A Trig Based Proof That Pi is Irrational

Timothy Jones

December 19, 2024

Abstract

It is shown that the limit of $\cos(j)$ and $\sin(j)$ as j goes to infinity does not exist. DeMoivre's theorem implies $\cos j! + i \sin j!$ raised to the $1/(j-1)!$ power equals $\cos j + i \sin j$. Assuming π is rational, its multiple can be expressed as a factorial. This implies that $\cos(j)$ and $\sin(j)$ converges, a contradiction.

Introduction

There are many proofs of the irrationality of π [3] [4]. Something like the holy grail of showing π is irrational is to reduce things to just trigonometric manipulations. Although the proof here doesn't quite do that – some limit ideas are involved – it gets close to this ideal; it might be simple enough for a freshman calculus student.

There are two pre-requisites. One consists of a simple proof that the limit of $\cos(j)$ and $\sin(j)$ as j, a natural number, goes to infinity does not exist. This is given in the next section: *Trigonometry*. The other is potentially a thing of mathematical analysis, but, in the opinion of the author, only if you need convincing concerning something that many would take as obvious and easy. This material is covered in Apostol's *Mathematical Analysis* in sections 8.20 (Double Sequences) and 9.12 (Uniform Convergence and Double Sequences): a definition and two theorems [1]. We present his ideas in the third section: *Double Sequences*. The large print: a good avenue is to assume the trig result and go right to the last *Proof* section.

In the conclusion, I review whether or not the magic chalice could be a myth. Are there mistakes in the reasoning?

Trigonometry

The following Lemma is taken from a youtube video: Dr. Barker video.

Lemma 1. *The limits*

$$\lim_{j\to\infty} \cos(j) \tag{1}$$

and

$$\lim_{j\to\infty} \sin(j) \tag{2}$$

don't exist.

Proof. Suppose (2) exists and equals L. Then

$$\lim_{j\to\infty} \sin(j+1) = \lim_{j\to\infty} \sin(j-1) = L.$$

Using trigonometric identities,

$$\sin(j+1) = \sin(j)\cos(1) + \cos(j)\sin(1) \tag{3}$$

and

$$\sin(j-1) = \sin(j)\cos(1) - \cos(j)\sin(1). \tag{4}$$

Using (3), we solve for $\cos(j)$:

$$\cos(j) = \frac{\sin(j+1) - \sin(j)\cos(1)}{\sin(1)}.$$

This gives

$$\lim_{j\to\infty} \cos(j) = \frac{L - L\cos(1)}{\sin(1)} = \frac{L(1-\cos(1))}{\sin(1)}, \tag{5}$$

implying the limit of $\cos(j)$ exists. Similarly, using (4), we solve for $\cos(j)$

$$\cos(j) = \frac{\sin(j)\cos(1) - \sin(j-1)}{\sin(1)}.$$

This gives

$$\lim_{j\to\infty} \cos(j) = \frac{L\cos(1) - L}{\sin(1)} = \frac{L(\cos(1)-1)}{\sin(1)}. \tag{6}$$

The only way (5) and (6) can be made consistent is if both limits are 0. Given that $1 - \cos(1) \neq 0$, this forces $L = 0$.

We now have

$$\lim_{j\to\infty} \sin(j) = \lim_{j\to\infty} \cos(j) = 0,$$

but this implies

$$\lim_{j\to\infty} \cos^2(j) + \sin^2(j) = 0,$$

contradicting the Pythagorean identity: giving $0 = 1$. Therefore the limits, (1) and (2) don't exist. $\square$

Note: $\pi \neq 1$ follows from the circumference of any circle is greater than its diameter: $2\pi r > 2r$ implies $\pi > 1$ and hence $\pi \neq 1$.

Double Sequences

The following is taken from Apostol [1].

Definition 1. *A function f whose domain is $\mathbf{Z}^+ \times \mathbf{Z}^+$ is called a double sequence.*

NOTE. We shall be interested only in real- or complex-valued double sequences.

Definition 2. *If $a \in \mathbf{C}$, we write $\lim_{p,q\to\infty} f(p,q) = a$ and we say that the double sequence f converges to a, provided that the following condition is satisfied: For every $\epsilon > 0$, there exists an N such that $|f(p,q) - a| < \epsilon$ whenever both $p > N$ and $q > N$.*

Lemma 2. *Assume that $\lim_{p,q\to\infty} f(p,q) = a$. For each fixed p, assume that the limit $\lim_{q\to\infty} f(p,q)$ exists. Then the limit $\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q))$ also exists and has the value a.*

NOTE. To distinguish $\lim_{p,q\to\infty} f(p,q)$ from $\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q))$, the first is called a *double limit*, the second an *iterated limit*.

