On the function $f(x) = x - \operatorname{arcsinh}(1 + \arcsin(x))$

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abstract

We give some properties of the function $f(x) = x - \operatorname{arcsinh}(1 + \arcsin(x))$

1. Introduction

Entry 1.

$$f(x) = x - \operatorname{arcsinh}(1 + \arcsin(x)) \ , -1 \le x \le 1$$

$$f(x) = \sum_{n=0}^{\infty} c(n)\, x^n$$

$$c(n) = \left\{ -\ln\left(1+\sqrt{2}\right), 1-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{8}, -\frac{5\sqrt{2}}{48}, \frac{13\sqrt{2}}{384}, -\frac{29\sqrt{2}}{768}, \frac{529\sqrt{2}}{46080}, -\frac{1747\sqrt{2}}{92160}, \ldots \right\}$$

Entry 2.

$$\frac{\pi}{12} = \cos(1)\sinh\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) + \sum_{n=1}^{\infty} (-1)^n c(n) \left(\sin(1)\cosh\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)\right)^n g(n)$$

$$g(n) = \sum_{k=0}^{\operatorname{floor}\left(\frac{n-1}{2}\right)} (-1)^k \binom{n}{2k+1} \left(\cot(1)\tanh\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)\right)^{2k+1}$$

Entry 3.

$$1 + \frac{\pi}{6} = \sinh\left(\frac{1}{2} - \sum_{n=0}^{\infty} c(n)\, 2^{-n}\right)$$

$$1 - \frac{\pi}{6} = \sinh\left(-\frac{1}{2} - \sum_{n=0}^{\infty} c(n)(-1)^n 2^{-n}\right)$$

Entry 4.

$$\frac{\pi}{12} = \sinh\left(\frac{\left(2-\sqrt{3}\right)\sqrt{6+\sqrt{3}}}{2}\right) - \sum_{n=0}^{\infty} c(2n+1)(-1)^n \cdot \left(\sinh\left(\frac{\left(2-\sqrt{3}\right)\sqrt{6+\sqrt{3}}}{2}\right)\right)^{2n+1}$$

Recall that:

