Newtonian knot in empty (2+1)-dimensional anti-de Sitter space-time

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We propose the existence of a topological object, a Newtonian knot, in the framework of an Abelian Chern-Simons action with a small negative cosmological constant (anti-de Sitter) in empty (2+1)-dimensional space-time.

Keywords: *empty (2+1)-dimensional gravity theory, anti de-Sitter space-time, weak-field limit, Abelian Chern-Simons action, knot.*

I. INTRODUCTION

It has been widely believed that topological objects can not exist in linear theories. Topological theories are inherently non-linear[1]. How, then, could a topological object, like a gravitational knot, exist in the linear theory, such as an Abelian Chern-Simons theory?

It is well known that the general theory of gravitation is identical to a gauge theory[2–6]. Maxwell's theory of electromagnetism and Einstein's theory of gravitation are identical where the gauge potential and the field strength tensor in Maxwell's theory (in general, a non-Abelian gauge theory, such as Yang-Mills theory) are identical to the connection and the curvature in general relativity, respectively[2]. Both theories are the gauge theories, where Maxwell's theory is an Abelian $U(1)$ gauge theory of internal space and general relativity can be treated as the gauge theory of translation of (3+1)-dimensional (external) space-time[3]. The vierbein formalism of general relativity (Cartan gravity) makes general relativity similar to a gauge theory[4]. Nevertheless, in the case of (3+1)-dimensional space-time, general relativity and a gauge theory are definitely not equivalent. But, they are precisely equivalent in (2+1)-dimensional space-time[4].

Roughly speaking, gravity theory in (2+1)-dimensional space-time is a simpler model than general relativity in (3+1)-dimensional space-time. Gravity theory in (2+1)-dimensional space-time shares the important conceptual features of general relativity in (3+1)-dimensional space-time while avoiding some of the computational difficulties. As a generally covariant theory of space-time, (2+1)-dimensional gravity has the same conceptual foundation as realistic (3+1)-dimensional general relativity[5].

With a few exceptions, (2+1)-dimensional solutions are physically quite different from those in 3+1 dimensions. The 2+1 dimensional model is not very helpful for understanding the dynamics of realistic quantum gravity. But for the analysis of conceptual problems e.g. the nature of time, the construction of states and observable, the role of topology and topology change, the model has proven highly instructive[5]. In (2+1)-dimensional space-time gravity, the dynamics is topology[7]. The (2+1)-dimensional gravity theory could be interpreted as a Chern-Simons three form[4]. The Chern-Simons theory is topological gauge theory in (2+1)-dimensional space-time[7], where the Chern-Simons action precisely coincides with the (2+1)-dimensional space-time of the Einstein-Hilbert action[4,6].

The Einstein-Hilbert action in (2+1)-dimensional space-time, without a cosmological constant, is equivalent to a gauge theory with gauge group ISO(2,1) and a pure Chern-Simons action[4]. If we include a cosmological constant in (2+1) general relativity, then Minkowski (flat) space-time is replaced by space-time with a constant curvature: de Sitter or anti-de Sitter depending on the sign of a cosmological constant (plus for de Sitter and minus for anti-de Sitter), and gauge group ISO(2,1) is replaced by SO(3,1) or SO(2,2)[4].

If the relation between general relativity and Chern-Simons gauge theory is valid at the quantum level, then there is a close relationship between general relativity and knot theory, at least in (2+1)-dimensional space-time, since Chern-Simons gauge theory in (2+1)-dimensional space-time is intimately connected with knot theory[4]. We consider the quantum level here to be related to the topological quantum condition, the discreteness.

The formulation of a gravitational knot for a non-Abelian Chern-Simons action in (2+1)-dimensional empty space-time has been proposed[4,6–9]. In this article, we propose the existence of a gravitational knot in the weak-field limit in (2+1)-dimensional empty space-time formulated as an Abelian Chern-Simons action with a small positive cosmological constant written using the Clebsch variables. To the best of our knowledge[1,4,6–16], the formulation of such weak gravitational knot has not been done yet.

We assume that a topological structure in three-dimensional gravity is similar to that in Maxwell's theory of vacuum space[1]. Analogous to the linearized Ricci curvature tensor in (3+1)-dimensional space-time, the linearized Ricci curvature tensor (with a small positive cosmological constant) in the case of the weak-field limit is assumed to be valid in (2+1)-dimensional space-time.

