The distribution of prime numbers: a novel approach

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Abstract

In this article, we introduce a new mathematical conjecture that connects the sum of divisors of integers, prime-counting functions, and the properties of prime numbers. The conjecture incorporates basic number-theoretic functions, including the sum of divisors of an integer n, denoted $\sigma(n)$, the sum of the squares of the divisors of n, denoted $\sigma_2(n)$, and the prime-counting function π(n), which counts the number of prime numbers less than or equal to n.

Definitions and introduction

Let us first recall the definitions of the functions involved, which are fundamental in analytic number theory.

1. $\sigma(n)$**:** the sum of the divisors of n. It is a classical function in number theory, defined as the sum of all divisors of n, including 1 and n itself:

$$\sigma(n) = \sum_{d|n} d$$

where d runs over all divisors of n. The function $\sigma(n)$ is crucial in the study of multiplicative functions and Dirichlet series [Apostol, 1976; Rademacher, 1954].

2. $\sigma_2(n)$: The sum of the squares of the divisors of n. This is another important arithmetic function that generalizes $\sigma(n)$ by summing the squares of the divisors of n:

$$\sigma_2(n) = \sum_{d|n} d^2$$

The study of $\sigma_2(n)$ is related to the properties of quadratic forms and is a topic of interest in both elementary and analytic number theory [Vinogradov, 1954].

3. π(n) : The prime-counting function, which gives the number of primes less than or equal to n. This function is central to the distribution of prime numbers and plays a key role in many areas of analytic number theory:

$$\pi(n) = \#\{p \mid p \leq n, \text{ where } p \text{ is prime}\}$$

The prime-counting function is studied extensively in the context of the Prime Number Theorem and the Riemann Hypothesis [Tenenbaum, 1995; Apostol, 1976].

The Conjecture

The conjecture can be expressed as follows:

For a given integer n, define A by the following expression:

$$A = \sigma_2(\pi(n) - \sigma(n+2))$$

We focus on values of A that end with the digit 2. After calculating A, we subtract 1 to obtain A−1, and then take the square root of A−1. If the square root of A−1 is an integer, then this integer is always a prime number.

Example

Let us illustrate this conjecture with an example. Suppose n = 100547. We have:

Let $n = 100547$, we have $A = \sigma_2(9639 - \sigma(100549)) = 8264809922$ We have $A - 1 = 8264809921$ and we calculate the square root of 8264809921 and we have $\sqrt{A-1} = \sqrt{8264809921} = 90911$ and 90911 is prime.

Computing

There is no counterexample up $2 \cdot 10^{12}$ [Daoudi, 2024].

The program: https://textup.fr/813621ac

Discussion

The conjecture discussed in this article appears to present a novel connection between classical number-theoretic functions like $\sigma(n)$, $\sigma_2(n)$, and $\pi(n)$, , and the properties of prime numbers. While similar conjectures exist, particularly those that explore the distribution of primes and divisor functions (e.g., Hardy and Wright's work on divisor sums and prime density), the specific form of the conjecture presented here is original.

The connection between divisor sums, prime counting function, and prime number properties is an intriguing area of study that could reveal new patterns in the distribution of primes. It would be interesting to explore whether this conjecture holds for a broader set of integers or whether it is tied to specific properties of numbers that end with 2.

Further investigation is needed to determine whether this conjecture holds in general or whether it might be a more specialized result. The pattern of primes arising from such expressions could open new pathways for research in prime number theory and arithmetic functions.

The current conjecture has been studied on mathoverflow and a partial proof has been found [Daoudi, 2024].

References

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