Sign normalization for higher Genus curves in Generalized Riemann hypothesis, and Generalized Birch and Swinnerton-Dyer conjecture

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[†]The author contributes to all the work in this research paper, which is dedicated to his premature daughter Jelena born 13 weeks early on Monday May 14, 2012. All generated or computed data for this paper are included. *Ethical approval*: Not applicable. *Conflict of Interest Statement*: There is no conflict of interest.

Abstract

This expository-styled paper contains interesting observations and conjectures about distribution of nontrivial zeros in L-functions; and [optional] use of Sign normalization when computing Hardy Z-function, including its relationship to Analytic rank and Symmetry type of L-functions. On the Sign normalization when applied to eligible L-functions, we posit its dependency on even-versus-odd Analytic ranks, degree of L-function, and the particular gamma factor present in functional equations for Genus 1 elliptic curves and higher Genus curves. The relevant mathematical arguments are postulated to satisfy Generalized Riemann hypothesis, and Generalized Birch and Swinnerton-Dyer conjecture. We explicitly mention their underlying proven/unproven hypotheses or conjectures.

Keywords: Analytic continuation; Analytic normalization; Analytic rank; Birch and Swinnerton-Dyer conjecture; Composite numbers; Euler product; Functional equation; Gamma factor; Generic L-function; Graph of Z-function; Polar graph; Prime numbers; Riemann hypothesis; Sign normalization

MSC Classification: 11M26 , 11A41

Contents

1 Introduction

An L-function is a meromorphic function on the complex plane, associated to one out of several categories of mathematical objects. For Generic L-functions [*aka* General L-functions] that include dual L-functions and self-dual L-functions theoretically arising from Maass forms, Genus 0, 1, 2, 3, 4, 5... curves, etc; we compare and contrast these two types of L-functions, and show different forms of symmetry being manifested by Z(t) plots of their nontrivial zeros (spectrum). *As with many problems in Number theory, the basic questions are easy to state but difficult to resolve.* This colloquial saying is prophetically true in mathematics especially for our outlined intractable open problems in Number theory, whereby we explicitly mention throughout this paper their underlying proven/unproven hypotheses or conjectures.

Genus of a connected, orientable surface is an integer representing the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected. Topologically, it is equal to number of "holes" (or "handles") on it. Alternatively, it is defined in terms of Euler characteristic χ via the relationship $\chi = 2 - 2g$ for closed surfaces where g is Genus. For surfaces with b boundary components, the equation reads $\chi = 2 - 2g - b$. Thus Genus 0, 1, 2, 3, 4, 5,... curves have 0, 1, 2, 3, 4, 5,... holes.

In classical algebraic geometry, the genus-degree formula relates the degree d of an irreducible plane curve C with its arithmetic genus g via the formula: $g = \frac{1}{2}(d-1)(d-2)$. Here "plane curve" means that C is a closed curve in projective plane $\mathbb{P}^2$. If the curve is non-singular the geometric genus and the arithmetic genus are equal, but if the curve is singular, with only ordinary singularities, the geometric genus is smaller. More precisely, an ordinary singularity of multiplicity r decreases the genus by $\frac{1}{2}r(r-1)$.

Definition 1. *Completely Unpredictable entities, Completely Predictable entities, and Incompletely Predictable entities with all three constituting countably infinite sets*

Predominantly based on p. 18 of [7], we provide formal definitions for three types of [infinitely-many] entities in a succinct manner: The *Completely Unpredictable* (non-deterministic) entities are defined as entities that are random and behave like one e.g. [true] random number generator that supply sequences of entities (as non-distinct Sets of numbers) that are not reproducible; viz, these entities do not contain any repeatable spatial or temporal patterns. The *Completely Predictable* (deterministic) entities are defined as entities that are actually NOT random and DO NOT behave like one e.g. distinct Set of Even numbers {0, 2, 4, 6, 8, 10,...} and Set of Odd numbers {1, 3, 5, 7, 9, 11,...}; viz, these entities are reproducible. The distinct Sets of trivial zeros in various L-functions [as infinitely-many negative integers] are other examples of Completely Predictable entities. The *Incompletely Predictable* [or *Pseudo-random*] (deterministic) entities are defined as entities that are actually NOT random but DO behave like one e.g. distinct Set of Prime numbers {2, 3, 5, 7, 11, 13,...} and Set of Composite numbers {4, 6, 8, 9, 10, 12,...}; viz, these entities are reproducible. Apart from integers, these entities are also constituted from other number systems e.g. distinct Sets of t-valued irrational (transcendental) numbers representing infinitely-many Incompletely Predictable nontrivial zeros (spectrum) of various L-functions are, as conjectured under Riemann hypothesis or Generalized Riemann hypothesis, *only* located on $\Re(s) = \frac{1}{2}$-Critical line or Analytically normalized $\Re(s) = \frac{1}{2}$-Critical line.

Lemma 1. *The plots of Z-function for general L-functions [and for the L-function from Riemann zeta-function] manifest unique distributions of both Z(t) positivity and Z(t) negativity that depend on the choice of sqrt(root number) being correctly and arbitrarily chosen from* $+1$ *or* -1 *value for even Analytic rank L-functions AND on the choice of sqrt(root number) being correctly and arbitrarily chosen from* $+i$ *or* $-i$ *value for odd Analytic rank L-functions.*

Proof. Riemann zeta function is Genus 0 curve having Analytic rank 0 [of degree 1]. Elliptic curves are Genus 1 curves having Analytic rank 0, 1, 2, 3, 4, 5,... [of degree 2], and there are other higher Genus 2, 3, 4, 5, 6... curves [of higher degree]. They all have associated self-dual L-functions generating unique nontrivial zeros (spectrum) with t values being fully independent of the chosen Z(t) positivity [or Z(t) negativity].

A product P, having positive ($+ve$) or negative ($-ve$) value, is the multiplication of two or more factors A, B, C, D, ...; viz, $P = A \times B \times C \times D \times \cdots$. Let $P = Z(t)$, $A = \bar{\varepsilon}^{\frac{1}{2}}$ {viz, [optional] "Sign normalization" from L-functions and modular forms database (LMFDB)}, $B = \frac{\gamma(\frac{1}{2}+it)}{|\gamma(\frac{1}{2}+it)|}$ and $C = L(\frac{1}{2}+it)$. LMFDB's $Z(t)$ is defined by $P = A \times B \times C$ whereby $A = \pm 1$ for L-functions having even Analytic rank 0, 2, 4, 6, 8, 10... and $A = \pm i$ for L-functions having odd Analytic rank 1, 3, 5, 7, 9, 11....

The epsilon (ϵ) [$= +1$ for even Analytic ranks $/ -1$ for odd Analytic ranks] is also known as Sign or Root number in the functional equation for an analytic L-function. Then the sqrt(root number) [or square root of epsilon] has values of $+1$ or -1 for even

Analytic ranks and $+i$ or $-i$ for odd Analytic ranks. This value is arbitrarily chosen under LMFDB's stated convention so that $Z(t) > 0$ for sufficiently small $t > 0$; viz, manifesting Z(t) positivity [$\equiv$ Sign normalization]. The corollary convention so that $Z(t) < 0$ for sufficiently small $t > 0$ refers to arbitrarily choosing this value to manifest Z(t) negativity. Thus, sqrt(root number) always give rise to two opposite choices for two complementary Z(t) plots of nontrivial zeros (spectrum) in both even and odd Analytic rank L-functions. Consequently, both Z(t) positivity and Z(t) negativity are inherently and validly present in Z(t) plots of L-functions when we use $\epsilon^{\frac{1}{2}} = \sqrt{\epsilon}$ in relevant Z-functions [instead of just using ϵ in relevant Z-functions], and with their unique distributions being specified by chosen $+1$ or -1 choices and $+i$ or $-i$ choices.

As per Axiom 1, Z(t) plots from L-functions using ϵ without adopting, or $\sqrt{\epsilon}$ with adopting, LMFDB's Sign normalization will NOT affect actual t values of nontrivial zeros (spectrum). Requiring confirmatory research studies, we intuitively propose here that these unique distributions have various [unknown] deterministic Incompletely Predictable properties. ***The proof is now complete for Lemma 1***$\square$***.***

Proposition 1. *The self-deal L-functions are special cases of dual L-functions, whereby they both have unique Z(t) plots of nontrivial zeros (spectrum) with LMFDB's enforced [optional] Sign normalization that manifest different forms of symmetry.*

Proof. Sign (root number) or epsilon for dual L-functions is a complex number $a+bi$ being the "Root of Unity". Then for self-dual L-functions [which can be usefully considered as simply special cases of dual L-functions], $\epsilon = +1 + 0i = +1$ with $\overline{\varepsilon}^{\frac{1}{2}} = \pm 1$ for even Analytic rank AND $\epsilon = -1+0i = -1$ with $\overline{\varepsilon}^{\frac{1}{2}} = \pm i$ for odd Analytic rank.

Both dual and self-dual concepts deal with the relationships between L-functions and their duals, and self-dual L-functions represent a stronger condition of symmetry:
[1] Symmetry. Dual L-functions may have a more general relation to their duals, while self-dual L-functions exhibit exact symmetry.
[2] Applications. Self-dual L-functions are often directly tied to important conjectures and results in Number theory.
[3] Examples. Many well-known L-functions associated with modular forms or Dirichlet characters can be self-dual, whereas others might not exhibit this property.

In Remark 1.1 below, we discuss using examples of Z(t) plots of nontrivial zeros (spectrum) that manifest unique types of individual/combined symmetry for both dual and self-dual L-functions. ***The proof is now complete for Proposition 1***$\square$***.***

Axiom 1. *The LMFDB's [optional] Sign normalization does not affect the actual values of nontrivial zeros (spectrum) from Z(t) plots of L-functions.*

Proof. Root number or Sign ϵ directly govern the functional equation; viz, [1] If $\epsilon = 1$, the L-function is symmetric (even functional equation); and [2] If $\epsilon = -1$, the L-function is anti-symmetric (odd functional equation). When constructing Z(t), the absolute value $|\Lambda(s)|$ eliminates the impact of ϵ as a Sign, so ϵ remains unchanged, and thus we strictly DO NOT need to square root the ϵ for Z(t) to be valid.

