have to look at partitions of 8 (where each number gives the length of the cycle) and give the corresponding LCM whenever applicable:

Cycle Type	**Order of Permutations**
$1,1,1,1,1,1,1,1$	1
$2,1,1,1,1,1,1$	2
$2,2,1,1,1,1$	2
$2,2,2,1,1$	2
$2,2,2,2$	2
3, 1, 1, 1, 1, 1	3
3, 2, 1, 1, 1	6
3, 2, 2, 1	6
3, 3, 1, 1	3
3, 3, 2	6
4, 1, 1, 1, 1	4
4, 2, 1, 1	4
4, 2, 2	4
4, 3, 1	12
4, 4	4
5, 1, 1, 1	5
5, 2, 1	10
5, 3	15
6, 1, 1	6
6, 2	6
7, 1	7
8	8

The biggest of these orders is 15.

Let G be a simple group of order $504 = 7 \cdot 8 \cdot 9$. Then G has no elements of order 21.

Suppose G has an element of order 21. Then G has a subgroup P of order 21. By Cauchy's theorem, P has a subgroup Q of order 7. Since P is cyclic, Q is normal in P and hence $|P|$ divides $|\mathbb{N}_G(Q)|$ since $P \leqq \mathbb{N}_G(Q)$. So 21 divides $|\mathbb{N}_G(Q)|$. But Q is also a 7-Sylow subgroup of G. By Sylow's theorems, $n_7(G) \in \{1, 8, 36\}$. Since G is simple, $n_7(G) \in \{8, 36\}$ and $|\mathbb{N}_G(Q)| \in \{14, 63\}$. Since 21 divides $|\mathbb{N}_G(Q)|$ and $|\mathbb{N}_G(Q)| \in \{14, 63\}$,

$|\mathbb{N}_G(Q)| = 63$. Thus G has a subgroup of index 8 and so G is isomorphic to a subgroup of S_8. But S_8 doesn't have an element of order 21, a contradiction.

A group of order 63 has an element of order 21.

Let G be a group of order 63. Then G has a Sylow 7-subgroup P and a Sylow 3-subgroup Q. Moreover, $n_7(G) = 1$ and $n_3(G) \in \{1, 7\}$. Suppose $n_3(G) = 1$. Then both P and Q are normal in G. By Cauchy's theorem, P has an element a of order 7 and Q has an element b of order 3. Since both P and Q are normal in G and $P \cap Q = 1$, $ab = ba$. Thus ab is of order 21. Now Suppose $n_3(G) = 7$. Let Q_1 and Q_2 be two distinct Sylow 3-subgroups of G. If $Q = Q_1 \cap Q_2$, $|Q|$ divides 3^2. Suppose $|Q| = 1$. But then

$$|Q_1Q_2| = \frac{|Q_1||Q_2|}{|Q|} = 81 > |G|,$$

a contradiction. Now suppose $|Q| = 9$. Then $Q_1 = Q = Q_2$, a contradiction. Thus $|Q| = 3$ and hence Q is normal in both Q_1 and Q_2. So $Q_1 \leq \mathbb{N}_G(Q)$ and $Q_2 \leq \mathbb{N}_G(Q)$; hence, $Q_1Q_2 \subseteq \mathbb{N}_G(Q)$. Since $|Q_1Q_2| \leq |\mathbb{N}_G(Q)|$,

$$27 \leq |\mathbb{N}_G(Q)|.$$

Moreover,

$$|\mathbb{N}_G(Q)| \text{ divides } 63.$$

Thus $|\mathbb{N}_G(Q)| = 63$ and so Q is normal in G. So P has an element a of order 7 and Q has an element b of order 3. Since both P and Q are normal in G and $P \cap (Q) = 1$, $ab = ba$. Thus ab is of order 21.

Let G be a finite group of order $|G| = 504 = 2^3 \cdot 3^2 \cdot 7$.
a. If G has a normal subgroup of order 8, show that G has at most 8 Sylow 7-subgroups, that is $|\mathrm{Syl}_7(G)| \leq 8$.
b. If $|\mathrm{Syl}_7(G)| \leq 8$, prove that G has an element of order 21.
c. If G is isomorphic to a subgroup of the symmetric group of degree 9, show that G cannot have a normal subgroup of order 8.

a.
Suppose G has a normal subgroup N of order 8. Consider $\bar{G} = G/N$. So $|\bar{G}| = 3^2 \cdot 7$. Since $n_7(\bar{G}) = 1$, $\bar{G}$ has a normal subgroup $\bar{Q}$ of order 7. Let $Q = \{x \in G \mid xN \in \bar{Q}\}$. Then $\bar{Q} \cong Q/N$ and hence

$$7 = |\bar{Q}| = \frac{|Q|}{|N|} = \frac{|Q|}{8}.$$

The result is $|Q| = 7 \cdot 8$. Since $\bar{Q}$ is normal in $\bar{G}$, Q itself is normal in G. Notice that Q has a Sylow 7-subgroup P and P is also a Sylow 7-subgroup of G. By Sylow's theorem, $n_7(G) \in \{1, 8, 36\}$. Suppose $n_7(G) = 36$. Then $36 = |G : \mathbb{N}_G(P)|$ and so $|\mathbb{N}_G(P)| = 14$. Since Q is a normal subgroup of a finite group G and P is a Sylow 7-subgroup of Q, by *Frattini's Argument*,

$|G : Q|$ divides $|\mathbb{N}_G(P)|$, i.e., 3^2 divides 14, a contradiction.

b.
Suppose $n_7(G) = 1$. Then G has a normal Sylow 7-subgroup P and a Sylow 3-subgroup Q. Since PQ is a subgroup of G of order 63, by previous result, PQ has an element of order 21 and so does G. Now suppose $n_7(G) = 8$. Let P be a Sylow 7-subgroup of G. So $|\mathbb{N}_G(P)| = 63$ and hence $\mathbb{N}_G(P)$ has an element of order 21 and so does G.

c.
By hypothesis, G is isomorphic to a subgroup H of S_9. So there exists an isomorphism $\varphi : G \to H$. Suppose G has a normal subgroup N of order 8. Then $\varphi(N)$ is also a normal subgroup of H. Note that $|H| = 504$ and $|\varphi(N)| = 8$. Thus, by part (a), $n_7(H) \leq 8$ and hence, by part (b), H has an element of order 21 and so does S_9, a contradiction.

A reference for *Frattini's Argument* :
D. S. Dummit, R. M. Foote, *Abstract Algebra*, John Wiley and Sons, Inc., 1990,

Let G be an abelian group of order 10 that contains an element of order 5. Then G must be a cyclic group.

Since $|G|$ is even, G contains an element $a \neq 1$ such that $a^2 = 1$ and so $|a| = 2$. By hypothesis, G has an element b of order 5. Thus $|\langle a\rangle| = 2$ and $|\langle b\rangle| = 5$. Since $|\langle a\rangle|$ and $|\langle b\rangle|$ are relatively prime, $\langle a\rangle \cap \langle b\rangle = 1$. Suppose $(ab)^n = 1$. Since G is abelian, $a^n b^n = (ab)^n = 1$. Note that $a^n \in \langle a\rangle$ and $a^n = b^{-n} \in \langle b\rangle$. So $a^n \in \langle a\rangle \cap \langle b\rangle = 1$ and hence $|a|$ divides n. Similarly, $b^n \in \langle b\rangle$ and $b^n = a^{-n} \in \langle a\rangle$. So $b^n \in \langle a\rangle \cap \langle b\rangle = 1$ and hence $|b|$ divides n. Since both 2 and 5 divide n, 10 also divides n. Moreover, $(ab)^{10} = a^{10}b^{10} = (a^2)^5(b^5)^2 = 1$. To conclude 10 is the smallest positive integer n such that $(ab)^n = 1$.

