Some hypergeometric formulas

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Abstract

In this note we give some formulas related to Pi

1. Introduction

Recall that (Vieta,~1579)

$$\frac{1}{\pi}=\frac{1}{2}\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

In this note we give some formulas for π constant.

Notation:

The generalized hypergeometric function is defined by

$${}_pF_q\left(\begin{matrix}a_1,\,a_2,\,\ldots,\,a_p\\ b_1,\,b_2,\,\ldots,\,b_q\end{matrix}\,\middle|\,z\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}$$

where

$$p,\,q\in\mathbb{N}\cup\{0\}$$

$$z\in\mathbb{C}$$

$$a_k\in\mathbb{C}\,,\;k=1,\,\ldots,\,p$$

$$b_k\in\mathbb{C}-\mathbb{Z}_0^-\,,\;k=1,\,\ldots,\,q$$

If a numerator parameter is in $\mathbb{Z}_0^-$, the series ${}_pF_q$ is found to terminate and becomes a polynomial in z .

For details see [3],[4] .

2. Formulas

Entry1.

$$\pi=\frac{1417}{450}\sum_{n=0}^{\infty}\left(-\frac{1}{27\,000}\right)^n\binom{2n}{n}{}_5F_4\left(\begin{matrix}-n,\,\frac{1}{2},\,1,\,1,\,\frac{239}{130}\\ \frac{109}{130},\,\frac{7}{6},\,\frac{11}{6},\,\frac{1}{2}-n\end{matrix}\,\middle|\,\frac{25}{12}\right)\tag{1}$$

Entry 2.

$$\pi=\frac{1}{105}\cdot\sum_{n=0}^{\infty}2^{-n}\left(175\,{}_3F_2\left(\begin{matrix}-n,\,\frac{1}{2},\,1\\ \frac{5}{4},\,\frac{7}{4}\end{matrix}\,\middle|\,\frac{1}{8}\right)-6\,n\,{}_3F_2\left(\begin{matrix}1-n,\,\frac{3}{2},\,2\\ \frac{9}{4},\,\frac{11}{4}\end{matrix}\,\middle|\,\frac{1}{8}\right)\right)\tag{2}$$

Entry 3.

$$\pi=\frac{121}{38}\sum_{n=0}^{\infty}\left(-\frac{1}{722}\right)^n\binom{2n}{n}{}_5F_4\left(\begin{matrix}-n,\,\frac{1}{2},\,1,\,1,\,\frac{25}{14}\\ \frac{11}{14},\,\frac{5}{4},\,\frac{7}{4},\,\frac{1}{2}-n\end{matrix}\,\middle|\,\frac{361}{96}\right)\tag{3}$$

Entry 4.

$$\pi=\frac{1}{52}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{2704}\right)^n\left(165\binom{2n}{n}{}_4F_3\left(\begin{matrix}-n,\,\frac{1}{2},\,1,\,1\\ \frac{5}{4},\,\frac{7}{4},\,\frac{1}{2}-n\end{matrix}\,\middle|\,-\frac{169}{12}\right)-2704\binom{2n-2}{n-1}{}_4F_3\left(\begin{matrix}1-n,\,\frac{3}{2},\,2,\,2\\ \frac{9}{4},\,\frac{11}{4},\,\frac{3}{2}-n\end{matrix}\,\middle|\,-\frac{169}{12}\right)\right)\tag{4}$$

Entry 5.

$$\pi = \frac{10}{3}\,{}_3F_2\left(\begin{matrix}\frac{1}{2},\,1,\,1\\ \frac{1}{4},\,\frac{3}{4}\end{matrix}\,\middle|\,-\frac{1}{8}\right) - 8\sum_{n=0}^{\infty}\frac{(-1)^n\left(71+128\,n+48\,n^2\right)2^n}{(4\,n+1)\,(4\,n+3)\,(4\,n+5)\,(4\,n+7)\binom{4\,n+4}{2\,n+2}}\,{}_3F_2\left(\begin{matrix}1,\,\frac{3}{2}+n,\,2+n\\ \frac{5}{4}+n,\,\frac{7}{4}+n\end{matrix}\,\middle|\,-\frac{1}{8}\right) \tag{5}$$