Why do LMFDB use $\sqrt{\epsilon}$ to obtain two choices [and arbitrarily choose one of the two choices under Sign normalization to manifest Z(t) positivity]? [1] Ensure symmetry: By incorporating $\sqrt{\epsilon}$ *whereby the required phase adjustment DOES NOT change the magnitude of* $\Lambda(E, s)$, the functional equation becomes symmetric $Z(-t) = Z(t)$, [2] Numerical stability: The square root ensures the phase of $\Lambda(s)$ along the Critical line

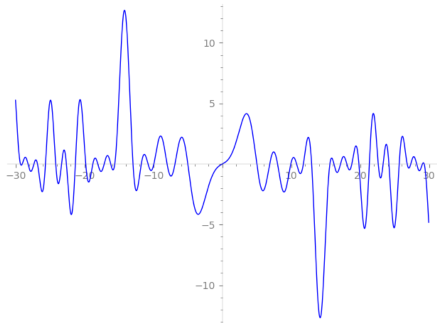

Fig. 1 Graph of Z-function along [Analytically normalized] $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in Genus 1 odd Analytic rank 1 semistable Elliptic curve 37.a1 of degree 2. Point Symmetry of Origin point, trajectory intersect Origin point, and manifest Z(t) positivity. Integral points (−1, 0), (−1, −1), (0, 0), (0, −1), (1, 0), (1, −1), (2, 2), (2, −3), (6, 14), (6, −15).

is correctly adjusted for numerical computations, [3] Nontrivial zeros being unaffected: The t values for infinitely-many nontrivial zeros (spectrum) are independent of using ϵ versus $\sqrt{\epsilon}$, and [4] Historical context: Similar constructions occur in Analytic Number theory e.g. for Riemann zeta function. ***The proof is now complete for Axiom 1***$\square$**.**

An old bug in the code for computing some of (Hardy or Riemann-Siegel) Z(t) plots in LMFDB website had previously resulted in a failure to follow the LMFDB's stated convention that Z(t) $>$ 0 as t $\rightarrow 0^+$. In particular, the Z(t) plots being affected by this bug are, firstly, from L-functions of all Genus 1 elliptic curves having (odd) Analytic rank 3 [except for the very first listed Analytic rank 3 Elliptic curve 5077.a1 being not affected]; and, secondly, from solitary L-function of the (non-elliptic) Genus 0 curve Riemann zeta function having (even) Analytic rank 0. We assign this stated convention as definition for 'Z(t) positivity' [hereby also called 'Sign normalization'] whereby the complementary 'Z(t) negativity' is defined by (corollary) convention Z(t) $<$ 0 as t $\rightarrow 0^+$. However we acknowledge this Sign normalization [so that Z(t) $>$ 0 for sufficiently small t $>$ 0] used in LMFDB (which is explicitly noted to be arbitrary) should not, in general, be used as a basis for definitive mathematical arguments.

As an obvious [randomly chosen] example correctly manifesting Z(t) positivity, the Z(t) plot of nontrivial zeros (spectrum) in Figure 1 for Degree 2 Genus 1 (odd) Analytic rank 1 Elliptic curve 37.a1 [NOT affected by the bug] is uniquely determined by choosing sqrt(root number) = $+i$ choice in self-dual L-function 2-37-1.1-c1-0-1.

Regarding these incorrectly depicted Z(t) plots manifesting Z(t) negativity [instead of Z(t) positivity] in affected Analytic rank 3 elliptic curves, they were first alerted by us in August 2024. The culprit bug in the code causing this problem was subsequently discovered by LMFDB Associate Editor Dr. Edgar Costa in conjunction with LMFDB Managing Editor Prof. Andrew Sutherland, and was largely fixed in October 2024.

As already mentioned: Apart from smallest conductor Degree 2 Analytic rank 3 Elliptic curve 5077.a1 with its Z(t) plot in Figure 2 of self-dual L-function 2-5077-1.1-c1-0-410 showing Z(t) positivity from arbitrarily choosing sqrt(root number) = $-i$ choice [NOT affected by the bug]; this bug affects Z(t) plots of self-dual L-functions derived from all other Degree 2 (odd) Analytic rank 3 elliptic curves e.g. randomly

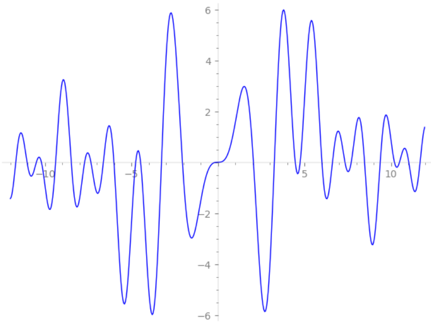

Fig. 2 Graph of Z-function along [Analytically normalized] $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in Genus 1 odd Analytic rank 3 semistable Elliptic curve 5077.a1 of degree 2. Point Symmetry of Origin point, trajectory intersect Origin point, and manifest Z(t) positivity as Pseudo-transitional curve. Integral points (−3, 0), (−3, −1), (−2, 3), (−2, −4), (−1, 3), (−1, −4), (0, 2), (0, −3), (1, 0), (1, −1), (2, 0), (2, −1), (3, 3), (3, −4), (4, 6), (4, −7), (8, 21), (8, −22), (11, 35), (11, −36), (14, 51), (14, −52), (21, 95), (21, −96), (37, 224), (37, −225), (52, 374), (52, −375), (93, 896), (93, −897), (342, 6324), (342, −6325), (406, 8180), (406, −8181), (816, 23309), (816, −23310).

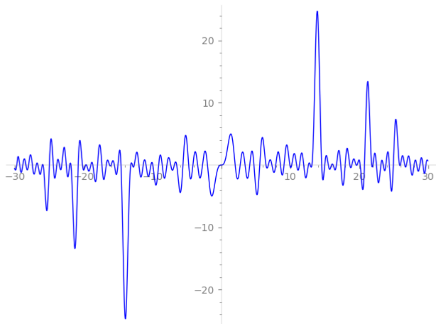

Fig. 3 Graph of Z-function along [Analytically normalized] $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in Genus 1 curve odd Analytic rank 3 semistable Elliptic curve 21858.a1 of degree 2. Point Symmetry of Origin point, trajectory intersect Origin point, and manifest Z(t) positivity [post-bug-fixing]. Integral points are (−7, 5), (−7, 2), (−6, 12), (−6, −6), (−4, 14), (−4, −10), (−2, 12), (−2, −10), (1, 5), (1, −6), (2, 2), (2, −4), (3, 0), (3, −3), (4, 2), (4, −6), (5, 5), (5, −10), (7, 12), (7, −19), (11, 29), (11, −40), (14, 44), (14, −58), (22, 92), (22, −114), (30, 150), (30, −180), (68, 530), (68, −598), (119, 1244), (119, −1363), (122, 1292), (122, −1414), (137, 1541), (137, −1678), (786, 21660), (786, −22446), (1069, 34437), (1069, −35506), (38746, 7607514), (38746, −7646260), (783868, 693616502), (783868, −694400370).

chosen self-dual L-function 2-21858-1.1-c1-0-3 of Elliptic curve 21858.a1 [depicted as correct Z(t) positivity version post-bug-fixing in Figure 3 and incorrect Z(t) negativity version pre-bug-fixing in Figure 4]. This bug has also affected Z(t) plot of self-dual L-function 2-11-1.1-c1-0-0 derived from Degree 1 (even) Analytic rank 0 (non-elliptic) Number field 1.1.1.1: $\mathbb{Q}$ / Riemann zeta function / Dirichlet eta function [depicted as correct Z(t) positivity version post-bug-fixing in Figure 5 and incorrect Z(t) negativity version pre-bug-fixing in Figure 6].

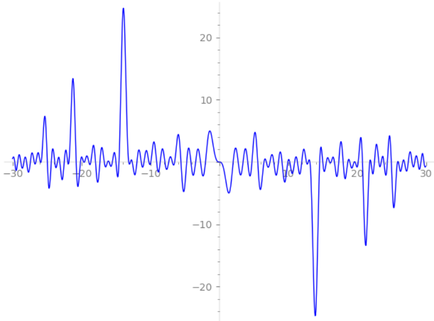

Fig. 4 Graph of Z-function along [Analytically normalized] $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in Genus 1 curve odd Analytic rank 3 semistable Elliptic curve 21858.a1 of degree 2. Point Symmetry of Origin point, trajectory intersect Origin point, and manifest Z(t) negativity [pre-bug-fixing]. Integral points are identical to that in Figure 3.

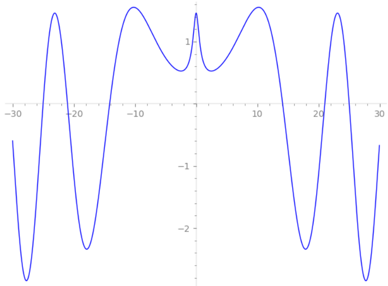

Fig. 5 Graph of Z-function along $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in even Analytic rank 0 Genus 0 Dirichlet eta function $\eta(s)$ of degree 1 over $K = \mathbb{Q}$ as *Analytic continuation* of Riemann zeta function $\zeta(s)$. Line Symmetry of vertical y-axis, trajectory DO NOT intersect Origin point, and manifest Z(t) positivity [post-bug-fixing]. Integral basis 1. [An integral basis of a number field K is a $\mathbb{Z}$-basis for ring of integers of K. This is also a $\mathbb{Q}$-basis for K.]

