A PROGRESS ON THE P VS NP PROBLEM

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ABSTRACT. This paper presents a significant advancement in understanding the P vs NP problem through the lens of problem theory. Using isotopes as a technical tool within this framework, we provide a solution to the problem, establishing that P = NP. The results demonstrate the effectiveness of the proposed theoretical framework in addressing fundamental problems in computational complexity.

1. Introduction and background

Problem theory have been developed by the author in [1],[2], [3],[4] and [5]. We provide a summary of the background of this theory.

Definition 1.1. Let X denotes a solution (resp. answer) to problem Y (resp. question). Then we call the collection of all the problems to be solved to provide the solution X to the problem Y the problem space induced by providing the solution X to problem Y. We denote this space by $\mathcal{P}_Y(X)$. If K is any subspace of the space $\mathcal{P}_Y(X)$, then we denote this relation by $K \subseteq \mathcal{P}_Y(X)$. If the space K is a subspace of the space $\mathcal{P}_Y(X)$ with $K \neq \mathcal{P}_Y(X)$, then we write $K \subset \mathcal{P}_Y(X)$. We say problem V is a sub-problem of problem Y if providing a solution to problem Y furnishes a solution to problem V. If V is a subproblem of the problem Y, then we write $V \leq Y$. If V is a subproblem of the problem Y and $V \neq Y$, then we write $V < Y$ and call V a proper subproblem of Y.

Definition 1.2. Let $\mathcal{P}_Y(X)$ be the problem space induced by providing the solution X to the problem Y. Then we call the number of problems in the space (size) the **complexity** of the space and denote by $\mathbb{C}[\mathcal{P}_Y(X)]$ the complexity of the space. We make the assignment $Z \in \mathcal{P}_Y(X)$ if problem Z is also a problem in this space.

Definition 1.3. Let X denotes a solution (resp. answer) to problem Y (resp. question). Then we call the collection of all solutions to problems obtained as a result of providing the solution X to the problem Y the solution space induced by providing the solution X to the problem Y. We denote this space by $\mathcal{S}_Y(X)$. If K is any subspace of the space $\mathcal{S}_Y(X)$, then we denote this relation by $K \subset \mathcal{S}_Y(X)$. We assign $T \in \mathcal{S}_Y(X)$ if the solution T is also a solution in this space.

Proposition 1.1. *Let $\mathcal{S}_Y(X)$ be the solution space induced by providing the solution X to the problem Y. Then $X \in \mathcal{S}_Y(X)$.*

Proof. This follows by virtue of Definition 1.3. □

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Definition 1.4. Let $\mathcal{S}_Y(X)$ be the solution space induced by providing the solution X to the problem Y. Then we call the number of solutions in the space (size) the **index** of the space and denote by $\mathbb{I}[\mathcal{S}_Y(X)]$ the index of this space.

Definition 1.5. Let $\mathcal{S}_Y(X)$ be the solution space induced by providing the solution X to the problem Y. Then by the **entropy** of the space, we mean the expression
$$\mathcal{E}[S] = \frac{1}{\mathbb{I}[\mathcal{S}_Y(X)]}.$$

In the sequel, we formalize the notion that the problem space induced by providing a solution to a problem should, by necessity, contain this solution. The argument is an iteration of a never-diminishing entropy of larger and larger solution spaces. We launch formally the following arguments.

Theorem 1.6. *Let $\mathcal{P}_Y(X)$ be the induced problem space of providing the solution X to the problem Y. Then $Y \in \mathcal{P}_Y(X)$.*

Proof. Let us suppose to the contrary that for any problem space $Y \not\in \mathcal{P}_Y(X)$. Since Y is a solved problem, it must belong to some problem space, say $\mathcal{P}_V(U)$. In particular, we have the containment
$$Y \in \mathcal{P}_V(U).$$
Since X is a solution to problem Y and V has a solution U, it follows that X is a solution obtained as a result of providing a solution U to problem V. It follows that $X \in \mathcal{S}_V(U)$ so that the embedding
$$\mathcal{S}_Y(X) \subset \mathcal{S}_V(U)$$
holds, since $X \in \mathcal{S}_Y(X)$. Again, $V \not\in \mathcal{P}_V(U)$ under the assumption, so that V belongs to some problem space, say $\mathcal{P}_K(L)$. That is, $V \in \mathcal{P}_K(L)$, a problem space induced by providing a solution L to problem K. Since U is a solution to problem V and K has solution L, it must be a problem solved as a result of providing a solution L to problem K. It follows that $U \in \mathcal{S}_K(L)$ and the embedding holds
$$\mathcal{S}_Y(X) \subset \mathcal{S}_V(U) \subset \mathcal{S}_K(L)$$
since $U \in \mathcal{S}_V(U)$. By iterating the argument in this manner under the assumption that $G \not\in \mathcal{P}_G(F)$ for an arbitrary problem space, we obtain the infinite embedding
$$\mathcal{S}_Y(X) \subset \mathcal{S}_V(U) \subset \mathcal{S}_K(L) \subset \cdots \subset \cdots.$$
It follows from this the following infinite decreasing sequence of the entropy of solution spaces towards zero
$$\frac{1}{\mathbb{I}[\mathcal{S}_Y(X)]} > \frac{1}{\mathbb{I}[\mathcal{S}_V(U)]} > \frac{1}{\mathbb{I}[\mathcal{S}_K(L)]} > \cdots > \cdots$$
which is not possible. This completes the proof of the theorem. □

Definition 1.7. Let Y and V be any two problems. Then we say problem Y is equivalent to problem V if providing a solution to problem Y also provides a solution to problem V and conversely providing a solution to problem V also provides a solution to problem Y. We denote the equivalence with $V \equiv Y$.

