Nonextensive Black Hole Entropies from truly Point Mass Sources

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Abstract

The authors [1] have recently constructed models of nonextensive black hole Thermodynamics from a generalized Wick's rotation procedure in the evaluation of the Euclidean path integral. We have explicitly shown in [6] how the Schwarzschild Black Hole Entropy (in all dimensions) emerges from truly point mass sources at $r = 0$ due to a non-vanishing scalar curvature involving the Dirac delta distribution. It is the density and *anisotropic* pressure components associated with the point mass delta function source at the origin $r = 0$ which furnish the Schwarzschild black hole entropy in all dimensions $D \geq 4$ after evaluating the non-vanishing Euclidean Einstein-Hilbert action. In this work we generalize our construction of the Euclidean Einstein-Hilbert action by following the generalized Wick's rotation procedure of [1] in order to construct the nonextensive Schwarzschild black hole entropies in all dimensions. The first law of Thermodynamics is obeyed and when the nonextensivity parameter is $\lambda < 0$, the nonextensive entropy is $finite$ at $T = 0$ despite that the Bekenstein-Hawking entropy $S_{BH}(\beta = \infty) = S_{BH}(T = 0) = \infty$ blows up violating the the third law of Thermodynamics.

Keywords : Nonextensive Statistics; Black Holes; Entropy.

1 Nonextensive Black Hole Entropy from truly Point Mass Sources

Recently, the authors [1] proposed to construct models of nonextensive black hole Thermodynamics from generalized Euclidean path integrals and Wick's rotation. Concretely, they introduced a generalized Wick's rotation from real time t to imaginary time τ such that $t \to -if_\lambda(\tau)$ where f_λ is a differentiable function and λ is a parameter related to the nonextensivity. After applying this generalized Euclidean path integral to black hole thermodynamics they derived the generalized Wick's rotation functions associated to a given nonextensive statistics [2].

The Black-Brane/Bose-Gas Duality and Third Law of Thermodynamics has been studied more recently by [3]. In the thermodynamics of black holes in asymptotically flat space the third law of thermodynamics is violated, and the black hole entropy cannot be consistently modeled through conventional statistical mechanics. Notably, the third law of thermodynamics is violated for the Schwarzschild black hole, and its entropy can only be described using an unconventional model, such as a Bose gas in negative dimensions [3]. For these reasons, instead of dealing with conventional statistics we shall be working with the q-entropy inspired from Tsallis statistics [9].

The interplay between nonextensive statistical mechanics and black hole entropy has been been previously analyzed by [4] and others. Using nonextensive statistical mechanics, the author [4] has shown that the Bekenstein-Hawking area law is obtained from microstates of black holes in Loop Quantum Gravity, for arbitrary real positive values of the Barbero-Immirzi parameter γ. The arbitrariness of γ was encoded in the strength of the "bias" created in the horizon microstates through the coupling with the quantum geometric fields exterior to the horizon. Majhi argued [4] that an experimental determination of γ will fix this coupling, leaving out the macroscopic area of the black hole to be the only free quantity of the theory. Hence, a key link between the microscopic theory of black holes in Loop Quantum Gravity (LQG) and the application of nonextensive statistics to black hole thermodynamics was found.

Recently, it was explicitly shown in [6] how the Schwarzschild Black Hole Entropy (in all dimensions) emerges from truly point mass sources at $r = 0$ due to a non-vanishing scalar curvature involving the Dirac delta distribution. In order to achieve this, one requires to *extend* the domain of r to *negative* values $-\infty \leq r \leq +\infty$. It is the density and *anisotropic* pressure components associated with the point mass delta function source at the origin $r = 0$ which furnish the Schwarzschild black hole entropy in all dimensions $D \geq 4$ after evaluating the non-vanishing Euclidean Einstein-Hilbert action. As usual, it was required to take the inverse Hawking temperature β_H as the length of the circle S^1_β obtained from a compactification of the Euclidean time in thermal field theory which results after a Wick rotation, $it = \tau$, to imaginary time.