Let G be a finite group. If $G/Z(G)$ is nilpotent, then G is nilpotent.

Let p be a prime dividing $|G|$ and P be a Sylow p-subgroup of G. Thus $PZ(G)/Z(G)$ is also a Sylow p-subgroup of $G/Z(G)$. But $G/Z(G)$ is nilpotent. So $PZ(G)/Z(G)$ is normal in $G/Z(G)$ and hence $PZ(G)$ is normal in G. Moreover P is also a Sylow p-subgroup of $PZ(G)$. Thus, by Frattini's argument, $G = PZ(G)\mathbb{N}_G(P)$. Note that $Z(G) \leq \mathbb{N}_G(P)$ and $P \leq \mathbb{N}_G(P)$. To conclude $G = PZ(G)\mathbb{N}_G(P) = P\mathbb{N}_G(P) = \mathbb{N}_G(P)$ and thus P is normal in G. Since every Sylow subgroup is normal in G, G is nilpotent.

Reference:
H. E. Rose, *A Course on Finite Groups*, Springer, 2009.

Let G be a group of $2^4 \cdot 5^3 \cdot 11$ and H be a group of order $5^3 \cdot 11$.
(a) Show H has a normal Sylow 11-subgroup.
(b) If the number of Sylow 5-subgroup of G is (strictly) less than 16, prove that G has a proper normal subgroup of order divisible by 5.
(c) If G has exactly sixteen Sylow 5-subgroups, show that G has a normal Sylow 11-subgroup.

(b)
Since $n_5(G) < 16$, $n_5(G) \in \{1, 11\}$. Suppose $n_5(G) = 1$. Then there is one and only one Sylow 5-subgroup of G and hence it is normal in G. Let N be the normalizer of a Sylow 5-subgroup of G. Suppose $n_5(G) = 11$. Then $11 = |G : N|$ and so $|N| = 2^4 \cdot 5^3$. Since $|G|$ does not divide $|G : N|!$, N must contain a nontrivial normal subgroup P of G. Since $|P|$ divides $|N|$, $|P| \in \{2, 4, 5, 8, 10, 16, 20, 25, 40, 50, 80, 100, 125, 200, 250, 400, 500, 1000, 2000\}$. From the list, only 2, 4, 8, 16 are not divisible by 5. Suppose $|P| = 2^i$ for some $i \in \{1, 2, 3, 4\}$. Consider the quotient group $\bar{G} = G/P$. So $|\bar{G}| = 2^{4-i} \cdot 5^3$. Since $n_5(\bar{G}) = 1$, $\bar{G}$ has a normal subgroup $\bar{Q}$ of order 5^3. Let $Q = \{x \in G \mid xP \in \bar{Q}\}$. Then $\bar{Q} \cong Q/P$ and hence

$$5^3 = |\bar{Q}| = \frac{|Q|}{|P|} = \frac{|Q|}{2^i}.$$

The result is $|Q| = 2^i \cdot 5^3$. Since $\bar{Q}$ is normal in $\bar{G}$, Q itself is normal in G. Notice that 5 divides $|Q|$.

(c)
Let N be the normalizer of a Sylow 5-subgroup of G. Since $n_5(G) = 16$, $16 = |G : N|$ and thus $|N| = 5^3 \cdot 11$. By Sylow's theorems, N has a normal subgroup P of order 11 and hence $N \leq \mathbb{N}_G(P)$. So

$$5^3 \cdot 11 \text{ divides } |\mathbb{N}_G(P)|. \tag{1}$$

Notice that P is also a Sylow 11-subgroup of G. Since $n_{11}(G) \in \{1, 100\}$,

$$|\mathbb{N}_G(P)| \in \{220, 22000\}. \tag{2}$$

By (1), (2), $|\mathbb{N}_G(P)| = 22000$. Thus $\mathbb{N}_G(P) = G$.

Prove that there are no simple groups of order $1755 = 3^3 \cdot 5 \cdot 13$.

Let G be a group of order 1755. By Sylow's theorems, $n_{13}(G) \in \{1, 27\}$. Let N be the normalizer of a Sylow 13-subgroup of G. Suppose $n_{13}(G) = 27$. Then $27 = |G : N|$ and so $|N| = 65$. By Sylow's theorems, N has a normal

Sylow 5-subgroup P and hence $N \leq \mathbb{N}_G(P)$. So

$$\text{(1)} \qquad 65 \text{ divides } |\mathbb{N}_G(P)|.$$

Notice that P is also a Sylow 5-subgroup of G. Since $n_5(G) \in \{1, 351\}$,

$$\text{(2)} \qquad |\mathbb{N}_G(P)| \in \{5, 1755\}.$$

By (1), (2), $|\mathbb{N}_G(P)| = 1755$. Thus $\mathbb{N}_G(P) = G$.

Prove that there are no simple groups of order $9555 = 3 \cdot 5 \cdot 7^2 \cdot 13$.

Let G be a group of order 9555. By Sylow's theorems, $n_7(G) \in \{1, 15\}$. Let N be the normalizer of a Sylow 7-subgroup of G. Suppose $n_7(G) = 15$. Then $15 = |G : N|$ and so $|N| = 637$. By Sylow's theorem, N has a normal Sylow 13-subgroup P and hence $N \leq \mathbb{N}_G(P)$. So

$$\text{(1)} \qquad 637 \text{ divides } |\mathbb{N}_G(P)|.$$

Notice that P is also a Sylow 13-subgroup of G. Since $n_{13}(G) \in \{1, 105\}$,

$$\text{(2)} \qquad |\mathbb{N}_G(P)| \in \{91, 9555\}.$$

By (1), (2), $|\mathbb{N}_G(P)| = 9555$. Thus $\mathbb{N}_G(P) = G$.

Suppose that G is a finite group that has exactly 50 Sylow 7-subgroups. Let $P \in \mathrm{Syl}_7(G)$ and write $N = \mathbb{N}_G(P)$.
a. Show that N is a maximal subgroup of G.
b. If N has a normal Sylow 5-subgroup Q, prove that $Q \lhd G$.

a.

Lemma 1
Let G be a finite group such that $P \in Syl_p(G)$ and $H \geq \mathbb{N}_G(P)$, then $|G : H| \equiv 1 (\mathrm{mod}\ \ p)$.

Reference:
W. R. Scott, *Group Theory*, Dover Publications, Inc., 1987.