Next, we present a simple criterion for creating a subspace of a problem space.

Proposition 1.2. *Let $X \in \mathcal{S}_V(U)$ and $Y \in \mathcal{P}_V(U)$. If X is a solution to the problem Y, then*
$$\mathcal{P}_Y(X) \subset \mathcal{P}_V(U).$$

Proof. Under the requirement $Y \in \mathcal{P}_V(U)$, then Y is a subproblem to be solved to provide a solution U to the problem V. Since $X \in \mathcal{S}_V(U)$, it follows that X is a solution obtained by providing a solution U to the problem V. Since X solves Y and $Y \in \mathcal{P}_Y(X)$, it follows that
$$\mathcal{P}_Y(X) \subset \mathcal{P}_V(U).$$
□

We use the following criterion to determine the solubility of a problem.

Proposition 1.3. *Let V be a problem with solution U. If $Y \in \mathcal{P}_V(U)$, then Y must have a solution.*

Proof. Clearly, problem V is solved by U with an induced problem space $\mathcal{P}_V(U)$. Since this space consists of all subproblems to be solved in order to provide a solution U to the problem V and $Y \in \mathcal{P}_V(U)$, then Y has a solution. □

Definition 1.8. Let V be a problem. Then we say V is reducible if there exists a proper subproblem of V with no proper subproblem. On the other hand, we say problem V is irreducible if every proper sub-problem of V has a proper sub-problem.

Definition 1.9. Let V be a problem and $\{Y_i\}_{i\geq 1}$ be the sequence of all the subproblems of V. Then we say V is regular if
$$\cdots \leq Y_3 \leq Y_2 \leq Y_1 \leq V.$$
We say it is irregular if there exist subproblems Y_j and Y_k of V such that $Y_j \not\leq Y_k$ and $Y_k \not\leq Y_j$.

De facto, regular problem can easily be solved as opposed to irregular problems, where a solution to one sub-problem cannot in anyway be modified and advanced to obtain a solution to other sub-problems. This makes the theory much more tractable with reducible problems.

1.1. Maximal and minimal sub-problems.

Definition 1.10. Let V be a problem and Y a proper subproblem of V. Then we say Y is the maximal subproblem of V if all other proper subproblems of V are subproblems of Y. We say it is the minimal subproblem of V if it is a subproblem of all other subproblems of V.

Next we relate the notion of minimal sub-problem to the notion of reducibility.

Proposition 1.4. *Let V be a problem. If there exists a minimal subproblem of V, then V must be reducible.*

Proof. Let Y be the minimal subproblem of problem V. Then Y has no proper subproblem. This implies that V must be reducible. □

In a similar fashion, we relate the notion of maximal sub-problem with the notion of regularity.

Theorem 1.11. *Let V be a problem. If every sub-problem of V has a maximal proper sub-problem, then V must be regular.*

Proof. Let Y be the maximal proper subproblem of V, since $V \leq V$. Then we have the relation $Y < V$ and every other proper sub-problem of V must be a sub-problem of Y. Since every sub-problem of V has a maximal sub-problem, we let Z be the maximal proper sub-problem of Y then $Z < Y$ and all other proper sub-problems of Y are subproblems of Z. Since the proper subproblems of V excluding Y are proper subproblems of Y and the remaining excluding Z are subproblems of Z, we obtain the chain of sub-problems

$$\cdots < Z < Y < V$$

and thus the chain contains all the subproblems of V. This shows that V must be a regular problem. □

2. **The time complexity**

In this section we study the notion of time complexity of problem and solution spaces.

Definition 2.1. The **resolution** complexity of problem T by providing a solution U that solves T is the **algorithmic** time required to generate the solution U for problem T. We denote this complexity by $\mathcal{C}_r(T, U)$.

Definition 2.2. The **verification** time complexity of a solution U to the problem T is the **algorithmic** time required to check the correctness of solution U. We denote this complexity by $\mathcal{C}_v(T, U)$.

Definition 2.3. Let T be a problem with solution U. We say the time complexity with respect to the problem T with solution U is in **equilibrium** if $\mathcal{C}_r(T, U) = \mathcal{C}_v(T, U)$.