In units $\hbar = c = k_B = 1$, the $4D$ scalar curvature and the Euclidean action I turned out to be [6]

$$\mathcal{R} \;=\; \frac{4GM\delta(r)}{r^2} \;\Rightarrow\; I \;=\; -\;\frac{i}{16\pi G}\int_0^{\beta_H} d\tau \int_0^{\infty} \mathcal{R}\; 4\pi r^2\; dr \qquad (1)$$

The magnitude of the integral (1) becomes (after inserting the inverse Hawking temperature $\beta = 8\pi GM$)

$$|I| = \frac{1}{2}\; M\; \beta_H \;=\; 4\pi GM^2 \;=\; \frac{4\pi(2GM)^2}{4G} \;=\; \frac{4\pi r_h^2}{4G} \;=\; \frac{Area}{4L_P^2} \qquad (2)$$

and which is the Schwarzschild black hole entropy in $D > 4$.

In higher dimensions $D > 4$, the scalar curvature is [6]

$$\mathcal{R} \ = \ 2 \ \frac{16\pi GM}{(D-2)\Omega_{D-2}} \ (D-3) \ \frac{\delta(r)}{|r|^{D-2}} \ = \ 2 \ r_h^{D-3} \ (D-3) \ \frac{\delta(r)}{|r|^{D-2}} \tag{3}$$

where $\Omega_{D-2} = 2\pi^{\frac{D-1}{2}}/\Gamma(\frac{D-1}{2})$ is the solid angle of the $D-2$-dim hypersphere. The horizon radius is given by

$$r_h \ = \ \left(\frac{16\pi GM}{(D-2) \ \Omega_{D-2}} \right)^{\frac{1}{D-3}} \tag{4}$$

and the magnitude of the Euclidean integral I

$$I \ = \ - \ \frac{i}{16\pi G} \ \int_0^\beta \ d\tau \ \int_0^\infty \ \mathcal{R} \ \Omega_{D-2} \ r^{D-2} \ dr \tag{5}$$

after inserting the inverse Hawking temperature $\beta = 4\pi r_h/(D-3)$, becomes

$$|I| \ = \ \frac{\Omega_{D-2} \ r_h^{D-2}}{\mathbf{4}G_D} \ = \ \frac{\Omega_{D-2}}{\mathbf{4}G_D} \ \left(\frac{16\pi G_D M}{(D-2) \ \Omega_{D-2}} \right)^{\frac{D-2}{D-3}} \tag{6}$$

which is the Schwarzschild black hole entropy in $D > 4$ dimensions given by one-quarter of the horizon area in Planck units. Essential in these findings was the result that $\int_0^\infty \delta(r)dr = \frac{1}{2}\int_{-\infty}^\infty \delta(r)dr = \frac{1}{2}$ resulting from the symmetry of the delta function $\delta(-r) = \delta(r)$.

Armed with the finding that the black hole entropy *is* the same as the Euclidean Einstein-Hilbert action associated to a point-mass singular source we shall be able to define the notion of q-entropy. Our procedure to construct the modified gravitational entropy for the $4D$ Schwarzschild black hole is very $different$ than the one undertaken by [1] based on the Euclidean path integral. We only need to focus on the modified Wick rotation procedure $t = -if_\lambda(\tau)$ leading to the modification of the expression $|I| = \frac{1}{2}M\beta_H$ found in eq-(2) by simply making the following replacement $\beta_H \rightarrow f_\lambda(\beta_H)$

$$|I| = \ \frac{1}{2} \ M \ \beta_H \ \rightarrow \ |I|_\lambda = \ \frac{1}{2} \ M \ f_\lambda(\beta_H) \tag{7}$$

and resulting from the imaginary time integration of the Euclidean Einstein-Hilbert action after imposing the modified Wick rotation procedure $t = -if_\lambda(\tau)$, and leading to [1]

$$I \ = \ -i \ \frac{1}{16\pi G} \ \int_0^{\beta_H} \ \frac{df_\lambda(\tau)}{d\tau} \ d\tau \ \int_0^\infty \ \mathcal{R} \ 4\pi r^2 \ dr \ = \ -\frac{i}{2} \ M \ (f_\lambda(\beta_H) - f_\lambda(0)) \tag{8}$$

[1]The i and $\frac{df_\lambda(\tau)}{d\tau}$ term result from evaluating the integral measure $\sqrt{-det(g_{\mu\nu})}$ after the Wick rotation.