Proof. Let $F(p) = \lim_{q\to\infty} f(p,q)$. Given $\epsilon > 0$, choose N_1 so that

$$|f(p,q) - a| < \frac{\epsilon}{2}, \text{ if } p > N_1 \text{ and } q > N_1. \tag{7}$$

For each p we can choose N_2, so that

$$|F(p) - f(p,q)| < \frac{\epsilon}{2}, \text{ if } q > N_2. \tag{8}$$

(Note that N_2 depends on p as well as on ϵ.) For each $p > N_1$ choose N_2, and then choose a fixed q greater than both N_1 and N_2. Then both (7) and (8) hold, and hence

$$|F(p) - a| < \epsilon, \text{ if } p > N_1.$$

Therefore, $\lim_{p\to\infty} F(p) = a$. □

NOTE. A similar result holds if we interchange the roles of p and q.

Thus the existence of the double limit $\lim_{p,q\to\infty} f(p,q)$ and of $\lim_{q\to\infty} f(p,q)$ implies the existence of the iterated limit

$$\lim_{p\to\infty}\left(\lim_{q\to\infty} f(p,q)\right).$$

The following example shows that the converse is not true.

Example 1. Let

$$f(p,q) = \frac{pq}{p^2+q^2}, \ (p = 1, 2, \ldots, \ q = 1, 2, \ldots).$$

Then $\lim_{q\to\infty} f(p,q) = 0$ and hence $\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q)) = 0$. But $f(p,q) = \frac{1}{2}$ when $p = q$ and $f(p,q) = \frac{2}{5}$ when $p = 2q$, and hence it is clear that the double limit cannot exist in this case.

A suitable converse to Lemma 2 can be established by introducing the notion of *uniform convergence*.

Lemma 3. *Let f be a double sequence and let $\mathbb{Z}^+$ denote the set of positive integers. For each $n = 1, 2, \ldots$, define a function g_n on $\mathbb{Z}^+$ as follows:*

$$g_n(m) = f(m,n), \quad \text{if } m \in \mathbb{Z}^+.$$

Assume that $g_n \to g$ uniformly on $\mathbb{Z}^+$, where $g(m) = \lim_{n\to\infty} f(m,n)$. If the iterated limit $\lim_{m\to\infty}(\lim_{n\to\infty} f(m,n))$ exists, then the double limit $\lim_{m,n\to\infty}$ also exists and has the same value.

Proof. Given $\epsilon > 0$, choose N_1 so that $n > N_1$ implies

$$|f(m,n) - g(m)| < \frac{\epsilon}{2}, \qquad \text{for every } m \in \mathbb{Z}^+.$$ [1]

Let $a = \lim_{m\to\infty}(\lim_{n\to\infty} f(m,n)) = \lim_{m\to\infty} g(m)$. For the same ϵ, choose N_2 so that $m > N_2$ implies $|g(m) - a| < \epsilon/2$. Then, if N is the larger of N_1 and N_2, we have $|f(m,n) - a| < \epsilon$ whenever both $m > N$ and $n > N$. In other words, $\lim_{m,n\to\infty} f(m,n) = a$. □

[1] Apostol doesn't define uniform converge explicitly for double sequences, but one gets the drift here.

Proof

We need a slight modification of DeMoivre's Theorem. First a review of complex roots [2].

Let $w = r(\cos\theta + i\sin\theta)$ be a complex number in polar form. If $w \neq 0$, w has n distinct complex nth roots given by the formula

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right], \tag{9}$$

where $k = 0, 1, 2, \ldots, n-1$. The principal root [5] is the root corresponding to the $k = 0$ case in (9).

Lemma 4. *For $n > 4$*

$$(\cos(n!) + i\sin(n!))^{1/(n-1)!} = \cos n + i\sin n \tag{10}$$

gives the principal root of the $(n-1)!$th root of $\cos(n!) + i\sin(n!)$.

