Group Theory: Problems and Solutions (Part 1)

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**Abstract.** There is nothing new about group theory in this paper. It presents group theory problems at undergraduate level and their solutions. In presenting the solutions, we avoid using advanced theorems from group theory but we tried to discuss the solutions using elementary facts in group theory.

Let $G$ be a simple group of order 60. Assume, by way of contradiction that $G$ has no subgroup of index 5.
(1) If $Q$ is a Sylow 2-subgroup of $G$, show that $Q$ is maximal.
(2) Conclude that $Q$ is a self-normalizing, so $G$ has 15 Sylow 2-subgroups.
(3) Show that there must exist two distinct Sylow 2-subgroups $Q_1$ and $Q_2$ which intersect nontrivially.
(4) If $X = Q_1 \cap Q_2$, get a contradiction by observing that the centralizer of $X$ contains both $Q_1$ and $Q_2$.

(1)
Let $P$ be a subgroup of $G$ such that $Q \leq P$. Since $|Q|$ divides $|P|$ and $|P|$ divides $|G|$,

$$|P| \in \{4, 12, 20, 60\}.$$

Suppose $P \neq Q$. Then $|P| \neq 4$. If $|P| = 12$, then $|G : P| = 5$, a contradiction since $G$ has no subgroup of index 5. So $|P| \neq 12$. If $|P| = 20$, then $|G : P| = 3$. Since $|G|$ does not divide $|G : P|!$, $P$ must contain a nontrivial normal subgroup of $G$, a contradiction since $G$ is simple. To conclude $|P| = 60$ and thus $P = G$.

(2)
Note that $Q \leq \mathbb{N}_G(Q) \leq G$. Since $Q$ is maximal by part (1), $\mathbb{N}_G(Q) = Q$ or $\mathbb{N}_G(Q) = G$. If $\mathbb{N}_G(Q) = G$, then $Q$ is normal in $G$, a contradiction since $G$ is simple. Thus $\mathbb{N}_G(Q) = Q$. Since $Q$ is a Sylow 2-subgroup of $G$, $n_2(G) = |G|/|\mathbb{N}_G(Q)| = 15$.

(3)
Since $G$ is simple, $n_5(G) = 6$. So there are $6 \cdot 4$ elements of order 5 in $G$. By part (2), $n_2(G) = 15$. Suppose any 2 distinct Sylow 2-subgroups intersect trivially. Then there are $15 \cdot (2^2 - 1)$ nonidentity elements in Sylow 2-subgroups of $G$ besides the other 24 elements of order 5 in $G$, a contradiction since they do not fit into $G$. Thus there exist two distinct Sylow 2-subgroups $Q_1$ and $Q_2$ with $Q_1 \cap Q_2 \neq 1$.

(4)
Since $|Q_1| = 4$, $Q_1$ is abelian. Thus $\mathbb{C}_G(Q_1) \geq Q_1$. Since $Q_1 \geq X$, $\mathbb{C}_G(X) \geq \mathbb{C}_G(Q_1)$. Thus $\mathbb{C}_G(X) \geq Q_1$ and hence $|\mathbb{C}_G(X)| \geq 4$. Suppose

$|\mathbb{C}_G(X)| = 4$. Then, since $\mathbb{C}_G(X) \geq Q_1$ and $|\mathbb{C}_G(X)| = |Q_1|$, $\mathbb{C}_G(X) = Q_1$. By the same token, $\mathbb{C}_G(X) = Q_2$. Thus $Q_1 = \mathbb{C}_G(X) = Q_2$, a contradiction since $Q_1$ and $Q_2$ are distinct. Thus

(1) $$|\mathbb{C}_G(X)| > 4.$$

Moreover $|Q_1|$ divides $|\mathbb{C}_G(X)|$. So

(2) $$4 \text{ divides } |\mathbb{C}_G(X)|.$$

Moreover, by Lagrange's theorem,

(3) $$|\mathbb{C}_G(X)| \text{ divides } 60.$$

By (1), (2), (3),

$$|\mathbb{C}_G(X)| \in \{12, 20, 60\}.$$

If $|\mathbb{C}_G(X)| = 20$, then $|G : \mathbb{C}_G(X)| = 3$. Since $|G|$ does not divide $|G : \mathbb{C}_G(X)|!$, $\mathbb{C}_G(X)$ must contain a nontrivial normal subgroup of $G$, a contradiction since $G$ is simple. If $|\mathbb{C}_G(X)| = 60$, then $\mathbb{C}_G(X) = G$ and so $X$ is a normal subgroup of $G$, a contradiction since $G$ is simple. If $|\mathbb{C}_G(X)| = 12$, then $|G : \mathbb{C}_G(X)| = 5$, a contradiction since $G$ has no subgroup of index 5.

Let $G$ be an abelian group and suppose $G$ has elements of orders $m$ and $n$. Prove that $G$ has an element whose order is the least common multiple of $m$ and $n$.

Let $a$ and $b$ be elements of $G$ of orders $m$ and $n$, respectively. Note that there are divisors $q$ of $m$ and $r$ of $n$ with $(q, r) = 1$ and $[m, n] = qr$. Let $c = m/q$ and $d = n/r$.

If $g$ is an element of a group and $|g| = n$, then $g^k, k \neq 0$, has order $n/(n, k)$ where $(n, k)$ is the greatest common divisor of $n$ and $k$.

So $|a^c| = m/(m, c) = q$ and $|b^d| = n/(n, d) = r$. Let $H$ and $K$ be the subgroups of $G$ generated by $a^c$ and $b^d$, respectively. Consider $H \times K$. Then

$$|(a^c, b^d)| = [|a^c|, |b^d|] = qr.$$

$G$ is abelian so $H$ and $K$ are normal in $G$. Moreover, $H \cap K = 1$. and $HK \cong H \times K$. Hence there is an isomorphism $\varphi : H \times K \to HK$ given by $\varphi((h, k)) = hk$ for $h \in H, k \in K$ and

$$|(a^c, b^d)| = |\varphi((a^c, b^d))| = |a^c b^d| = qr = [m, n].$$

Reference:

K. H. Rosen, Elementary number theory and its application. 5th edition, Pearson Education Inc. Press, 2005, p. 119.

If $P$ is a Sylow $p$-subgroup of $G$ and $a \in N(P)$ such that $a^{p^k} = 1$, then $a \in P$.

Since $a^{p^k} = 1$, $|a| \mid p^k$ and hence $\langle a \rangle$ is a $p$-subgroup of $N(P)$. Since $P$ is normal in $N(P)$, $\langle a \rangle P$ is a subgroup of $N(P)$. Since

$$|\langle a \rangle P| = \frac{|\langle a \rangle||P|}{|\langle a \rangle \cap P|},$$

$\langle a \rangle P$ is a $p$-subgroup of $N(P)$. Then $\langle a \rangle P$ is contained in some $p$-Sylow subgroup $Q$ of $G$ and hence $|\langle a \rangle P| \leq |Q| = |P|$. Since $P \leq \langle a \rangle P$, $|P| \leq |\langle a \rangle P|$. Since $|P| \leq |\langle a \rangle P|$ and $|\langle a \rangle P| \leq |P|$, $|P| = |\langle a \rangle P|$ and hence $P = \langle a \rangle P$. It is obvious that $a \in \langle a \rangle P = P$.

Let $\theta$ be an automorphism of a group $G$. If $M$ is a maximal subgroup of $G$, then $\theta(M)$ is also a maximal subgroup of $G$.

Let $N$ be a subgroup of $G$ such that $\theta(M) \subseteq N \subseteq G$ but $\theta(M) \neq N$. Hence $M \subseteq \theta^{-1}(N)$ and $M \neq \theta^{-1}(N)$. But $M$ is maximal in $G$ and hence $\theta^{-1}(N) = G$. To conclude $N = \theta(G) = G$.