The choice of $f_\lambda(0) = 0$ yields the result in eq-(7). Hence, by equating the modified entropy (8) to the λ-entropy $S_\lambda(\beta_H)$ associated with non-extensive statistics, one can read-off the functional form of the function $f_\lambda(\beta)$

$$|I|_\lambda = \frac{1}{2}\ M\ f_\lambda(\beta_H)\ =\ S_\lambda(\beta_H)\ \Rightarrow$$

$$f_\lambda(\beta_H)\ =\ \frac{2}{M}\ S_\lambda(\beta_H)\ =\ \frac{16\pi G}{\beta_H}\ S_\lambda(\beta_H),\ \ M\ =\ \frac{\beta_H}{8\pi G} \tag{9}$$

Inspired by the q entropy proposed by Tsallis [9]

$$S_q\ =\ \frac{1-\sum_i^\Omega p_i^q}{q-1} \tag{10}$$

where the q parameter is called the entropic index and can be $q > 1$ or $q < 1$; p_i is the probability for the i-th microstate; Ω is the total number of microstates of the system, we may find the functional form $f_\lambda(\beta_H)$ corresponding to the Tsallis entropy.

The q-entropy of a black hole

$$S_q^{(1/2)}\ =\ \frac{2^{(1-q)N}-1}{1-q}\ =\ \frac{e^{(1-q)Nln2}-1}{1-q}\ =\ \frac{e^{(1-q)ln(2^N)}-1}{1-q} \tag{11}$$

was derived in [5] by counting black holes microstates in Loop Quantum Gravity after setting all the spins to $s = \frac{1}{2}$ for the black hole horizon punctures. In view of the result found in eq-(11), after noticing that $S = ln(\Omega) = ln(2^N)$, we may *define* the q-entropy of the Schwarzschild black hole as follows

$$S_q\ \equiv\ \frac{e^{(1-q)S_{BH}}\ -\ 1}{1-q}\ =\ \frac{e^{((1-q)\beta^2/16\pi G)}\ -\ 1}{1-q} \tag{12}$$

Defining $\lambda \equiv 1-q$ one may rewrite (12) as

$$S_\lambda(\beta_H)\ =\ \frac{e^{\lambda S_{BH}}\ -\ 1}{\lambda}\ =\ \frac{e^{(\lambda\beta_H^2/16\pi G)}\ -\ 1}{\lambda},\quad -\infty < \lambda < 1 \tag{13}$$

One may notice that the expression (13) involves the *same* functional relation between the λ-entropy S_λ, and the black hole entropy S_{BH}, as the relation between the Tsallis S_T and Renyi entropy S_R given by

$$S_T\ =\ \frac{e^{\lambda S_R}\ -\ 1}{\lambda}\ =\ \frac{1\ -\ e^{-(q-1)S_R}}{q-1},\ \ \lambda = 1-q \tag{14}$$

The Tsallis and Renyi entropies are defined, respectively, as follows

$$S_T\ =\ \frac{1}{q-1}\left(1\ -\ \sum_{i=1}^N p_i^q\right),\ \ S_R\ =\ -\ \frac{1}{q-1}\ ln\left(\sum_{i=1}^N p_i^q\right),\ \ 0 < q < \infty \tag{15}$$

Eliminating the term $\sum_{i=1}^N p_i^q$ in eq-(15) leads to the relation (14). When $\lambda = 0$ $(q = 1)$, $S_T = S_R = -\sum_{i=1}^N p_i ln(p_i)$ and one recovers the Shannon entropy

($k_B = 1$) after using L'Hopital's rule. A fundamental property of the Tsallis entropies is the fact that they are *not* additive (non-extensive) for independent subsystems [13].

Consequently, given eqs-(9,13) one then arrives at the expression for $f_\lambda(\beta_H)$

$$f_\lambda(\beta_H) \ = \ \frac{2}{M} \ S_\lambda(\beta_H) \ = \ \frac{16\pi G}{\beta_H} \ S_\lambda(\beta_H) \ = \ \frac{16\pi G}{\beta_H} \ \frac{e^{(\lambda\beta_H^2/16\pi G)} \ - \ 1}{\lambda} \tag{16}$$

and such that the generalized Wick rotation procedure $t = -if_\lambda(\tau)$ is captured by the function

$$f_\lambda(\tau) \ = \ \frac{16\pi G}{\tau} \ \frac{e^{(\lambda\tau^2/16\pi G)} \ - \ 1}{\lambda} \tag{17a}$$

One may verify that $f_\lambda(\tau = 0) = 0$ which is a sign of consistency, and when $\lambda \to 0 \Rightarrow f_\lambda(\tau) \to \tau$, and $S_\lambda(\beta_H) \to S_{BH} = \frac{\beta_H^2}{16\pi G}$.