It is important to declare that the time complexity is not unique to problems and solutions. More precisely, it is indeed possible that the resolution time complexity and the verification time complexity may differ quite significantly among equivalent problems and alternative solutions. Consequently, it may not be possible to extend an equilibrium to equivalent problems and alternative solutions. Suppose that $\mathcal{C}_r(T_1, U_1) < \infty$ and $\mathcal{C}_v(T_1, U_1) < \infty$ with $T_1 \equiv T_2$ (equivalent problems) then $U_1 \perp U_2$ (alternative solution). It is possible that

$$\mathcal{C}_r(T_1, U_1) \neq \mathcal{C}_r(T_2, U_1)$$

and

$$\mathcal{C}_v(T_1, U_1) \neq \mathcal{C}_v(T_2, U_1)$$

and similarly

$$\mathcal{C}_r(T_2, U_2) \neq \mathcal{C}_r(T_2, U_1)$$

and

$$\mathcal{C}_v(T_2, U_2) \neq \mathcal{C}_v(T_2, U_1).$$

Hence if $\mathcal{C}_r(T_1, U_1) = \mathcal{C}_v(T_1, U_1)$ and $T_1 \equiv T_2$ then the equilibrium

$$\mathcal{C}_r(T_2, U_2) = \mathcal{C}_v(T_2, U_2)$$

may only hold under certain condition. We begin by verifying that time complexity can be ordered up to sub-problems and sub-solutions of a given problem.

Proposition 2.1. *Let T be a problem with solution U. Let $\{T_i\}_{i\geq 1}$ and $\{U_i\}_{i\geq 1}$ denote the sequence of all subproblems and subsolutions of T and U, respectively. If $\mathcal{C}_r(T,U)<\infty$ and $\mathcal{C}_v(T,U)<\infty$, then we have*

$$\mathcal{C}_r(T_i,U_i)<\mathcal{C}_r(T,U)$$

and

$$\mathcal{C}_v(T_i,U_i)<\mathcal{C}_v(T,U)$$

for each $i\geq 1$.

Proof. Since $\mathcal{C}_r(T,U)<\infty$ and $\mathcal{C}_v(T,U)<\infty$ and

$$\mathcal{C}_r(T,U):=\sum_{i\geq 1}\mathcal{C}_r(T_i,U_i)$$

and

$$\mathcal{C}_v(T,U):=\sum_{i\geq 1}\mathcal{C}_v(T_i,U_i)$$

the inequality follows easily. □

Remark 2.4. In cases where we do not want to make reference to the solution and a problem in the notation of resolution and verification time complexity, we will write for simplicity $\mathcal{C}_r(T)$ and $C_v(U)$. We will adopt this notation in situations where a reference to a problem or a solution turns out to be irrelevant.

Proving the existence of an equilibrium of time complexity of problems is by no means an easy endeavor. In the sequel, we prove that assuming equilibrium in the time complexity can be passed down to subproblems and subsolutions. We make these ideas formal in the following proposition.

Proposition 2.2. *Let T be a regular problem with solution U such that for any sub-problems T_i,T_j with $i\neq j$, then $\mathcal{C}_r(T_i,U_i)\neq\mathcal{C}_v(T_j,U_j)$. If $\mathcal{C}_r(T,U)=\mathcal{C}_v(T,U)$, then there exists $Q\leq T$ (Q a sub-problem of T) and $L\leq U$ (L a sub-solution of U) that solves Q such that $\mathcal{C}_r(Q,L)=\mathcal{C}_v(Q,L)$.*

Proof. Suppose T is a regular problem with a solution U. Let $\{T_i\}_{i\geq 1}$ be the sequence of all subproblems of T with the corresponding sequence of solutions $\{U_i\}_{i\geq 1}$. Suppose on the contrary that $\mathcal{C}_r(T_i,U_i)=\mathcal{C}_v(T_i,U_i)$ for each $i\geq 1$. By virtue of the regularity of T, we can arrange the sequence of subproblems and subsolutions in the following way $T_1\geq T_2\geq\cdots$ and the corresponding sequence of subsolutions $U_1\geq U_2\geq\cdots$, where each preceding T_i is a subproblem of T_{i-1} and similarly each U_i is a subsolution for U_{i-1}. Since problem T is said to be solved by providing a solution to each of the sub-problems, we find under the assumption $\mathcal{C}_r(T,U)=\mathcal{C}_v(T,U)$, that

$$\mathcal{C}_r(T,U)=\sum_{i\geq 1}\mathcal{C}_r(T_i,U_i)=\sum_{i\geq 1}\mathcal{C}_v(T_i,U_i)=\mathcal{C}_v(T,U).$$