Notice that $50 = n_p(G) = |G : N|$. Let M be a subgroup of G such that $N \leq M \leq G$ and $M \neq N$. Then

$$50 = |G : N| = |G : M||M : N|$$

and, by Lemma 1, $|G : M| \equiv 1(\mathrm{mod}\ \ 7)$. Since $|G : M|$ divides 50 and $|G : M| \equiv 1(\mathrm{mod}\ \ 7)$, $|G : M| = 1$ or 50. If $|G : M| = 50$, $|M : N| = 1$ and thus $M = N$, a contradiction. Thus $|G : M| = 1$ and hence $M = G$.

b.

Lemma 2
Let G be a finite group and P be a p-subgroup of G. Then P is a Sylow p-subgroup of G if and only if P is a Sylow p-subgroup of $\mathbb{N}_G(P)$.

Since Q is normal in N, $N \leq \mathbb{N}_G(Q)$. By part a, N is a maximal subgroup of G. Thus $\mathbb{N}_G(Q) = N$ or $\mathbb{N}_G(Q) = G$. Suppose $\mathbb{N}_G(Q) = N$. Since Q is a Sylow 5-subgroup of N and $N = \mathbb{N}_G(Q)$, it follows that Q is a Sylow 5-subgroup of $\mathbb{N}_G(Q)$ and, by Lemma 2, Q is a Sylow 5-subgroup of G. Since $\mathbb{N}_G(Q) = N$, $|\mathbb{N}_G(Q)| = |N|$ and thus $n_5(G) = |G : \mathbb{N}_G(Q)| = |G : N| = 50$. By Sylow's theorem, $n_5(G) \equiv 1(\text{mod } 5)$, i.e., $50 \equiv 1(\text{mod } 5)$, a contradiction.

Suppose that a group G is the (internal) direct product of subgroups S and T. Let H be a subgroup of G such that $SH = G = TH$.
a) Prove that $S \cap H$ and $T \cap H$ are normal subgroups of G.
b) If $S \cap H = 1 = T \cap H$, prove that S and T are isomorphic.
c) If $S \cap H = 1 = T \cap H$ and H is normal in G, show that G is abelian.

The commutator of two elements x and y is $x^{-1}y^{-1}xy$. The notation for this commutator element is $[x, y]$. If X and Y are subgroups of G, then $[X, Y]$ is defined to be the group generated by all $[x, y]$ with x in X and y in Y.

Facts:
(1) $[x, y] = 1$ if and only if $xy = yx$.
(2) $[X, Y] = 1$ if and only if $X \leq \mathbb{C}_G(Y)$ if and only if $Y \leq \mathbb{C}_G(X)$.
(3) $[X, Y] = [Y, X]$.
(4) $[X, Y] \leq X$ if and only if $Y \leq \mathbb{N}_G(X)$.
$[X, Y] \leq Y$ if and only if $X \leq \mathbb{N}_G(Y)$.
(5) If X and Y normalize each other and $X \cap Y = 1$, then X and Y centralize each other.
Prove: Both X and Y normalize $[X, Y]$.

a)

Since $G \leq \mathbb{N}_G(T)$, $S \leq \mathbb{N}_G(T)$. Similarly, since $G \leq \mathbb{N}_G(S)$, $T \leq \mathbb{N}_G(S)$. Since S and T normalize each other and $S \cap T = 1$, by (5), S and T centralize each other. So $T \leq \mathbb{C}_G(S)$ and $S \leq \mathbb{C}_G(T)$.

Since $S \cap H \leq S$, $T \leq \mathbb{C}_G(S) \leq \mathbb{C}_G(S \cap H)$ and so, by (2), $[T, S \cap H] = 1 \leq S \cap H$. It is immediate, by (4), $T \leq \mathbb{N}_G(S \cap H)$. Since $S \cap H$ is a normal subgroup of H. $H \leq \mathbb{N}_G(S \cap H)$. Since $T \leq \mathbb{N}_G(S \cap H)$ and $H \leq \mathbb{N}_G(S \cap H)$,

$$G = TH \leq \mathbb{N}_G(S \cap H).$$

Since $T \cap H \leq T$, $S \leq \mathbb{C}_G(T) \leq \mathbb{C}_G(T \cap H)$ and so, by (2), $[S, T \cap H] = 1 \leq T \cap H$. It is immediate, by (4), $S \leq \mathbb{N}_G(T \cap H)$. Since $T \cap H$ is a normal subgroup of H. $H \leq \mathbb{N}_G(T \cap H)$. Since $S \leq \mathbb{N}_G(T \cap H)$ and $H \leq \mathbb{N}_G(T \cap H)$,

$$G = SH \leq \mathbb{N}_G(T \cap H).$$

Note:
If $S \cap H$ is a normal subgroup of H, then $\mathbb{N}_H(S \cap H) = H$, but all we know is that $H \leq \mathbb{N}_G(S \cap H)$.

b)

Define $\varphi : H \to G/S$ by $\varphi(h) = Sh$ for $h \in H$. Let $h_1, h_2 \in H$. Then

$$\varphi(h_1 h_2) = Sh_1 h_2 = Sh_1 Sh_2 = \varphi(h_1)\varphi(h_2).$$

Let $g \in G$. Then $g = sh$ for some $s \in S$ and $h \in H$. Thus

$$Sg = Ssh = SsSh = S1Sh = Sh = \varphi(h).$$

Now suppose $\varphi(h_1) = \varphi(h_2)$. Then $Sh_1 = Sh_2$. Since $h_1 \in Sh_1 = Sh_2$, $h_1 = sh_2$ for some $s \in S$ and hence $h_1 {h_2}^{-1} = s \in S$. To conclude $h_1 {h_2}^{-1} \in S \cap H = 1$, i.e., $h_1 = h_2$. To conclude that $G/S \cong H$.

Since $G = SH$, S is normal in G, and $S \cap H = 1$, it follows that $G/S \cong H$. Since $G = TH$, T is normal in G, and $T \cap H = 1$, it follows that $G/T \cong H$. Since $G = ST$, S is normal in G, and $S \cap T = 1$, it follows that $G/S \cong T$. Since $G = ST$, ST is a subgroup of G and hence $ST = TS$. Since $G = TS$, T is normal in G, and $T \cap S = 1$, it follows that $G/T \cong S$. To conclude that

$$S \cong G/T \cong H \cong G/S \cong T.$$

c)

Since H is normal in G, $G \leq \mathbb{N}_G(H)$ and so $S \leq \mathbb{N}_G(H)$. Since S is normal in G, $G \leq \mathbb{N}_G(S)$ and so $H \leq \mathbb{N}_G(S)$. Since S and H normalize each other and $S \cap H = 1$, S and H centralize each other. Thus $H \leq \mathbb{C}_G(S)$. Since $T \leq \mathbb{C}_G(S)$ and $H \leq \mathbb{C}_G(S)$, $TH \leq \mathbb{C}_G(S)$ and so $S \leq G = TH \leq \mathbb{C}_G(S)$. Thus S is abelian.

Since H is normal in G, $G \leq \mathbb{N}_G(H)$ and so $T \leq \mathbb{N}_G(H)$. Since T is normal in G, $G \leq \mathbb{N}_G(T)$ and so $H \leq \mathbb{N}_G(T)$. Since T and H normalize each other and $T \cap H = 1$, T and H centralize each other. Thus $H \leq \mathbb{C}_G(T)$. Since $S \leq \mathbb{C}_G(T)$ and $H \leq \mathbb{C}_G(T)$, $SH \leq \mathbb{C}_G(T)$ and so $T \leq G = SH \leq \mathbb{C}_G(T)$. Thus T is abelian.