Let $\theta$ be an automorphism of a group $G$ and let $\mathscr{M}$ be the set of all maximal subgroups of $G$. Then

$$\mathscr{M} = \theta(\mathscr{M}).$$

Note that $\theta^{-1}$ is also an automorphism of $G$. To show the first containment, let $M \in \mathscr{M}$. Since $\theta^{-1}(M) \in \mathscr{M}$,

$$M = \theta(\theta^{-1}(M)) \in \theta(\mathscr{M}).$$

For the second containment, it is obvious.

Prove that the Frattini subgroup of $G$ (denoted by $\Phi(G)$) is a characteristic subgroup of $G$.

Let $\mathscr{M}$ be the set of all maximal subgroups of $G$ and let $M \in \mathscr{M}$. Then $\Phi(G) \leq M$. If $\theta$ is an automorphism of $G$, then $\theta(\Phi(G)) \leq \theta(M)$ and hence

$$\theta(\Phi(G)) \leq \bigcap_{M \in \mathscr{M}} \theta(M) = \bigcap_{M \in \mathscr{M}} M = \Phi(G).$$

Moreover, $\theta^{-1}(\Phi(G)) \leq \Phi(G)$. To conclude

$$\Phi(G) = \theta(\theta^{-1}(\Phi(G))) \leq \theta(\Phi(G)).$$

Prove that there are no simple groups of order $90 = 2 \cdot 3^2 \cdot 5$.

Let $G$ be a group of order 90. By Sylow's theorems, $n_3(G) \in \{1, 10\}$

and $n_5(G) \in \{1,6\}$. Consider $n_3(G) = 10$ and $n_5(G) = 6$. So there are $6 \cdot 4$ elements of order 5 in $G$. Suppose any 2 distinct Sylow 3-subgroups intersect trivially. Then there are $10 \cdot (3^2 - 1)$ nonidentity elements in Sylow 3-subgroups of $G$ besides the other 24 elements of order 5 in $G$, a contradiction since they do not fit into $G$. Thus there exist two distinct Sylow 3-subgroups $Q_1$ and $Q_2$ with $Q_1 \cap Q_2 \neq 1$. Thus $|Q_1 \cap Q_2| \in \{3, 9\}$. If $|Q_1 \cap Q_2| = 9$, then $Q_1 = Q_1 \cap Q_2 = Q_2$, a contradiction since $Q_1$ and $Q_2$ are distinct. Thus $|Q_1 \cap Q_2| = 3$. Since the index of $Q_1 \cap Q_2$ in both $Q_1$ and $Q_2$ is 3, $Q_1 \cap Q_2$ is normal in both $Q_1$ and $Q_2$. Thus $Q_1 \leq \mathbb{N}_G(Q_1 \cap Q_2)$ and $Q_2 \leq \mathbb{N}_G(Q_1 \cap Q_2)$. As the result, $Q_1Q_2 \subseteq \mathbb{N}_G(Q_1 \cap Q_2)$. Since $|Q_1Q_2| = \frac{|Q_1||Q_2|}{|Q_1 \cap Q_2|} = 27$,

$$27 \leq |\mathbb{N}_G(Q_1 \cap Q_2)|. \tag{1}$$

Moreover $|Q_1|$ divides $|\mathbb{N}_G(Q_1 \cap Q_2)|$ and so

$$9 \text{ divides } |\mathbb{N}_G(Q_1 \cap Q_2)|. \tag{2}$$

By Lagrange's theorem,

$$|\mathbb{N}_G(Q_1 \cap Q_2)| \text{ divides } 90. \tag{3}$$

By (1), (2), (3),

$$|\mathbb{N}_G(Q_1 \cap Q_2)| \in \{45, 90\}.$$

If $|G : \mathbb{N}_G(Q_1 \cap Q_2)| = 2$, then $\mathbb{N}_G(Q_1 \cap Q_2)$ is normal in $G$. If $|\mathbb{N}_G(Q_1 \cap Q_2)| = 90$, then $\mathbb{N}_G(Q_1 \cap Q_2) = G$ and so $Q_1 \cap Q_2$ is a normal subgroup of $G$.

Prove that there are no simple groups of order $180 = 2^2 \cdot 3^2 \cdot 5$.

Let $G$ be a group of order 180. By Sylow's theorems, $n_3(G) \in \{1, 4, 10\}$. Let $N$ be the normalizer of a Sylow 3-subgroup of $G$. Suppose $n_3(G) = 4$. Then $|G : N| = 4$. Since $|G|$ does not divide $|G : N|!$, $N$ must contain a nontrivial normal subgroup of $G$. By Sylow's theorems, $n_5(G) \in \{1, 6, 36\}$. Let $N$ be the normalizer of a Sylow 5-subgroup of $G$. Suppose $n_5(G) = 6$. Then $|N| = |G|/n_5(G) = 30$. So $N$ has a normal subgroup $K$ of order 3. Since $K$ is normal in $N$, $N \leq \mathbb{N}_G(K)$. So

$$30 \text{ divides } |\mathbb{N}_G(K)|. \tag{1}$$

Note that $K \leq P$ where $P$ is some Sylow 3-subgroup of $G$. Thus $K$ is normal in $P$ and hence $P \leq \mathbb{N}_G(K)$. So

$$9 \text{ divides } |\mathbb{N}_G(K)|. \tag{2}$$

Moreover, by Lagrange's theorem,

$$|\mathbb{N}_G(K)| \text{ divides } 180. \tag{3}$$

By (1), (2) and (3),

$$|\mathbb{N}_G(K)| \in \{90, 180\}.$$

If $|\mathbb{N}_G(K)| = 90$, then $|G : \mathbb{N}_G(K)| = 2$ and so $\mathbb{N}_G(K)$ is a normal subgroup of $G$. If $|\mathbb{N}_G(K)| = 180$, then $\mathbb{N}_G(K) = G$ and so $K$ is a normal subgroup of

$G$. Now consider $n_3(G) = 10$ and $n_5(G) = 36$. So there are $36 \cdot 4$ elements of order 5 in $G$. Suppose any 2 distinct Sylow 3-subgroups intersect trivially. Then there are $10 \cdot (3^2 - 1)$ nonidentity elements in Sylow 3-subgroups of $G$ besides the other 144 elements of order 5 in $G$, a contradiction since they do not fit into $G$. Thus there exist two distinct Sylow 3-subgroups $Q_1$ and $Q_2$ with $Q_1 \cap Q_2 \neq 1$. Thus $|Q_1 \cap Q_2| \in \{3, 9\}$. If $|Q_1 \cap Q_2| = 9$, then $Q_1 = Q_1 \cap Q_2 = Q_2$, a contradiction since $Q_1$ and $Q_2$ are distinct. Thus $|Q_1 \cap Q_2| = 3$. Since the index of $Q_1 \cap Q_2$ in both $Q_1$ and $Q_2$ is 3, $Q_1 \cap Q_2$ is normal in both $Q_1$ and $Q_2$. Thus $Q_1 \leq \mathbb{N}_G(Q_1 \cap Q_2)$ and $Q_2 \leq \mathbb{N}_G(Q_1 \cap Q_2)$. As the result, $Q_1 Q_2 \subseteq \mathbb{N}_G(Q_1 \cap Q_2)$. Since $|Q_1 Q_2| = \frac{|Q_1||Q_2|}{|Q_1 \cap Q_2|} = 27$,

$$27 \leq |\mathbb{N}_G(Q_1 \cap Q_2)|. \tag{1}$$

Moreover $|Q_1|$ divides $|\mathbb{N}_G(Q_1 \cap Q_2)|$ and so

$$9 \text{ divides } |\mathbb{N}_G(Q_1 \cap Q_2)|. \tag{2}$$

By Lagrange's theorem,

$$|\mathbb{N}_G(Q_1 \cap Q_2)| \text{ divides } 180. \tag{3}$$

By (1), (2), (3),

$$|\mathbb{N}_G(Q_1 \cap Q_2)| \in \{36, 45, 90, 180\}.$$

If $|G : \mathbb{N}_G(Q_1 \cap Q_2)| \in \{2, 4, 5\}$, then $|G|$ does not divide $|G : \mathbb{N}_G(Q_1 \cap Q_2)|!$ and hence $\mathbb{N}_G(Q_1 \cap Q_2)$ must contain a nontrivial normal subgroup of $G$. If $|\mathbb{N}_G(Q_1 \cap Q_2)| = 180$, then $\mathbb{N}_G(Q_1 \cap Q_2) = G$ and so $Q_1 \cap Q_2$ is a normal subgroup of $G$.