Now suppose on the contrary that $\mathcal{C}_r(T_1,U_1)\neq\mathcal{C}_v(T_1,U_1)$, then under the regularity condition, it follows that

$$\sum_{i\geq 2}\mathcal{C}_r(T_i,U_i)\neq\sum_{i\geq 2}\mathcal{C}_v(T_i,U_i)$$

since providing a solution to all sub-problems of T_2 solves problem T_2. Under the requirement that $\mathcal{C}_r(T_i,U_i) \neq \mathcal{C}_v(T_j,U_j)$ for all $i \neq j$, it follows that

$$\mathcal{C}_r(T,U) = \sum_{i\geq 1}\mathcal{C}_r(T_i,U_i) \neq \sum_{i\geq 1}\mathcal{C}_v(T_i,U_i) = \mathcal{C}_v(T,U)$$

violating the assumption that $\mathcal{C}_r(T,U) = \mathcal{C}_v(T,U)$. □

Theorem 2.5. *Let T be a regular problem with a solution K. If M is the maximal subproblem of T with a solution L and $\mathcal{C}_r(M,L) \ll$ polynomial time and $\mathcal{C}_r(T,K) = \mathcal{C}_v(T,K)$, then $\mathcal{C}_v(T,K) \ll$ polynomial time.*

Proof. Suppose T is a regular problem and let $\{T_i\}_{i\geq 1}$ denote the sequence of all subproblems of T with corresponding sequence of subsolutions $\{K_i\}_{i\geq 1}$ where each K_i solves T_i. We can arrange the sequence of subproblems in the following way: $T_1 \geq T_2 \geq \cdots$ where $T_1 := M$ is the maximal subproblem of T and where each subproblem T_i is a subproblem of T_{i-1} for $i \geq 2$. Since problem T is solved by solving each of the sub-problems in the sequence, we can write

$$\begin{aligned} C_r(T,K) &= \sum_{i\geq 1} C_r(T_i,K_i) \\ &= C_r(T_1,K_1) + \sum_{i\geq 2} C_r(T_i,K_i). \end{aligned}$$

By the regularity of problem T, we see that

$$\sum_{i\geq 2} C_r(T_i,K_i) = C_r(T_1,K_1) \ll \textit{polynomial time}.$$

Thus $C_r(T,K) \ll$ *polynomial time*. Under the equality $C_r(T,K) = C_v(T,K)$, we deduce that $C_v(T,K) \ll$ *polynomial time*, which completes the proof of the theorem. □

Remark 2.6. Theorem 2.5 is an important ingredient in exploring a deep understanding of the P vs NP problem. It purports that once there exists an equilibrium of time complexity of a given problem, it suffices to only investigate the resolution complexity of the maximal sub-problem for a class of well-behaved problems which we refer to as regular problems, introduced and studied in [2].

Although the task of proving equilibrium of resolution and verification time complexity can be very hard, we can often carry out this process from the bottom up. That is to say, proving the equilibrium of time complexity for sub-problems can be extended to the time complexity equilibrium of the actual problem. The following proposition exemplifies that principle.

Proposition 2.3. *Let Y be a problem with solution X and let $\{Y_i\}_{i\geq 1}$ and $\{X_i\}_{i\geq 1}$ denote the sequence of all proper sub-problems and a solution to sub-problems of Y. If $\mathcal{C}_r(Y_i,X_i) = \mathcal{C}_v(Y_i,X_i)$ for each $i \geq 1$, then $\mathcal{C}_r(Y,X) = \mathcal{C}_v(Y,X)$.*

Proof. The sequences $\{Y_i\}_{i\geq 1}$ and $\{X_i\}_{i\geq 1}$ denote the sequence of all proper subproblems and solutions to subproblems of Y, respectively. Since the solution to problem Y is furnished solving each of the sub-problems in $\{Y_i\}_{1\geq 1}$, it follows under the assumption $\mathcal{C}_r(Y_i, X_i) = \mathcal{C}_v(Y_i, X_i)$ for each $i \geq 1$ that

$$\mathcal{C}_r(Y, X) = \sum_{i\geq 1} \mathcal{C}_r(Y_i, X_i) = \sum_{i\geq 1} \mathcal{C}_v(Y_i, X_i) = \mathcal{C}_v(Y, X).$$

□

We now obtain an important characterization of irreducible problems.

Theorem 2.7. *If X is an irreducible problem, then $\mathcal{C}_r(X) = \infty$ or X is not solvable.*

Proof. Suppose X is an irreducible problem and assume the contrary that $\mathcal{C}_r(X) < \infty$ and X is solvable. Since X is irreducible, each subproblem $X_j \leq X$ has a proper subproblem, and the problem X has infinitely many proper subproblems $X_i < X$. Thus

$$\mathcal{C}_r(X) := \sum_{i=1}^{\infty} \mathcal{C}_r(X_i) < \infty$$

since the problem X is solved by providing a solution to each of the subproblems. This implies that for any $\epsilon > 0$, there exists some $N := N(\epsilon)$ such that for all $i \geq N$ we have

$$\sum_{i=N}^{\infty} \mathcal{C}_r(X_i) < \epsilon.$$

That is, $\mathcal{C}_r(X_i) \longrightarrow 0$ as $i \longrightarrow \infty$. This means that the algorithmic time required to solve infinitely many proper subproblems of problem X converges to zero, which violates the assumption that X is solvable. □

The difficulty of proving the equilibrium of time complexity of a given problem may be made easier depending on its structure. Irregular problems seem to be very difficult to understand, and unfortunately, most problems fall into this category. However, it is much easier to establish an equilibrium for a class of well-behaved problems that fall into the category of reducible and regular problems. It turns out that once equilibrium is reached for the finest form of this problem, equilibrium will certainly be achieved for the actual problem. We make this discussion formal in the following results.