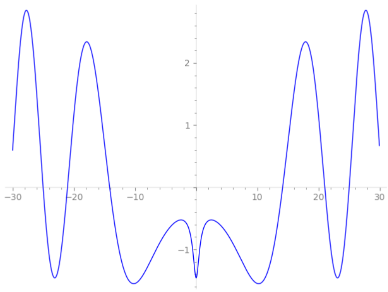

Fig. 6 Graph of Z-function along $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in even Analytic rank 0 Genus 0 Dirichlet eta function $\eta(s)$ of degree 1 over $K = \mathbb{Q}$ as *Analytic continuation* of Riemann zeta function $\zeta(s)$. Line Symmetry of vertical y-axis, trajectory DO NOT intersect Origin point, and manifest Z(t) negativity [pre-bug-fixing] as *Pseudo*-transitional curve. This is the complementary $Z(t)$ plot of nontrivial zeros (spectrum) to that depicted by Figure 5.

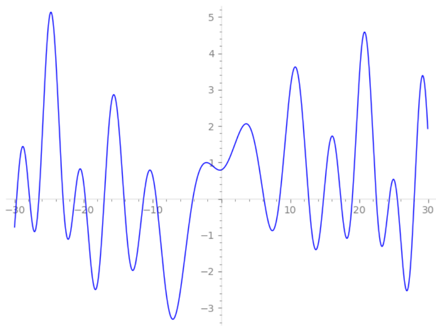

Fig. 7 Graph of Z-function along $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in even Analytic rank 0 Genus 0 Dirichlet character $\chi_5(2, \cdot)$ with odd Parity. There is neither Line symmetry nor Point symmetry being manifested. The trajectory DO NOT intersect the Origin point. This is the complementary $Z(t)$ plot to that depicted by Figure 8.

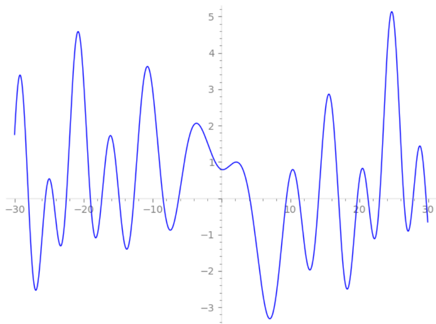

Fig. 8 Graph of Z-function along $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in even Analytic rank 0 Genus 0 Dirichlet character $\chi_5(3, \cdot)$ with odd Parity. There is neither Line symmetry nor Point symmetry being manifested. The trajectory DO NOT intersect the Origin point. This is the complementary $Z(t)$ plot to that depicted by Figure 7.

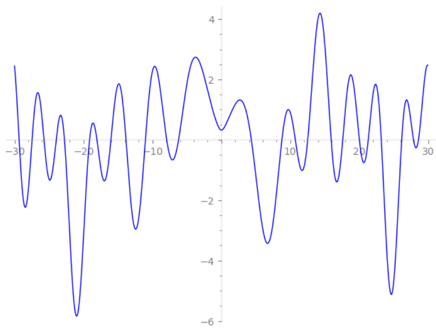

Fig. 9 Graph of Z-function along $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in even Analytic rank 0 Genus 0 Dirichlet character $\chi_7(2, \cdot)$ with even Parity. There is neither Line symmetry nor Point symmetry being manifested. The trajectory DO NOT intersect the Origin point. This is the complementary $Z(t)$ plot to that depicted by Figure 10.

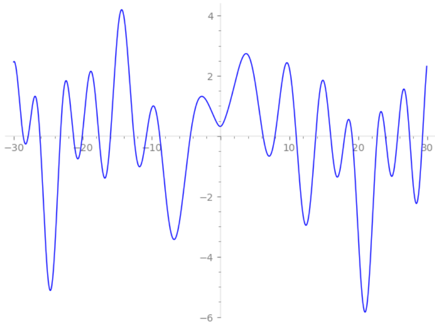

Fig. 10 Graph of Z-function along $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in even Analytic rank 0 Genus 0 Dirichlet character $\chi_7(4, \cdot)$ with even Parity. There is neither Line symmetry nor Point symmetry being manifested. The trajectory DO NOT intersect the Origin point. This is the complementary $Z(t)$ plot to that depicted by Figure 9.

Remark 1.1. In relation to self-dual L-functions, we see the horizontal x-axis acting as Line Symmetry for [combined] Figure 3 and Figure 4 with their [paired] sqrt(root number) given by $\pm i$ (for odd Analytic ranks), and [combined] Figure 5 and Figure 6 with their [paired] sqrt(root number) given by ± 1 (for even Analytic ranks).

On randomly chosen examples of Analytic rank 0 dual L-functions from Dirichlet characters having [paired] Sign given by complex number and its conjugate, we depict them via [combined] Figure 7 (L-function 1-5-5.2-r1-0-0) and Figure 8 (L-function 1-5-5.3-r1-0-0) as [respectively] odd Parity $\chi_5(2, \cdot)$ with Sign: $0.850+0.525i$ and odd Parity $\chi_5(3, \cdot)$ with Sign: $0.850-0.525i$; and via [combined] Figure 9 (L-function 1-7-7.2-r0-0-0) and Figure 10 (L-function 1-7-7.4-r0-0-0) as [respectively] even Parity $\chi_7(2, \cdot)$ with Sign: $0.895-0.444i$ and even Parity $\chi_7(4, \cdot)$ with Sign: $0.895+0.444i$. In contrast to self-dual L-functions, we instead see the vertical y-axis acting as Line Symmetry for these complementary-paired [with "conjugate Signs"] dual L-functions having either even or odd parity, and having [combined] "reverse" patterns of nontrivial zeros (spectrum).

Using the very definition of Z(t) for an L-function whereby we [optionally] adopt LMFDB's sqrt(root number) that always provide two choices, we unambiguously obtain valid mathematical statements in Lemma 1, Proposition 1 and Axiom 1. These statements are rigorously proven to be true using simple mathematical arguments.

2 Generalized Riemann hypothesis, and Generalized Birch and Swinnerton-Dyer conjecture

Proposed in 1859 by German mathematician Bernhard Riemann (September 17, 1826 – July 20, 1866), Riemann hypothesis (RH) refers to a famous conjecture on all nontrivial zeros in (self-dual) L-function from Genus 0 curve known as Riemann zeta function [and, through Analytic continuation, its *proxy* Dirichlet eta function]. Then our posited Generalized RH simply refers to this same conjecture on higher Genus 1, 2, 3, 4, 5... curves [and also, with "overlap", on lower Genus 0 curves]. Proposed during the early 1960's by two British mathematicians Bryan John Birch and Peter Swinnerton-Dyer, Birch and Swinnerton-Dyer (BSD) conjecture refers to a famous conjecture on Analytic ranks of (self-dual) L-functions from Genus 1 curves known as elliptic curves. Then our posited Generalized BSD simply refers to this same conjecture on higher Genus 2, 3, 4, 5, 6... curves [and also on lower Genus 0 curves and, with "overlap", on Genus 1 curves].

Widely studied diverse L-functions [e.g. having to be entire with poles on edge of $0 < \Re(s) < 1$-Critical strip or in other locations] are those arising from arithmetic objects such as elliptic and higher-genus curves, holomorphic cusp or modular forms, Maass forms, number fields with their Hecke characters, Artin representations, Galois representations, and motives. Two characterizations of such L-functions are in terms of Dirichlet coefficients and spectral parameters. That every Galois representation arises from an automorphic representation is known as Modularity Conjecture. Sometimes an L-function may arise from > 1 source e.g. L-functions associated with elliptic curves are also associated with weight 2 cusp forms. A big goal of Langlands program ostensibly is to prove any degree d L-function is associated with an automorphic form on GL(d). Because of this representation theoretic genesis, one can associate an L-function not only to an automorphic representation but also to symmetric powers, or exterior powers of that representation, or to the tensor product of two representations (the Rankin-Selberg product of two L-functions).

Of relevance to (Analytically normalized) $\sigma = \frac{1}{2}$-Critical Line when referenced to positive $0 < t < +\infty$ range in complex variable $s = \sigma \pm it$, the LMFDB's Sign normalization is applicable to eligible L-functions having Z(t) plots of '*OUTPUTS*' as infinitely-many *Incompletely Predictable* nontrivial zeros (spectrum). Using the vast [albeit limited] catalogues in LMFDB website[5] for our observational study, we propose LMFDB's Sign normalization is ubiquitously satisfied by Genus 0, 1, 2, 3, 4, 5... curves e.g. Genus 1 elliptic curves over Number field K = Rational number $\mathbb{Q}$, real and imaginary quadratic fields; Genus 2 curves over $K = \mathbb{Q}$; etc. We confirm LMFDB's Sign normalization [$\equiv$ Z(t) positivity] will always, *by default*, be "standardized" on an individual case-by-case basis through arbitrarily applying this normalization in a correct manner [so that Z(t) > 0 for sufficiently small $t > 0$]. As further outlined in section 3, Analytic rank 0 Genus 0 curves of degree 1 and Analytic rank 3 Genus 1 curves of degree 2 have (respective) Pseudo-transitional curves [pre-bug-fixing]: non-elliptic curve Riemann zeta function/Dirichlet eta function and Elliptic curve 5077.a1.

One could adopt Selberg class $\mathcal{S}$ as the set of all Dirichlet series ('Generic L-functions') $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ that satisfy four Selberg class axioms [whereby it is often practical to regard Axioms I, II and III to be "essential", and Axiom IV to be "optional"]. As opposed to the very particular cuspidal automorphic representations of GL(n) by Langlands, this set $\mathcal{S}$ contains very general analytic axioms defined by Atle Selberg who conjectured its elements all satisfy (Generalized) Riemann hypothesis.

· Axiom I. Analyticity: $(s-1)^m F(s)$ is an entire function of finite order for some non-negative integer m.