Prove that there are no simple groups of order $144 = 2^4 \cdot 3^2$.

Let $G$ be a group of order 144. By Sylow's theorems, $n_3(G) \in \{1, 4, 16\}$. If $n_3(G) = 1$, then $G$ is not simple. Let $N$ be the normalizer of a Sylow 3-subgroup of $G$. If $n_3(G) = 4$, then $|G : N| = 4$. Since $|G|$ does not divide $|G : N|!$, $N$ must contain a nontrivial normal subgroup of $G$. Now suppose $n_3(G) = 16$. Since $n_3(G) \not\equiv 1 (\text{mod } 3^2)$, there exist two distinct Sylow 3-subgroups $Q_1$ and $Q_2$ of $G$ such that $Q_1 \cap Q_2$ is of index 3 in both $Q_1$ and $Q_2$. Since $Q_1 \cap Q_2$ is normal in both $Q_1$ and $Q_2$, $Q_1 \leq \mathbb{N}_G(Q_1 \cap Q_2)$ and $Q_2 \leq \mathbb{N}_G(Q_1 \cap Q_2)$. As the result, $Q_1 Q_2 \subseteq \mathbb{N}_G(Q_1 \cap Q_2)$. Since $|Q_1 Q_2| = \frac{|Q_1||Q_2|}{|Q_1 \cap Q_2|} = 27$,

$$27 \leq |\mathbb{N}_G(Q_1 \cap Q_2)|. \tag{1}$$

Moreover $|Q_1|$ divides $|\mathbb{N}_G(Q_1 \cap Q_2)|$ and so

$$9 \text{ divides } |\mathbb{N}_G(Q_1 \cap Q_2)|. \tag{2}$$

By Lagrange's theorem,

$$|\mathbb{N}_G(Q_1 \cap Q_2)| \text{ divides } 144. \tag{3}$$

By (1), (2), (3),

$$|\mathbb{N}_G(Q_1 \cap Q_2)| \in \{36, 72, 144\}.$$

If $|G : \mathbb{N}_G(Q_1 \cap Q_2)| \in \{2, 4\}$, then $|G|$ does not divide $|G : \mathbb{N}_G(Q_1 \cap Q_2)|!$ and hence $\mathbb{N}_G(Q_1 \cap Q_2)$ must contain a nontrivial normal subgroup of $G$. If $|\mathbb{N}_G(Q_1 \cap Q_2)| = 144$, then $\mathbb{N}_G(Q_1 \cap Q_2) = G$ and so $Q_1 \cap Q_2$ is a normal subgroup of $G$.

Prove that there are no simple groups of order $792 = 2^3 \cdot 3^2 \cdot 11$.

Let $G$ be a group of order 792. By Sylow's theorems, $n_{11}(G) \in \{1, 12\}$. let $N$ be the normalizer of a Sylow 11-subgroup of $G$. Suppose $n_{11}(G) = 12$. Then $|N| = |G|/n_{11}(G) = 66$. So $N$ has a normal subgroup $H$ of order 11 and a subgroup $K$ of order 3. Let $\theta \in \mathrm{Aut}(HK)$. Then $\theta(K)$ is also a subgroup of $HK$ of order 3. By Sylow's theorems, there is one and only one subgroup of order 3 in $HK$. Thus $\theta(K) = K$ and hence $K$ is a characteristic subgroup of $HK$. Since $|N : HK| = 2$, $HK$ is normal in $N$. Since $K$ is a characteristic subgroup of $HK$ and $HK$ is normal in $N$, $K$ is normal in $N$ and hence $N \leq \mathbb{N}_G(K)$. So

(1) $$66 \text{ divides } |\mathbb{N}_G(K)|.$$

Note that $K \leq P$ where $P$ is some Sylow 3-subgroup of $G$. Thus $K$ is normal in $P$ and hence $P \leq \mathbb{N}_G(K)$. So

(2) $$9 \text{ divides } |\mathbb{N}_G(K)|.$$

Moreover, by Lagrange's theorem,

(3) $$|\mathbb{N}_G(K)| \text{ divides } 792.$$

By (1), (2), (3),

$$|\mathbb{N}_G(K)| \in \{198, 396, 792\}.$$

If $|G : \mathbb{N}_G(K)| \in \{2, 4\}$, then $|G|$ does not divide $|G : \mathbb{N}_G(K)|!$ and hence $\mathbb{N}_G(K)$ must contain a nontrivial normal subgroup of $G$. If $|\mathbb{N}_G(K)| = 792$, then $\mathbb{N}_G(K) = G$ and so $K$ is a normal subgroup of $G$.

Prove that there are no simple groups of order $2025 = 3^4 \cdot 5^2$.

Let $G$ be a group of order 2025. By Sylow's theorems, $n_5(G) \in \{1, 81\}$. If $n_5(G) = 1$, then $G$ is not simple. Now suppose $n_5(G) = 81$. Since $n_5(G) \not\equiv 1 (\mathrm{mod}\ 5^2)$, there exist two distinct Sylow 5-subgroups $Q_1$ and $Q_2$ of $G$ such that $Q_1 \cap Q_2$ is of index 5 in both $Q_1$ and $Q_2$. Since $Q_1 \cap Q_2$ is normal in both $Q_1$ and $Q_2$, $Q_1 \leq \mathbb{N}_G(Q_1 \cap Q_2)$ and $Q_2 \leq \mathbb{N}_G(Q_1 \cap Q_2)$. As the result, $Q_1Q_2 \subseteq \mathbb{N}_G(Q_1 \cap Q_2)$. Since $|Q_1Q_2| = \frac{|Q_1||Q_2|}{|Q_1 \cap Q_2|} = 125$,

(1) $$125 \leq |\mathbb{N}_G(Q_1 \cap Q_2)|.$$

Moreover $|Q_1|$ divides $|\mathbb{N}_G(Q_1 \cap Q_2)|$ and so

(2) $$25 \text{ divides } |\mathbb{N}_G(Q_1 \cap Q_2)|.$$

By Lagrange's theorem,

(3) $$|\mathbb{N}_G(Q_1 \cap Q_2)| \text{ divides } 2025.$$

By (1), (2), (3),

$$|\mathbb{N}_G(Q_1 \cap Q_2)| \in \{225, 675, 2025\}.$$

If $|G : \mathbb{N}_G(Q_1 \cap Q_2)| \in \{3, 9\}$, then $|G|$ does not divide $|G : \mathbb{N}_G(Q_1 \cap Q_2)|!$ and hence $\mathbb{N}_G(Q_1 \cap Q_2)$ must contain a nontrivial normal subgroup of $G$. If $|\mathbb{N}_G(Q_1 \cap Q_2)| = 2025$, then $\mathbb{N}_G(Q_1 \cap Q_2) = G$ and so $Q_1 \cap Q_2$ is a normal subgroup of $G$.

Prove that there are no simple groups of order $3159 = 3^5 \cdot 13$.

Let $G$ be a group of order 3159. By Sylow's theorems, $n_3(G) \in \{1, 13\}$. If $n_3(G) = 1$, then $G$ is not simple. Suppose $n_3(G) = 13$. Since $n_3(G) \not\equiv 1 \pmod{3^2}$, there are two distinct Sylow 3-subgroups $P$ and $Q$ in $G$ such that $P \cap Q$ is of index 3 in both $P$ and $Q$ (hence is normal in each). Thus $P \leq \mathbb{N}_G(P \cap Q)$. Similarly, $Q \leq \mathbb{N}_G(P \cap Q)$. As the result, $PQ \subseteq \mathbb{N}_G(P \cap Q)$. Since $|PQ| = \frac{|P||Q|}{|P \cap Q|} = \frac{3^5 \cdot 3^5}{3^4} = 3^6$,

(1) $$3^6 \leq |\mathbb{N}_G(P \cap Q)|.$$

Moreover, $|P|$ divides $|\mathbb{N}_G(P \cap Q)|$ and so

(2) $$3^5 \text{ divides } |\mathbb{N}_G(P \cap Q)|.$$

By Lagrange's theorem,

(3) $$|\mathbb{N}_G(P \cap Q)| \text{ divides } 3159.$$

By (1), (2), (3), $|\mathbb{N}_G(P \cap Q)| = 3159$ and thus $\mathbb{N}_G(P \cap Q) = G$.