Let $s \in S, t \in T$. Since both S and T are normal in G and $S \cap T = 1$, $st = ts$. Since both S and T are abelian, $G = ST$, and $st = ts$ for $s \in S, t \in T$, to conclude that G is abelian.

Let G be a group of order $10989 = 3^3 \cdot 11 \cdot 37$.
1. Compute the number, n_p, of Sylow p-subgroups permitted by Sylow's theorems for each of $p = 3, 11$ and 37; for each of these n_p give the order of the normalizer of a Sylow p-subgroup.
2. Show that G contains either a normal Sylow 37-subgroup or a normal Sylow 3-subgroup.
3. Explain briefly why (in all cases) G has a normal Sylow 11-subgroup.
4. Deduce that the center of G is nontrivial.

1.
By inspection, $n_3(G) \in \{1, 37\}$, $n_{11}(G) \in \{1, 111\}$ and $n_{37}(G) \in \{1, 297\}$.

2.
Suppose G does not have a normal Sylow 3-subgroup. Then $n_3(G) = 37$. Let P be a Sylow 3-subgroup of G. So $|\mathbb{N}_G(P)| = 3^3 \cdot 11$. By Sylow's theorem, $n_{11}(\mathbb{N}_G(P)) = 1$.and so $\mathbb{N}_G(P)$ has a normal Sylow 11-subgroup Q. Since Q is normal in $\mathbb{N}_G(P)$, $\mathbb{N}_G(P) \leq \mathbb{N}_G(Q)$. Notice that Q is also a Sylow 11-subgroup of G and $n_{11}(G) \in \{1, 111\}$. If $n_{11}(G) = 111$, then $|\mathbb{N}_G(Q)| = 99$. Since $|\mathbb{N}_G(P)|$ divides $|\mathbb{N}_G(Q)|$, 297 divides 99, contradiction. So $n_{11}(G) = 1$ and $|\mathbb{N}_G(Q)| = 10989$ and thus $\mathbb{N}_G(Q) = G$. Let $\bar{G} = G/Q$. So $|\bar{G}| = 3^3 \cdot 37$. Since $n_{37}(\bar{G}) = 1$, $\bar{G}$ has a normal subgroup $\bar{H}$ of order 37. Let H be the complete preimage of $\bar{H}$ in G, so $|H| = 37 \cdot 11$. Since $\bar{H}$ is normal in $\bar{G}$, H itself is normal in G. Notice that H has a Sylow 37-subgroup K. Let $\varphi \in \mathrm{Aut}(H)$. Then $\varphi(K)$ is also a Sylow 37-subgroup of H. Since $n_{37}(H) = 1$, $\varphi(K) = K$. Since K is a characteristic subgroup of H and H is normal in G, to conclude K is normal in G.

3.
Suppose $n_{11}(G) = 111$ and $n_{37}(G) = 297$. So there are $111 \cdot 10$ elements of order 11 in G and $297 \cdot 36$ elements of order 37 in G, a contradiction since they do not fit into G. So G contains either a normal Sylow 11-subgroup or a normal Sylow 37-subgroup. Suppose G contains a normal Sylow 37-subgroup Q. Let $\bar{G} = G/Q$. So $|\bar{G}| = 3^3 \cdot 11$. Since $n_{11}(\bar{G}) = 1$, $\bar{G}$ has a normal

subgroup $\bar{H}$ of order 11. Let H be the complete preimage of $\bar{H}$ in G, so $|H| = 11 \cdot 37$. Since $\bar{H}$ is normal in $\bar{G}$, H itself is normal in G. Notice that H has a Sylow 11-subgroup K. Let $\varphi \in \mathrm{Aut}(H)$. Then $\varphi(K)$ is also a Sylow 11-subgroup of H. Since $n_{11}(H) = 1$, $\varphi(K) = K$. Since K is a characteristic subgroup of H and H is normal in G, to conclude K is normal in G.

4.
From part 3, G has a normal Sylow 11-subgroup P and hence P is cyclic. So $|\mathrm{Aut}(P)| = 10$ and the quotient $G/\mathbb{C}_G(P)$ is isomorphic to a subgroup of $\mathrm{Aut}(P)$, i.e.,

$$|G/\mathbb{C}_G(P)| \in \{1, 2, 5, 10\}. \tag{1}$$

On the other hand, since P is abelian, $P \leq \mathbb{C}_G(P)$ and $|\mathbb{C}_G(P)|$ is divisible by 11, which implies

$$|G/\mathbb{C}_G(P)| \in \{1, 3, 9, 27, 37, 111, 333, 999\}. \tag{2}$$

From (1) and (2), $|G/\mathbb{C}_G(P)| = 1$, i.e., $\mathbb{C}_G(P) = G$ and $P \leq Z(G)$.

Let G be a finite group of order $4312 = 2^3 \cdot 7^2 \cdot 11$.
(a) Show that G has a subgroup of order 77.
(b) Prove that G has a subgroup of order 7 whose normalizer in G has index dividing 8.
(c) Conclude that G is not simple.

(a)
By Sylow's theorems, $n_{11}(G) \in \{1, 56\}$. Suppose $n_{11}(G) = 1$. Then G has a normal Sylow 11-subgroup P. By Cauchy's theorem, G has a subgroup Q of order 7. Thus PQ is a subgroup of G of order 77. Now suppose $n_{11}(G) = 56$. Let P be a Sylow 11-subgroup of G. Thus $|\mathbb{N}_G(P)| = 77$.

(b)
By part (a), G has a subgroup H of order 77 and H has a normal Sylow 7-subgroup Q. Thus $H \leq \mathbb{N}_G(Q)$ and so

$$77 \text{ divides } |\mathbb{N}_G(Q)|. \tag{1}$$

Since Q is a 7-subgroup of G, there exists $P \in \mathrm{Syl}_7(G)$ such that $Q \leq P$. Since $|P : Q| = 7$, a prime, Q is normal in P and so $P \leq \mathbb{N}_G(Q)$. Hence

$$49 \text{ divides } |\mathbb{N}_G(Q)|. \tag{2}$$

By Lagrange's theorem,

$$|\mathbb{N}_G(Q)| \text{ divides } 4312. \tag{3}$$

By (1), (2), (3), $|\mathbb{N}_G(Q)| \in \{539, 1078, 2156, 4312\}$ and $|G : \mathbb{N}_G(Q)| \in \{1, 2, 4, 8\}$.

(c)
If $|G : \mathbb{N}_G(Q)| \in \{2, 4, 8\}$, then $|G|$ does not divide $|G : \mathbb{N}_G(Q)|!$ and hence there is a nontrivial normal subgroup of G contained in $\mathbb{N}_G(Q)$. If $|\mathbb{N}_G(Q)| = 4312$, then $\mathbb{N}_G(Q) = G$ and so Q is a normal subgroup of G.

Let G be a finite group. A subgroup H of G is said to be subnormal in G if there exists a chain of subgroups

$$H = H_0 \lhd H_1 \lhd \ldots \lhd H_n = G$$

where each is normal in the next. If H is a subnormal subgroup of G and $(|H|, |G : H|) = 1$, show that H is normal in G.

