Evaluating an Almost Impossible Integral

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Abstract

This paper investigates a complex integral that has long been considered insurmountable due to the challenging nature of its structure. Through a novel approach, we aim to deconstruct the integral into manageable parts and uncover insights into its underlying properties.

1 Introduction

Integrals are the lifeblood of mathematical analysis, serving as gateways to understanding complex systems and abstract relationships. However, not all integrals yield their secrets easily. Some, due to their intricate behavior, have defied solution for decades, leaving mathematicians both inspired and perplexed. Among these, the integral we explore here stands out as a formidable challenge. Its complexity is so daunting that few have dared to tackle it in full.

In this paper, we aim to confront this near-impossible integral. By utilizing a combination of advanced techniques from calculus, complex analysis, and asymptotic expansion, we strive to illuminate a solution pathway. Though the journey may be fraught with challenges, the insights gained could open doors to new methods and interpretations in mathematical theory.

2 Evaluation

Observe:

$$\ln x \ln^3\left(\frac{x}{1+x}\right) = \ln x \left[\ln x - \ln(1+x)\right]^3$$

$$= \ln x \left[\ln^3 x - 3\ln^2 x \ln(1+x) + 3\ln x \ln^2(1+x) - \ln^3(1+x)\right]$$

$$= \ln x \ln^3 x - 3\ln^2 x \ln x \ln(1+x) + 3\ln^2 x \ln^2(1+x) - \ln x \ln^3(1+x)$$

$$\therefore 3\ln^2 x \ln(1+x) = \ln x \ln^3\left(\frac{x}{1+x}\right) - \ln^4 x + 3\ln^3(x)\ln(1+x) + \ln x \ln^3(1+x)$$

$$\Rightarrow I = \frac{1}{3}\int_0^1 \frac{x\ln x \ln^3\left(\frac{x}{1+x}\right)}{1-x^2}dx - \frac{1}{3}\int_0^1 \frac{x\ln^4 x}{1-x^2}dx + \frac{1}{3}\int_0^1 \frac{x\ln^3(1+x)\ln(x)}{1-x^2}dx + \int_0^1 \frac{x\ln^3 x\ln(1+x)}{1-x^2}dx$$

Now,

$$\frac{1}{1-x^2}=\frac{1}{2}\left(\frac{1}{1-x}+\frac{1}{1+x}\right),\quad \frac{x}{1-x^2}=\frac{1}{2}\left(\frac{1}{1-x}-\frac{1}{1+x}\right)$$

$$\therefore I=\frac{1}{6}\int_0^1\frac{x\ln(x)\ln^3\left(\frac{x}{1+x}\right)}{1+x}dx+\frac{1}{6}\int_0^1\frac{x\ln(x)\ln^3\left(\frac{x}{1+x}\right)}{1-x}dx$$

$$+\frac{1}{6}\int_0^1\frac{\ln^4 x}{1+x}dx-\frac{1}{6}\int_0^1\frac{\ln^4 x}{1-x}dx-\frac{1}{6}\int_0^1\frac{\ln x\ln^3(1+x)}{1+x}dx+\frac{1}{6}\int_0^1\frac{\ln x\ln^3(1+x)}{1-x}dx$$

$$-\frac{1}{2}\int_0^1\frac{\ln^3 x\ln(1+x)}{1+x}dx+\frac{1}{2}\int_0^1\frac{\ln^3 x\ln(1+x)}{1-x}dx$$

$$I=\frac{1}{6}I_1+\frac{1}{6}I_2+\frac{1}{6}I_3-\frac{1}{6}I_4-\frac{1}{6}I_5+\frac{1}{6}I_6-\frac{1}{2}I_7+\frac{1}{2}I_8$$

$$I_1=\int_0^1\frac{x\ln x\ln^3\left(\frac{x}{1+x}\right)}{1+x}dx,\quad I_2=\int_0^1\frac{x\ln x\ln^3\left(\frac{x}{1+x}\right)}{1-x}dx$$

$$I_3=\int_0^1\frac{\ln^4 x}{1+x}dx,\quad I_4=\int_0^1\frac{\ln^4 x}{1-x}dx,\quad I_5=\int_0^1\frac{\ln^3 x\ln(1+x)}{1+x}dx$$

$$I_6=\int_0^1\frac{\ln^3 x\ln(1+x)}{1-x}dx,\quad I_7=\int_0^1\frac{\ln^3 x\ln(1+x)}{1+x}dx,\quad I_8=\int_0^1\frac{\ln^3 x\ln(1+x)}{1-x}dx$$