Prove that there are no simple groups of order $432 = 2^4 \cdot 3^3$.

Let $G$ be a group of order 432. By Sylow's theorems, $n_3(G) \in \{1, 4, 16\}$. If $n_3(G) = 1$, then $G$ is not simple. Let $N$ be the normalizer of a Sylow 3-subgroup of $G$. If $n_3(G) = 4$, then $|G : N| = 4$. Since $|G|$ does not divide $|G : N|!$, $N$ must contain a nontrivial normal subgroup of $G$. Suppose $n_3(G) = 16$. Since $n_3(G) \not\equiv 1 \pmod{3^2}$, there are two distinct Sylow 3-subgroups $P$ and $Q$ in $G$ such that $P \cap Q$ is of index 3 in both $P$ and $Q$ (hence is normal in each). Thus $P \leq \mathbb{N}_G(P \cap Q)$. Similarly, $Q \leq \mathbb{N}_G(P \cap Q)$. As the result, $PQ \subseteq \mathbb{N}_G(P \cap Q)$. Since $|PQ| = \frac{|P||Q|}{|P \cap Q|} = \frac{3^3 \cdot 3^3}{3^2} = 3^4$,

(1) $$3^4 \leq |\mathbb{N}_G(P \cap Q)|.$$

Moreover, $|P|$ divides $|\mathbb{N}_G(P \cap Q)|$ and so

(2) $$3^3 \text{ divides } |\mathbb{N}_G(P \cap Q)|.$$

By Lagrange's theorem,

(3) $$|\mathbb{N}_G(P \cap Q)| \text{ divides } 432.$$

By (1), (2), (3),

$$|\mathbb{N}_G(P \cap Q)| \in \{108, 216, 432\}.$$

If $|\mathbb{N}_G(P \cap Q)| = 108$, then $|G : \mathbb{N}_G(P \cap Q)| = 4$. Since $|G|$ does not divide $|G : \mathbb{N}_G(P \cap Q)|!$, $\mathbb{N}_G(P \cap Q)$ must contain a nontrivial normal subgroup of $G$. If $|\mathbb{N}_G(P \cap Q)| = 216$, then $|G : \mathbb{N}_G(P \cap Q)| = 2$ and so $\mathbb{N}_G(P \cap Q)$ is a normal subgroup of $G$. If $|\mathbb{N}_G(P \cap Q)| = 432$, then $\mathbb{N}_G(P \cap Q) = G$ and so $P \cap Q$ is a normal subgroup of $G$.

Prove that there are no simple groups of order $252 = 2^2 \cdot 3^2 \cdot 7$.

Let $G$ be a group of order 252. By Sylow's theorems, $n_3(G) \in \{1, 4, 7, 28\}$ and $n_7(G) \in \{1, 36\}$. Let $N$ be the normalizer of a Sylow 3-subgroup of $G$. If $n_3(G) = 4$, then $|G : N| = 4$. Since $|G|$ does not divide $|G : N|!$, $N$ must contain a nontrivial normal subgroup of $G$. Now consider $n_3(G) \in \{7, 28\}$ and $n_7(G) = 36$. So there are $36 \cdot 6$ elements of order 7 in $G$. Suppose any 2 distinct Sylow 3-subgroups intersect trivially. Then there are $n_3(G) \cdot (3^2 - 1)$ nonidentity elements in Sylow 3-subgroups of $G$ besides the other 216 elements of order 7 in $G$, a contradiction since they do not fit into $G$. Thus there exist two distinct Sylow 3-subgroups $Q_1$ and $Q_2$ with $Q_1 \cap Q_2 \neq 1$. Thus $|Q_1 \cap Q_2| \in \{3, 9\}$. If $|Q_1 \cap Q_2| = 9$, then $Q_1 = Q_1 \cap Q_2 = Q_2$, a contradiction since $Q_1$ and $Q_2$ are distinct. Thus $|Q_1 \cap Q_2| = 3$. Since the index of $Q_1 \cap Q_2$ in both $Q_1$ and $Q_2$ is 3, $Q_1 \cap Q_2$ is normal in both $Q_1$ and $Q_2$. Thus $Q_1 \leq \mathbb{N}_G(Q_1 \cap Q_2)$ and $Q_2 \leq \mathbb{N}_G(Q_1 \cap Q_2)$. As the result, $Q_1 Q_2 \subseteq \mathbb{N}_G(Q_1 \cap Q_2)$. Since $|Q_1 Q_2| = \frac{|Q_1||Q_2|}{|Q_1 \cap Q_2|} = 27$,

(1) $$27 \leq |\mathbb{N}_G(Q_1 \cap Q_2)|.$$

Moreover $|Q_1|$ divides $|\mathbb{N}_G(Q_1 \cap Q_2)|$ and so

(2) $$9 \text{ divides } |\mathbb{N}_G(Q_1 \cap Q_2)|.$$

By Lagrange's theorem,

(3) $$|\mathbb{N}_G(Q_1 \cap Q_2)| \text{ divides } 252.$$

By (1), (2), (3),

$$|\mathbb{N}_G(Q_1 \cap Q_2)| \in \{36, 63, 126, 252\}.$$

Suppose $|\mathbb{N}_G(Q_1 \cap Q_2)| = 36$. There are $36 \cdot 6$ elements of order 7 in $G$ and the remaining 36 elements should belong to $\mathbb{N}_G(Q_1 \cap Q_2)$. So $\mathbb{N}_G(Q_1 \cap Q_2)$ is the only subgroup of order 36 in $G$ and hence it is normal in $G$. If

$|G : \mathbb{N}_G(Q_1 \cap Q_2)| \in \{2, 4\}$, then $|G|$ does not divide $|G : \mathbb{N}_G(Q_1 \cap Q_2)|!$ and hence $\mathbb{N}_G(Q_1 \cap Q_2)$ must contain a nontrivial normal subgroup of $G$. If $|\mathbb{N}_G(Q_1 \cap Q_2)| = 252$, then $\mathbb{N}_G(Q_1 \cap Q_2) = G$ and so $Q_1 \cap Q_2$ is a normal subgroup of $G$.

Prove that there are no simple groups of order $540 = 2^2 \cdot 3^3 \cdot 5$.

Let $G$ be a group of order 540. By Sylow's theorems, $n_3(G) \in \{1, 4, 10\}$. Let $H$ be the normalizer of a Sylow 3-subgroup of $G$. If $n_3(G) = 4$, then $|G : H| = 4$. Since $|G|$ does not divide $|G : H|!$, $H$ must contain a nontrivial normal subgroup of $G$. By Sylow's theorems, $n_5(G) \in \{1, 6, 36\}$. Let $K$ be the normalizer of a Sylow 5-subgroup of $G$. If $n_5(G) = 6$, then $|G : K| = 6$. Since $|G|$ does not divide $|G : K|!$, $K$ must contain a nontrivial normal subgroup of $G$. Now consider $n_3(G) = 10$ and $n_5(G) = 36$. Then $|H| = |G|/n_3(G) = 54$. and $|K| = |G|/n_5(G) = 15$. By Lagrange's theorem, $|H \cap K| \in \{1, 3\}$. Suppose $|H \cap K| = 1$. Then $|HK| = \frac{|H||K|}{|H \cap K|} = 810$, a contradiction since the elements of $HK$ cannot fit into $G$. Thus $|H \cap K| = 3$. Note that $H \cap K < P$ where $P$ is some Sylow 3-subgroup of $G$. Since $H \cap K < \mathbb{N}_P(H \cap K)$,