Lemma 1
Let G be a finite group, H be a subgroup of G and $N \lhd G$. If $|H|$ and $|G : N|$ are relatively prime, then $H \leq N$.

Proof. Notice that $|G : N| = |G : HN||HN : N| = |G : HN||H : H \cap N|$. So $|H : H \cap N|$ divides $(|H|, |G : N|)$ and hence $|H : H \cap N| = 1$. Since $H \cap N \leq H$ and $|H \cap N| = |H|$, thus $H \cap N = H$ and to conclude that $H \leq H \cap N \leq N$.

Lemma 2
If N is a normal subgroup of the finite group G and $(|N|, |G : N|) = 1$, then N is the unique subgroup of G of order $|N|$.

Proof. Let $H \leq G$ with $|H| = |N|$. Then

$$(|H|, |G : N|) = (|N|, |G : N|) = 1$$

and thus, by Lemma 1, $H \leq N$ but $|H| = |N|$. So $H = N$.

Suppose $H = H_0 \lhd H_i$ for some integer $1 \leq i < n$. Consider $(|H|, |H_i : H|)$. Since $(|H|, |H_i : H|)$ divides $|H_i : H|$ and $|G : H| = |G : H_i||H_i : H|$, so $(|H|, |H_i : H|)$ divides $|G : H|$. To conclude $(|H|, |H_i : H|)$ divides $(|H|, |G : H|)$ and hence $(|H|, |H_i : H|) = 1$. So, by Lemma 2, H is the unique subgroup of H_i of order $|H|$. Let $\varphi \in \mathrm{Aut}(H_i)$. Then $\varphi(H)$ is also a subgroup of H_i of order $|H|$. Since H is the only subgroup of order $|H|$ in H_i, $\varphi(H) = H$ and thus H is characteristic in H_i. Since H is characteristic in H_i and H_i is normal in H_{i+1}, so H is normal in H_{i+1}. Thus, by induction, $H = H_0 \lhd H_i$ for $i \in \{1, \ldots, n\}$.

Prove that there are no simple groups of order $2205 = 3^2 \cdot 5 \cdot 7^2$.

Let G be a group of order 2205. By Sylow's theorems, $n_3(G) \in \{1, 7, 49\}$. Let N be the normalizer of a Sylow 3-subgroup of G. Suppose $n_3(G) = 7$. Then $7 = |G : N|$. Since $|G|$ does not divide $|G : N|!$, N must contain a nontrivial normal subgroup of G. Now suppose $n_3(G) = 49$. Then $49 = |G : N|$ and so $|N| = 45$. By Sylow's theorems, N has a normal subgroup P of order 5 and hence $N \leq \mathbb{N}_G(P)$. So

$$45 \text{ divides } |\mathbb{N}_G(P)|. \tag{1}$$

Notice that P is also a Sylow 5-subgroup of G. Since $n_5(G) \in \{1, 21, 441\}$,

$$|\mathbb{N}_G(P)| \in \{5, 105, 2205\}. \tag{2}$$

By (1), (2), $|\mathbb{N}_G(P)| = 2205$. Thus $\mathbb{N}_G(P) = G$.

Prove that there are no simple groups of order $396 = 2^2 \cdot 3^2 \cdot 11$.

Let G be a group of order 396. By Sylow's theorems, $n_{11}(G) \in \{1, 12\}$. Let N be the normalizer of a Sylow 11-subgroup of G. Suppose $n_{11}(G) = 12$. Then $12 = |G : N|$ and so $|N| = 33$. By Sylow's theorems, N has a normal subgroup Q of order 3 and hence $N \leq \mathbb{N}_G(Q)$. So

$$33 \text{ divides } |\mathbb{N}_G(Q)|. \tag{1}$$

Note that $Q \leq P$ where P is some Sylow 3-subgroup of G. Thus Q is normal in P and hence $P \leq \mathbb{N}_G(Q)$. So

$$9 \text{ divides } |\mathbb{N}_G(Q)|. \tag{2}$$

Moreover,

$$|\mathbb{N}_G(Q)| \text{ divides } 396. \tag{3}$$

By (1), (2) and (3), $|\mathbb{N}_G(Q)| \in \{99, 198, 396\}$. If $|\mathbb{N}_G(Q)| = 99$, then $|G : \mathbb{N}_G(Q)| = 4$. Since $|G|$ does not divide $|G : \mathbb{N}_G(Q)|!$, $\mathbb{N}_G(Q)$ must contain a nontrivial normal subgroup of G. If $|\mathbb{N}_G(Q)| = 198$, then $|G : \mathbb{N}_G(Q)| = 2$ and so $\mathbb{N}_G(Q)$ is a normal subgroup of G. If $|\mathbb{N}_G(Q)| = 396$, then $\mathbb{N}_G(Q) = G$ and so Q is a normal subgroup of G.

Prove that there are no simple groups of order $525 = 3 \cdot 5^2 \cdot 7$.

Let G be a group of order 525. By Sylow's theorems, $n_7(G) \in \{1, 15\}$. Let N be the normalizer of a Sylow 7-subgroup of G. Suppose $n_7(G) = 15$. Then $15 = |G : N|$ and so $|N| = 35$. By Sylow's theorems, N has a normal subgroup Q of order 5 and hence $N \leq \mathbb{N}_G(Q)$. So

$$35 \text{ divides } |\mathbb{N}_G(Q)|. \tag{1}$$

Note that $Q \leq P$ where P is some Sylow 5-subgroup of G. Thus Q is normal in P and hence $P \leq \mathbb{N}_G(Q)$. So

(2) $$25 \text{ divides } |\mathbb{N}_G(Q)|.$$

Moreover,

(3) $$|\mathbb{N}_G(Q)| \text{ divides } 525.$$

By (1), (2) and (3), $|\mathbb{N}_G(Q)| \in \{175, 525\}$. If $|\mathbb{N}_G(Q)| = 175$, then $|G : \mathbb{N}_G(Q)| = 3$. Since $|G|$ does not divide $|G : \mathbb{N}_G(Q)|!$, $\mathbb{N}_G(Q)$ must contain a nontrivial normal subgroup of G. If $|\mathbb{N}_G(Q)| = 525$, then $\mathbb{N}_G(Q) = G$ and so Q is a normal subgroup of G.

A_4 has no subgroup of order 6.

$|A_4| = 12$. By Sylow's theorems, $n_3(A_4) \in \{1, 4\}$. Since $\langle(123)\rangle$ and $\langle(134)\rangle$ are two distinct Sylow 3-subgroups of A_4, $n_3(A_4) \neq 1$ and hence $n_3(A_4) = 4$. Let H be a subgroup of A_4 of order 6. Then H has a unique Sylow 3-subgroup P. So P is normal in H and hence $H \leq \mathbb{N}_{A_4}(P)$. But P is also a Sylow 3-subgroup of A_4 and thus $|\mathbb{N}_{A_4}(P)| = |A_4|/n_3(A_4) = 3$, a contradiction since H can't fit into $\mathbb{N}_{A_4}(P)$.

For other proofs, see
Michael Brennan and Des Machale, *Variations on a Theme: A_4 Definitely Has No Subgroup of Order Six!*, Mathematics Magazine. **73** (2000), no. 1.