In analogy to Maxwell's theory of vacuum space where the field strength tensor could consist of a set of subset fields[1,17], complex scalar fields, we propose that the Ricci curvature tensor (the set of the solutions of Einstein field equations) could consist of a set of curvature components, complex scalar potentials. This set of curvature components, such as a set of subset fields, satisfies the non-trivial Hopf maps. This means that non-trivial Hopf maps can describe the properties of a set of curvature components.

A set of curvature components is locally equal to the linearized Ricci curvature tensor, i.e. the linearized Ricci curvature tensor can be obtained by patching together a set of curvature components (except in a zero-measure set) but globally different. The difference is global, instead of local, since a set of curvature components obeys the topological quantum condition, but the linearized Ricci curvature tensor does not. The linearized Ricci curvature tensor satisfies the linear Ricci theory, but a set of curvature components satisfies the non-linear Ricci theory. Both, the linearized Ricci curvature tensor and a set of curvature components, satisfy the linear Ricci theory in the case of a weak-field limit. This means that, in the case of a weak-field limit, the non-linear Ricci theory is reduced to the linear Ricci theory.

This article is organized as follows. In Section II, we discuss in brief the (3+1)-dimensional gravity in the case of sourceless and without a cosmological constant. In Section III, the (2+1)-dimensional gravity in the case of sourceless, without and with a cosmological constant. In Section IV, we identify the relation between the Einstein-Hilbert and the Chern-Simons actions in (2+1)-dimensional space-time. In Section V, linearized metric perturbations, scalar, and vector potentials are discussed. In Section VI, we discuss in brief a set of curvature components and Hopf maps. In Section VII, the Hopf invariant, Hopf index, and Chern-Simons action are discussed. Section VIII, we formulate the non-linear and linear Ricci theories using complex scalar potentials and vector potential in terms of the Clebsch variables. In Section IX, the relation between the gauge potential and the gauge fields is described. We formulate the gauge fields in terms of Clebsch variables. In Section X, we formulate the weak gravitational knot. In Section XI, a discussion and conclusion are given.

II. (3+1) GRAVITY

The Einstein field equations in (3+1)-dimensional space-time can be written as

$$G_{\mu\nu} + \Lambda \; g_{\mu\nu} = -8\pi G \; T_{\mu\nu} \tag{1}$$

where

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \; R \tag{2}$$

$G_{\mu\nu}$ is Einstein tensor, $R_{\mu\nu}$ is the Ricci curvature tensor, $g_{\mu\nu}$ is metric tensor, R is the Ricci scalar curvature, Λ is a cosmological constant, G is the gravitational coupling constant (the generalization to other dimensions of Newton's constant)[8], $T_{\mu\nu}$ is the energy-momentum tensor of matter.

A. $T_{\mu\nu} = 0$, $\Lambda = 0$

What we mean with an empty space-time is a vacuum space-time, $R_{\mu\nu} = 0$, where there is no matter source present, $T_{\mu\nu} = 0$, and there exists no physical fields except the gravitational field[18]. This gravitational field does not disturb the emptyness, but other fields do[18]. Einstein assumed that in (3+1)-dimensional empty space-time, it constitutes his law of gravitation[18].

In the absence of matter and without cosmological constant, the Einstein field equations (1), (2), read[9]

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \; R = 0 \tag{3}$$

In general, the vanishing of $G_{\mu\nu}$, hence of $R_{\mu\nu}$ and R, does not imply that the Riemann curvature tensor is zero, i.e. the space-time need not be flat[9]. However, in (2+1)-dimensional space-time the situation is different.

B. $T_{\mu\nu} = 0$, $\Lambda \neq 0$

In this article, we will not discuss the gravity theory with a non-zero cosmological constant in (3+1)-dimensional empty space-time.

III. (2+1) GRAVITY

In (2+1)-dimensional space-time manifold, M, Einstein-Hilbert action for gravity coupled to matter can be written as[5,19]

$$I_{\text{EH}} = \frac{1}{16\pi G} \int_M d^{2+1}x \; \sqrt{-g} \; (R - 2\Lambda) + I_{\text{matter}} \tag{4}$$

where $g = \det(g_{\mu\nu})$ is the determinant of the metric tensor matrix.

Equation of motion for the action (4) are[5,19]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \; R + \Lambda \; g_{\mu\nu} = -8\pi G \; T_{\mu\nu} \tag{5}$$

We see that the equation of motion (5) is the same as (1). Eq.(5) are generally covariant, i.e. they are invariant under the action of the group of diffeomorphisms (which can be viewed as a gauge group) of the space-time[5]. Roughly speaking, we could say that a diffeomorphism is a smooth (continuously differentiable), reversible transformation between spaces or shapes that preserves their smooth structure whereas an isometry[20] (distance-preserving mapping) is a special case of a diffeomorphism.