Entry 6.

$$\pi = \frac{1}{6}\cdot\sum_{n=0}^{\infty}\left(-\frac{1}{576}\right)^n\binom{2\,n}{n}\left(9\,{}_3F_2\left(\begin{matrix}-n,\,\frac{1}{2},\,\frac{1}{2}\\ \frac{3}{2},\,\frac{1}{2}-n\end{matrix}\,\middle|\,-\frac{11\,664}{145}\right)+8\,{}_3F_2\left(\begin{matrix}-n,\,\frac{1}{2},\,\frac{1}{2}\\ \frac{3}{2},\,\frac{1}{2}-n\end{matrix}\,\middle|\,-\frac{9216}{145}\right)\right) \tag{6}$$

$$\pi = \frac{1}{35}\cdot\sum_{n=0}^{\infty}\left(-\frac{1}{19\,600}\right)^n\binom{2\,n}{n}\left(50\,{}_3F_2\left(\begin{matrix}-n,\,\frac{1}{2},\,\frac{1}{2}\\ \frac{3}{2},\,\frac{1}{2}-n\end{matrix}\,\middle|\,-\frac{12\,250\,000}{4901}\right)+49\,{}_3F_2\left(\begin{matrix}-n,\,\frac{1}{2},\,\frac{1}{2}\\ \frac{3}{2},\,\frac{1}{2}-n\end{matrix}\,\middle|\,-\frac{11\,764\,900}{4901}\right)\right) \tag{7}$$

Entry 7.

$$\pi = 3\cdot\sum_{n=0}^{\infty}2^{-4\,n}\binom{2\,n}{n}\,{}_3F_2\left(\begin{matrix}-n,\,\frac{1}{2},\,1\\ \frac{3}{2},\,\frac{1}{2}-n\end{matrix}\,\middle|\,-\frac{4}{3}\right) \tag{8}$$

Entry 8.

$$\pi = \sum_{n=0}^{\infty}2^{-3\,n-1}\binom{2\,n}{n}\left(4\,{}_3F_2\left(\begin{matrix}-n,\,\frac{1}{2},\,1\\ \frac{3}{2},\,\frac{1}{2}-n\end{matrix}\,\middle|\,-1\right)+{}_3F_2\left(\begin{matrix}-n,\,\frac{1}{2},\,1\\ \frac{3}{2},\,\frac{1}{2}-n\end{matrix}\,\middle|\,-\frac{1}{4}\right)\right) \tag{9}$$

$$\pi = \sum_{n=0}^{\infty}2^{-2\,n+1}\cdot 7^{-2\,n-1}\binom{2\,n}{n}\left(8\,{}_3F_2\left(\begin{matrix}-n,\,\frac{1}{2},\,1\\ \frac{3}{2},\,\frac{1}{2}-n\end{matrix}\,\middle|\,-\frac{49}{27}\right)+3\,{}_3F_2\left(\begin{matrix}-n,\,\frac{1}{2},\,1\\ \frac{3}{2},\,\frac{1}{2}-n\end{matrix}\,\middle|\,-\frac{49}{48}\right)\right) \tag{10}$$

Entry 9.

$$\pi = \frac{2}{45}\cdot\sum_{n=0}^{\infty}6^{-2\,n}\binom{2\,n}{n}\left(60\,{}_3F_2\left(\begin{matrix}-n,\,\frac{1}{6},\,1\\ \frac{7}{6},\,\frac{1}{2}-n\end{matrix}\,\middle|\,-\frac{9}{8}\right)+5\,{}_3F_2\left(\begin{matrix}-n,\,\frac{1}{2},\,1\\ \frac{3}{2},\,\frac{1}{2}-n\end{matrix}\,\middle|\,-\frac{9}{8}\right)+3\,{}_3F_2\left(\begin{matrix}-n,\,\frac{5}{6},\,1\\ \frac{11}{6},\,\frac{1}{2}-n\end{matrix}\,\middle|\,-\frac{9}{8}\right)\right) \tag{11}$$