· Axiom II. Functional equation: there is a function $\gamma_F(s)$ of form $\gamma_F(s) =$

$\epsilon Q^s \prod_{i=1}^{k} \Gamma(\lambda_i s + \mu_i)$ where $|\epsilon| = 1$, $Q > 0$, $\lambda_i > 0$, and $\text{Re}(\mu_i) \geq 0$ such that $\Lambda(s) = \gamma_F(s)F(s)$ satisfies $\Lambda(s) = \overline{\Lambda}(1-s)$ where $\overline{\Lambda}(s) = \overline{\Lambda(\overline{s})}$.

· Axiom III. Euler product: $a_1 = 1$, and $\log F(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ where $b_n = 0$ unless n is a positive power of a prime and $b_n \ll n^\theta$ for some $\theta < \frac{1}{2}$.

· Axiom IV. Ramanujan hypothesis: $a_n = O_\epsilon(n^\epsilon)$ for any fixed $\epsilon > 0$.

In [2], we encounter an attractively useful but *unavoidably* complex and *inherently quasi*-complete classification on universal 'Generic L-functions' as provided by Prof. David Farmer and his colleagues via dividing Analytic L-functions and $\mathbb{Q}$-automorphic L-functions into arithmetic type and algebraic type based on [extra] collection of axioms. Conjecturally, all four resulting sets of L-functions are equal arising from arithmetic objects of pure motives and geometric Galois representations.

Imperfect commonly accepted scheme on modern classification (taxonomy) is never a mutually exclusive classification system for *Living Things* = *Life* → *Domain* → *Kingdom* → *Phyllum* → *Class* → *Order* → *Family* → *Genus* → *Species*. It is strongly influenced by modern technology of e.g. Artificial Intelligence (AI) software, DNA sequencing, bioinformatics, databases, imaging, etc. Likewise for our primitive and arbitrary but insightful and practical (lineage) classification of *Scientific Knowledge* = *Science* → *Mathematics* → *Number theory*: Algebraic, Analytic or Geometric → *Genus curves*: having different polynomial-degree → *Generic L-functions*: dual and self-dual L-functions having different degree, Euler product and gamma factors in functional equations → *Analytic ranks*: even versus odd with Line symmetry versus Point symmetry in Z(t) plots of nontrivial zeros (spectrum) → *Sign normalization*: adopting the arbitrary decision to have Z(t) positivity → *Isogeny class over a field K*: e.g. elliptic curves over $\mathbb{Q}$ either have or have not rational isogeny, two elliptic curves are twists if an only if they have same j-invariant, etc.

Remark 2.1. Many of the L-functions we consider in this paper (including those associated to curves of Genus > 1) are not known to admit an Analytic continuation or satisfy a functional equation. To discuss nontrivial zeros on the Critical Line and in the Hardy Z-function; we therefore need to, at least, assume the Hasse-Weil conjecture.

Taking Remark 2.1 into full and perspective consideration; all correct and complete mathematical arguments in this paper are assumed to comply with two conditions below whereby the "Analytic rank 0" component is present in both conditions:

Condition 1. Generalized Riemann hypothesis (RH): All nontrivial zeros (spectrum) of Generic L-functions from Genus 0, 1, 2, 3, 4, 5... curves with Analytic rank 0, 1, 2, 3, 4, 5... lie on the $\sigma = \frac{1}{2}$-Critical Line or the Analytically normalized $\sigma = \frac{1}{2}$-Critical Line. The 'special case' (*simplest*) RH[7] refers to [Analytic rank 0] Genus 0 non-elliptic curve called Riemann zeta function/Dirichlet eta function.

Condition 2. Generalized Birch and Swinnerton-Dyer (BSD) conjecture: All Generic L-functions from Genus 0, 1, 2, 3, 4, 5... curves satisfy Algebraic (Mordell-Weil) rank = Analytic rank [for even Analytic rank 0, 2, 4, 6, 8, 10... and odd Analytic rank 1, 3, 5, 7, 9, 11...]. The 'special case' (*simplest*) BSD conjecture refers to Genus 1 elliptic curves; expressed as *weak form* and *strong form* of BSD conjecture (see Remark 2.2).

Analogy for (Generalized) Riemann hypothesis: Let $\delta = \frac{1}{\infty}$ [an infinitesimal small number value], Geometrical 0-dimensional $\sigma = \frac{1}{2}$-Origin point $\equiv$ Mathematical 1-dimensional $\sigma = \frac{1}{2}$-Critical Line, and Origin point intercept $\equiv$ nontrivial zeros. Always having Origin point intercept $\Leftrightarrow \sin x = \cos(Ax - \frac{C\pi}{2})$ uniquely when $C = 1$. Never having Origin point intercept $\Leftrightarrow \sin x \neq \cos(Ax - \frac{C\pi}{2})$ non-uniquely when $C = 1 \pm \delta$. Assigned values for A is "inconsequential" in the sense solitary $A = 1$ value $\Longrightarrow$ 'special case' Riemann hypothesis [involving Genus 0 curve], and multiple $A \neq 1$ values $\Longrightarrow$ Generalized Riemann hypothesis [involving Genus 1, 2, 3, 4, 5... curves].

Under Generalized Riemann hypothesis, nontrivial zeros [as actual $\mathbb{C}$ s-values] are conventionally denoted by $\mathbb{R}$ t-values in $0 < t < +\infty$ range, and lie on Critical Line $\Re(s) = \frac{1}{2}$ (in Analytic normalization). The lowest nontrivial zero of an L-function $L(s)$ is the least $t > 0$ for which $L(\frac{1}{2} + it) = 0$. Even when $L(\frac{1}{2}) = 0$, the lowest nontrivial zero is by "traditional" definition a positive t-valued real number. As functions of complex variable s, L-functions for elliptic curves are denoted by $L(E, s)$ or $L_E(s)$, with these symbols often used interchangeably. *They have Analytic rank of zero integer value [whereby $L(1) \neq 0$ and $t \neq 0$ for first nontrivial zero] or non-zero integer values [whereby $L(1) = 0$ and $t = 0$ for first nontrivial zero].* Analytic rank = 0 $\Longrightarrow$ associated L-functions for elliptic [and non-elliptic] curves NEVER have first nontrivial zero given by ($\mathbb{R}$-valued) variable $t = 0$. Analytic rank ≥ 1 [viz, 1, 2, 3, 4, 5... up to an arbitrarily large number value] $\Longrightarrow$ associated L-functions for elliptic [and non-elliptic] curves ALWAYS have first nontrivial zero given by ($\mathbb{R}$-valued) variable $t = 0$.

In (Generalized) BSD conjecture, Generic L-functions and associated modular forms are usefully regarded as types of infinite series. The 2001 modularity theorem states that elliptic curves, with their $L_E(s)$, over $\mathbb{Q}$ are uniquely related to [*weight 2 for $\Gamma_0(N)$ classical*] modular form in a particular way. The rank of a number field K is size of any set of fundamental units of K. It is equal to $r = r_1 + r_2 - 1$ where r_1 is number of real embeddings of K into $\mathbb{C}$ and $2r_2$ is number of complex embeddings of K into $\mathbb{C}$. The analytic rank of an abelian variety is analytic rank of its L-function $L(A, s)$. The analytic rank of a curve is analytic rank of its Jacobian. The weak form of BSD conjecture $\Longrightarrow$ Analytic rank = Rank of Mordell-Weil group of abelian variety. Analytic ranks are always computed under assumption that $L(A, s)$ satisfies Hasse-Weil conjecture [they are not necessarily well-defined otherwise]. When A is defined over $\mathbb{Q}$, parity of analytic rank is always compatible with sign of functional equation. In general, analytic ranks stored in LMFDB are only upper bounds on true analytic rank [they could be incorrect if $L(A, s)$ has a zero very close to but not on the central point]. For the abelian varieties over $\mathbb{Q}$ of analytic rank < 2 this upper bound is necessarily tight, due to parity. The rank of an elliptic curve E defined over a number field K is rank of its Mordell-Weil group $E(K)$.

Mordell-Weil theorem states that the set of rational points on an abelian variety over a number field forms a finitely generated abelian group, hence isomorphic to a group of form $T \oplus \mathbb{Z}^r$ where T is a finite torsion group. The integer $r \geq 0$ is Mordell-Weil rank of abelian variety. Phrased in another way: This theorem says that $E(K)$ is a finitely-generated abelian group, hence $E(K) \cong E(K)_{\mathrm{tor}} \times \mathbb{Z}^r$ where $E(K)_{\mathrm{tor}}$ is finite torsion subgroup of $E(K)$, and $r \geq 0$ is the rank. Rank is an isogeny invariant: all curves in an isogeny class have the same rank.

A p-adic field (or local number field) is a finite extension of $\mathbb{Q}_p$, equivalently, a nonarchimedean local field of characteristic zero. A p-group is a group whose order is a power of a prime p. A result of Higman and Sims shows that the number of groups of order p^k is $p^{(2/27+o(1))k^3}$, and this can be combined with a result of Pyber to show that, asymptotically, 100% of groups are p-groups. For p-groups, the rank can be computed by taking the $\mathbb{F}_p$-dimension of the quotient by the Frattini subgroup. Let $A/\mathbb{F}_q$ be an abelian variety where $q = p^r$. The p-rank of an abelian variety is the dimension of the geometric p-torsion as a $\mathbb{F}_p$-vector space: $p\text{-rank}(A) = \dim_{\mathbb{F}_p}(A(\overline{\mathbb{F}}_p)[p])$. The p-rank is at most the dimension of A, with equality if and only if A is ordinary; the difference between the two is the p-rank deficit of A.

Remark 2.2. Formal statements on Birch and Swinnerton-Dyer conjecture: The central value of an L-function is its value at central point of Critical Strip. The central point of an L-function is the point on real axis of Critical Line. Equivalently, it is the fixed point of functional equation. In its Arithmetic normalization, an L-function $L(s)$ of weight w has its central value at $s = \dfrac{w+1}{2}$ and functional equation relates s to $1+w-s$. For L-functions defined by an Euler product $\prod_p L_p(s)^{-1}$ where coefficients of L_p are algebraic integers, this is the usual normalization implied by definition. The Analytic normalization of an L-function is defined by $L_{an}(s) := L(s + \dfrac{w}{2})$, where $L(s)$ is L-function in its arithmetic normalization. This moves the central value to $s = \dfrac{1}{2}$, and the functional equation of $L_{an}(s)$ relates s to $1 - s$.