Now,

$$I_1=\int_0^1\frac{x\ln x\ln^3\left(\frac{x}{1+x}\right)}{1+x}dx$$

Let $y=\frac{x}{1+x}$, so $dx=\frac{dy}{(1-y)^2}$.

$$=\int_0^{1/2}\frac{y\ln\left(\frac{y}{1-y}\right)\ln^3 y}{(1-y)^2}dy=\int_0^{1/2}\frac{\ln\left(\frac{y}{1-y}\right)\ln^3(y)}{(1-y)^2}dy-\int_0^{1/2}\frac{\ln\left(\frac{y}{1-y}\right)\ln^3(y)}{(1-y)}dy$$

$$=-\int_0^{1/2}\frac{\ln^4 y}{1-y}dy+\int_0^{1/2}\frac{\ln(1-y)\ln^3 y}{(1-y)^2}dy+\int_0^{1/2}\frac{\ln^4 y}{(1-y)^2}dy-\int_0^{1/2}\frac{\ln(1-y)\ln^3 y}{(1-y)^2}dy$$
$$=-J_1+J_2+J_3-J_4$$

Now,

$$J_1=\int_0^{1/2}\frac{\ln^4 y}{1-y}dy=\sum_{r=0}^{\infty}\int_0^{1/2}y^r\ln^4 y dy$$

$$=\sum_{r=0}^{\infty}\frac{24}{2^{r+1}(r+1)^5}-\frac{\ln^4 2}{2^{r+1}(r+1)}+\frac{4\ln^3(2)}{2^{r+1}(r+1)^2}+\frac{12\ln^2 2}{2^{r+1}(r+1)^3}-\frac{24\ln 2}{2^{r+1}(r+1)^4}$$

$$=24\sum_{r=1}^{\infty}\frac{1}{2^r r^5}-\ln^4 2\sum_{r=1}^{\infty}\frac{1}{2^r r}+4\ln^3 2\sum_{r=1}^{\infty}\frac{1}{2^r r^2}+12\ln^2 2\sum_{r=1}^{\infty}\frac{1}{2^r r^3}-24ln2\sum_{r=1}^{\infty}\frac{1}{2^r r^4}$$

Hence,

$$J_1 = 24\mathrm{Li}_5\left(\frac{1}{2}\right) + 24\ln(2)\mathrm{Li}_4\left(\frac{1}{2}\right) + 12\ln^2 2\mathrm{Li}_3\left(\frac{1}{2}\right)$$
$$+4\ln^3 2\mathrm{Li}_2\left(\frac{1}{2}\right) + \ln^5 2$$

$$J_2 = \int_0^{1/2} \frac{\ln(1-y)\ln^3 y}{1-y}dy = -\frac{1}{2}\ln^2(1-y)\ln^3 y\Big|_0^{1/2} + \frac{3}{2}\int_0^{1/2}\frac{\ln^2(1-y)\ln^2 y}{y}dy$$
$$= \frac{\ln^5 2}{2} + \frac{3}{2}\left[2\sum_{n=1}^{\infty}\frac{H_{n-1}}{n}\int_0^{1/2} y^{n-1}\ln^2(y)dy\right]$$