Theorem 2.8 (extension principle)**.** *Let T be a regular and reducible problem with a solution U. If T_k is a subproblem of T with solution U_k such that there exists no $T_j \in \{T_i\}_{i\geq 1}$ with $T_j \not< T_k$ and that $\mathcal{C}_r(T_k, U_k) = \mathcal{C}_v(T_k, U_k)$, then $\mathcal{C}_r(T, U) = \mathcal{C}_v(T, U)$.*

Proof. Suppose T is a regular problem with a solution U and let $\{T_i\}_{i\geq 1}$ be the sequence of all subproblems of T with the corresponding sequence of solutions $\{U_i\}_{i\geq 1}$, where each U_i solves T_i for each $i \geq 1$. Since T is reducible, it has a subproblem without a proper subproblem. Let T_k be this sub-problem of T, then by the regularity of T, we can arrange the sequence of all sub-problems of T in the following way:

$$T_k \leq T_{k-1} \leq T_{k-2} \leq \cdots \leq T_1$$

with

$$U_k \leq U_{k-1} \leq U_{k-2} \leq \cdots U_1$$

where each T_i is a sub-problem of T_{i-1} and U_i is a sub-solution of U_{i-1}. Under equilibrium $\mathcal{C}_r(T_k, U_k) = \mathcal{C}_v(T_k, U_k)$ and since problem T_{k-1} is solved by providing a solution to all its proper sub-problems, it follows that $\mathcal{C}_r(T_{k-1}, U_{k-1}) = \mathcal{C}_v(T_{k-1}, U_{k-1})$. Similarly, problem T_{k-2} is solved by providing a solution to all of its sub-problems and it follows that

$$\begin{aligned}\mathcal{C}_r(T_{k-2}, U_{k-2}) &= \mathcal{C}_r(T_k, U_k) + \mathcal{C}_r(T_{k-1}, U_{k-1}) \\ &= \mathcal{C}_v(T_k, U_k) + \mathcal{C}_v(T_{k-1}, U_{k-1}) \\ &= \mathcal{C}_v(T_{k-2}, U_{k-2}).\end{aligned}$$

We can iterate this process to reach the equilibrium $\mathcal{C}_r(T, U) = \mathcal{C}_v(T, U)$. □

Corollary 2.1. Let T be a regular and reducible problem with a solution U. Let T_k be a subproblem of T with solution U_k such that there exists no $T_j \in \{T_i\}_{i\geq 1}$ with $T_j \not< T_k$ and that $\mathcal{C}_r(T_k, U_k) = \mathcal{C}_v(T_k, U_k)$. If $\mathcal{C}_v(T, U) \ll$ *polynomial time* then $\mathcal{C}_r(T, U) \ll$ *polynomial time*.

Proof. It follows from Theorem 2.8 that $\mathcal{C}_r(T, U) = C_v(T, U)$ so that under hypothesis $\mathcal{C}_v(T, U) \ll$ *polynomial time* then $\mathcal{C}_r(T, U) \ll$ *polynomial time*. □

Remark 2.9. Corollary 2.1 suggests that under a certain mild condition, if a certain class of well-behaved problems have a solution that is easy to verify for correctness, then they must also be easy to solve at the same level.

2.1. **The time complexity of problem and solution spaces.** In this section, we study the notion of time complexity on problem and solutions spaces, as opposed to a specific problem and its solution.

Definition 2.10. Let $\mathcal{P}_Y(X)$ and $\mathcal{S}_Y(X)$ be the problem and solution spaces induced by providing the solution X to problem Y. Then by the resolution complexity of the problem space $\mathcal{P}_Y(X)$, we mean the sum of each resolution complexity of each problem in the space. For each problem $T \in \mathcal{P}_Y(X)$ there exists a solution $L \in \mathcal{S}_Y(X)$ that solves T. We denote the resolution complexity of the space with

$$\mathcal{P}_Y^r(X) := \sum_{\substack{T\in\mathcal{P}_Y(X)\\ L\in\mathcal{S}_Y(X)}} \mathcal{C}_r(T, L)$$

and the verification complexity with

$$\mathcal{S}_Y^v(X) := \sum_{\substack{L\in\mathcal{S}_Y(X)\\ T\in\mathcal{P}_Y(X)}} \mathcal{C}_v(T, L)$$

where $T < Y$ and $L < X$.

Proposition 2.4. *Let $\mathcal{P}_Y(X)$ and $\mathcal{S}_Y(X)$ be the problem and solution spaces induced by providing the solution X to the problem Y. If for each $T \in \mathcal{P}_Y(X)$ and each $L \in \mathcal{S}_Y(X)$ that solves T, $\mathcal{C}_r(T, L) = \mathcal{C}_v(T, L)$ then $\mathcal{P}_Y^r(X) = \mathcal{S}_Y^v(X)$.*

Proof. This follows trivially from the proof of Proposition 2.3. □

3. Analysis on the topology of problem spaces

In this section, we introduce and develop the analysis of the theory of problem and their solution spaces. We adapt some classical concepts in functional analysis to study problems and their corresponding solution spaces.