Rodriguez-Villegas and Zagier[6] have proven a formula, conjectured by Gross and Zagier[3], for central value of $L(s, \chi^{2n-1})$, namely $L(\dfrac{1}{2}, \chi^{2n-1}) = 2\dfrac{(2\pi\sqrt{7})^n \Omega^{2n-1} A(n)}{(n-1)!}$ where $\Omega = \dfrac{\Gamma(\frac{1}{7})\Gamma(\frac{2}{7})\Gamma(\frac{4}{7})}{4\pi^2}$. By the functional equation $A(n) = 0$ whenever n is even. For odd n Gross and Zagier conjectured that $A(n)$ is a square [and provide tabulated values using their notation]. Rodriguez-Villegas and Zagier then prove that $A(n) = B(n)^2$ where $B(1) = \dfrac{1}{2}$ and $B(n)$ is an integer for $n > 1$; and that $A(n)$ is given by a remarkable recursion formula [not stated in this paper]. The accompanying incredible [derived] result of "for odd n, $B(n) \equiv -n \mod 4$", in one fell swoop, proves the non-vanishing of $L(\dfrac{1}{2}, \chi^{2n-1})$ for all odd n.

BSD conjecture relates the order of vanishing (or analytic rank) and the leading coefficient of the L-function associated to an elliptic curve E defined over a number field K at central point $s = 1$ to certain arithmetic data, the BSD invariants of

E. It is usually stated as two forms. (1) The *weak* form of BSD conjecture states just that the analytic rank r_{an} [that is, the order of vanishing of $L(E,s)$ at $s = 1$], is equal to the rank r of E/K. (2) The *strong* form of BSD conjecture states also that the leading coefficient of the L-function is given by the formula $\frac{1}{r!}L^{(r)}(E,1) = |d_K|^{1/2} \cdot \frac{\#\text{Ш}(E/K)\cdot\Omega(E/K)\cdot\text{Reg}(E/K)\cdot\prod_{\mathfrak{p}} c_{\mathfrak{p}}}{\#E(K)^2_{\text{tor}}}$. The quantities appearing in this formula are: d_K is discriminant of K; r is rank of $E(K)$; $\text{Ш}(E/K)$ is Tate-Shafarevich group of E/K; $\text{Reg}(E/K)$ is regulator of E/K; $\Omega(E/K)$ is global period of E/K; $c_{\mathfrak{p}}$ is Tamagawa number of E at each prime $\mathfrak{p}$ of K; $E(K)_{\text{tor}}$ is torsion order of $E(K)$.

For elliptic curves over $\mathbb{Q}$, a natural normalization for its L-function is the one that yields a functional equation $s \leftrightarrow 2 - s$. As stated above, this is known as arithmetic normalization, because Dirichlet coefficients are rational integers. We emphasize here that arithmetic normalization is being used by writing L-function as $L(E,s)$. In this notation, the central point is at $s = 1$. The "Special value" in LMFDB is the first non-zero value among $L(E,1)$, $L'(E,1)$, $L''(E,1)$, $L'''(E,1)$, $L''''(E,1)$, $L'''''(E,1)$,... *as (correspondingly) listed for Analytic rank 0, 1, 2, 3, 4, 5...* elliptic curves.

Let $A/\mathbb{F}_q$ be an abelian variety of dimension g defined over a finite field. Its L-polynomial is the polynomial $P(A/\mathbb{F}_q,t) = \det(1 - tF_q|H^1((A_{\overline{\mathbb{F}}_q})_{et},\mathbb{Q}_l))$, where F_q is the inverse of Frobenius acting on cohomology. This is a polynomial of degree $2g$ with integer coefficients. By a theorem of Weil, the complex roots of this polynomial all have norm $1/\sqrt{q}$; this means that there are only finitely many L-polynomials for any fixed pair (q,g). The L-polynomial of A is the reverse of Weil polynomial. Let $K = F_q$ be the finite field with q elements and E an elliptic curve defined over K. By Hasse's theorem on elliptic curves, the precise number of rational points $\#E(K)$ of E; will comply with inequality $|\#E(K) - (q+1)| \leq 2\sqrt{q}$. Implicit in the strong form of BSD conjecture is that the Tate-Sharafevich group $\text{Ш}(E/K)$ is finite. There is a similar conjecture for abelian varieties over number fields.

3 Pseudo-transitional curves: Genus 0 Riemann zeta function and Genus 1 elliptic 5077.a1

Preliminary note: Mathematical arguments in this section are [falsely] true to the extent if there was [incorrectly] "never a bug in the code for computing Z(t) plots in LMFDB website, whereby Z(t) negativity do exist in some of Z(t) plots irrespective of the LMFDB's stated convention to follow Z(t) positivity [≡ Sign normalization]".

In reference to Z(t) plots of nontrivial zeros (spectrum) as '*OUTPUTS*' from L-functions: Analytic rank 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11... have corresponding Sign +1, +1, +1, −1, +1, +1, +1, −1, +1, +1, +1, −1... with our [incorrectly] derived **Sign normalization** here being (conjecturally) ONLY satisfied by Genus 1 elliptic curves over $\mathbb{Q}$. However by conforming with the [incorrect] *liberalized* Sign normalization, we cautiously devise the following conditions:

[#1.] We expect all even Analytic rank 0, 2, 4, 6, 8, 10... Genus 0, 1, 2, 3, 4, 5... curves to always manifest Z(t) positivity; viz, having Sign +1, +1, +1, +1, +1....

[#2.] We expect all odd Analytic rank 1, 3, 5, 7, 9, 11... Genus 0, 1, 2, 3, 4, 5... curves to always manifest alternating Z(t) positivity and Z(t) negativity; viz, having Sign +1, −1, +1, −1, +1....

Denote r = Analytic rank. Then our [incorrect] Sign normalization is [falsely] represented by $(1)^{r-1}$ for even r with $\epsilon = 1$ and resulting in +1; and by $(i)^{r-1}$ for odd r with $\epsilon = i$ [that satisfies $(r-1)^{th}$ "Root of Unity"] resulting in ±1. Intuitively, one anticipate Sign changes to occur exactly when $r \equiv 1, 2 \pmod 4$ but this is not true: [I] For even $r = 0, 2, 4, 6, 8, 10...$; $1^{r-1} = (1)^{-1}, (1)^1, (1)^3, (1)^5, (1)^7...$ = same +1 sign [of +1, +1, +1, +1, +1,...]. c.f. [II] For odd $r = 1, 3, 5, 7, 9, 11...$; $i^{r-1} = (i)^0, (i)^2, (i)^4, (i)^6, (i)^8...$ = alternating ±1 sign [of +1, −1, +1, −1, +1,...]. Combined signs = +1, +1, +1, −1, +1, +1, +1, −1, +1, +1, +1, −1,... for r = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.... NOTE: In Z(t) plots: Number of nontrivial zeros with '0' value of {0, 1, 2, 3, 4, 5...} = r of {0, 1, 2, 3, 4, 5...} ∝ width of $Z(t) = 0$ value [which is of equal length to the $-ve$ left and $+ve$ right of Origin point in self-dual L-functions].

All (even) Analytic rank 0 Genus 1 elliptic curves manifest Z(t) positivity without exception. But (even) Analytic rank 0 Genus 0 non-elliptic curve Riemann zeta function / Dirichlet eta function manifest Z(t) negativity [pre-bug-fixing], and is called a *Pseudo*-transitional curve (see Figure 6).

[Definition: An elliptic curve is *semistable* if it has multiplicative reduction at every "bad" prime.] All (odd) Analytic rank 3 Genus 1 elliptic curves manifest Z(t) negativity [pre-bug-fixing] but we observe an exception for *Pseudo*-transitional curve (see Figure 2) of semistable elliptic curve 5077.a1 $\{y^2 + y = x^3 - 7x + 6\}$ that [instead] manifests Z(t) positivity, has smallest conductor 5077 amongst elliptic curves over $\mathbb{Q}$ of Analytic rank 3, 36 Integral points, one "bad" prime at $p = 5077 \equiv F_pT = 1 + O(T)$, Mordell-Weil group structure $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, Infinite order Mordell-Weil generators P = (1, 0), (2, 0), (0, 2), Endomorphism ring $\mathbb{Z}$ that is NOT larger than $\mathbb{Z} \Longrightarrow$ DO NOT have Complex Multiplication. With associated L-function of degree 2, elliptic curve 5077.a1 has no rational isogenies and its isogeny class 5077.a consists of this elliptic curve only. This elliptic curve is its own minimal quadratic twist.

History of Gauss elliptic curve 5077.a1: In 1985, Buhler, Gross and Zagier used the celebrated Gross-Zagier Theorem on heights of Heegner points (see [4]) to prove L-function of this curve has a zero of order 3 at its critical point $s = 1$, thus establishing first part of BSD conjecture for this curve (see [1]). This was first time that BSD had been established for any elliptic curve of rank 3. To this day, it is not possible, even in principle, to establish BSD for any curve of rank ≥ 4 since there is no known method for rigorously establishing the value of Analytic rank when it is > 3. *We anticipate future Z(t) plots of nontrivial zeros (spectrum) for (odd) Analytic rank 5, 7, 9, 11... elliptic curves over $\mathbb{Q}$, when available, should definitively (dis)prove our [incorrect] Sign normalization*. Via Goldfeld's method, which required use of an L-function of Analytic rank at least 3, elliptic curve 5077.a1 also found an application in context of obtaining explicit lower bounds for the class numbers of imaginary quadratic fields. This solved Gauss's Class Number Problem first posed by Gauss in 1801 in his book Disquisitiones Arithmeticae (Section V, Articles 303 and 304).