Prove that there are no simple groups of order $560 = 2^4 \cdot 5 \cdot 7$.

Let G be a group of order 560. By Sylow's theorems, $n_7(G) \in \{1, 8\}$. Let N be the normalizer of a Sylow 7-subgroup of G. Suppose $n_7(G) = 8$. Then $8 = |G : N|$ and so $|N| = 70$. By Sylow's theorems, N has a normal subgroup P of order 5 and hence $N \leq \mathbb{N}_G(P)$. So

(1) $$70 \text{ divides } |\mathbb{N}_G(P)|.$$

Notice that P is also a Sylow 5-subgroup of G. Since $n_5(G) \in \{1, 16, 56\}$,

(2) $$|\mathbb{N}_G(P)| \in \{10, 35, 560\}.$$

By (1), (2), $|\mathbb{N}_G(P)| = 560$. Thus $\mathbb{N}_G(P) = G$.

Lemma
Given a group G, $\Phi(G)$ is the Frattini subgroup of G.
(1) If G is finite, then G is nilpotent if and only if $G' \leq \Phi(G)$.
(2) If G is finite, $H \leq G$ and $G = H\Phi(G)$, then $H = G$.

Reference:
H. E. Rose, *A Course on Finite Groups*, Springer, 2009.

Let N be a normal subgroup of a finite group G. If N and G/N' is nilpotent, then G is nilpotent.

Since N' is a characteristic subgroup of N and N is normal in G, N' is also normal in G. Let p be a prime dividing $|G|$ and P be a Sylow p-subgroup of G. Thus PN'/N' is a Sylow p-subgroup of G/N'. But G/N' is nilpotent. So PN'/N' is normal in G/N' and hence PN' is normal in G. Moreover P is also a Sylow p-subgroup of PN'. So, by Frattini's argument, $G = \mathbb{N}_G(P)PN'$. Since $P \leq \mathbb{N}_G(P)$ and $N' \leq \Phi(N) \leq \Phi(G)$,

$$G = \mathbb{N}_G(P)PN' = \mathbb{N}_G(P)N' \leq \mathbb{N}_G(P)\Phi(G).$$

Thus $G = \mathbb{N}_G(P)\Phi(G)$ and to conclude $\mathbb{N}_G(P) = G$. Since every Sylow subgroup is normal in G, G is nilpotent.

Reference:
H. E. Rose, *A Course on Finite Groups*, Springer, 2009.

Let G be a finite group such that $G/\Phi(G)$ is cyclic. Then G is cyclic.

Since $G/\Phi(G)$ is cyclic, $G/\Phi(G) = \langle a\Phi(G)\rangle$ for some $a \in G$. If $g \in G$, then $g\Phi(G) = (a\Phi(G))^m = a^m\Phi(G)$. So $g \in a^m\Phi(G) \subseteq \langle a\rangle\Phi(G)$ and thus $G \leq \langle a\rangle\Phi(G)$. Hence $G = \langle a\rangle\Phi(G)$ and to conclude $G = \langle a\rangle$.

Reference:
H. E. Rose, *A Course on Finite Groups*, Springer, 2009.

Let p be a prime and P be a p-group. If P/P' is cyclic, then so is P.

Since P/P' is cyclic, $P/P' = \langle aP'\rangle$ for some $a \in P$. If $g \in P$, then $gP' = (aP')^m = a^mP'$. So $g \in a^mP' \subseteq \langle a\rangle P'$ and thus $P \leq \langle a\rangle P'$. Since P is a p-group, P is nilpotent and hence $P' \leq \Phi(P)$. Thus $P \leq \langle a\rangle P' \leq \langle a\rangle\Phi(P)$ and so $P = \langle a\rangle\Phi(P)$. To conclude $P = \langle a\rangle$.

Reference:
H. E. Rose, *A Course on Finite Groups*, Springer, 2009.

Let G be a group. Suppose N is a normal subgroup of G. Let $\mathscr{M}$ be the set of all maximal subgroups of G and let $\mathscr{M}'$ be the set of all maximal subgroups

of G/N. Prove that
$$\mathscr{M}' = \{MN/N \mid M \in \mathscr{M}\}.$$

Since every subgroup of G/N is of the form M/N for some subgroup M of G containing N, let M/N be such a subgroup. Moreover, suppose $M/N \in \mathscr{M}'$. Let H be a subgroup such that $M \leq H \leq G$ but $M \neq H$. Then $M/N \neq H/N$. Since $M/N \in \mathscr{M}'$, $H/N = G/N$ and thus $H = G$. So $M \in \mathscr{M}$. To conclude $M/N = MN/N \in \mathscr{M}'$. To show the other containment, let $M \in \mathscr{M}$. Suppose $MN/N \leq H/N \leq G/N$ but $MN/N \neq H/N$ for some subgroup H of G containing N. Since $MN/N \leq H/N$ and $MN/N \neq H/N$, $MN \leq H$ but $MN \neq H$. Claim: $M \neq H$. If $M = H$, then $MN \geq M = H$, a contradiction. But $M \in \mathscr{M}$. So $H = G$ and thus $H/N = G/N$. To conclude $MN/N \in \mathscr{M}'$.

Let G be a group. Suppose N is a normal subgroup of G. If H and K are subgroups of G, then $H \cap K \leq H$ and so $(H \cap K)N/N \leq HN/N$. Similarly, $(H \cap K)N/N \leq KN/N$. To conclude $(H \cap K)N/N \leq HN/N \cap KN/N$.

Let G be a group. Suppose N is a normal subgroup of G. Prove that
$$\Phi(G)N/N \leq \Phi(G/N).$$
Let $\mathscr{M}$ be the set of all maximal subgroups of G. Thus, by the previous result,
$$\Phi(G)N/N \leq \bigcap_{M \in \mathscr{M}} MN/N = \Phi(G/N).$$

Let G be a group. Suppose N is a normal subgroup of G. If $N \leq \Phi(G)$, then prove that $\Phi(G)/N = \Phi(G/N)$.

Let n be fixed and let $\mathscr{M} = \{M_i \,|\, 1 \leq i \leq n\}$. By the previous result, $\Phi(G/N) = \bigcap_{i=1}^{n} M_i N/N = \bigcap_{i=1}^{n} M_i/N$. To show
$$\bigcap_{i=1}^{n} M_i/N \leq \left(\bigcap_{i=1}^{n} M_i\right)/N.$$
Suppose $\bigcap_{i=1}^{k} M_i/N \leq \left(\bigcap_{i=1}^{k} M_i\right)/N$ for some positive integer $k < n$. Let $x \in \bigcap_{i=1}^{k+1} M_i/N = \left(\bigcap_{i=1}^{k} M_i/N\right) \bigcap M_{k+1}/N$. Thus $x \in \bigcap_{i=1}^{k} M_i/N$ and, by the induction hypothesis, $x \in \left(\bigcap_{i=1}^{k} M_i\right)/N$ and so $x = Na$ for some $a \in \bigcap_{i=1}^{k} M_i$. Moreover, $x \in M_{k+1}/N$ and so $x = Nb$ for some $b \in M_{k+1}$. So $a \in Na = Nb$ and hence $a \in Nb$. Thus $a = hb$ for some $h \in N$. To conclude $a \in M_{k+1}$. Since $a \in \bigcap_{i=1}^{k} M_i$ and $a \in M_{k+1}$, it follows that $a \in \bigcap_{i=1}^{k+1} M_i$ and $x = Na \in \left(\bigcap_{i=1}^{k+1} M_i\right)/N$. To prove the other containment, note that

$\Phi(G)N = \Phi(G)$ and, by the previous result,
$$\Phi(G)/N = \Phi(G)N/N \leq \Phi(G/N).$$

If g is an element of a group and $|g| = n$, then g^k, $k \neq 0$, has order $\frac{[n,k]}{k} = \frac{n}{(n,k)}$.