It is important to emphasize that the Wick rotation function of eq-(17a) $differs$ from the Wick rotation functions found in [1] for a variety of black holes under different statistics. However, the Taylor expansion of (17a)

$$f_\lambda(\tau) \ = \ \tau \ + \frac{1}{32\pi G} \ \lambda \ \tau^3 \ + \ \frac{1}{6(16\pi G)^2} \ \lambda^2 \ \tau^5 \ + \ \ldots \tag{17b}$$

does capture a similar behavior in powers of τ as the Wick rotation functions found in [1] up to numerical factors and signs. Such Wick rotation functions were obtained after using the saddle point method to evaluate approximately the Euclidean gravitational path integral.

The physical relevance of using the λ-entropy in the case that $\lambda < 0$ is that $S_\lambda(\beta_H = \infty) = S_\lambda(T_H = 0) = -\frac{1}{\lambda}$ is $finite$, and positive ($\lambda < 0$), whereas the Bekenstein-Hawking entropy $S_{BH}(\beta_H = \infty) = S_{BH}(T_H = 0) = \infty$ blows up and violates the third law of Thermodynamics. On the other hand, when $\lambda > 0$ the λ-entropy $S_\lambda(\beta_H = \infty) = S_\lambda(T_H = 0)$ still diverges violating the third law.

In higher dimensions $D > 4$, the modified Wick rotation $t = -if_\lambda(\tau)$ procedure leads to an Euclidean Einstein-Hilbert action whose modified magnitude is given by

$$|I|_\lambda \ = \ \frac{1}{16\pi G_D} \ f_\lambda(\beta_H) \ (D-3) \ \Omega_{D-2} \ r_h^{D-3} \tag{18}$$

Upon equating $|I|_\lambda$ with the λ-entropy $S_\lambda(\beta_H)$

$$|I|_\lambda \ = \ S_\lambda(\beta_H) \ = \ \frac{e^{\lambda S_{BH}(\beta_H)} - 1}{\lambda} \tag{19}$$

one can read-off the expression for $f_\lambda(\beta_H)$

$$f_\lambda(\beta_H) \ = \ \frac{16\pi G_D}{(D-3) \ \Omega_{D-2} \ B_D \ \beta_H^{D-3}} \ \frac{e^{\lambda(A_D\beta_H^{D-2})} - 1}{\lambda} \tag{20}$$

after rewriting

$$S_{BH}(\beta_H) \;=\; \frac{\Omega_{D-2}\; r_h^{D-2}}{4G_D} \;=\; A_D\; \beta_H^{D-2},\;\; r_h^{D-3} \;=\; [\frac{\beta_H(D-3)}{4\pi}]^{D-3} \;=\; B_D\; \beta_H^{D-3} \tag{21}$$

in terms of powers of β_H, and the dimension-dependent numerical coefficients A_D, B_D

$$A_D \;\equiv\; \frac{\Omega_{D-2}}{4G_D}\; [\frac{D-3}{4\pi}]^{D-2},\;\; B_D \;\equiv\; [\frac{D-3}{4\pi}]^{D-3} \tag{22}$$

and which required relating the expression for the black hole horizon r_h in terms of the inverse Hawking temperature $\beta_H = 4\pi r_h/(D-3) \Rightarrow r_h = \frac{\beta_H(D-3)}{4\pi}$.

Therefore, the modified Wick rotation $t = -if_\lambda(\tau)$ is captured by the function

$$f_\lambda(\tau) \;=\; \frac{16\pi G_D}{(D-3)\;\Omega_{D-2}\;B_D\;\tau^{D-3}}\;\frac{e^{\lambda(A_D\tau^{D-2})}-1}{\lambda},\;\; D \geq 4 \tag{23}$$

One can verify that $f_\lambda(\tau=0)=0$, and when $\lambda = 0 \Rightarrow f_\lambda(\tau) \rightarrow \tau$.