Elliptic curves over Number field $\mathbb{Q}$ are classical 2-variable "mixed"-polynomial-degree 3 Genus 1 curves having degree 2 L-functions of Analytic rank 0, 1, 2, 3, 4, 5.... The Number field $\mathbb{Q}$ represented by Normalized defining polynomial $\pm x$ [or simply x]

is the "simplest" 1-variable polynomial-degree 1 Genus 0 (non-elliptic) curve having L-function of Analytic rank 0. This curve is represented by Analytically continued (self-dual) L-function [LMFDB Number field 1.1.1.1: $\mathbb{Q}$] of Dirichlet eta function $\eta(s)$, which is derived from Riemann zeta function $\zeta(s)$; and DO NOT respect Z(t) positivity under our [incorrect] Sign normalization [see Figure 6]. $\zeta(s)$ is the prototypical L-function, the only L-function of degree 1 and conductor 1, and (conjecturally) the only primitive L-function with a pole. It is a self-dual L-function that originated from Dirichlet character $\chi_1(1, \cdot)$ having even parity. Its unique pole is located at $s = 1$. The first nontrivial zero of Analytic rank 0 $\eta(s)$ [*proxy* function for $\zeta(s)$], at *height* $\approx$*14.134, is higher than that of any other algebraic L-function.* Then *any other algebraic L-function [with Analytic rank 0, 1, 2, 3, 4, 5...] will comparatively have more frequent nontrivial zeros that first occur at a relatively lower height [for L-functions with Analytic rank 0], up to and including (lowest) height of 0 [for L-functions with Analytic rank 1 or higher]*. *As an example of Analytic rank 0 Genus 0 curves of degree 1 respecting Z(t) positivity without exception: LMFDB Analytic rank 0 L-function 1-5-5.4-r0-0-0 Genus 0 curve of degree 1 that originated from Dirichlet character $\chi_5(4, \cdot)$ clearly manifests Z(t) positivity. It has functional equation $\Lambda(s) = 5^{s/2}\Gamma_{\mathbb{R}}(s)L(s) = \Lambda(1-s)$. After Riemann zeta function, the analytic conductor in this self-dual L-function (of even parity with Sign: +1) is the smallest among L-functions of degree 1.*

4 Functional equations of Generic L-functions and their associated Gamma factors

An (analytic) L-function is a Dirichlet series that has an Euler product and satisfies a certain type of functional equation, and allows analytic continuation. Then this L-function is also called Dirichlet L-function, associated with its Dirichlet L-series, which can be meromorphically continued to the complex plane, have an Euler product $L(s, \chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}$, and satisfy a functional equation of the form $\Lambda(s, \chi) = q^{\frac{s}{2}}\Gamma_{\mathbb{R}}(s)L(s, \chi) = \varepsilon_\chi \overline{\Lambda}(1-s)$, where q is the conductor of χ.

The two complex functions $\Gamma_{\mathbb{R}}(s) := \pi^{-\frac{s}{2}}\Gamma(\frac{S}{2})$ and $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$ that appear in functional equation of an L-function are known as gamma factors. Here $\Gamma(s) := \int_0^\infty e^{-t}t^{s-1}dt$ is Euler's gamma function, with poles located at $s = 0, -1, -2, -3, -4, -5....$ The gamma factors satisfy $\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)$ and is also viewed as "missing" factors of Euler product of an L-function corresponding to (real or complex) archimedean places. Completely Predictable *trivial zeros* are zeros of an L-function $L(s)$ that occur at poles of its gamma factors. An L-function $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is called arithmetic if its Dirichlet coefficients a_n are algebraic numbers. Thus for arithmetic L-functions, the poles are at certain negative integers.

All known analytic L-functions have functional equations that can be written in the form [whereby $\Lambda(s)$ is now called the *completed L-function*] $\Lambda(s) := N^{\frac{s}{2}} \prod_{j=1}^{J} \Gamma_{\mathbb{R}}(s+$

$\mu_j) \prod_{k=1}^{K} \Gamma_{\mathbb{C}}(s+\nu_k) \cdot L(s) = \varepsilon\overline{\Lambda}(1-s)$ where N is an integer, $\Gamma_{\mathbb{R}}$ and $\Gamma_{\mathbb{C}}$ are defined in terms of the Γ-function, $\mathrm{Re}(\mu_j) = 0$ or 1 (assuming Selberg's eigenvalue conjecture), and $\mathrm{Re}(\nu_k)$ is a positive integer or half-integer, $\sum \mu_j + 2\sum \nu_k$ is real, and ε *is the Sign of functional equation.* With these restrictions on spectral parameters [viz, the numbers μ_j and ν_k that appear as shifts in gamma factors $\Gamma_{\mathbb{R}}$ and $\Gamma_{\mathbb{C}}$ (respectively)], the data in the functional equation is specified uniquely. *The integer* $d = J+2K$ *is the degree of the L-function. The integer* N *is the conductor (or level) of the L-function. The pair* $[J, K]$ *is the signature of the L-function.* **The Sign ε, as complex number, appears as the fourth component of the Selberg data of $L(s)$; viz, $(d, N, (\mu_1, \ldots, \mu_J : \nu_1, \ldots, \nu_K), \varepsilon)$. If all of the coefficients of the Dirichlet series defining $L(s)$ are real, then necessarily $\varepsilon = \pm 1$. If the coefficients are real and $\varepsilon = -1$, then $L(\frac{1}{2}) = 0$**.

The axioms of the Selberg class are less restrictive than given above. Note that the functional equation above has the central point at $s = \frac{1}{2}$, and relates $s \leftrightarrow 1-s$. As already stated previously, for many L-functions there is another normalization which is natural. The corresponding functional equation relates $s \leftrightarrow w+1-s$ for some positive integer w, called the motivic weight of the L-function. The central point is at $s = \frac{(w+1)}{2}$, and the arithmetically normalized Dirichlet coefficients $a_n n^{w/2}$ are algebraic integers.

The gamma factor $\Gamma_{\mathbb{R}}(s)$ in functional equation for even Analytic rank 0 polynomial-degree 1 Genus 0 curve with L-function of degree 1 $\eta(s)$ / $\zeta(s)$ over Number field $K = \mathbb{Q}$ as given by Normalized defining polynomial $\pm x$ / x [of polynomial-degree 1] is $\Lambda(s) = \Gamma_{\mathbb{R}}(s)L(s) = \Lambda(1-s)$

An L-function $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is called arithmetic if its Dirichlet coefficients a_n are algebraic numbers. A rational L-function $L(s)$ is an arithmetic L-function with coefficient field $\mathbb{Q}$; equivalently, its Euler product in the arithmetic normalization can be written as a product over rational primes $L(s) = \prod_p L_p(p^{-s})^{-1}$ with $L_p \in \mathbb{Z}[T]$.

The gamma factor $\Gamma_{\mathbb{R}}(s)$ is present in functional equations for Degree 3 Conductor 1 Sign 1 Genus 0 curve via (even) Analytic rank 0 dual L-functions 3-1-1.1-r0e3-m0.24m25.28p25.52-0 AND its "counterpart" related object 3/1/1.1/r0e3/p0.24p25.28m25.52/0 whereby both of these dual L-functions originated from e.g. GL3 Maass form that are NOT self-dual / NOT rational / NOT arithmetic. Their respective functional equations consist of $\Lambda(s) = \Gamma_{\mathbb{R}}(s-25.2i)\Gamma_{\mathbb{R}}(s-0.243i)\Gamma_{\mathbb{R}}(s+25.5i)L(s) = \overline{\Lambda}(1-s)$ AND $\Lambda(s) = \Gamma_{\mathbb{R}}(s+25.2i)\Gamma_{\mathbb{R}}(s+0.243i)\Gamma_{\mathbb{R}}(s-25.5i)L(s) = \overline{\Lambda}(1-s)$. The t-valued [infinitely-many] nontrivial zeros (spectrum) for them[5] as transcendental (irrational) numbers are ...-22.812865, -19.882193, -17.687387, -16.327596, -14.304332, -12.718105, -9.820639, -7.744307, -6.757323, -3.647261, 2.721292, 5.404222, 8.838084, 10.034902, 11.938378, 13.965832, 16.042992, 18.823934, 19.919083, 22.010794... AND [with "reverse" pattern] ...-22.010794, -19.919083, -18.823934, -16.042992, -13.965832, -11.938378, -10.034902, -8.838084, -5.404222, -2.721292, 3.647261,

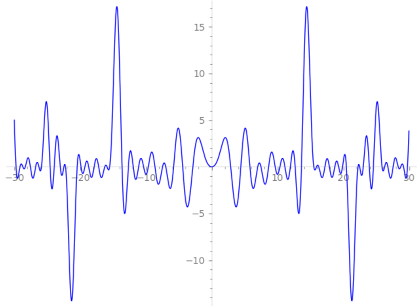

Fig. 11 Graph of Z-function along [Analytically normalized] $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ depicting UNIQUE nontrivial zeros spectrum for Elliptic curve LMFDB label 389.a1 [with non-zero Analytic rank of 2]. Note **Line Symmetry of vertical y-axis and trajectory intersecting Origin point. Integral points are (−2, 0), (−2, −1), (−1, 1), (−1, −2), (0, 0), (0, −1), (1, 0), (1, −1), (3, 5), (3, −6), (4, 8), (4, −9), (6, 15), (6, −16), (39, 246), (39, −247), (133, 1539), (133, −1540), (188, 2584), (188, −2585).

6.757323, 7.744307, 9.820639, 12.718105, 14.304332, 16.327596, 17.687387, 19.882193, 22.812865... resulting in individual Z(t) plots having Z(t) positivity but manifesting neither Line Symmetry nor Point Symmetry. However, they manifest the [combined] Line Symmetry of vertical y-axis.