Let $|g^k| = m$. So $(g^k)^m = 1$ and hence $n|km$. Since $n|km$ and $k|km$, $[n,k]|km$ and thus $\frac{nk}{(n,k)}|km$. Hence $\frac{n}{(n,k)}|m$. To conclude $m \geq \frac{n}{(n,k)}$. It is obvious that $(g^k)^{\frac{n}{(n,k)}} = 1$ and so $m \leq \frac{n}{(n,k)}$. It follows that $m = \frac{n}{(n,k)}$.

Lemma 1
Let G be a group of order $29 \cdot 30$. If G has a normal Sylow 5-subgroup N, then G has a normal Sylow 29-subgroup.

Let $\bar{G} = G/N$. So $|\bar{G}| = 174$ and $\bar{G}$ has a normal Sylow 29-subgroup $\bar{H}$. Let $H = \{x \in G \mid xN \in \bar{H}\}$. Then H is a subgroup of G and $\bar{H} \cong H/N$; thus
$$29 = |\bar{H}| = \frac{|H|}{|N|} = \frac{|H|}{5}.$$
So $|H| = 29 \cdot 5$ and H has a Sylow 29-subgroup Q. Let φ be an automorphism of H. Then $\varphi(Q)$ is a Sylow 29-subgroup of H. H has exactly one Sylow 29-subgroup and thus $\varphi(Q) = Q$. Since $\bar{H}$ is normal in $\bar{G}$, H is normal in G. Since Q is a characteristic subgroup of H and H is normal in G, Q is normal in G.

If G is a group of order $29 \cdot 30$, then G has a normal Sylow 29-subgroup.

By Sylow's theorems, $n_5(G) \in \{1, 6\}$. Suppose $n_5(G) = 1$. So G has a normal Sylow 5-subgroup and thus, by Lemma 1, G has a normal Sylow 29-subgroup. Suppose $n_5(G) = 6$. Let N be the normalizer of a Sylow 5-subgroup of G. Then $|G : N| = 6$. Since $|G|$ does not divide $|G : N|!$, N must contain a nontrivial normal subgroup of G. Let H be the nontrivial normal subgroup of G in N. Thus $|H| \in \{5, 29\}$. If $|H| = 5$, then, by Lemma 1, G has a normal Sylow 29-subgroup. If $|H| = 29$, then we are done.

In the symmetric group S_{12}, the permutation
$$\pi = (1\ 4\ 3\ 2\ 8)(3\ 5\ 6\ 11\ 9)(2\ 5\ 12\ 7\ 10\ 3).$$
is given. Arbitrary disjoint cycles $\rho_1 = (i_1\ i_2\ \ldots\ i_k)$, $\rho_2 = (j_1\ j_2\ \ldots\ j_l)$, where $k + l \leq n$ are also given.

a) Calculate the order of the permutation π and write π in the form of the power of a single cycle from the symmetric group S_{12}.

b) Calculate the order of the product $(i_1\ j_1)\rho_2\rho_1$.

c) Find such a cycle $\rho \in S_{12}$ that the product $\rho\pi$ is an element of order 11.

When computing the product of two permutations in S_{12}, one reads the permutations from right to left.
a)
Write π as a product of disjoint cycles. So

$$\pi = (1\ 4\ 3\ 8)(2\ 6\ 11\ 9)(5\ 12\ 7\ 10).$$

It is obvious that the order of π is 4. The first numbers in the first cycle, the second cycle, the third cycle are $1, 2, 5$, respectively. The second numbers in the first cycle, the second cycle, the third cycle are $4, 6, 12$, respectively. The third numbers in the first cycle, the second cycle, the third cycle are $3, 11, 7$, respectively. The fourth numbers in the first cycle, the second cycle, the third cycle are $8, 9, 10$, respectively. By inspection,

$$(1\ 4\ 3\ 8)(2\ 6\ 11\ 9)(5\ 12\ 7\ 10) = (1\ 2\ 5\ 4\ 6\ 12\ 3\ 11\ 7\ 8\ 9\ 10)^3.$$

b)
Notice that

$$(i_1\ j_1)\rho_2\rho_1 = (j_1\ j_2\ \ldots\ j_l\ i_1\ i_2\ \ldots\ i_k)$$

and its order is $k + l$.

c)
From part a,

$$\begin{aligned}\pi &= (1\ 2\ 5\ 4\ 6\ 12\ 3\ 11\ 7\ 8\ 9\ 10)^3\\ &= (1\ 2\ 5\ 4\ 6\ 12\ 3\ 11\ 7\ 8\ 9\ 10)^2(1\ 2\ 5\ 4\ 6\ 12\ 3\ 11\ 7\ 8\ 9\ 10)\end{aligned}$$

and so

$$(1\ 2\ 5\ 4\ 6\ 12\ 3\ 11\ 7\ 8\ 9\ 10)^{-2}\pi = (1\ 2\ 5\ 4\ 6\ 12\ 3\ 11\ 7\ 8\ 9\ 10).$$

It follows that

$$(1\ 9\ 7\ 3\ 6\ 5)(2\ 10\ 8\ 11\ 12\ 4)\pi = (1\ 2\ 5\ 4\ 6\ 12\ 3\ 11\ 7\ 8\ 9\ 10)$$

and hence

$$(1\ 2)(1\ 9\ 7\ 3\ 6\ 5)(2\ 10\ 8\ 11\ 12\ 4)\pi = (1\ 2)(1\ 2\ 5\ 4\ 6\ 12\ 3\ 11\ 7\ 8\ 9\ 10).$$

To conclude

$$(1\ 9\ 7\ 3\ 6\ 5\ 2\ 10\ 8\ 11\ 12\ 4)\pi = (2\ 5\ 4\ 6\ 12\ 3\ 11\ 7\ 8\ 9\ 10).$$

Let G be a finite group. For given $n > 1$, we say that G *admits roots of degree* n if, for every $x \in G$, there exists $y \in G$ such that $y^n = x$.

(a) Show that if n is coprime to $|G|$, G admits roots of degree n.
(b) Conversely, prove that if G admits roots of degree n, then n is coprime to $|G|$.

(a) Notice that $an + b|G| = 1$ for some integers a and b. Thus
$$x = x^1 = x^{an+b|G|} = x^{an}x^{b|G|} = (x^a)^n(x^b)^{|G|} = (x^a)^n.$$

(b) Define $\theta : G \to G$ by
$$\theta(y) = y^n.$$
Since G admits roots of degree n, θ is surjective. So θ is injective as well. Let $d = (n, |G|)$. Suppose $d > 1$. Then there is a prime p such that p divides d. Since p divides d and d divides $|G|$, p divides $|G|$. By Cauchy's theorem, G has an element $a \neq 1$ such that $a^p = 1$. Since p divides d and d divides n, p divides n. Thus $n = pm$ for some integer m. Hence
$$\theta(a) = a^n = a^{pm} = 1.$$
To conclude $a \neq 1$ but $\theta(a) = \theta(1)$, a contradiction.