As,

$$\ln^2(1-x) = 2\sum_{n=1}^{\infty}\frac{H_{n-1}}{n}x^n$$

we have:

$$J_2 = \frac{\ln^5 2}{2} + 3\ln^2 2\sum_{n=1}^{\infty}\frac{H_n}{n^2 2^n} - 3\ln^2 2\sum_{n=1}^{\infty}\frac{1}{2^n n^3} + 6\ln 2\sum_{n=1}^{\infty}\frac{H_n}{n^3 2^n}$$
$$-6\ln 2\sum_{n=1}^{\infty}\frac{H_n}{n^4 2^n} + 6\sum_{n=1}^{\infty}\frac{H_n}{n^4 2^n} - 6\sum_{n=1}^{\infty}\frac{H_n}{n^5 2^n}$$

Using the definition of harmonic numbers $H_{n-1} = H_n - \frac{1}{n}$, we obtain this Thus,

$$J_2 = \frac{21}{8}\zeta(3)\ln^2(2) - \zeta(2)\ln^3(2) + 6\ln(2)\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{3}{16}\zeta(5) - 3\zeta(2)\zeta(3)$$
$$+6\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{2}{5}\ln^5(2)$$

$$J_3 = \int_0^{1/2}\frac{\ln^4(y)}{(1-y)^2}\,dy = \sum_{n=1}^{\infty} n\int_0^{1/2} y^{n-1}\ln y\,dy$$
$$= \sum_{n=1}^{\infty}\frac{1}{n^4}\left[\frac{n^4\ln^4(2)}{2^n} + \frac{4n^3\ln^3(2)}{2^n} + \frac{12n^2\ln^2(2)}{2^n} + 24\ln 2\frac{n}{2^n} + 24\frac{1}{2^n}\right]$$
$$= 24\mathrm{Li}_4\left(\frac{1}{2}\right) + 21\zeta(3)\ln 2 + 3\ln^4(2) - 6\zeta(2)\ln^2(2)$$

$$J_4 = \int_0^{1/2}\frac{\ln(1-y)\ln^3 y}{(1-y)^2}dy$$

After By-parts,

$$J_4=\int_0^{1/2}\frac{\ln(1-y)\ln^3(y)}{(1-y)^2}\,dy=\frac{\ln(1-y)\ln^3(y)}{1-y}\Big|_0^{1/2}+\int_0^{1/2}\frac{\ln^3(y)}{(1-y)^2}\,dy-3\int_0^{1/2}\frac{\ln(1-y)\ln^2(y)}{y(1-y)}\,dy$$

$$=2\ln^4(2)+\int_0^{1/2}\frac{\ln^3(y)}{(1-y)^2}\,dy-3\int_0^{1/2}\frac{\ln(1-y)\ln^2(y)}{y}\,dy+3\int_0^{1/2}\frac{\ln(1-y)\ln^2(y)}{1-y}\,dy$$

$$=2\ln^4(2)+\left[\frac{\ln^3(y)}{1-y}\right]\Big|_0^{1/2}-3\int_0^{1/2}\frac{\ln^2(y)}{y(1-y)}\,dy+3\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{1/2}y^{n-1}\ln^2(y)\,dy$$

$$=2\ln^4(2)+\left[-2\ln^3(2)-3\int_0^{1/2}\frac{\ln^2(y)}{1-y}\,dy-3\int_0^{1/2}\frac{\ln^2(y)}{y}\,dy\right]+3\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{\ln^2(y)}{n\cdot 2^n}+\frac{2\ln(2)}{n^2\cdot 2^n}\right]$$

$$+3\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{2}{2^n\cdot n^3}\right]-3\sum_{n=1}^{\infty}H_{n-1}\int_0^{1/2}y^{n-1}\ln^2(y)\,dy$$

$$=2\ln^4(2)+\left[-2\ln^3(2)-3\left(\frac{7}{4}\zeta(3)+\frac{\ln^3(2)}{3}\right)\right]+3\ln^2(2)\sum_{n=1}^{\infty}\frac{1}{2^n\cdot n^2}+6\ln(2)\sum_{n=1}^{\infty}\frac{1}{n^3\cdot 2^n}$$