In (2+1)-dimensional space-time, the relation between the Einstein tensor and the Riemann curvature tensor can be written as[9]

$$G^{\mu}{}_{\nu} = -\frac{1}{4}\varepsilon^{\mu\alpha\beta}\ \varepsilon_{\nu\gamma\delta}\ R_{\alpha\beta}{}^{\gamma\delta} \tag{6}$$

where $\varepsilon^{\mu\alpha\beta}$ is Levi-Civita symbols, μ, α, β, denote space-time coordinates. Eq.(6) may be inverted as[8]

$$R^{\alpha\mu}_{\ \ \beta\nu} = \varepsilon^{\alpha\mu\gamma}\ \varepsilon_{\beta\nu\delta}\ G^{\delta}{}_{\gamma} \tag{7}$$

Eqs.(6),(7) are the identities linking Einstein tensor and the Riemann curvature tensor.

A. $T_{\mu\nu} = 0$, $\Lambda = 0$

Eq.(7) without a cosmological constant implies that if the Einstein tensor vanishes (as a consequence of the absence of matter) then the Riemann curvature tensor vanishes. In turn, the vanishing Riemann curvature tensor implies that the Ricci curvature tensor and the Ricci curvature scalar are equal to zero. So, the solution of eq.(7) is flat space-time. We call the theory trivial, i.e. it does not possess any propagating degrees of freedom[9].

B. $T_{\mu\nu} = 0$, $\Lambda \neq 0$

In the case of an empty space-time and a non-zero cosmological constant, eq.(5) can be replaced by

$$G_{\mu\nu} + \Lambda\ g_{\mu\nu} = 0 \tag{8}$$

and by substituting eq.(8) into eq.(7), we obtain[9]

$$R_{\alpha\mu\beta\nu} = -\Lambda(g_{\alpha\beta}\ g_{\mu\nu} - g_{\alpha\nu}\ g_{\beta\mu}) \tag{9}$$

which shows that without sources, all spaces that solve (8) are of constant curvature: a closed de Sitter space for $\Lambda > 0$ or a hyperbolic anti-de Sitter space for $\Lambda < 0$[9]. We consider that the constant curvature indicates that the geometry of space-time is locally homogeneous[4] and isotropic in the sense that curvature is uniform.

Eq.(8) implies that the Ricci curvature tensor can be written as

$$R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}\ R - \Lambda\ g_{\mu\nu} \tag{10}$$

It means that the Ricci curvature tensor is not simply proportional to the metric tensor, $g_{\mu\nu}$, scaled by a constant R, but also has an additional term involving the cosmological constant.

IV. EINSTEIN-HILBERT ACTION AS CHERN-SIMONS ACTION

A. The Einstein-Hilbert action without a cosmological constant

The Einstein-Hilbert action without a cosmological constant in (2+1)-dimensional space-time manifold would be[4]

$$I_{\rm EH} = \frac{1}{2}\int_M \varepsilon^{\mu\nu\rho}\ \varepsilon_{abc}\ e^a_\mu\left(\partial_\nu\omega^{bc}_\rho - \partial_\rho\omega^{bc}_\nu + [\omega_\nu, \omega_\rho]^{bc}\right) d^{2+1}x \tag{11}$$

where e^a_μ is a dreibein, ω^{bc}_ρ is a spin connection, ε_{abc} is the Levi-Civita symbols in the internal (local Lorentz frame) space. If a dreibein and a spin connection are interpreted as gauge fields, it might conceivably to interpreted (11) as a Chern-Simons action[4].

From eq.(11), the Ricci curvature tensor can be written as

$$R_{\nu\rho}{}^{bc} = \partial_\nu\omega_\rho{}^{bc} - \partial_\rho\omega_\nu{}^{bc} + [\omega_\nu, \omega_\rho]^{bc} \tag{12}$$

Eq.(12) is a non-linear equation. The nonlinearity is shown by the commutation relation in the third term of the right-hand side, $[\omega_\nu, \omega_\rho]^{bc}$. This commutation term represents the self-interaction of the spin connection.