3.1. **Bounded problem and solution spaces.** In this section we study the notion of *bounded* problem and solution spaces.

Definition 3.1. Let $\mathcal{P}_X(Y)$ be a problem space induced by providing the solution Y to the problem X. We say the space $\mathcal{P}_X(Y)$ is bounded if and only if it has finite complexity. If we denote the complexity of the space with $\mathbb{C}[\mathcal{P}_X(Y)]$, then we say $\mathcal{P}_X(Y)$ is bounded if and only if $\mathbb{C}[\mathcal{P}_X(Y)] < \infty$. Similarly, we say the corresponding solution space $\mathcal{S}_X(Y)$ is bounded if only if it has a finite index. If we denote the index of this space by $\mathbb{I}[\mathcal{S}_X(Y)]$, then $\mathcal{S}_X(Y)$ is bounded if and only if $\mathbb{I}[\mathcal{S}_X(Y)] < \infty$.

Proposition 3.1. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing the solution Y to the problem X. If $\mathbb{C}[\mathcal{P}_X(Y)] < \infty$, then $\mathcal{P}_X(Y)$ contains a reducible problem.*

Proof. Suppose each problem $X_i \in \mathcal{P}_X(Y)$ is irreducible, then we can construct the infinite nested sequence of sub-problem spaces $\cdots \subset \mathcal{P}_{X_2}(Y_2) \subset \mathcal{P}_{X_1}(Y_1) \subset \mathcal{P}_X(Y)$ with $X_1 > X_2 > \cdots$, where $X_{j+1} < X_j$ indicates that X_{j+1} is a proper sub-problem of X_j. This implies that the space $\mathcal{P}_X(Y)$ contains infinitely many problems and thus $\mathbb{C}[\mathcal{P}_X(Y)] = \infty$. □

3.2. **Maps between problem and solution spaces.** In this section, we study the analysis of map between between problem spaces and solution spaces. We examine how the notion of *boundedness* and *compactness* are preserved under the map.

Definition 3.2. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces. We say f is *bounded* if $f(\mathcal{P}_U(T))$ is a finite subset of problems in $\mathcal{P}_S(T)$ for each bounded $\mathcal{P}_U(T) \subset \mathcal{P}_X(Y)$.

Definition 3.3. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces. We say f is *compact* if and only if $f(\mathcal{P}_X(Y))$ is *compact.*

We expose the fact that *compactness* of a map between problem spaces can be inherited from the compactness of the space on which it acts.

Theorem 3.4 (Stability theorem)**.** *Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces. If $\mathcal{P}_X(Y)$ is compact, then f is compact.*

Proof. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces and suppose that the space $\mathcal{P}_X(Y)$ is *compact.* Then there exists a finite number of problems spaces $\mathcal{P}_{K_1}(L_1), \cdots, \mathcal{P}_{K_n}(L_n)$ such that

$$\mathcal{P}_X(Y) \subset \mathcal{P}_{K_1}(L_1) \cup \cdots \cup \mathcal{P}_{K_n}(L_n).$$

We observe that $f(\mathcal{P}_X(Y) \cap \mathcal{P}_{K_1}(L_1)) \subseteq f(\mathcal{P}_{K_1}(L_1)))$. Using this relation, we can put

$$f(\mathcal{P}_X(Y)) \subseteq \bigcup_{j=1}^{n} f(\mathcal{P}_X(Y) \cap \mathcal{P}_{K_j}(L_j)) \subseteq \bigcup_{j=1}^{n} f(\mathcal{P}_{K_j}(L_j)).$$

This proves that the range $f(\mathcal{P}_X(Y))$ is *compact* and hence f is also compact. □

4. **Isotope and Isotope problem and solution spaces**

In this section we study the notion of an *isotope* of problem and solution spaces.

Definition 4.1. Let V and U be any two problems. We say V and U are *compatible* if there exists a problem space $\mathcal{P}_X(Y)$ such that $V, U \in \mathcal{P}_X(Y)$. We denote this compatibility by $V \diamond U$ or $U \diamond V$. Similarly, we say two solutions R, S to some (possibly) distinct problems are compatible if there exists a solution space $\mathcal{S}_X(Y)$ such that $R, S \in \mathcal{S}_X(Y)$. We denote this compatibility by $R \diamond S$ or $S \diamond R$.