$$+6\sum_{n=1}^{\infty}\frac{1}{n^4\cdot 2^n}\quad-3\sum_{n=1}^{\infty}\left(H_n-\frac{1}{n}\right)\left(\frac{\ln^2(2)}{n\cdot 2^n}+\frac{2\ln(2)}{n^2\cdot 2^n}+\frac{2}{n^3\cdot 2^n}\right)$$

$$=2\ln^4(2)+\left[-2\ln^3(2)-\frac{21}{4}\zeta(3)-\ln^3(2)\right]+3\ln^2(2)\mathrm{Li}_2\left(\frac{1}{2}\right)+6\ln(2)\mathrm{Li}_3\left(\frac{1}{2}\right)$$

$$+6\mathrm{Li4}\left(\frac{1}{2}\right)-3\ln^2(2)\sum_{n=1}^{\infty}\frac{H_n}{n^2\cdot 2^n}-6\ln(2)\sum_{n=1}^{\infty}\frac{H_n}{n^2 2^n}-6\sum_{n=1}^{\infty}\frac{H_n}{n^3 2^n}+3\ln^2(2)\mathrm{Li}_2\left(\frac{1}{2}\right)+6\mathrm{Li4}\left(\frac{1}{2}\right)$$

$$=2\ln^4(2)-3\ln^3(2)-\frac{21}{4}\zeta(3)+6\ln^2(2)\mathrm{Li}_2\left(\frac{1}{2}\right)+12\ln(2)\mathrm{Li}_3\left(\frac{1}{2}\right)+12\mathrm{Li}_4\left(\frac{1}{2}\right)$$

$$-3\ln^2(2)\sum_{n=1}^{\infty}\frac{H_n}{n2^n}-6\ln(2)\sum_{n=1}^{\infty}\frac{H_n}{n^2 2^n}-6\sum_{n=1}^{\infty}\frac{H_n}{n^3 2^n}$$

Now,

$$S_1=\frac{\zeta(2)}{2},\quad S_2=\zeta(3)-\frac{\zeta(2)}{2}\ln(2),\quad S_3=\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{\zeta(4)}{8}-\frac{1}{8}\zeta(3)\ln(2)+\frac{\ln^4(2)}{24}$$

and

$$\mathrm{Li}_4\left(\frac{1}{2}\right)=\frac{\zeta(2)}{2}-\frac{\ln^2(2)}{2},\quad \mathrm{Li}_3\left(\frac{1}{2}\right)=\frac{7}{8}\zeta(3)+\frac{\ln^3(2)}{6}-\frac{\zeta(2)}{2}\ln(2)$$

Thus,

$$J_4 = 6\text{Li}_4\left(\frac{1}{2}\right) + \frac{21}{4}\zeta(3)\ln(2) - \frac{21}{4}\zeta(3) + \frac{9}{4}\ln^4(2) - 2\ln^3(2) + \frac{3}{4}\zeta(4) - \frac{3}{2}\zeta(2)\ln^2(2)$$

Now,

$$I_1 = -J_1 + J_2 + J_3 - J_4$$

$$I_1 = -18\text{Li}_5\left(\frac{1}{2}\right) - 18\ln(2)\text{Li}_4\left(\frac{1}{2}\right) - \frac{63}{8}\zeta(3)\ln^2(2) - \frac{3}{5}\ln^5(2) + 3\zeta(2)\ln^3(2)$$

$$+\frac{3}{16}\zeta(5) - 3\zeta(2)\zeta(3) + 18\text{Li}_4\left(\frac{1}{2}\right) + \frac{63}{4}\zeta(3)\ln(2) + \frac{3}{4}\ln^4(2) - \frac{9}{2}\zeta(2)\ln^2(2)$$

$$+\frac{21}{4}\zeta(3) + 2\ln^3(2) - \frac{3}{4}\zeta(4)$$

$$I_2 = \int_0^1 x\ln x\ln^3\left(\frac{x}{1+x}\right)\frac{dx}{1-x}$$

Let

$$\frac{1-x}{1+x} = y \quad \text{(wirestrass)}$$

$$= \int_0^1 (1-y)\left[\ln(1-y) - \ln(1+y)\right]\left[\ln(1-y) - \ln 2\right]^3 \frac{dy}{y(1+y)^2}$$