B. The Chern-Simons action without a cosmological constant

The Chern-Simons action in (2+1)-dimensional space-time manifold can be written as[4,8]

$$I_{\rm CS} = \int_M \varepsilon^{\mu\nu\rho}\ e_{\mu a}\left(\partial_\nu\omega_\rho{}^a - \partial_\rho\omega_\nu{}^a + \varepsilon^a{}_{bc}\ \omega_\nu{}^b\omega_\rho{}^c\right) d^{2+1}x \tag{13}$$

We raise the a index in $\varepsilon^a{}_{bc}$ to show explicitly the contraction of the index. The Chern-Simons action (13) precisely coincides with the Einstein-Hilbert action (11)[4].

C. The Einstein-Hilbert action with a cosmological constant

The generalized Einstein-Hilbert action in (2+1)-dimensional space-time with a non-zero (a small negative) cosmological constant can be written as[4]

$$I_{\rm EH} = \int_M \varepsilon^{\mu\nu\rho}\left\{e_{\mu a}\left(\partial_\nu\omega^a_\rho - \partial_\rho\omega^a_\nu\right) + \varepsilon_{abc}\ e^a_\mu\ \omega^b_\nu\ \omega^c_\rho - \frac{\Lambda}{3}\ \varepsilon_{abc}\ e^a_\mu\ e^b_\nu\ e^c_\rho\right\} d^{2+1}x \tag{14}$$

The equations of motion (14) say that space-time is not flat but locally homogeneous with the curvature proportional to a cosmological constant[4].

The simply connected covering space of such a space-time is not a portion of Minkowski space, but a portion of de Sitter or anti-de Sitter space. The spaces of de Sitter and anti-de Sitter have for their symmetries SO(3,1) and SO(2,2), respectively. It is different from a flat space-time of Minkowski which has for its symmetry ISO(2,1)[4]. Thus, it is reasonable to guess that if the gravity theory without a cosmological constant in (2+1)-dimensional space-time is related to the gauge theory of ISO(2,1), then the gravity theory with a cosmological constant in (2+1)-dimensional space-time will be related to gauge theory of SO(3,1) and SO(2,2)[4].

We see from eq.(14), the Ricci curvature tensor could be written as[6]

$$R^a_{\nu\rho} = \partial_\nu \omega^a_\rho - \partial_\rho \omega^a_\nu + \varepsilon^a{}_{bc}\ \omega^b_\nu\ \omega^c_\rho - \frac{\Lambda}{3}\ \varepsilon^a{}_{bc}\ e^b_\nu\ e^c_\rho \quad (15)$$

In terms of the spin connection, eq.(15) is a non-linear equation due to there exists the multiplication term of the spin connection in the third term of the right-hand side, $\omega^a_\nu\ \omega^a_\rho$, such as in eq.(12).

D. The weak-field limit and a small negative cosmological constant

In the case of the weak-field limit of the gauge fields and a small negative cosmological constant, $\Lambda < 0$, $|\Lambda| << 1$, eq.(15) reduces to a linearized Ricci curvature tensor written below

$$R^a_{\nu\rho} = \partial_\nu \omega^a_\rho - \partial_\rho \omega^a_\nu - \frac{\Lambda}{3}\ \varepsilon^a{}_{bc}\ e^b_\nu\ e^c_\rho \quad (16)$$

In terms of the spin connection, eq.(16) is a linear equation. There is no self-interaction of the spin connection. Although, at first sight, eq.(16) looks like a non-linear equation, because there exists a quadratic form (as a product of the dreibein components) in the third term of eq.(16). Here, the dreibein components can be viewed as the fixed fields i.e. the fields that are considered given or fixed externally, parameters. They are not variables being solved for. The fixed dreibein fields due to a cosmological constant introduce a source term that is imposed on the curvature.

E. The Chern-Simons action with a cosmological constant

The generalized (non-Abelian) Chern-Simons action with a non-zero (a small negative) cosmological constant could be written as[4,6]

$$I_{\text{CS}} = \int_M \varepsilon^{\mu\nu\rho}\ e_{\mu a} \times \left\{ \partial_\nu \omega^a_\rho - \partial_\rho \omega^a_\nu + \varepsilon^a{}_{bc} \left(\omega^b_\nu\ \omega^c_\rho - \frac{\Lambda}{3}\ e^b_\nu\ e^c_\rho \right) \right\} d^{2+1}x \quad (17)$$

From eq.(17), the Ricci curvature tensor can be written as

$$R^a_{\nu\rho} = \partial_\nu \omega^a_\rho - \partial_\rho \omega^a_\nu + \varepsilon^a{}_{bc} \left(\omega^b_\nu\ \omega^c_\rho - \frac{\Lambda}{3}\ e^b_\nu\ e^c_\rho \right) \quad (18)$$

We see that eq.(18) is equivalent to eq.(15).