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right)$$

2. Graphics

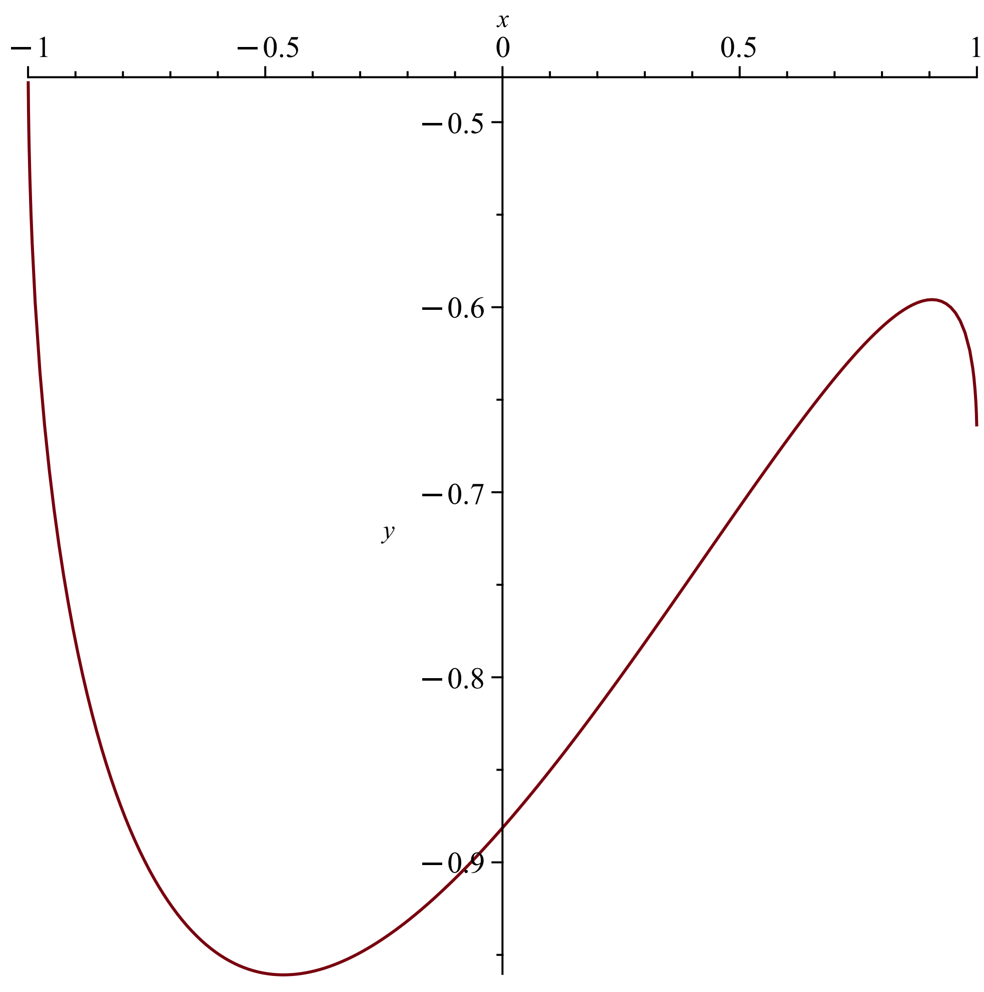

Fig. 1: $f(x) = x - \operatorname{arcsinh}(1 + \arcsin(x))$, $-1 \leq x \leq 1$

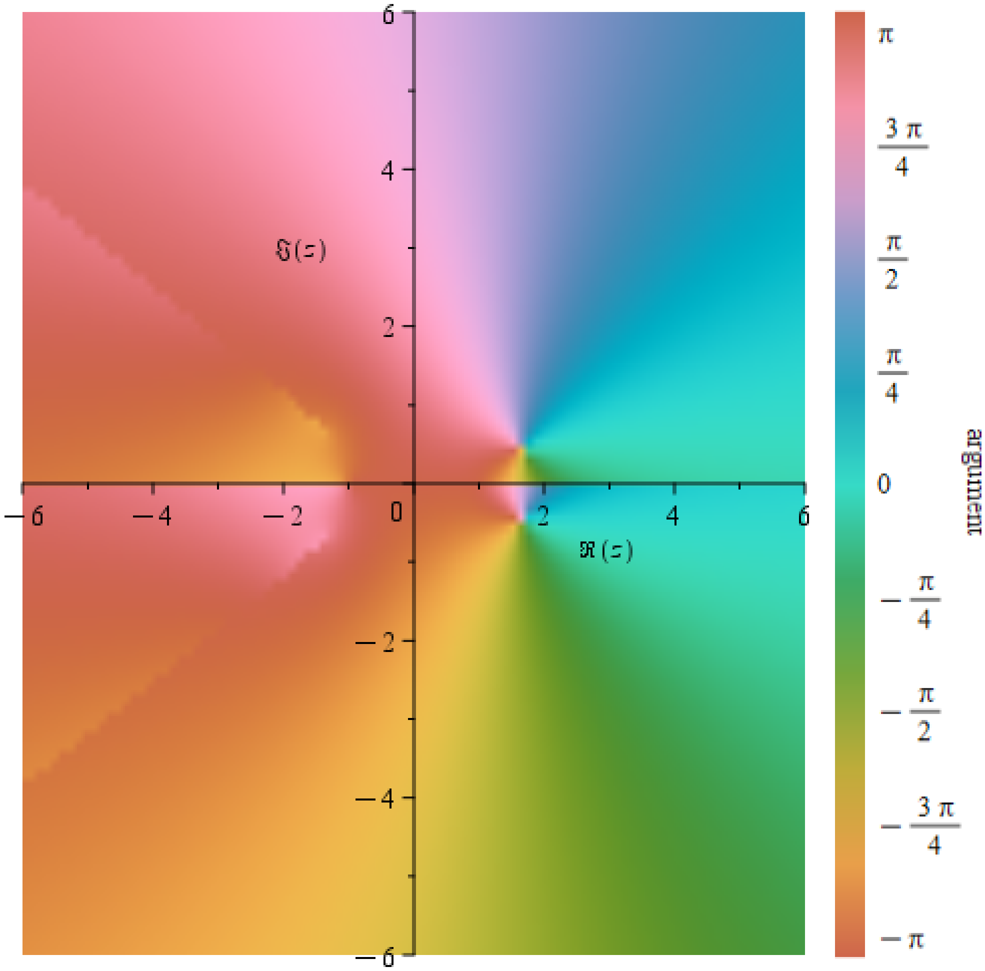

Fig. 2 : complex plot, $f(z) = z - \operatorname{arcsinh}(1 + \arcsin(z))$, $z \in (-6 - 6i, 6 + 6i)$