There are many other examples of modified entropies which can be obtained by recurring to the modified Wick rotation procedure and the Euclidean Einstein-Hilbert action associated with a truly point mass source with a delta function scalar curvature singularity at the origin. For example, given the Kaniadakis entropy [7] in $D = 4$

$$S_K \;=\; \frac{1}{\lambda}\; sinh(\lambda S_{BH}) \;=\; \frac{1}{\lambda}\; sinh\left(\lambda\beta_H^2/16\pi G\right) \tag{24}$$

upon equating S_K and $\frac{M}{2} f_\lambda(\beta_H) = \frac{\beta_H}{16\pi G} f_\lambda(\beta_H)$ allows us to read-off the expression for $f_\lambda(\beta_H)$

$$f_\lambda(\beta_H) \;=\; \frac{16\pi G}{\beta_H}\;\frac{1}{\lambda}\; sinh\left(\lambda\beta_H^2/16\pi G\right) \;\;\Rightarrow$$

$$f_\lambda(\tau) \;=\; \frac{16\pi G}{\tau}\;\frac{1}{\lambda}\; sinh\left(\lambda\tau^2/16\pi G\right) \tag{25}$$

and, which in turn, furnishes the functional form of the modified Wick rotation function. Once more, one can verify that $f_\lambda(\tau = 0) = 0$, and when $\lambda = 0 \Rightarrow f_\lambda(\tau) = \tau$.

2 First Law of Black Hole Nonextensive Thermodynamics

A careful inspection of eq-(9) reveals that one could perform a modification of the black hole's mass $M \rightarrow M_\lambda$ (black hole's area $A \rightarrow A_\lambda$), and inverse Hawking temperature $\beta_H \rightarrow \beta_{\lambda,H}$, with the provision that $\beta_{\lambda,H} = 8\pi G M_\lambda$

in order to maintain the *periodicity* in τ and avoid a conical singularity for the Euclidean metric near the horizon. Furthermore, one can then rewrite the λ-entropy $S_\lambda(\beta)$ in $4D$ as follows

$$S_\lambda(\beta_H) \;=\; \frac{e^{(\lambda\beta_H^2/16\pi G)} \;-\; 1}{\lambda} \;=\; \frac{1}{2}\; M\; f_\lambda(\beta_H) \;=$$
$$S_{BH}(\beta_{\lambda,H}) \;=\; \frac{A_\lambda}{4G} \;=\; \frac{1}{2}\; M_\lambda\; \beta_{\lambda,H} \;=\; \frac{\beta_{\lambda,H}^2}{16\pi G} \tag{26}$$

and from which one can read-off the sought-after functional form of $\beta_{\lambda,H}$, and M_λ in terms of M and the Wick rotation function $f_\lambda(\beta_H)$ (20) of the previous section, as follows

$$\beta_{\lambda,H} \;=\; \sqrt{8\pi G M f_\lambda(\beta_H)} \;=\; \sqrt{\beta_H f_\lambda(\beta_H)};\;\; M_\lambda \;=\; \frac{1}{8\pi G}\; \sqrt{8\pi G M f_\lambda(\beta_H)} \tag{27}$$

and one then finds that the deformed inverse Hawking temperature $\beta_{\lambda,H}$ is the geometric mean of β_H and $f_\lambda(\beta_H)$. The latter $f_\lambda(\beta_H)$ could be interpreted as an "effective" inverse Hawking temperature.

The reason of introducing the modified mass M_λ and inverse temperature $\beta_{\lambda,H}$ is to ensure that the first law of black hole non-extensive Thermodynamics is obeyed. From the second line of eq-(26) one can verify that

$$\frac{\partial S_\lambda(\beta_H)}{\partial M_\lambda} \;=\; \frac{\partial \beta_{\lambda,H}}{\partial M_\lambda}\; \frac{\partial S_{BH}(\beta_{\lambda,H})}{\partial \beta_{\lambda,H}} \;=\; \beta_{\lambda,H}, \quad \lambda \neq 0 \tag{28}$$

It is important to mention that one must *exclude* the other two possibilities

$$\frac{\partial S_\lambda(\beta_H)}{\partial M_\lambda} \;=\; \beta_H, \;\; \frac{\partial S_\lambda(\beta_H)}{\partial M} \;=\; \beta_{\lambda,H} \tag{29}$$

because the pair of values (M_λ, β_H) and $(M, \beta_{\lambda,H})$ do not obey the condition required to maintain the *periodicity* in τ and avoid a conical singularity for the Euclidean metric near the horizon. In ordinary Gibbs-Boltzman extensive thermodynamics (statistics), the first law of black-hole Thermodynamics yields $\frac{\partial S_{BH}}{\partial M} = \beta_H$. In the modified entropy case one must have $\frac{\partial S_\lambda}{\partial M_\lambda} = \beta_{\lambda,H}$, instead.