$$|\mathbb{N}_P(H \cap K)| \in \{9, 27\}$$

and so 9 divides $|\mathbb{N}_P(H \cap K)|$. But $\mathbb{N}_P(H \cap K) \leq \mathbb{N}_G(H \cap K)$ and so 9 divides $|\mathbb{N}_G(H \cap K)|$. Since $|K| = 15$, $K$ is cyclic and hence abelian. So $H \cap K$ is normal in $K$ and $K \leq \mathbb{N}_G(H \cap K)$. Thus 15 divides $|\mathbb{N}_G(H \cap K)|$. Both 9 and 15 divide $|\mathbb{N}_G(H \cap K)|$ and hence

$$45 \text{ divides } |\mathbb{N}_G(H \cap K)|. \tag{1}$$

By Lagrange's theorem,

$$|\mathbb{N}_G(H \cap K)| \text{ divides } 540. \tag{2}$$

By (1), (2),

$$|\mathbb{N}_G(H \cap K)| \in \{45, 90, 135, 180, 270, 540\}.$$

Suppose $|\mathbb{N}_G(H \cap K)| = 45$. By Sylow's theorems, $\mathbb{N}_G(H \cap K)$ has a normal subgroup $Q$ of order 5 and hence $\mathbb{N}_G(H \cap K) \leq \mathbb{N}_G(Q)$. Notice that $Q$ is also a Sylow 5-subgroup of $G$. Since $n_5(G) = 36$, $|\mathbb{N}_G(Q)| = 15$. But $|\mathbb{N}_G(H \cap K)|$ divides $|\mathbb{N}_G(Q)|$ and so 45 divides 15, a contradiction. Thus $|\mathbb{N}_G(H \cap K)| = 45$ is not an actual possibility. If $|G : \mathbb{N}_G(H \cap K)| \in \{2, 3, 4, 6\}$, then $|G|$ does not divide $|G : \mathbb{N}_G(H \cap K)|!$ and hence there is a nontrivial normal subgroup of $G$ contained in $\mathbb{N}_G(H \cap K)$. If $|\mathbb{N}_G(H \cap K)| = 540$, then $\mathbb{N}_G(H \cap K) = G$ and so $H \cap K$ is a normal subgroup of $G$.

Let $G$ be a group of order $120 = 2^3 \cdot 3 \cdot 5$. Then $G$ has a subgroup of index 3 or a subgroup of index 5 (or both).

By Sylow's theorems, $n_2(G) \in \{1, 3, 5, 15\}$. Suppose $n_2(G) = 1$. Then

$G$ has a normal Sylow 2-subgroup $P$. Let $Q$ be a subgroup of $G$ such that $|Q| \in \{3,5\}$. Thus $|G : PQ| \in \{3,5\}$. Let $N$ be the normalizer of a Sylow 2-subgroup of $G$. If $n_2(G) = 3$, then $|G : N| = 3$. If $n_2(G) = 5$, then $|G : N| = 5$. Now suppose $n_2(G) = 15$. Since $n_2(G) \not\equiv 1 \pmod{2^2}$, there are two distinct Sylow 2-subgroups $P$ and $Q$ in $G$ such that $P \cap Q$ is of index 2 in both $P$ and $Q$ (hence is normal in each). Thus $P \leq \mathbb{N}_G(P \cap Q)$ and $Q \leq \mathbb{N}_G(P \cap Q)$. So $PQ \subseteq \mathbb{N}_G(P \cap Q)$. Since $|PQ| = \frac{|P||Q|}{|P \cap Q|} = \frac{2^3 \cdot 2^3}{2^2} = 2^4$,

$$2^4 \leq |\mathbb{N}_G(P \cap Q)|. \tag{1}$$

Moreover, $|P|$ divides $|\mathbb{N}_G(P \cap Q)|$ and so

$$2^3 \text{ divides } |\mathbb{N}_G(P \cap Q)|. \tag{2}$$

By Lagrange's theorem,

$$|\mathbb{N}_G(P \cap Q)| \text{ divides } 120. \tag{3}$$

By (1), (2), (3),

$$|\mathbb{N}_G(P \cap Q)| \in \{24, 40, 120\}.$$

If $|\mathbb{N}_G(P \cap Q)| = 24$, then $|G : \mathbb{N}_G(P \cap Q)| = 5$. If $|\mathbb{N}_G(P \cap Q)| = 40$, then $|G : \mathbb{N}_G(P \cap Q)| = 3$. If $|\mathbb{N}_G(P \cap Q)| = 120$, then $\mathbb{N}_G(P \cap Q) = G$ and so $P \cap Q$ is a normal subgroup of $G$. Let $N = P \cap Q$. Consider $\overline{G} = G/N$. So $|\overline{G}| = 120/2^2 = 30$. Note that $\overline{G}$ has subgroups of order 6 and 10 and their complete preimages in $G$ are subgroups of $G$ of order 24 and 40, respectively. So in this case $G$ has subgroups of index 3 and 5.

Prove that there are no simple groups of order $240 = 2^4 \cdot 3 \cdot 5$.

Let $G$ be a group of order 240. By Sylow's theorems, $n_2(G) \in \{1, 3, 5, 15\}$. If $n_2(G) = 1$, then $G$ is not simple. Let $N$ be the normalizer of a Sylow 2-subgroup of $G$. If $n_2(G) = 3$, then $|G : N| = 3$. Since $|G|$ does not divide $|G : N|!$, $N$ must contain a nontrivial normal subgroup of $G$. If $n_2(G) = 5$, then $|G : N| = 5$. Since $|G|$ does not divide $|G : N|!$, $N$ must contain a nontrivial normal subgroup of $G$. Suppose $n_2(G) = 15$. Since $n_2(G) \not\equiv 1 \pmod{2^2}$, there are two distinct Sylow 2-subgroups $P$ and $Q$ in $G$ such that $P \cap Q$ is of index 2 in both $P$ and $Q$ (hence is normal in each). Thus $P \leq \mathbb{N}_G(P \cap Q)$ and $Q \leq \mathbb{N}_G(P \cap Q)$. As the result, $PQ \subseteq \mathbb{N}_G(P \cap Q)$. Since $|PQ| = \frac{|P||Q|}{|P \cap Q|} = \frac{2^4 \cdot 2^4}{2^3} = 2^5$,

$$2^5 \leq |\mathbb{N}_G(P \cap Q)|. \tag{1}$$

Moreover, $|P|$ divides $|\mathbb{N}_G(P \cap Q)|$ and so

$$2^4 \text{ divides } |\mathbb{N}_G(P \cap Q)|. \tag{2}$$

By Lagrange's theorem,

$$|\mathbb{N}_G(P \cap Q)| \text{ divides } 240. \tag{3}$$

By (1), (2), (3),

$$|\mathbb{N}_G(P\cap Q)| \in \{48, 80, 240\}.$$

If $|G : \mathbb{N}_G(P\cap Q)| \in \{3,5\}$, then $|G|$ does not divide $|G : \mathbb{N}_G(P\cap Q)|!$ and hence there is a nontrivial normal subgroup of $G$ contained in $\mathbb{N}_G(P\cap Q)$. If $|\mathbb{N}_G(P\cap Q)| = 240$, then $\mathbb{N}_G(P\cap Q) = G$ and so $P\cap Q$ is a normal subgroup of $G$.

Prove that there are no simple groups of order $280 = 2^3 \cdot 5 \cdot 7$.

Let $G$ be a group of order 280. By Sylow's theorems, $n_7(G) \in \{1,8\}$. Let $N$ be the normalizer of a Sylow 7-subgroup of $G$. Suppose $n_7(G) = 8$. Then $8 = |G : N|$ and so $|N| = 35$. By Sylow's theorems, $N$ has a normal subgroup $P$ of order 5 and hence $N \leq \mathbb{N}_G(P)$. So

(1) $$35 \text{ divides } |\mathbb{N}_G(P)|.$$

Notice that $P$ is also a Sylow 5-subgroup of $G$. Since $n_5(G) \in \{1, 56\}$,

(2) $$|\mathbb{N}_G(P)| \in \{5, 280\}.$$

By (1), (2), $|\mathbb{N}_G(P)| = 280$. Thus $\mathbb{N}_G(P) = G$.