The gamma factor $\Gamma_{\mathbb{C}}(s)$ in [Analytically normalized] functional equations for polynomial-degree 3 Genus 1 elliptic curves with self-dual L-functions of degree 2 over Number field $K = \mathbb{Q}$ e.g.:
even Analytic rank 2 E 389.a1 $\{y^2 + y = x^3 + x^2 - 2x\}$ [see Figure 11] is $\Lambda(s) = 389^{s/2}\Gamma_{\mathbb{C}}(s + 1/2)L(s) = \Lambda(2 - s)$
odd Analytic rank 3 E 5077.a1 $\{y^2 + y = x^3 - 7x + 6\}$ [see Figure 2] is $\Lambda(s) = 5077^{s/2}\Gamma_{\mathbb{C}}(s + 1/2)L(s) = -\Lambda(2 - s)$
odd Analytic rank 3 E 21858.a1 $\{y^2 + xy = x^3 + x^2 - 32x + 60\}$ [see Figure 3] is $\Lambda(s) = 21858^{s/2}\Gamma_{\mathbb{C}}(s + 1/2)L(s) = -\Lambda(2 - s)$

The gamma factor $\Gamma_{\mathbb{C}}(s)$ in [Analytically normalized] functional equation for odd Analytic rank 1 polynomial-degree 3 Genus 1 E 14.1-b6 $\{y^2 + xy + y = x^3 - 2731x - 55146\}$ with self-dual L-function of degree 4 over Real quadratic field $K = \mathbb{Q}(\sqrt{7})$ is $\Lambda(s) = 10976^{s/2}\Gamma_{\mathbb{C}}(s + 1/2)^2 L(s) = -\Lambda(2 - s)$

The gamma factor $\Gamma_{\mathbb{C}}(s)$ in [Analytically normalized] functional equation for odd Analytic rank 3 polynomial-degree 3 Genus 1 E 44563.1-a1 $\{y^2 + axy + ay = x^3 - x^2 + (-2a + 1)\,x\}$ with self-dual L-function of degree 4 over Imaginary quadratic field $K = \mathbb{Q}(\sqrt{-3})$ is $\Lambda(s) = 401067^{s/2}\Gamma_{\mathbb{C}}(s + 1/2)^2 L(s) = -\Lambda(2 - s)$

The gamma factor $\Gamma_{\mathbb{C}}(s)$ in [Analytically normalized] functional equation for odd Analytic rank 3 polynomial-degree 4 Genus 2 curve 35131.a.35131.1 $\{y^2 + x^3 y = x^4 - 3x^3 + 4x^2 - 3x + 1\}$ with self-dual L-function of degree 4 over Number field $K = \mathbb{Q}$ is $\Lambda(s) = 35131^{s/2}\Gamma_{\mathbb{C}}(s + 1/2)^2 L(s) = -\Lambda(2 - s)$

Remark 4.1. We document some analyzed Genus 0, 1 and 2 curves with minimal Weierstrass equations. Akin to satisfying unitary pairing condition at prime p e.g. $\Gamma_{\mathbb{R}}(s - 0.2)\Gamma_{\mathbb{R}}(s + 0.2)\Gamma_{\mathbb{R}}(s)^3\Gamma_{\mathbb{R}}(s + 0.9)\Gamma_{\mathbb{R}}(s + 1.1) \times \Gamma_{\mathbb{C}}(s + 0.7)\Gamma_{\mathbb{C}}(s + 1.3)^2\Gamma_{\mathbb{C}}(s + 1.7)\Gamma_{\mathbb{C}}(s + 7)$ and $\Gamma_{\mathbb{R}}(s - 0.2 + 3i)\Gamma_{\mathbb{R}}(s + 0.2 + 3i)\Gamma_{\mathbb{R}}(s + 1)\Gamma_{\mathbb{R}}(s + 1 - 8i) \times \Gamma_{\mathbb{C}}(s +$

0.7)$\Gamma_{\mathbb{C}}(s+1.3)\Gamma_{\mathbb{C}}(s+1.3-7i)\Gamma_{\mathbb{C}}(s+1.7-7i)$: $\Gamma_{\mathbb{R}}(s) \Leftrightarrow$ "good" primes and $\Gamma_{\mathbb{C}}(s) \Leftrightarrow$ "bad" primes[2]. Involving gamma factor $\Gamma_{\mathbb{C}}(s)$, *all Analytic rank 0, 1, 2, 3, 4, 5... e.g. the polynomial-degree 3 Genus 1 elliptic curves with L-functions of degree 2 over* $\mathbb{Q}$ *satisfy Sign normalization* which will likely depend on even-versus-odd Analytic ranks, (BSD) Invariants, degree of L-function, Special value, etc.

5 Sign normalization on Graphs of Z-function as the Z(t) plots of Nontrivial zeros

We adopt the traditional anti-clockwise notation of Quadrant (Q) I, II, III and IV. We deduce our Q I Z(t) positivity / Q IV Z(t) negativity in Graphs of Z-function can be further shortened, without ambiguity, to Z(t) positivity / Z(t) negativity for range $0 < t < +\infty$. *The solutions to* $\sqrt{x}$ *become "larger values" for x sufficiently close to 0 e.g.* $\sqrt{9} = 3$, $\sqrt{4} = 2$, $\sqrt{0.002}$ = *"larger value" 0.0447213...,* $\sqrt{0.0002}$ = *"larger value" 0.014142..., etc.* Although this statement is true *per se*, it is not the reason for performing LMFDB's Sign normalization on Z(t) plots (see Axiom 1).

Analytic rank r of elliptic curves E consist of even r = 0, 2, 4, 6, 8, 10... [with '$+ve$ even' Line symmetry, $\varepsilon = 1$ and $\varepsilon^{\frac{1}{2}} = +1$ or -1 that can arbitrarily be chosen to display Z(t) plots in two reciprocal manners "+1 Z(t)" or "-1 Z(t)"], and odd r = 1, 3, 5, 7, 9... [with '$-ve$ odd' Point symmetry, $\varepsilon = -1$ and $\varepsilon^{\frac{1}{2}} = +i$ or $-i$ that can arbitrarily be chosen to display Z(t) plots in two reciprocal manners "$+i$ Z(t)" or "$-i$ Z(t)"]. Note: $r = 0$ for (non-elliptic) Riemann zeta function $\zeta(s)$ / Dirichlet eta function $\eta(s)$. Polar graphs e.g. all Analytically normalized $\sigma = \frac{1}{2}$-Critical Line Polar graphs of E, Polar graph Figure 12 on $\sigma = \frac{1}{2}$-Critical Line for (non-elliptic) $\zeta(s)$ / $\eta(s)$, etc manifest features of even functions [when having even r] and odd functions [when having odd r]. Caveat: The horizontal x-axis and vertical y-axis are arbitrarily chosen such that Line Symmetry is [dependently] the horizontal x-axis for Polar graphs having even r, but Point Symmetry is [independently] the Origin point for Polar graphs having odd r. Cf: Line Symmetry is [dependently] the vertical y-axis for Graphs of Z-functions having even r, but Point Symmetry is [independently] the Origin point for Graphs of Z-functions having odd r. Considering $0 < t < +\infty$ range in plotted trajectory of Polar graph or Graph of Z-function, let distance d = difference between P_1 (trajectory initially intersecting horizontal x-axis of Polar graph / vertical y-axis in Graph of Z-function) and P_2 (trajectory initially intersecting Origin point of Polar graph / Graph of Z-function). Then (i) $d = P_2 - P_1 \neq 0$ for $r = 0$ $\zeta(s)$ / $\eta(s)$ and for $r = 0$ E, and (ii) $d = P_2 - P_1 = 0$ for r = 1, 2, 3, 4, 5... E [with these findings being equally valid for $-\infty < t < 0$ range].

Axes definitions for Polar graph VERSUS Graph of Z-function. Complex variable $s = \sigma \pm it$ for range $-\infty < t < +\infty$. For complete validity, we notationally replace $\zeta(s)$ (having Convergence for $\sigma > 1$) with $\eta(s)$ (having Convergence for $\sigma > 0$) since nontrivial zeros only occur at $\sigma = \frac{1}{2}$-Critical Line [whereby for elliptic curves, this require Analytic normalization]. Polar graphs [e.g. represented by Figure 12, whereby

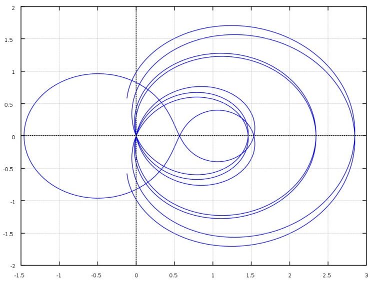

Fig. 12 OUTPUT at $\sigma = \frac{1}{2}$-Critical Line. Polar graph of $\zeta(\frac{1}{2} + \imath t)$ / $\eta(\frac{1}{2} + \imath t)$ plotted for real values of t between -30 and $+30$ [viz, for $s = \sigma \pm \imath t$ range]. Horizontal axis: $Re\{\eta(\frac{1}{2} + \imath t)\}$. Vertical axis: $Im\{\eta(\frac{1}{2} + \imath t)\}$. Origin intercept points $\equiv$ nontrivial zeros are present. There is manifested perfect Mirror symmetry about horizontal x-axis acting as Line Symmetry.

its 0-dimensional $\sigma = \dfrac{1}{2}$-Origin point $\equiv$ 1-dimensional $\sigma = \dfrac{1}{2}$-Critical Line]: Horizontal axis is $Re\{\eta(\dfrac{1}{2} \pm it)\}$. Vertical axis is $Im\{\eta(\dfrac{1}{2} \pm it)\}$. Graph of Z-function: Horizontal axis is variable t. Vertical axis is $Z(t)$. We use $Z(t) = \overline{\varepsilon}^{\frac{1}{2}} \dfrac{\gamma(\frac{1}{2} + it)}{|\gamma(\frac{1}{2} + it)|} L(\dfrac{1}{2} + it)$ [with $\sqrt{\epsilon}$; viz, with LMFDB's Sign normalization]. One could also use $Z(t) = \varepsilon \dfrac{\gamma(\frac{1}{2} + it)}{|\gamma(\frac{1}{2} + it)|} L(\dfrac{1}{2} + it)$ [without $\sqrt{\epsilon}$; viz, without LMFDB's Sign normalization].