Entry 10.

$$\pi = \frac{3}{2}\cdot\sum_{n=0}^{\infty}2^{-4\,n}\cdot\sum_{k=0}^{n}2^{3\,k}\binom{2\,n-2\,k}{n-k}\,{}_2F_1\left(\begin{matrix}-k,\,\frac{1}{2}\\ \frac{3}{2}\end{matrix}\,\middle|\,\frac{1}{3}\right) \tag{12}$$

$$\pi = \frac{3}{2}\cdot\sum_{n=0}^{\infty}2^{-n}\cdot\sum_{k=0}^{n}2^{-3\,k}\binom{2\,k}{k}\,{}_2F_1\left(\begin{matrix}k-n,\,\frac{1}{2}\\ \frac{3}{2}\end{matrix}\,\middle|\,\frac{1}{3}\right) \tag{13}$$

Entry 11.

$$\pi = \frac{9}{4}\cdot\sum_{n=0}^{\infty}2^{-4\,n}\,(2\,n+1)\binom{2\,n}{n}\,{}_3F_2\left(\begin{matrix}-n,\,\frac{1}{2},\,1\\ \frac{3}{2},\,\frac{3}{2}\end{matrix}\,\middle|\,1\right) \tag{14}$$

$$\pi = 4\cdot\sum_{n=0}^{\infty}(-1)^n\,12^{-n}\binom{2\,n}{n}\,{}_3F_2\left(\begin{matrix}-n,\,\frac{1}{2},\,1\\ \frac{3}{2},\,\frac{1}{2}-n\end{matrix}\,\middle|\,1\right) \tag{15}$$

Entry 12.

$$\pi = \frac{9}{35}\sum_{n=0}^{\infty}(-1)^n\,50^{-n}\binom{2\,n}{n}\left(6\,{}_3F_2\left(\begin{matrix}-n,\,\frac{1}{2},\,1\\ \frac{3}{2},\,\frac{1}{2}-n\end{matrix}\,\middle|\,\frac{75}{98}\right)+7\,{}_3F_2\left(\begin{matrix}-n,\,\frac{1}{2},\,1\\ \frac{3}{2},\,\frac{1}{2}-n\end{matrix}\,\middle|\,\frac{25}{24}\right)\right) \tag{16}$$

Entry 13.

$$\pi = \sum_{n=0}^{\infty}(-1)^n\,2^{-4\,n}\binom{2\,n}{n}\left(2\,{}_3F_2\left(\begin{matrix}-n,\,\frac{1}{2},\,\frac{1}{2}\\ \frac{3}{2},\,\frac{1}{2}-n\end{matrix}\,\middle|\,-\frac{16}{5}\right)+{}_3F_2\left(\begin{matrix}-n,\,\frac{1}{2},\,\frac{1}{2}\\ \frac{3}{2},\,\frac{1}{2}-n\end{matrix}\,\middle|\,-\frac{4}{5}\right)\right) \tag{17}$$

$$\pi = \frac{1}{35}\cdot\sum_{n=0}^{\infty} 7^{-2n}\binom{2n}{n}\left(12\,{}_3F_2\left(\left.\begin{matrix}-n,\ \frac{1}{2},\ \frac{1}{2}\\ \frac{3}{2},\ \frac{1}{2}-n\end{matrix}\right|\frac{49}{125}\right)+90\,{}_3F_2\left(\left.\begin{matrix}-n,\ \frac{1}{2},\ \frac{1}{2}\\ \frac{3}{2},\ \frac{1}{2}-n\end{matrix}\right|\frac{49}{20}\right)\right) \tag{18}$$

Entry 14.

$$\pi = \frac{1}{12}\cdot\sum_{n=0}^{\infty}\binom{2n}{n}\left(32\cdot 6^{-2n}\,{}_3F_2\left(\left.\begin{matrix}-n,\ \frac{1}{2},\ \frac{1}{2}\\ \frac{3}{2},\ \frac{1}{2}-n\end{matrix}\right|\frac{9}{8}\right)+3\cdot 16^{-2n}\,{}_3F_2\left(\left.\begin{matrix}-n,\ \frac{1}{2},\ 1\\ \frac{3}{2},\ \frac{1}{2}-n\end{matrix}\right|-\frac{64}{63}\right)\right) \tag{19}$$