Let $G$ be a group of order $2^3{\cdot}3{\cdot}23$. Then its Sylow 23-subgroup is normal in $G$.

By Sylow's theorems, $n_3(G) \in \{1,4,46,184\}$ and $n_{23}(G) \in \{1,24\}$. Suppose $n_3(G) = 1$. Thus $G$ has a normal Sylow 3-subgroup $N$. Since $|G/N| = 184$, $G/N$ has a normal subgroup of order 23 of the form $H/N$ for some subgroup $H$ of $G$ containing $N$. Thus $|H| = 23 \cdot 3 = 69$. Since $H/N$ is normal in $G/N$, $H$ is also normal in $G$. Note that $H$ has a subgroup $P$ of order 23. Let $\theta \in \mathrm{Aut}(H)$. Then $\theta(P)$ is also a subgroup of $H$ of order 23. By Sylow's theorems, there is one and only one subgroup of order 23 in $H$. Thus $\theta(P) = P$ and hence $P$ is a characteristic subgroup of $H$. Since $P$ is a characteristic subgroup of $H$ and $H$ is normal in $G$, $P$ is normal in $G$. If $n_{23}(G) = 1$, then $G$ has a normal Sylow 23-subgroup. Let $N$ be the normalizer of a Sylow 3-subgroup of $G$. Suppose $n_3(G) = 4$. Then $4 = |G : N|$ and so $|N| = 138$. By Sylow's theorems, $N$ has a normal subgroup $P$ of order 23 and hence $N \leq \mathbb{N}_G(P)$. So

(1) $$138 \text{ divides } |\mathbb{N}_G(P)|.$$

Notice that $P$ is also a Sylow 23-subgroup of $G$. Since $n_{23}(G) \in \{1, 24\}$,

(2) $$|\mathbb{N}_G(P)| \in \{23, 552\}.$$

By (1), (2), $|\mathbb{N}_G(P)| = 552$. Thus $\mathbb{N}_G(P) = G$. Consider $n_3(G) \in \{46, 184\}$ and $n_{23}(G) = 24$. So there are $n_3(G) \cdot 2$ elements of order 3 in $G$ and $24 \cdot 22$ elements of order 23 in $G$, a contradiction since they do not fit into $G$.

**Lemma 0**
Let $G$ be a group of order $23 \cdot 24$. If $G$ has a normal subgroup $N$ with $|N| \in \{2, 3, 6\}$, then $G$ has a normal Sylow 23-subgroup.

Let $\bar{G} = G/N$. So $|\bar{G}| \in \{92, 184, 276\}$ and $\bar{G}$ has a normal Sylow 23-subgroup $\bar{H}$. Let $H = \{x \in G \mid xN \in \bar{H}\}$. Then $H$ is a subgroup of $G$ and $\bar{H} \cong H/N$; thus

$$23 = |\bar{H}| = \frac{|H|}{|N|}.$$

So $|H| \in \{46, 69, 138\}$ and $H$ has a Sylow 23-subgroup $Q$. Let $\varphi$ be an automorphism of $H$. Then $\varphi(Q)$ is a Sylow 23-subgroup of $H$. $H$ has exactly one Sylow 23-subgroup and thus $\varphi(Q) = Q$. Since $\bar{H}$ is normal in $\bar{G}$, $H$ is normal in $G$. Since $Q$ is a characteristic subgroup of $H$ and $H$ is normal in $G$, $Q$ is normal in $G$.

If $G$ is a group of order $23 \cdot 24$, then $G$ has a normal Sylow 23-subgroup.

By Sylow's theorems, $n_3(G) \in \{1, 4, 46, 184\}$ and $n_{23}(G) \in \{1, 24\}$. If $n_3(G) \in \{1, 4, 46, 184\}$ and $n_{23}(G) = 1$, then G has a normal Sylow 23-subgroup. Now suppose $n_3(G) = 1$ and $n_{23}(G) = 24$. So G has a normal Sylow 3-subgroup and thus, by Lemma 0, $G$ has a normal Sylow 23-subgroup. Suppose $n_3(G) = 4$ and $n_{23}(G) = 24$. Let $N$ be the normalizer of a Sylow 3-subgroup of $G$. Then $|G : N| = 4$. Since $|G|$ does not divide $|G : N|!$, $N$ must contain a nontrivial normal subgroup of $G$. Let $H$ be the nontrivial normal subgroup of $G$ in $N$. Thus $|H| \in \{2, 3, 6, 23, 46, 69\}$. If $|H| \in \{2, 3, 6\}$, then, by Lemma 0, $G$ has a normal Sylow 23-subgroup. If $|H| = 23$, then there is nothing to prove. If $|H| \in \{46, 69\}$, then $H$ has a Sylow 23-subgroup $Q$. Let $\varphi$ be an automorphism of $H$. Then $\varphi(Q)$ is a Sylow 23-subgroup of $H$. $H$ has exactly one Sylow 23-subgroup and thus $\varphi(Q) = Q$. Since $Q$ is a characteristic subgroup of $H$ and $H$ is normal in $G$, $Q$ is normal in $G$. Now suppose $n_3(G) \in \{46, 184\}$ and $n_{23}(G) = 24$. So there are $24 \cdot 22$ elements of order 23 in $G$ and $n_3(G) \cdot (3 - 1)$ elements of order 3 in $G$, a contradiction since they do not fit into $G$.

Prove that there are no simple groups of order $2376 = 2^3 \cdot 3^3 \cdot 11$.

Let $G$ be a group of order 2376. By Sylow's theorems, $n_{11}(G) \in \{1, 12\}$. let $N$ be the normalizer of a Sylow 11-subgroup of $G$. Suppose $n_{11}(G) = 12$. Then $|N| = |G|/n_{11}(G) = 198$. So $N$ has a normal subgroup $H$ of order 11 and a subgroup $K$ of order $3^2$. Let $\theta \in \mathrm{Aut}(HK)$. Then $\theta(K)$ is also a subgroup of $HK$ of order $3^2$. By Sylow's theorems, there is one and only one subgroup of order $3^2$ in $HK$. Thus $\theta(K) = K$ and hence $K$ is a characteristic subgroup of $HK$. Since $|N : HK| = 2$, $HK$ is normal in $N$. Since $K$ is a characteristic subgroup of $HK$ and $HK$ is normal in $N$, $K$ is normal in $N$ and hence $N \leq \mathbb{N}_G(K)$. So

$$(1) \qquad 198 \text{ divides } |\mathbb{N}_G(K)|.$$

Note that $K < P$ where $P$ is some Sylow 3-subgroup of $G$. Thus $K$ is normal in $P$ and hence $P \leq \mathbb{N}_G(K)$. So

$$(2) \qquad 3^3 \text{ divides } |\mathbb{N}_G(K)|.$$

Moreover, by Lagrange's theorem,

$$(3) \qquad |\mathbb{N}_G(K)| \text{ divides } 2376.$$

By (1), (2), (3),

$$|\mathbb{N}_G(K)| \in \{594, 1188, 2376\}.$$

If $|G : \mathbb{N}_G(K)| \in \{2, 4\}$, then $|G|$ does not divide $|G : \mathbb{N}_G(K)|!$ and hence there is a nontrivial normal subgroup of $G$ contained in $\mathbb{N}_G(K)$. If $|\mathbb{N}_G(K)| = 2376$, then $\mathbb{N}_G(K) = G$ and so $K$ is a normal subgroup of $G$.

Prove that there are no simple groups of order $300 = 2^2 \cdot 3 \cdot 5^2$.

Let $G$ be a group of order 300. By Sylow's theorems, $n_5(G) \in \{1, 6\}$. If $n_5(G) = 1$, then $G$ is not simple. Let $N$ be the normalizer of a Sylow 5-subgroup of $G$. If $n_5(G) = 6$, then $|G : N| = 6$. Since $|G|$ does not divide $|G : N|!$, $N$ must contain a nontrivial normal subgroup of $G$.

Prove that there are no simple groups of order $600 = 2^3 \cdot 3 \cdot 5^2$.