Definition 4.2. Let U and V be compatible problems. We say V and U admit a *merger* in the space $\mathcal{P}_X(Y)$ if there exists a problem $S \in \mathcal{P}_X(Y)$ such that $V < S$ and $U < S$ and V, U are the only maximal subproblem of S. In notation, we write $V \bowtie U = S \in \mathcal{P}_X(Y)$ or $U \bowtie V = S \in \mathcal{P}_X(Y)$. Similarly, let R and T be compatible solutions. We say R and T admit a *merger* in the space $\mathcal{S}_X(Y)$ if there exists a solution $W \in \mathcal{S}_X(Y)$ such that $R < W$ and $T < W$ and R, T are the only maximal sub-solutions of W. In notation, we write $R \bowtie T = W \in \mathcal{S}_X(Y)$ or $R \bowtie T = W \in \mathcal{P}_X(Y)$

We now introduce the notion of an *isotope*.

Definition 4.3. Let $\mathcal{P}_X(Y)$ and $\mathcal{S}_X(Y)$ be the problem space and the corresponding solution space, induced by assigning solution Y to problem X. We denote an *isotope* on $\mathcal{P}_X(Y)$ as the map $\mathrm{Iso} : \mathcal{P}_X(Y) \longrightarrow \mathbb{R}$ such that

(i) $\mathrm{Iso}(V) \geq 0$ for each $V \in \mathcal{P}_X(Y)$ and
(ii) $\mathrm{Iso}(V \bowtie U) \leq \mathrm{Iso}(V) + \mathrm{Iso}(U)$ provided $U, V \in \mathcal{P}_X(Y)$ admits a merger.

A similar axiom also holds for solution spaces.

The notion of an *isotope* may not be viewed as an abstract notion. For example, if we consider a problem $V \in \mathcal{P}_X(Y)$ with a solution $U \in \mathcal{S}_X(Y)$ and an induced problem space $\mathcal{P}_V(U) \subset \mathcal{P}_X(Y)$, then we can associate a number to problem V to be

$$(\mathbb{C}[\mathcal{P}_V(U)])^{\frac{1}{\mathbb{C}[\mathcal{P}_V(U)]}-1}$$

where $\mathbb{C}[\mathcal{P}_V(U)]$ as usual denotes the complexity of the space. Similarly for a solution U in the solution space $\mathcal{S}_X(Y)$, we can assign a number to the solution U to be

$$(\mathbb{I}[\mathcal{S}_V(U)])^{\frac{1}{\mathbb{I}[\mathcal{P}_V(U)]}-1}$$

where $\mathbb{I}[\mathcal{P}_V(U)]$ as usual denotes the index of the space. We could verify that these two maps satisfy the axioms of an *isotope*. In particular, an isotope is a pseudo semi-norm.

Definition 4.4. Let $\mathcal{P}_X(Y)$ and $\mathcal{S}_X(Y)$ be a problem and a corresponding solution space whose topology admits an *isotope*. A problem (resp. solution) space equipped with an isotope is an isotope problem (resp. isotope solution) space. We denote these spaces by $(\mathcal{P}_X(Y), \mathrm{Iso}(\cdot))$ and $(\mathcal{S}_X(Y), \mathrm{Iso}(\cdot))$, respectively.

Definition 4.5. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between isotope problem spaces. We put the isotope of f, denoted $\mathrm{Iso}(f)$, to be

$$\mathrm{Iso}(f) := \sup_{\substack{V \in \mathcal{P}_X(Y) \\ \mathrm{Iso}(V) \neq 0}} \frac{\mathrm{Iso}(f(V))}{\mathrm{Iso}(V)}.$$

We say f is bounded if $\text{Iso}(f) < \infty$. A similar characterization also holds for solution spaces.

Proposition 4.1. *Let* $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ *be a map between problem spaces. Then* $\text{Iso}(f) < \infty$ *if and only if there exists an absolute constant* $c > 0$ *such that* $\text{Iso}(f(V)) \leq c \, \text{Iso}(V)$ *for all* $V \in \mathcal{P}_X(Y)$*.*

Proof. Suppose $\text{Iso}(f) < \infty$ then by definition 4.5 there exists an absolute constant $c > 0$ such that $\frac{\text{Iso}(f(V))}{\text{Iso}(V)} \leq c$ for all $V \in \mathcal{P}_X(Y)$. It implies immediately that $\text{Iso}(f(V)) \leq c \, \text{Iso}(V)$ for all $V \in \mathcal{P}_X(Y)$. Conversely, suppose $\text{Iso}(f(V)) \leq c \, \text{Iso}(V)$ for all $V \in \mathcal{P}_X(Y)$ then

$$\text{Iso}(f) := \sup_{\substack{V \in \mathcal{P}_X(Y) \\ \text{Iso}(V) \neq 0}} \frac{\text{Iso}(f(V))}{\text{Iso}(V)} < \infty.$$

□

4.1. **Bounded isotope problem spaces.** In this section, we introduce and study the notion of a *bounded* isotope problem and solution spaces.

Definition 4.6. Let $\mathcal{P}_X(Y)$ be an isotope problem space induced by providing solution Y to problem X. We say the space $\mathcal{P}_X(Y)$ is bounded if $\text{Iso}(V) < \infty$ for all $V \in \mathcal{P}_X(Y)$.

Remark 4.7. We now show that a bounded map between problem spaces maps bounded subspaces to a bounded set of problems.