$$= \int_0^1 \frac{1-y}{y(1+y)^2}\Big[\ln^4(1-y) - 3\ln(2)\ln^3(1-y) - \ln^3(2)\ln(1-y) + 3\ln^2(2)\ln^2(1-y)$$

$$- \ln(1+y)\ln^3(1-y) + \ln^3(2)\ln(1+y) + 3\ln(2)\ln^2(1-y)\ln(1+y) - 3\ln^2(2)\ln(1-y)\ln(1+y)\Big]\,dy$$

Now,

1)

$$\int_0^1 \frac{1-y}{y(1+y)^2}\ln^4(1-y)dy = -24\text{Li}_4\left(\frac{1}{2}\right) + 24\zeta(5) - 24\text{Li}_5\left(\frac{1}{2}\right)$$

2)

$$\int_0^1 \frac{1-y}{y(1+y)^2}\ln^3(1-y)dy$$

$$= 6\text{Li}_4\left(\frac{1}{2}\right) + \frac{21}{4}\zeta(3) - 6\zeta(4) + \ln^3(2) - 3\zeta(2)\ln(2)$$

3)

$$\int_0^1 \frac{1+y}{y(1+y)^2}\ln(1-y)dy = \frac{\ln(2)}{2}$$

4)

$$\int_0^1 \frac{1-y}{y(1+y)^2}\ln^2(1-y)dy = \frac{\zeta(3)}{4} - \frac{1}{3}\ln^3(2) - \ln^2(2) + \zeta(2)\ln(2) - \zeta(2)$$

5)

$$\int_0^1 \frac{1-y}{y(1+y)^2}\ln(1+y)\ln^3(1-y)\,dy$$

$$= \int_0^1 \frac{\ln(1+y)\ln^3(1-y)}{y(1+y)^2}\,dy - \int_0^1 \frac{\ln(1+y)\ln^3(1-y)}{(1+y)^2}\,dy$$

$$= -\int_0^1 \frac{\ln(1+y)\ln^3(1-y)}{1+y}\,dy - 2\int_0^1 \frac{\ln(1+y)\ln^3(1-y)}{(1+y)^2}\,dy + \int_0^1 \frac{\ln(1+y)\ln^3(1-y)}{y}\,dy$$

$$= -\left[6\operatorname{Li}_5\left(\frac{1}{2}\right) - \zeta(2)\ln^3(2) + 3\zeta(3)\ln^2(2) - \frac{3}{4}\zeta(4)\ln(2) - 3\zeta(2)\zeta(3) + \frac{3}{16}\zeta(5)\right.$$

$$+\frac{3}{20}\ln^5(2) - 2\left(-\frac{21}{8}\zeta(3)\ln^2(2) - \frac{3}{8}\ln^4(2) + \frac{3}{2}\zeta(3)\ln(2) - \frac{21}{8}\zeta(3) - \frac{1}{2}\ln^3 2 + \frac{3}{2}\zeta(2)\ln(2)\right)$$

$$\left.-\frac{1}{20}\ln^2(2) + \frac{3}{2}\zeta(3)\ln(2)\right] + \left[6\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + 6\operatorname{Li}_5\left(\frac{1}{4}\right) - \frac{81}{16}\zeta(5) - \right.$$

$$\left.\frac{21}{8}\zeta(2)\zeta(3) + \frac{21}{8}\zeta(3)\ln^2(2) - \zeta(2)\ln^3(2) + \frac{1}{5}\ln^5(2)\right]$$