These results can be extended to higher dimensions $D > 4$. After deforming the mass, M_λ, and the inverse Hawking temperature $\beta_{\lambda,H}$, one arrives at the relationship among $\beta_H, f_\lambda(\beta_H)$ and $\beta_{\lambda,H}$

$$A_D\; \beta_{\lambda,H}^{D-2} \;=\; \frac{1}{16\pi G_D}\; f_\lambda(\beta_H)\; (D-3)\; \Omega_{D-2}\; B_D\; \beta_H^{D-3} \tag{30}$$

with A_D, B_D and $f_\lambda(\beta_H)$ given explicitly by eqs-(20,22). Once more, in order to avoid a conical singularity the deformed mass M_λ and $\beta_{\lambda,H}$ must obey the relation

$$\frac{16\pi G_D M_\lambda}{(D-2)\Omega_{D-2}} \;=\; \left(\frac{\beta_{\lambda,H}(D-3)}{4\pi}\right)^{D-3} \tag{31}$$

To sum up, the physical relevance of these results is three-fold : (i) In $D = 4$, the λ-entropy agrees with the BH entropy $\frac{A_\lambda}{4G}$ associated to a deformation of the area A_λ resulting from modifying the mass M_λ *and* the upper τ integration limit of the Euclidean Einstein-Hilbert action with the provision that $\beta_{\lambda,H} = 8\pi G M_\lambda$ in order to maintain the *periodicity* in τ and avoid a conical singularity for the Euclidean metric near the horizon. These results can be extended to higher dimensions $D > 4$. (ii) The first law of Thermodynamics is obeyed as displayed in eq-(28), and (iii) when $\lambda < 0$, the λ-entropy is $finite$ and positive $S_\lambda(\beta = \infty) = -\frac{1}{\lambda}$ despite that the Bekenstein-Hawking entropy $S_{BH}(\beta = \infty) = S_{BH}(T = 0) = \infty$ blows up violating the the third law of Thermodynamics.

In essence, by introducing the λ-entropy S_λ, one has "regularized" the value of $\beta = \infty$ furnishing an infinite entropy at zero temperature to a finite value $\beta_{\lambda,0} = \sqrt{(16\pi G/|\lambda|)}$ leading to a bounded S_λ entropy when $\lambda < 0$. There are systems with non-zero and bounded entropy at absolute zero. The extremal Reissner-Nordstrom and Kerr-Newman black holes have *zero* temperature but non-vanishing entropy given by one-quarter of the horizon area. The authors [3] have pointed out that for charged black branes the entropy at zero temperature is bounded but not zero. This is not common in thermodynamic systems and seems to imply a highly degenerate ground state [14].

To conclude, we should mention that Kaniadakis [8] has shown that it is possible to select generalized statistical theories in which the twofold link between entropy and the distribution function continues to hold, such as in the case of ordinary statistical mechanics. Within this scenario, apart from the standard logarithmic-exponential functions that define ordinary statistical mechanics, there emerge other new couples of direct-inverse functions, i.e. generalized logarithms and generalized exponentials, defining coherent and self-consistent generalized statistical theories. Interestingly, Kanadiakis found [8] that all these theories preserve the main features of ordinary statistical mechanics, and predict distribution functions presenting power-law tails. Furthermore, the obtained generalized entropies are both thermodynamically and Lesche stable.

The relevant three-parameter entropy associated with a generalized logarithm [8] contains as special cases *all* the one-parameter and two-parameter trace form entropies appearing in the literature. The ensuing three-parameter probability distribution functions represent the minimal deformation of the Maxwell-Boltzmann exponential distribution compatible with the maximum entropy principle and exhibit power-law tails. Sharma-Mittal entropies [11],[12] are two-parameter families of entropic forms which contain many of the entropies like the Tsallis [9], Kaniadakis [7], Abe [10] entropies as special cases for the values of the two paramters [13].

Therefore, it is warranted to explore if a three-parameter family of Wick rotation functions $f_{\lambda_1\lambda_2,\lambda_3}(\tau)$ can reproduce all the different modified black hole entropies mentioned above when one evaluates the Euclidean Einstein-Hilbert actions involving a delta function scalar curvature singularity at $r = 0$.

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