Let $G$ be a group of order 600. By Sylow's theorems, $n_5(G) \in \{1, 6\}$. If $n_5(G) = 1$, then $G$ is not simple. Let $N$ be the normalizer of a Sylow 5-subgroup of $G$. If $n_5(G) = 6$, then $|G : N| = 6$. Since $|G|$ does not divide $|G : N|!$, $N$ must contain a nontrivial normal subgroup of $G$.

Prove that there are no simple groups of order $900 = 2^2 \cdot 3^2 \cdot 5^2$.

Let $G$ be a group of order 900. By Sylow's theorems, $n_5(G) \in \{1, 6, 36\}$. If $n_5(G) = 1$, then $G$ is not simple. Let $N$ be the normalizer of a Sylow 5-subgroup of $G$. If $n_5(G) = 6$, then $|G : N| = 6$. Since $|G|$ does not divide $|G : N|!$, $N$ must contain a nontrivial normal subgroup of $G$. Suppose $n_5(G) = 36$. Since $n_5(G) \not\equiv 1 (\text{mod } 5^2)$, there are two distinct Sylow 5-subgroups $P$ and $Q$ in $G$ such that $P \cap Q$ is of index 5 in both $P$ and $Q$ (hence is normal in each). Thus $P \leq \mathbb{N}_G(P \cap Q)$. Similarly, $Q \leq \mathbb{N}_G(P \cap Q)$. As the result, $PQ \subseteq \mathbb{N}_G(P \cap Q)$. Since $|PQ| = \frac{|P||Q|}{|P \cap Q|} = \frac{5^2 \cdot 5^2}{5} = 125$,

$$(1) \qquad 125 \leq |\mathbb{N}_G(P \cap Q)|.$$

Moreover, $|P|$ divides $|\mathbb{N}_G(P \cap Q)|$ and so

(2) $$5^2 \text{ divides } |\mathbb{N}_G(P \cap Q)|.$$

By Lagrange's theorem,

(3) $$|\mathbb{N}_G(P \cap Q)| \text{ divides } 900.$$

By (1), (2), (3),

$$|\mathbb{N}_G(P \cap Q)| \in \{150, 225, 300, 450, 900\}.$$

If $|G : \mathbb{N}_G(P \cap Q)| \in \{2, 3, 4, 6\}$, then $|G|$ does not divide $|G : \mathbb{N}_G(P \cap Q)|!$ and hence there is a nontrivial normal subgroup of $G$ contained in $\mathbb{N}_G(P \cap Q)$. If $|\mathbb{N}_G(P \cap Q)| = 900$, then $\mathbb{N}_G(P \cap Q) = G$ and so $P \cap Q$ is a normal subgroup of $G$.

Prove that there are no simple groups of order $2^n \cdot 3 \cdot 5$ for $n \geq 4$.

Let $G$ be a group of order $2^n \cdot 3 \cdot 5$ for $n \geq 4$. By Sylow's theorems, $n_2(G) \in \{1, 3, 5, 15\}$. If $n_2(G) = 1$, then $G$ is not simple. Let $N$ be the normalizer of a Sylow 2-subgroup of $G$. If $n_2(G) = 3$, then $|G : N| = 3$. Since $|G|$ does not divide $|G : N|!$, $N$ must contain a nontrivial normal subgroup of $G$. If $n_2(G) = 5$, then $|G : N| = 5$. Since $|G|$ does not divide $|G : N|!$, $N$ must contain a nontrivial normal subgroup of $G$. Suppose $n_2(G) = 15$. Since $n_2(G) \not\equiv 1 \pmod{2^2}$, there are two distinct Sylow 2-subgroups $P$ and $Q$ in $G$ such that $P \cap Q$ is of index 2 in both $P$ and $Q$ (hence is normal in each). Thus $P \leq \mathbb{N}_G(P \cap Q)$ and $Q \leq \mathbb{N}_G(P \cap Q)$. As the result, $PQ \subseteq \mathbb{N}_G(P \cap Q)$. Since $|PQ| = \frac{|P||Q|}{|P \cap Q|} = \frac{2^n \cdot 2^n}{2^{n-1}} = 2^{n+1}$,

(1) $$2^{n+1} \leq |\mathbb{N}_G(P \cap Q)|.$$

Moreover, $|P|$ divides $|\mathbb{N}_G(P \cap Q)|$ and so

(2) $$2^n \text{ divides } |\mathbb{N}_G(P \cap Q)|.$$

By Lagrange's theorem,

(3) $$|\mathbb{N}_G(P \cap Q)| \text{ divides } 2^n \cdot 3 \cdot 5.$$

By (1), (2), (3),

$$|\mathbb{N}_G(P \cap Q)| \in \{2^n \cdot 3, 2^n \cdot 5, 2^n \cdot 3 \cdot 5\}.$$

If $|G : \mathbb{N}_G(P \cap Q)| \in \{3, 5\}$, then $|G|$ does not divide $|G : \mathbb{N}_G(P \cap Q)|!$ and hence there is a nontrivial normal subgroup of $G$ contained in $\mathbb{N}_G(P \cap Q)$. If $|\mathbb{N}_G(P \cap Q)| = 2^n \cdot 3 \cdot 5$, then $\mathbb{N}_G(P \cap Q) = G$ and so $P \cap Q$ is a normal subgroup of $G$.

Prove that there are no simple groups of order $324 = 2^2 \cdot 3^4$.

Let $G$ be a group of order 324. Thus $G$ has a Sylow 3-subgroup $H$.

Since $|G|$ does not divide $|G : H|!$, $H$ must contain a nontrivial normal subgroup of $G$.

Prove that there are no simple groups of order $648 = 2^3 \cdot 3^4$.

Let $G$ be a group of order 648. Thus $G$ has a Sylow 3-subgroup $H$. Since $|G|$ does not divide $|G : H|!$, $H$ must contain a nontrivial normal subgroup of $G$.

Prove that there are no simple groups of order $1176 = 2^3 \cdot 3 \cdot 7^2$.

Let $G$ be a group of order 1176. By Sylow's theorems, $n_7(G) \in \{1, 8\}$. If $n_7(G) = 1$, then $G$ is not simple. Suppose $n_7(G) = 8$. Since $n_7(G) \not\equiv 1 (\bmod 7^2)$, there are two distinct Sylow 7-subgroups $P$ and $Q$ in $G$ such that $P \cap Q$ is of index 7 in both $P$ and $Q$ (hence is normal in each). Thus $P \leq \mathbb{N}_G(P \cap Q)$ and $Q \leq \mathbb{N}_G(P \cap Q)$. As the result, $PQ \subseteq \mathbb{N}_G(P \cap Q)$. Since $|PQ| = \frac{|P||Q|}{|P \cap Q|} = \frac{7^2 \cdot 7^2}{7} = 343$,

$$343 \leq |\mathbb{N}_G(P \cap Q)|. \tag{1}$$

Moreover, $|P|$ divides $|\mathbb{N}_G(P \cap Q)|$ and so

$$7^2 \text{ divides } |\mathbb{N}_G(P \cap Q)|. \tag{2}$$

By Lagrange's theorem,

$$|\mathbb{N}_G(P \cap Q)| \text{ divides } 1176. \tag{3}$$

By (1), (2), (3),

$$|\mathbb{N}_G(P \cap Q)| \in \{392, 588, 1176\}.$$

If $|G : \mathbb{N}_G(P \cap Q)| \in \{2, 3\}$, then $|G|$ does not divide $|G : \mathbb{N}_G(P \cap Q)|!$ and hence there is a nontrivial normal subgroup of $G$ contained in $\mathbb{N}_G(P \cap Q)$. If $|\mathbb{N}_G(P \cap Q)| = 1176$, then $\mathbb{N}_G(P \cap Q) = G$ and so $P \cap Q$ is a normal subgroup of $G$.

**Lemma 1**
Let $a$ and $b$ be positive integers such that $a \mid b$. Then $a \leq b$.
*Proof.*
Since $a \mid b$, $b = ma$ for some integer $m$. Suppose $m \leq 0$. Then $ma \leq 0$ and hence $b \leq 0$, a contradiction. So $m > 0$ and then $m \geq 1$. Thus $ma \geq a$ and to conclude $b \geq a$.