Proof. The $k = 0$ case of

$$\cos\left(\frac{n! + 2k\pi}{(n-1)!}\right) + i\sin\left(\frac{n! + 2k\pi}{(n-1)!}\right)$$

is the right hand side of (10). This gives

$$e^{in} = e^{i\frac{n!}{(n-1)!}} = (e^{in!})^{\frac{1}{(n-1)!}} = (\cos(n!) + i\sin(n!))^{\frac{1}{(n-1)!}}$$

which is (10). □

Theorem 1. *π is irrational.*

Proof. Assume $\pi = p/q$, then there exists a first $k!$ that has a factor of p in it and

$$k!\frac{q}{p} = 2n$$

for some integer n. This means for all $j \geq k$, $j! = 2m\pi$, for some integer m, giving $\cos(j!) = 1$ and $\sin(j!) = 0$. This makes

$$\lim_{j,k\to\infty}(\cos(j!) + i\sin(k!)) = 1 + i0,$$

trivially uniform – meaning $j = k$ doesn't change this limit. See the *Double Sequence* section. The iterated limit

$$\lim_{q\to\infty}(\lim_{r\to\infty}(\cos(r!) + i\sin(r!))^{1/(q-1)!})$$

also exists. Note: $1^{1/m} = 1$, for all $m \in \mathbb{Z}^+$. Therefore the double sequence

$$\lim_{j,k\to\infty} (\cos(j!) + i\sin(j!))^{1/(k-1)!} = 1 + 0i \tag{11}$$

exists and yields the same result when $j = k$: Lemma 3. But then (11) is identical to

$$\lim_{j\to\infty} (\cos(j) + i\sin(j)) = 1 + 0i,$$

using Lemma 4. This implies $\lim_{j\to\infty} \cos(j)$ and $\lim_{j\to\infty} \sin(j)$ both exist and equal 1 and 0 respectively, contradicting Lemma 1. See the *Trigonometry* section. □

Exploration

We can make some inferences from the assumption that π is rational. Let's suppose that $\pi = 6$ and naturally $2\pi = 12$ and this gives us the other circle of interest: the clock. We can say that in one sense $\cos n$ and $\sin n$ diverge, but in another sense they converge (*are*) multi-valued (or piece-wise defined) functions given by modulo classes from $[0]$ to $[11]$. So, for example $\sin(24)$ is $[0]$ as 24 is a multiple of 6; $\sin(15)$ is $[3]$ as $15 \equiv 3$ mod 6. Per the *Trigonometry* Section is this a contradiction, proving that π must be irrational?

My feeling is yes. The proof that $\cos n$ and $\sin n$ did not require any assumptions about π's rationality or irrationality. As per a good *by contradiction* proof, the assumption to give a contradiction resides on one side of a ledger and the consequences of not making the assumption rests on another.

If π is assumed to be irrational then $\cos n$ and $\sin n$ will always be different; the periodicity of these forces this conclusion. They don't converge to a multi-valued function.

The easy refrain is (10) isn't true. But the use of this identity is not to establish roots of a complex number. Multi-valued items can and are also single valued. For example, I think I can say $\sqrt{4}$ is 2 without clamoring that no its not. Its

$$\sqrt[2]{4}\left[\cos\left(\frac{0+2\pi k}{2}\right) + i\sin\left(\frac{0+2\pi k}{2}\right)\right] = 2 \text{ and } -2,$$

when $k = 0$ and $k = 1$. I can say $(e^{z_1})^{z_2} = e^{z_1 z_2}$ without detours into what the complex numbers z_1 and z_2 are. We are working within the constraint of principal values and single values.

Conclusion

The details concerning double sequences given in the third section seem unnecessary given the sequence in question turns into a constant: for some N, all $n > N$ make $a_n = b$, b a constant. Instead of uniform convergence one could term such convergence trivial. The word *trivial* in connection with proving π is irrational raises an alarm bell. Is there an error?

References

[1] Apostol, T. M. (1974). *Mathematical Analysis*, 2nd ed. Reading, Massachusetts: Addison-Wesley.

[2] Blitzer, R. (2010). *Algebra and Trignometry*, 3rd ed., Pearson.

[3] L. Berggren, J. Borwein, and P. Borwein, *Pi: A Source Book*, 3rd ed., Springer, New York, 2004.

[4] P. Eymard and J.-P. Lafon, *The Number* π, American Mathematical Society, Providence, RI, 2004.

[5] M. R. Spiegel, *Complex Variables*, 2nd ed., Schaums Outline Series, McGraw-Hill: New York (2009)