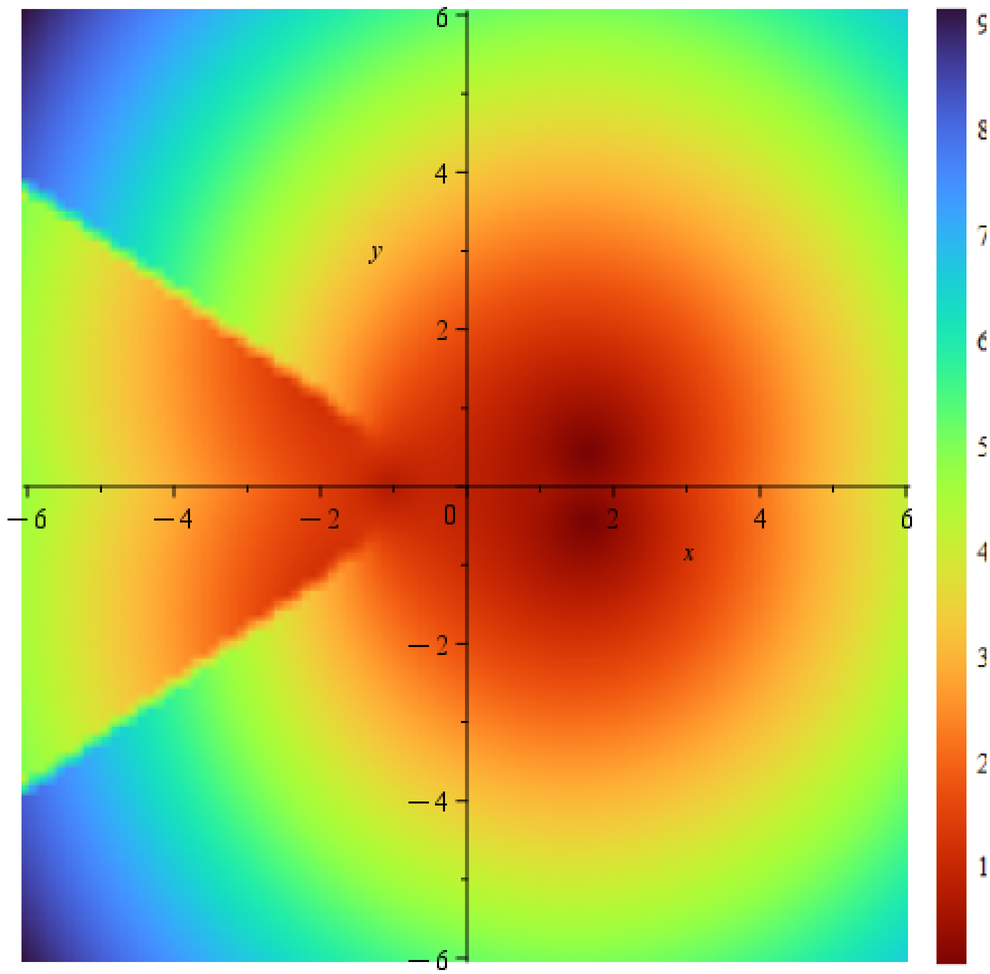

Fig. 3 : density plot, $|f(x+y\mathrm{I})|$, $-6 \le x, y \le 6$

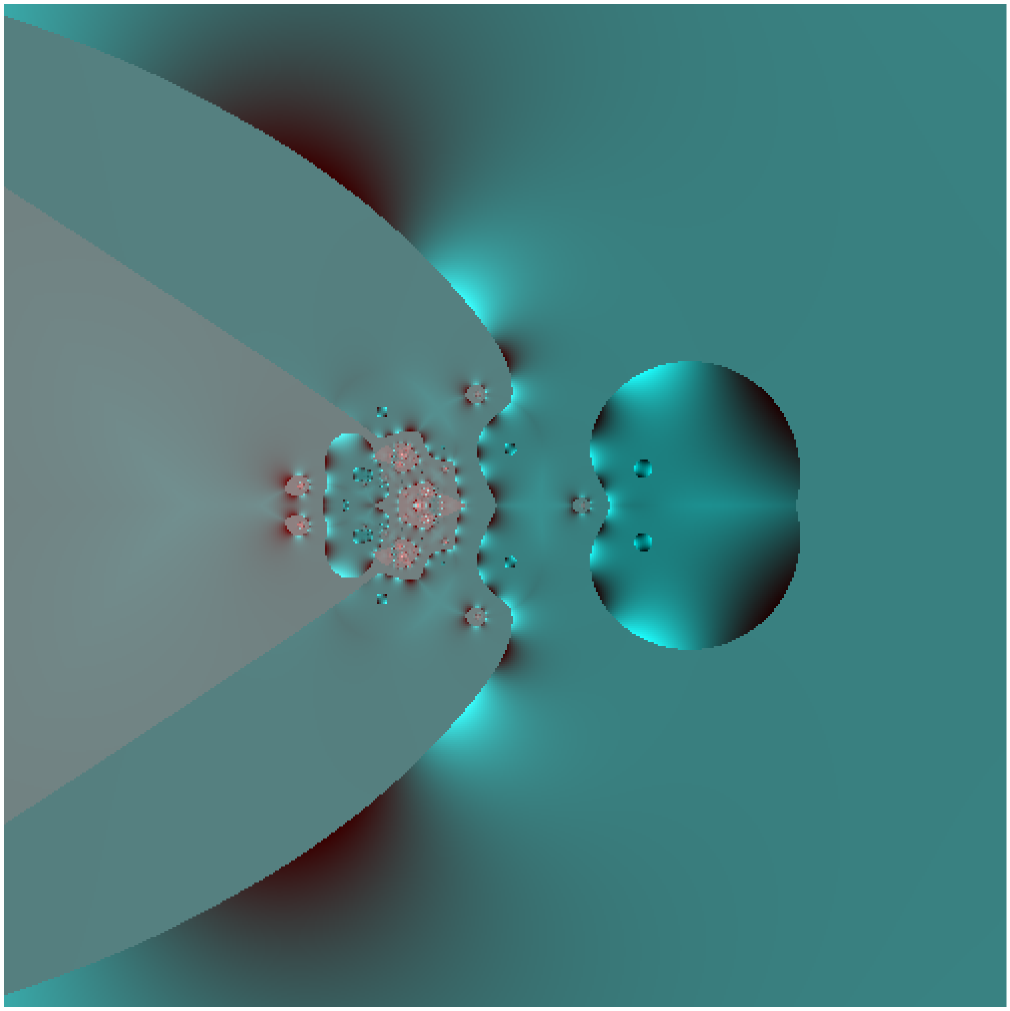

Fig. 4 : Newton fractal, $f(z) = z - \operatorname{arcsinh}(1 + \arcsin(z))$, $z \in (-6 - 6i, 6 + 6i)$

3. References

[1] F.W.J. Olver, et al.: NIST Handbook of Mathematical Functions. Cambridge University Press, 2010.