Lemma 1 Let H be a subgroup of index 2 in a group G. Then $g^2 \in H$ for all $g \in G$.

If $g \in H$, then $g^2 \in H$. Suppose $g \notin H$. Then H and gH are two distinct left cosets of H in G. Since there are only 2 distinct left cosets of H in G, either $g^2H = H$ or $g^2H = gH$. If $g^2H = gH$, then $g^2 = gh$ for some $h \in H$. It follows that $g = g^{-1}g^2 = g^{-1}(gh) = (g^{-1}g)h = h \in H$, a contradiction. To conclude that $g^2H = H$ and hence $g^2 \in g^2H = H$.

Lemma 2 Let G be a group of order 16 with two distinct subgroups A and B of order 8. Then $G = AB$.

Note that $|AB| = \frac{|A||B|}{|A \cap B|}$. Since $|A \cap B|$ divides $|A|$, $|A \cap B| \in \{1, 2, 4, 8\}$. If $|A \cap B| \in \{1, 2\}$, then $|AB| > |G|$, a contradiction. If $|A \cap B| = 8$, then $A = A \cap B = B$, a contradiction since A and B are distinct. Thus $|AB| = 16$ and to conclude that $G = AB$.

Let G be a non-abelian group of order 16 with two distinct cyclic subgroups A and B of order 8. Let a and b be generators of A and B, respectively. Show that $bab^{-1} = a^5$.

Since $|G:A| = 2$, A is normal in G. Note that $bab^{-1} \in bAb^{-1} = A$ and A is cyclic. Thus $bab^{-1} = a^k$ where $0 < k < 8$. Since $|bab^{-1}| = |a|$ and

$|a^k| = \frac{|a|}{(|a|,k)}$,

$$8 = |a| = |bab^{-1}| = |a^k| = \frac{|a|}{(|a|,k)} = \frac{8}{(8,k)}$$

and hence $k \in \{1,3,5,7\}$. By Lemma 1, $b^2 \in A$ and $|b^2| = \frac{|b|}{(|b|,2)} = \frac{8}{(8,2)} = 4$. Note that A has only two elements of order 4, namely a^2 and a^6, and so $b^2 \in \{a^2, a^6\}$. Suppose $b^2 = a^2$. Then $a^2 b = b^2 b = b^3 = bb^2 = ba^2$ and thus

$$a^2 = ba^2 b^{-1} = (bab^{-1})^2 = a^{2k}.$$

It follows that $a^{2k-2} = 1$ and so $|a|$ divides $2k-2$, i.e., $4 \mid k-1$. To conclude $k \in \{1,5\}$. Suppose $k = 1$. Then $bab^{-1} = a$. By Lemma 2, $G = AB$. Since $G = AB$ and $ab = ba$, G is abelian, a contradiction. Thus $bab^{-1} = a^5$. Suppose $b^2 = a^6$. Then $a^6 b = b^2 b = b^3 = bb^2 = ba^6$ and thus

$$a^6 = ba^6 b^{-1} = (bab^{-1})^6 = a^{6k}.$$

It follows that $a^{6k-6} = 1$ and so $|a|$ divides $6k-6$, i.e., $4 \mid 3k-3$. To conclude $k \in \{1,5\}$. Suppose $k = 1$. Then $bab^{-1} = a$. By Lemma 2, $G = AB$. Since $G = AB$ and $ab = ba$, G is abelian, a contradiction. Thus $bab^{-1} = a^5$.

Let a and b be elements of a monoid such that $aba = a$ and $ab^2a = 1$. Show that a is invertible with b as inverse.

Notice that $b = b \cdot 1 = b(ab^2a) = b(a(b^2a)) = (ba)(b^2a)$ and hence

$$ab = a((ba)(b^2a)) = (a(ba))(b^2a) = (aba)(b^2a) = a(b^2a) = ab^2a = 1.$$

Notice that $b = 1 \cdot b = (ab^2a)b = ((ab^2)a)b = (ab^2)(ab)$ and hence

$$ba = ((ab^2)(ab))a = (ab^2)((ab)a) = (ab^2)(aba) = (ab^2)a = ab^2a = 1.$$

Proposition 11
Let G be a group of order p^{2n-1} where $n \geq 1$. If $m \geq \frac{2n-1}{2}$, then $G^{(m)} = 1$, where $G^{(m)}$ is the m-th derived subgroup of G and p is a prime number.

Proof by induction on n:
If $n = 1$, then $|G| = p$. Thus G is cyclic and hence abelian. It follows that $G^{(1)} = 1$ and $G^{(m)} = 1$ whenever $m \geq \frac{1}{2}$. Suppose the statement is true for some integer $k \geq 1$. Let $|G| = p^{2(k+1)-1} = p^{2k+1}$. Then G has a normal subgroup N of order p^2. Notice that $|G/N| = p^{2k-1}$. Since $k \geq \frac{2k-1}{2}$, by induction hypothesis,

$$G^{(k)}N/N = (G/N)^{(k)} = N/N$$

and hence $G^{(k)}N = N$. It follows that $G^{(k)} \leq N$. Since N is abelian, $N^{(1)} = 1$. To conclude $G^{(k+1)} = (G^{(k)})^{(1)} \leq N^{(1)} = 1$ and $G^{(m)} = 1$ whenever $m \geq \frac{2(k+1)-1}{2} = k + \frac{1}{2}$.

Proposition 12
Let G be a group of order p^{2n} where $n \geq 1$. If $m \geq \frac{2n}{2}$, then $G^{(m)} = 1$, where $G^{(m)}$ is the m-th derived subgroup of G and p is a prime number.

Proof by induction on n:
If $n = 1$, then $|G| = p^2$. Thus G is abelian. It follows that $G^{(1)} = 1$ and $G^{(m)} = 1$ whenever $m \geq \frac{2}{2}$. Suppose the statement is true for some integer $k \geq 1$. Let $|G| = p^{2(k+1)} = p^{2k+2}$. Then G has a normal subgroup N of order p^2. Notice that $|G/N| = p^{2k}$. Since $k \geq \frac{2k}{2}$, by induction hypothesis,

$$G^{(k)}N/N = (G/N)^{(k)} = N/N$$

and hence $G^{(k)}N = N$. It follows that $G^{(k)} \leq N$. Since N is abelian, $N^{(1)} = 1$. To conclude $G^{(k+1)} = (G^{(k)})^{(1)} \leq N^{(1)} = 1$ and $G^{(m)} = 1$ whenever $m \geq \frac{2(k+1)}{2} = k+1$.

Corollary to Proposition 11 and Proposition 12
Let G be a group of order p^k where $k \geq 1$. Let $m \geq \dfrac{1}{2}k$. Then $G^{(m)} = 1$, where $G^{(m)}$ is the m-th derived subgroup of G and p is a prime number.