Let $\delta = \dfrac{1}{\infty}$ [an infinitesimal small number value]. We select the square root that makes Z(δ) $+ve$ for very small $+ve$ δ. Viz, eventhough it is a completely arbitrary choice, we will always achieve [inevitable] standardization by choosing whichever square root makes Z(δ) $> 0 \equiv$ LMFDB's Sign normalization. Then this Sign normalization $\equiv$ resultant manifestation of Z(t) positivity.

Let r = Analytic rank. Which square root we take; viz, $\sqrt{-1} = +i$ or $-i$ for odd r and $\sqrt{+1} = +1$ or -1 for even r is exactly the one needed to make Z(δ) > 0. Example 1: Manifesting Z(t) positivity, the $r = 1$ self-dual L-function from semistable elliptic curve 37.a1 (see Figure 1) requires $\sqrt{\epsilon} = +i$. By way of note, this elliptic curve is of minimal conductor with positive rank. It is also a model for quotient of modular curve $X_0(37)$ by its Fricke involution w_{37}; this quotient is also denoted $X_0^+(37)$. This is the smallest prime $N \in \mathbb{N}$ such that $X_0(N)/\langle w_N \rangle$ is of positive genus. Example 2: Manifesting Z(t) positivity, the $r = 3$ self-dual L-function from semistable elliptic curve 5077.a1 (see Figure 2 and its famous history in section 3) requires $\sqrt{\epsilon} = -i$.

Recall the following: The Sign (root number) of functional equation of an analytic L-function is complex number ε that appears in the functional equation of $\Lambda(s) = \varepsilon\overline{\Lambda}(1-s)$. An L-function $L(s) = \displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is called self-dual if its Dirichlet coefficients

a_n are real. Thus self-dual L-functions with odd Analytic rank must have Sign (root number) -1, and with even Analytic rank must have Sign (root number) $+1$.

A character has odd/even parity if it is odd/even as a function. The dual of an L-function $L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is the complex conjugate $\bar{L}(s) = \sum_{n=1}^{\infty} \frac{\bar{a_n}}{n^s}$. A Dirichlet character $\chi\colon \mathbb{Z} \to \mathbb{C}$ is odd if $\chi(-1) = -1$ and even if $\chi(-1) = 1$. The L-function 1-5-5.2-r1-0-0, as an example of Genus 0 curve dual L-function of Analytic rank 0 degree 1 odd parity, originate from Dirichlet character $\chi_5(2, \cdot)$ [see Figure 7] with having its functional equation as $\Lambda(s) = 5^{s/2}\Gamma_{\mathbb{R}}(s+1)L(s) = (0.850 + 0.525i)\overline{\Lambda}(1-s)$. Here, the Sign (root number) of $0.850 + 0.525i$ for [NOT self-dual] L-function 1-5-5.2-r1-0-0 could be anything of modulus 1. In contrast: Respectively, the Analytic rank 0 degree 1 L-function 1-2e2-4.3-r1-0-0 having odd parity and 1-2e3-8.5-r0-0-0 having even parity as two examples of Genus 0 curve, originating from Dirichlet character $\chi_4(3, \cdot)$ and $\chi_8(5, \cdot)$, have functional equations $\Lambda(s) = 4^{s/2}\Gamma_{\mathbb{R}}(s+1)L(s) = \Lambda(1-s)$ and $\Lambda(s) = 8^{s/2}\Gamma_{\mathbb{R}}(s+1)L(s) = \Lambda(1-s)$. The Sign (root number) ϵ is $+1$ because both [even Analytic rank 0] L-functions are self-dual.

The nontrivial zeros, as denoted by $+ve$ $\mathbb{R}$ γ values, of an L-function $L(s)$ are complex numbers ρ for which $L(\rho) = L(\frac{1}{2} + i\gamma) = 0$. (Hardy or Riemann-Siegel) *Z-function* for Genus 0 curve Riemann zeta-function $\zeta(s)$ / Dirichlet eta function $\eta(s)$ is a real-valued function defined in terms of values of $\zeta(s)$ / $\eta(s)$ on Critical Line via formula $Z(t) := e^{i\theta(t)}\zeta\left(\frac{1}{2} + it\right)$ / $Z(t) := e^{i\theta(t)}\eta\left(\frac{1}{2} + it\right)$, where $\theta(t)$ is *Riemann-Siegel theta function* $\theta(t) := \arg\left(\Gamma\left(\frac{2it+1}{4}\right)\right) - \frac{\log\pi}{2}t$. There is a bijection between zeros t_0 of $Z(t)$ and zeros $\frac{1}{2} + it_0$ of $\zeta(s)$ / $\eta(s)$. Here, $\zeta(s) = \frac{\eta(s)}{\gamma} \equiv \eta(s) = \gamma \cdot \zeta(s)$ whereby $\gamma = (1 - 2^{1-s})$ is the proportionality factor.

Z-function of a general L-function is a smooth real-valued function of a real variable t such that $|Z(t)| = |L(\frac{1}{2} + it)|$. Specifically, if we write the completed L-function as $\Lambda(s) = \gamma(s)L(s)$, where $\Lambda(s)$ satisfies functional equation $\Lambda(s) = \varepsilon\overline{\Lambda}(1-s)$, then $Z(t)$ is defined by $Z(t) = \varepsilon^{\frac{1}{2}}\frac{\gamma(\frac{1}{2}+it)}{|\gamma(\frac{1}{2}+it)|}L(\frac{1}{2} + it)$. Reiterating: In portion $\epsilon^{\frac{1}{2}} = \sqrt{\epsilon}$, the square root is chosen so that $Z(t) > 0$ for sufficiently small $t > 0 \equiv$ *LMFDB's Sign normalization*. The multiset of zeros of [*perpetual oscillatory function*] $Z(t)$ matches that of $L(\frac{1}{2} + it)$ and $Z(t)$ changes sign [*for infinitely-many times*] at zeros of $L(\frac{1}{2} + it)$ of odd multiplicity.

Analogical concepts for LMFDB's Sign normalization: Recall the parity of (simple) polynomial functions to be EITHER $\pm$ even functions OR $\pm$ odd functions: [I] e.g. $y = \pm x^{0,2,4,6,8,10\ldots}$ being even functions with corresponding entire functions of $-\infty < x < +\infty$ range are located in Quadrant I and II when "y is a $+ve$ function" and in Quadrant III and IV when "y is a $-ve$ function". [II] e.g. $y = \pm x^{1,3,5,7,9,11\ldots}$ being odd functions with corresponding entire functions of $-\infty < x < +\infty$ range are located in Quadrant I and III when "y is a $+ve$ function" and in Quadrant II and IV when "y

is a $-ve$ function". Nomenclature: Let elliptic curve be denoted by E. Let y and its exponents be denoted by ±Z(t) and r. We analyze $0 < t < +\infty$ range utilizing the [so-called] "first sinusoidal wave" of plotted Z-function for E whereby we arbitrarily choose in a consistent *de-facto* manner +Z(t) in even r [viz, Q I Z(t) positivity], and +Z(t) in odd r [viz, Q I Z(t) positivity]. Our analogical equivalent approach to Sign normalization is valid despite Z(t) plots perpetually oscillating above/below horizontal t axis an infinite number of times after the "first sinusoidal wave".

Additionally through various Incompletely Predictable *complex interactions*, we intuitively expect frequency and complexity of nontrivial zeros (spectrum) and the integer N values of conductor (or level) in self-dual L-functions of elliptic curves to be empirically correlated with Analytic rank 0, 1, 2, 3, 4, 5....

6 Conclusions

Irrespective of L-function source and always with one UNIQUE set of nontrivial zeros per each L-function, the infinitely-many nontrivial zeros as Incompletely Predictable entities are ONLY located on (Analytically normalized) $\sigma = \frac{1}{2}$*-Critical Line.* With respecting Remark 2.1, this profound statement insightfully describe intractable open problem in Number theory of (Generalized) Riemann hypothesis. Graphs of Z-function on Genus 1 elliptic curves with non-zero Analytic rank 1, 2, 3, 4, 5... have trajectories that intersect Origin point. Graphs of Z-function on Genus 1 elliptic curves with Analytic rank 0 [viz, having zero independent basis point (with infinite order) which are associated with either finitely many or zero $E(\mathbb{Q})$ solutions] DO NOT have trajectories that intersect Origin point. *Ditto* for Graph of Z-function on Genus 0 (non-elliptic) Riemann zeta function / Dirichlet eta function with Analytic rank 0 [viz, it DOES NOT have trajectory that intersect Origin point]. *This implies "simplest version" of BSD conjecture to be true*; and simultaneously *implies "simplest version" of Riemann hypothesis to be true (with rigorous proof previously provided in [7]). Adopting the Z(t) positivity* in Graphs of Z-function as part of LMFDB's Sign normalization occurs for both odd and even Analytic rank elliptic curves. Geometrically studying non-trivial zeros (spectrum) using *Graphs of Z-function plots versus Polar graphs plots* to detect altered patterns, symmetry, frequency, etc promises to be a useful [*experimental*] method to characterize L-functions of Analytic rank 0, 1, 2, 3, 4, 5....

L-functions literally encode *arithmetic information* e.g. Riemann zeta function connects through values at $+ve$ even integers (and $-ve$ odd integers) to Bernoulli numbers, with appropriate generalization of this phenomenon obtained via p-adic L-functions, which describe certain Galois modules. The distribution of nontrivial zeros (spectrum), orders, and conductors, often manifesting as self-similarity or large fractal dimension, are theoretically connected to Chaos theory and Fractal geometry, random matrix theory and quantum chaos.

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