Now, putting all together,

$$\begin{aligned} I_2 &= \int_0^1 \frac{\ln(x)\ln^3\left(\frac{x}{1+x}\right)}{1-x}\,dx \\ &= -24\operatorname{Li}_5\left(\frac{1}{2}\right) - 18\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{117}{4}\zeta(5) - 18\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right) \\ &\quad - \frac{63}{4}\zeta(3)\ln(2) + \frac{129}{8}\zeta(4)\ln(2) - \frac{3}{2}\ln^4(2) + \frac{9}{2}\zeta(2)\ln^2(2) \\ &\quad + \frac{21}{8}\zeta(3)\ln^2(2) - \frac{11}{20}\ln^5(2) + 4\zeta(2)\ln^3(2) + \frac{3}{4}\zeta(4) \\ &\quad - \frac{21}{4}\zeta(3) - \frac{3}{8}\zeta(2)\zeta(3) - 2\ln^3(2) \end{aligned}$$

$$I_3 = \int_0^1 \frac{\ln^4 x}{1+x}\,dx = \sum_{r=0}^{\infty}(-1)^r \int_0^1 x^r \ln^4(x)\,dx = \sum_{r=0}^{\infty} \frac{(-1)^r 24}{(r+1)^5} = -\sum_{r=0}^{\infty} \frac{(-1)^r 24}{r^5}$$

$$or \quad I_3 = \frac{45}{2}\zeta(5)$$

$$I_4=\int_0^1 \frac{\ln^4 x}{1-x}\,dx=\sum_{r=0}^{\infty}\int_0^1 x^r\ln^4(x)\,dx=\sum_{r=0}^{\infty}\frac{24}{(r+1)^5}=24\zeta(5),\quad I_4=24\zeta(5)$$

$$I_5=\int_0^1 \frac{\ln(x)\ln^3(1+x)}{1+x}\,dx$$

$$=\frac{1}{4}\ln(x)\ln^4(x+1)\Big|_0^1-\frac{1}{4}\int_0^1\frac{\ln^4(1+x)}{x}\,dx$$

$$=-\frac{1}{4}K_1$$

where, $K_1=\int_0^1 \frac{\ln^4(1+x)}{1-x}\,dx$.
We know,

$$\int_0^1 \frac{\ln^a(x)\,dx}{x}=\frac{\ln^{a+1}(2)}{a+1}+\Gamma(a+1)\zeta(a+1)-(-1)^a\int_0^{1/2}\frac{\ln^a(x)}{1-x}\,dx$$

$$\therefore\quad \int_0^1 \frac{\ln^4(x)\,dx}{x}=\frac{\ln^5(2)}{5}+24\zeta(5)-\int_0^{1/2}\frac{\ln^4(1+x)}{1-x}\,dx$$

$$\Rightarrow K_1=\frac{\ln^5(2)}{5}+24\zeta(5)-K_2$$

Now, $K_2=\int_0^{1/2}\frac{\ln^4(x)}{1-x}\,dx$.
Let $2x=p$,

$$K_2=\frac{1}{2}\int_0^1\frac{\ln^4\left(\frac{p}{2}\right)}{1-\frac{p}{2}}\,dp=\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\int_0^1 p^n\ln^4\left(\frac{p}{2}\right)\,dp$$

$$K_2=24\,\mathrm{Li}_5\left(\frac{1}{2}\right)+24\ln(2)\,\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{21}{2}\zeta(3)\ln^2(2)+\ln^5(2)+4\zeta(2)\ln^3(2)$$

$$\therefore\quad K_1=\frac{\ln^5(2)}{5}+24\zeta(5)-K_2$$

Plugging in,

$$K_1=24\zeta(5)-24\,\mathrm{Li}_4\left(\frac{1}{2}\right)-24\,\mathrm{Li}_5\left(\frac{1}{2}\right)-\frac{21}{2}\zeta(3)\ln^2(2)-\frac{4}{5}\ln(2)+4\zeta(2)\ln^3(2)$$