Entry 15.

$$\pi = \frac{2}{63}\sum_{n=0}^{\infty}(-1)^n\,14^{-4n}\binom{2n}{n}\left({}_4F_3\left(\left.\begin{matrix}-n,\ \frac{1}{2},\ 1,\ 1\\ \frac{1}{6},\ \frac{11}{6},\ \frac{1}{2}-n\end{matrix}\right|-\frac{98}{27}\right)+95\,{}_4F_3\left(\left.\begin{matrix}-n,\ \frac{1}{2},\ 1,\ 1\\ \frac{5}{6},\ \frac{7}{6},\ \frac{1}{2}-n\end{matrix}\right|-\frac{98}{27}\right)\right) \tag{20}$$

$$\pi = \frac{1}{208}\cdot\sum_{n=0}^{\infty}52^{-2n}\binom{2n}{n}\left(655\,{}_4F_3\left(\left.\begin{matrix}-n,\ \frac{1}{2},\ 1,\ 1\\ \frac{5}{6},\ \frac{7}{6},\ \frac{1}{2}-n\end{matrix}\right|-\frac{169}{81}\right)-{}_4F_3\left(\left.\begin{matrix}-n,\ \frac{1}{2},\ 1,\ 1\\ \frac{1}{6},\ \frac{11}{6},\ \frac{1}{2}-n\end{matrix}\right|-\frac{169}{81}\right)\right) \tag{21}$$

Entry 16.

$$\begin{aligned}\pi = \frac{41}{2^4\cdot 5^3\cdot 71}\sum_{n=0}^{\infty}(-1)^n\,2^{-n}\cdot 71^{-2n}\binom{2n}{n}\Bigg(&10464\,{}_6F_5\left(\left.\begin{matrix}-n,\ \frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ 1,\ 1\\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5},\ \frac{6}{5},\ \frac{1}{2}-n\end{matrix}\right|-\frac{45369}{12500}\right)+342\,{}_6F_5\left(\left.\begin{matrix}-n,\ \frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ 1,\ 1\\ \frac{1}{5},\ \frac{3}{5},\ \frac{4}{5},\ \frac{7}{5},\ \frac{1}{2}-n\end{matrix}\right|-\frac{45369}{12500}\right)+\\ &48\,{}_6F_5\left(\left.\begin{matrix}-n,\ \frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ 1,\ 1\\ \frac{1}{5},\ \frac{2}{5},\ \frac{4}{5},\ \frac{8}{5},\ \frac{1}{2}-n\end{matrix}\right|-\frac{45369}{12500}\right)+21\,{}_6F_5\left(\left.\begin{matrix}-n,\ \frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ 1,\ 1\\ \frac{1}{5},\ \frac{2}{5},\ \frac{3}{5},\ \frac{9}{5},\ \frac{1}{2}-n\end{matrix}\right|-\frac{45369}{12500}\right)\Bigg)\end{aligned} \tag{22}$$

Entry 17.

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-1)^n\,3^{-n}}{2n+1}\left(\frac{2\,(2n)!}{(5/3)_{2n}}\,{}_2F_1\left(\left.\begin{matrix}-\frac{1}{3},\ 2n+1\\ 2n+\frac{5}{3}\end{matrix}\right|-1\right)+\frac{(2n)!}{(7/3)_{2n}}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\ 2n+1\\ 2n+\frac{7}{3}\end{matrix}\right|-1\right)\right)=\\ &\frac{8}{\sqrt{3}}\,\pi\tanh^{-1}\left(1-\frac{1}{3}\left\{2+\frac{1}{36}\left(2+\frac{1}{36}\left(2+\frac{1}{36}\,(2+\ldots)^3\right)^3\right)^3\right\}\right)\end{aligned} \tag{23}$$

3. References

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