**Lemma 2**
Let $a, b, m, r$ be integers such that $b > 0$ and $0 < r < b$. If $a = mb + r$, then $b \nmid a$.
*Proof.*

Suppose $b \mid a$. Then $b \mid (a - mb)$ and thus $b \mid r$. Since $b$ and $r$ are positive integers such that $b \mid r$, $b \leq r$, a contradiction.

**Lemma 3**

If $p$ is a prime number, then, for all integers $n \geqq 2$, $p^n \nmid p!$.

*Proof.*

Since $p > 1, 0 < (p-1)p < p^2$. Since $(p-1)! \equiv -1 \bmod p$ and $-1 \equiv p - 1 \bmod p, (p-1)! \equiv p - 1 \bmod p$ and thus $p! \equiv (p-1)p \bmod p^2$. So $p! = kp^2 + (p-1)p$ for some integer $k$. By Lemma 2, $p^2 \nmid p!$. Now let $n > 2$. Suppose $p^n \mid p!$. Since $p^2 \mid p^n$ and $p^n \mid p!$, $p^2 \mid p!$ which is a contradiction.

**Lemma 4**

If $G$ is a finite group and $H \neq G$ is a subgroup of $G$ such that $|G| \nmid |G : H|!$, then $H$ must contain a nontrivial normal subgroup of $G$.

**Theorem 1**

Any subgroup of order $p^{n-1}$ in a group $G$ of order $p^n, p$ a prime number, is normal in $G$.

*Proof.*

Suppose the result is true for $n - 1$. To show that it then must follow for $n$. Let $G$ be a group of order $p^n$ and $H$ be its subgroup of order $p^{n-1}$. Since $|G| \nmid |G : H|!$, that is $p^n \nmid p!$, by Lemma 4, $H$ must contain a normal subgroup $N \neq 1$ of $G$ and hence $N \cap Z(G) \neq 1$. Since $p$ divides $|N \cap Z(G)|$, by Cauchy's theorem, $N \cap Z(G)$ has an element $b$ of order $p$. Let $B$ be the subgroup of $G$ generated by $b$. So $|B| = p$. Since $b \in Z(G), B$ must be normal in $G$. Since $G/B$ is a group of order $p^{n-1}$ and $H/B$ is its subgroup of order $p^{(n-1)-1}$, by the induction hypothesis $H/B$ is normal in $G/B$. To conclude $H$ is normal in $G$.

Let $G$ be a group of order $p^n$ and $H \neq G$ be a subgroup of $G$. Then there is a subgroup of $G$ that contains the subgroup $H$ of index $p$.

Suppose the result is true for $n - 1$. To show that it then must follow for $n$. Let $G$ be a group of order $p^n$ and $H \neq G$ be a subgroup of $G$. Since $Z(G) \neq 1$, $p$ divides $|Z(G)|$ and so $Z(G)$ has an element $a$ of order $p$. Let $N$ be the subgroup of $G$ generated by $a$. Since $a \in Z(G)$, $N$ must be normal in $G$. Moreover, $|N \cap H|$ divides $|N|$. So $|N \cap H|$ divides $p$. Thus $|N \cap H| = 1$ or $p$. Suppose $|N \cap H| = 1$. Then $NH$ is a subgroup of $G$ and

$$|NH| = \frac{|N||H|}{|N \cap H|} = p|H|.$$

Thus $H < NH$ and $|NH : H| = p$. Suppose $|N \cap H| = p$. Since $N \cap H \leq N$ and $|N \cap H| = |N|$, it follows that $N \cap H = N$ and hence $N \leq H$. Since $N$ is normal in $G$, $N$ must be normal in $H$ as well. Define $\bar{G} = G/N$ and $\bar{H} = H/N$. So $|\bar{G}| = p^{n-1}$. Moreover, $\bar{H}$ is a subgroup of $\bar{G}$. If $\bar{H} = \bar{G}$, then $|\bar{H}| = |\bar{G}|$

and hence $|H| = |G|$, a contradiction since $|H| < |G|$. Thus $\bar{H} \neq \bar{G}$. So, by induction hypothesis, $\bar{G}$ has a subgroup $\bar{P}$ that contains $\bar{H}$ and $|\bar{P} : \bar{H}| = p$. Let $P$ be the complete preimage of $\bar{P}$ in $G$. Let $h \in H$. Then $Nh \in \bar{H}$. But $\bar{P}$ contains $\bar{H}$ and thus $Nh \in \bar{P}$. So $h \in P$. To conclude that $P$ contains $H$ and $|P : H| = |\bar{P} : \bar{H}| = p$.

The commutator of two elements $x$ and $y$ is $x^{-1}y^{-1}xy$. The notation for this commutator element is $[x, y]$. If $X$ and $Y$ are subgroups of $G$, then $[X, Y]$ is defined to be the group generated by all $[x, y]$ with $x$ in $X$ and $y$ in $Y$.

Facts:
(1) $[x, y] = 1$ if and only if $xy = yx$.
(2) $[X, Y] = 1$ if and only if $X \leq \mathbb{C}_G(Y)$ if and only if $Y \leq \mathbb{C}_G(X)$.
(3) $[X, Y] = [Y, X]$.
(4) $[X, Y] \leq X$ if and only if $Y \leq \mathbb{N}_G(X)$.
$[X, Y] \leq Y$ if and only if $X \leq \mathbb{N}_G(Y)$.
(5) If $X$ and $Y$ normalize each other and $X \cap Y = 1$, then $X$ and $Y$ centralize each other.

Let $H$ and $K$ be subgroups of a group $G$ such that $H$ is normal in $K$ and $K$ is normal in $G$. Suppose there exists a normal subgroup $N$ of $G$ with $K \cap N = 1$ and $KN = G$. Then $H$ is normal in $G$.

Since $H$ is normal in $K$, $K = \mathbb{N}_K(H) \leq \mathbb{N}_G(H)$. Since both $K$ and $N$ are normal in $G$, $N \leq KN = G = \mathbb{N}_G(K)$ and $K \leq KN = G = \mathbb{N}_G(N)$. Thus $K$ and $N$ normalize each other. But $K \cap N = 1$ and so $K$ and $N$ centralize each other. Hence $N \leq \mathbb{C}_G(K) \leq \mathbb{C}_G(H)$ and so $[N, H] = 1 \leq H$. Thus $N \leq \mathbb{N}_G(H)$. Since $K \leq \mathbb{N}_G(H)$ and $N \leq \mathbb{N}_G(H)$, $G = KN \leq \mathbb{N}_G(H)$. To conclude $\mathbb{N}_G(H) = G$.

Let $G$ be a group, and let the prime $p$ divide $|G|$. Suppose further that $|G| < p^2$. Show that $G$ has a normal subgroup of order $p$.

Since $p$ divides $|G|$, $G$ has an element $x$ of order $p$ and the order of $\langle x \rangle$ is $p$. Suppose $H, K$ are subgroups of order $p$. By Lagrange's theorem, $|H \cap K|$ divides $|H|$ and so $|H \cap K| = 1$ or $p$. Suppose $|H \cap K| = 1$. Then

$$|HK| = \frac{|H||K|}{|H \cap K|} = p^2,$$

a contradiction since $HK$ is a subset of $G$ but $|G| < p^2$. Thus $|H \cap K| = p$. Since $H \cap K$ is a subset of $H$ and $|H \cap K| = |H|$. $H \cap K = H$ and thus $H \subset K$. Similarly, since $H \cap K$ is a subset of $K$ and $|H \cap K| = |K|$. $H \cap K = K$ and thus

$K \subset H$. Since $H \subset K$ and $K \subset H$, $H = K$. So there is exactly one subgroup of $G$ of order $p$. Let $g \in G$. Then $g\langle x\rangle g^{-1}$ is also a subgroup of $G$ of order $p$. To conclude $g\langle x\rangle g^{-1} = \langle x\rangle$.