Now,

$$I_5=-\frac{1}{4}K_1,\Rightarrow I_5=-6\zeta(5)+6\ln(2)\,\mathrm{Li}_4\left(\frac{1}{2}\right)+6\,\mathrm{Li}_5\left(\frac{1}{2}\right)+\frac{21}{8}\zeta(3)\ln^2(2)$$

$$+\frac{1}{5}\ln^5(2)-\zeta(2)\ln^3(2)$$

$$I_6=\int_0^1\frac{\ln(x)\ln^3(1+x)}{1-x}\,dx\quad\text{let } y=\frac{1-x}{1+x}$$

$$=-\int_1^0\frac{\ln\left(\frac{1-y}{1+y}\right)\ln\left(\frac{2}{1+y}\right)}{y(1+y)}\,dy$$

$$=\int_0^1\frac{1}{y(1+y)}\left[(\ln(1-y)-\ln(1+y))(\ln(2)-\ln(1+y))^3\right]\,dy$$

after separating and partial fraction decomposition,

$$\Rightarrow I_6=\frac{117}{4}\zeta(5)-\frac{21}{8}\zeta(2)\zeta(3)+\frac{63}{8}\zeta(3)\ln^2(2)-2\zeta(2)\ln^3(2)-\frac{1}{20}\ln^5(2)$$

$$-6\ln(2)\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{154}{8}\zeta(4)\ln(2)-24\mathrm{Li}_5\left(\frac{1}{2}\right)$$

$$I_7=\int_0^1\frac{\ln^3(x)\ln(1+x)}{1+x}\,dx=-\sum_{n=1}^{\infty}(-1)^nH_n\int_0^1x^n\ln^3(x)\,dx=-\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{(n+1)^4}\cdot 6$$

$$\Rightarrow I_7=-6\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{(n+1)^4}=-6\left[\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^4}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}\right]$$

$$\Rightarrow I_7=-6\left[\frac{1}{2}\zeta(2)\zeta(3)-\frac{59}{32}\zeta(5)\right]+6\left[-\frac{15}{16}\zeta(5)\right]$$

$$\therefore I_7=-3\zeta(2)\zeta(3)+\frac{87}{16}\zeta(5)$$

$$I_8=\int_0^1\frac{\ln(1+y)\ln^3(y)}{1-y}\,dy=-\ln(1-y)\ln(1+y)\ln^3(y)\Big|_0^1+\int_0^1\frac{\ln(1-y)\ln^3(y)}{1+y}\,dy$$

$$+3\int_0^1\frac{\ln(1+y)\ln(1-y)\ln^2(y)}{y}\,dy$$

$$\therefore I_8=Q_1+3Q_2,\quad Q_1=\int_0^1\frac{\ln(1-y)\ln^3(y)}{1+y}\,dy,\quad Q_2=3\int_0^1\frac{\ln(1+y)\ln(1-y)\ln^2(y)}{y}\,dy$$

Now,

$$Q_1=\int_0^1\frac{\ln(1-y)\ln^3(y)}{1+y}\,dy=\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1y^{n-1}\ln(1-y)\ln^3(y)\,dy$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1\frac{d^3}{dn^3}\left[y^{n-1}\ln(1-y)\right]dy=\sum_{n=1}^{\infty}\frac{d^3}{dn^3}\left[\frac{H_n-H_1}{n}\right]$$

$$= -\sum_{n=1}^{\infty}(-1)^{n-1}\frac{d^3}{dn^3}\left[\frac{H_n}{n}\right]$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{6H_n}{n^4}+\frac{6H_n^{(2)}}{n^3}+\frac{6H_n^{(3)}}{n^2}+\frac{6H_n^{(4)}}{n}-\frac{6\zeta(2)}{n^3}-\frac{6\zeta(3)}{n^2}-\frac{6\zeta(4)}{n}\right]$$

$$= -6\sum_{n=1}^{\infty}(-1)^n\frac{H_n}{n^4}-6\sum_{n=1}^{\infty}(-1)^n\frac{H_n^{(2)}}{n^3}-6\sum_{n=1}^{\infty}(-1)^n\frac{H_n^{(3)}}{n^2}$$

$$+6\zeta(2)\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^3}+6\zeta(3)\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}+6\zeta(4)\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

$$= Q_1 = \frac{273}{16}\zeta(5)-\frac{9}{2}\zeta(2)\zeta(3)-\frac{45}{4}\zeta(4)\ln(2)$$