Proposition 4.2. *Let* $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ *be a map between isotope problem spaces. Suppose* $\mathcal{P}_K(L) \subset \mathcal{P}_X(Y)$ *is a bounded sub-problem space. If* $\text{Iso}(f) < \infty$*, then* $f(\mathcal{P}_K(L))$ *is bounded in* $\mathcal{P}_S(T)$*.*

Proof. Consider the map $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ such that $\text{Iso}(f) < \infty$. Then there exists an absolute constant $c > 0$ such that Iso(f(V)) $\leq c \, \text{Iso}(V)$ for all $V \in \mathcal{P}_X(Y)$. The requirement that $\mathcal{P}_K(L)$ is bounded implies that $\text{Iso}(V) < \infty$ for all $V \in \mathcal{P}_K(L)$. This implies that $\text{Iso}(f(V)) \leq d$ for all $V \in \mathcal{P}_K(L)$. This shows that $f(\mathcal{P}_K(L))$ is bounded in $\mathcal{P}_S(T)$. □

A similar characterization could be performed and proofs can be constructed by replacing the problem spaces $\mathcal{P}_K(L)$ with the corresponding induced solution spaces $\mathcal{S}_K(L)$.

5. **Application to the P vs NP problem**

In this section, we provide a sketch solution to the P vs NP problem. We show that P = NP by employing ideas drawn from problem theory. We begin as follows:

5.1. **A sketch solution.** Let p be a problem in NP, then the solution q to the problem p is verifiable in polynomial time. In keeping with the notation of the theory, we let $\mathcal{P}_p(q)$ and $\mathcal{S}_p(q)$ denote the problem and the solution spaces induced equipped with an **isotope**. That is, we work in the isotope problem and solution spaces $(\mathcal{P}_p(q), \mathrm{Iso}(\cdot))$ and $(\mathcal{S}_p(q), \mathrm{Iso}(\cdot))$. We obtain for the corresponding resolution and verification complexity

$$\mathcal{P}_p^r(q) := \sum_{\substack{u\in\mathcal{P}_p(q)\\ k\in\mathcal{S}_p(q)}} \mathcal{C}_r(u,k) = \mathcal{C}_r(p,q)$$

and the verification complexity with

$$\mathcal{S}_p^v(q) := \sum_{\substack{k\in\mathcal{S}_p(q)\\ u\in\mathcal{P}_p(q)}} \mathcal{C}_v(u,k) = \mathcal{C}_v(p,q)$$

where $\mathcal{C}_r(u,k)$ and $\mathcal{C}_v(u,k)$ denotes the resolution and the verification time complexity each problem u in the problem space with solution q in the corresponding solution space. Because we have assumed that $p \in \mathrm{NP}$, it follows that for the verification time complexity $\mathcal{C}_v(p,q) \ll$ **polynomial time**. Hence, for each $u \in \mathcal{P}_p(q)$ with solution $k \in \mathcal{S}_p(q)$, we must have $\mathcal{C}_v(u,k) \ll$ **polynomial time**. It follows necessarily that for the index of the solution space $\mathcal{S}_p(q)$, we must have $\mathbb{I}[\mathcal{S}_p(q)] < \infty$. Consequently, we have, for the complexity of the problem space $\mathbb{C}[\mathcal{P}_p(q)] < \infty$.

Now consider a surjective map $f : \mathcal{P}_x(y) \longrightarrow \mathcal{P}_p(q)$ for $x \in \mathrm{P}$ and $p \in \mathrm{NP}$. Since $f(\mathcal{P}_x(y)) = \mathcal{P}_p(q)$ and $\mathbb{C}[\mathcal{P}_p(q)] < \infty$, it implies that f is **bounded**. It follows that $\mathrm{Iso}(f) < \infty$, where Iso denotes the **isotope** of the map f. Therefore, there exists an absolute constant $c_1 > 0$ such that $\mathrm{Iso}(f(l)) \leq c_1 \mathrm{Iso}(l)$ for all $l \in \mathcal{P}_x(y)$.

Now, we define $\mathrm{Iso} : \mathcal{P}_x(y) \longrightarrow \mathbb{R}$ by

$$\mathrm{Iso}(l) := \sum_{\substack{w\in\mathcal{S}_l(t)\\ z\in\mathcal{P}_l(t)}} \mathcal{C}_r(z,w).$$

It is easy to check that this definition satisfies the axioms of an **Isotope**. Because $x \in \mathrm{P}$, it follows that $\mathcal{C}_r(x,y) \ll$ **polynomial time** and hence $\mathrm{Iso}(x) \ll$ **polynomial time**. It follows that

$$\mathrm{Iso}(p) := \mathcal{C}_r(p,q) := \sum_{\substack{k\in\mathcal{S}_p(q)\\ u\in\mathcal{P}_p(q)}} \mathcal{C}_r(u,k) \leq c_1 \mathrm{Iso}(h) \ll \textbf{polynomial time}$$

for some $h \in \mathcal{P}_x(y)$ such that $f(h) = p \in \mathcal{P}_p(q)$. This proves $p \in \mathrm{P}$.

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