Given,

$$Q_2 = \int_0^1 \ln(1+y)\ln(1-y)\frac{\ln^2(y)}{y}\,dy$$

$$= -\sum_{n=1}^{\infty}\left(\frac{H_{2n}-H_n}{n}+\frac{1}{2n^2}\right)\left(\frac{1}{4n^3}\right)$$

Thus,

$$Q_2 = -\frac{1}{4}\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n)^4}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{H_n}{n^4}-\frac{1}{8}\sum_{n=1}^{\infty}\frac{1}{n^5}$$

$$= -\frac{16}{4}\sum_{n=1}^{\infty}\frac{H_{2n}}{n^4}+\frac{4}{4}\sum_{n=1}^{\infty}\frac{H_n}{n^4}-\frac{1}{8}\sum_{n=1}^{\infty}\frac{1}{n^5}$$

$$= -4\left[\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_{2n}}{n^4}+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{H_n}{n^4}\right]+\frac{1}{4}\sum_{n=1}^{\infty}\frac{H_n}{n^4}-\frac{1}{8}\sum_{n=1}^{\infty}\frac{1}{n^5}$$

$$= -\frac{7}{4}\sum_{n=1}^{\infty}\frac{H_n}{n^4}-2\sum_{n=1}^{\infty}(-1)^n\frac{H_n}{n^4}-\frac{1}{8}\sum_{n=1}^{\infty}\frac{1}{n^5}$$

$$= -\frac{7}{4}\left[3\zeta(5)-8\zeta(2)\zeta(3)\right]-2\left[\frac{1}{2}\zeta(2)\zeta(3)-\frac{59}{32}\zeta(5)\right]-\frac{1}{8}\zeta(5)$$

$$\Rightarrow Q_2 = \frac{3}{4}\zeta(2)\zeta(3)-\frac{27}{16}\zeta(5)$$

Therefore,

$$I_8 = \frac{273}{16}\zeta(5)-\frac{9}{2}\zeta(2)\zeta(3)-\frac{45}{4}\zeta(4)\ln(2)+\frac{9}{4}\zeta(2)\zeta(3)-\frac{243}{16}\zeta(5)$$

$$I_8 = \frac{30}{16}\zeta(5)-\frac{9}{4}\zeta(2)\zeta(3)-\frac{45}{4}\zeta(4)\ln(2)$$

Now, it's time to compile everything,

$$I = \frac{1}{6}I_1 + \frac{1}{6}I_2 + \frac{1}{6}I_3 - \frac{1}{6}I_4 - \frac{1}{6}I_5 + \frac{1}{6}I_6 - \frac{1}{2}I_7 + \frac{1}{2}I_8$$

$$\boxed{\begin{aligned}\int_0^1 \frac{x\ln(x)\ln^2(1+x)}{1-x^2}\,dx = &-12\operatorname{Li}_5\left(\frac{1}{2}\right) - 8\ln(2)\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{7}{30}\ln^5(2) \\ &+ \frac{221}{16}\zeta(5) - \frac{5}{8}\zeta(2)\zeta(3) - \frac{149}{16}\zeta(4)\ln(2) \\ &+ \zeta(2)\ln^3(2)\end{aligned}} \tag{1}$$

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3 Conclusion

In this paper, we have undertaken the daunting task of examining an integral of substantial complexity, employing a variety of analytical tools and approaches. The journey through this intricate landscape of mathematical analysis has not only deepened our understanding of advanced integration techniques but also highlighted the resilience required to confront such challenges.

While the integral in question has resisted complete simplification, the insights gained through this exploration provide a foundation for further study and potential breakthroughs in related fields. The process of wrestling with such an enigmatic problem underscores the beauty and rigor of mathematics—reminding us that, while solutions are valuable, the pursuit of understanding itself is profoundly rewarding.

We extend our sincere thanks to the readers who have joined us in navigating this intricate problem. Your dedication to understanding complex mathematics, as demonstrated by your engagement with this paper, is invaluable. It is through such collective curiosity and effort that mathematics continues to grow, evolve, and inspire.