In the case of the weak-field limit of the gauge fields and a small negative cosmological constant, eq.(18) reduces to a linearized Ricci curvature tensor written below

$$R^a_{\nu\rho} = \partial_\nu \omega^a_\rho - \partial_\rho \omega^a_\nu - \frac{\Lambda}{3}\ \varepsilon^a{}_{bc}\ e^b_\nu\ e^c_\rho \quad (19)$$

where $\varepsilon^a{}_{bc}$ is the Levi-Civita symbol which has a role as the structure constants (the structure coefficients)[21]. Eq.(19) is a linear equation in terms of the spin connection. The reason is analogous to eq.(16).

Previously, we have formulated the linearized Ricci curvature tensor in the case of the Newtonian limit and a small positive cosmological constant, $\Lambda > 0$, $|\Lambda| << 1$ (de Sitter space)[34]. The extension to the Newtonian limit with a small negative cosmological constant, $\Lambda < 0$, $|\Lambda| << 1$ (anti-de Sitter space) could be worked by neglecting time derivatives in the linearized Ricci curvature tensor. We obtain the linearized Ricci curvature tensor as written below

$$R^a{}_{jk} = \partial_j \omega^a_k - \partial_k \omega^a_j - \frac{\Lambda}{3}\ \varepsilon^a{}_{bc}\ e^b_j\ e^c_k \quad (20)$$

where $j, k = 1, 2$, denote spatial indices, $\varepsilon^a{}_{bc}$ is the Levi-Civita symbol which has a role as the structure constants (the structure coefficients)[21] showing explicitly there exists an interaction between the dreibein.

F. The Abelian Chern-Simons action with a cosmological constant

The Abelian Chern-Simons action with a small negative cosmological constant can be obtained from eq.(17) by replacing the Ricci curvature tensor (18) with a linearized Ricci curvature tensor (20), we have

$$I_{\text{CS}} = \int_M \varepsilon^{\mu\nu\rho}\ e_{ia} \left(\partial_j \omega^a_k - \partial_k \omega^a_j - \frac{\Lambda}{3}\ \varepsilon^a{}_{bc}\ e^b_j\ e^c_k \right) d^{2+1}x \quad (21)$$

where we replace $e_{\mu a}$ with e_{ia}. We will use this Abelian Chern-Simons action (21) to formulate the Newtonian knot.

V. LINEARIZED METRIC PERTURBATIONS, SCALAR AND VECTOR POTENTIALS

The linearized (small) metric perturbations can be written as

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \quad (22)$$

where $\eta_{\mu\nu}$ is the metric of Minkowski (flat) space-time. The small metric perturbations means that $|h_{\mu\nu}| << 1$ for all μ and ν.

In the language of a wave, the linearized metric perturbations can be written as[22]

$$h_{\mu\nu} = \rho_{\mu\nu}\ e^{i\vec{k}\cdot\vec{r}} \tag{23}$$

where $\rho_{\mu\nu}$ is the amplitude as a function of space-time, $\vec{k}$ is the wave vector, $\vec{r}$ is the position vector, and $\vec{k}\cdot\vec{r}$ is the phase, a function of space-time. In an empty space-time, the amplitude is constant.

In the linearized gravity theory, the linearized metric perturbations take a role as "potentials"[22]. We consider the linearized metric perturbations analogous to a set of curvature components, the scalar potentials, which could be complex, written as[23]

$$h = \rho\ e^{iq}, \quad h^* = \rho\ e^{-iq} \tag{24}$$

where ρ is the amplitude, q is the phase, h^* is the complex conjugate of h, i is an imaginary number. Both, ρ and q, are the functions of space-time.

The related (real) vector potential could be written as

$$h_\mu = f\ \partial_\mu q \tag{25}$$

where the Greek index, μ, denotes the spatial index, f is the amplitude function written as below

$$f = -1/\left\{2\pi(1+\rho^2)\right\} \tag{26}$$

Here f and q are the Clebsch variables[24] or the Gaussian potentials[8,25]. Both, f and q, are scalars.

In the following, we will interpret the gauge potential in a gauge theory as the gauge fields (the dreibein and the spin connection) in general relativity and reformulate the relation between the gauge potential and the gauge fields written using the Clebsch scalar variables. First, it is necessary to show that a set of curvature components satisfies the non-trivial Hopf maps.

VI. A SET OF CURVATURE COMPONENTS AND HOPF MAPS

The properties of the complex scalar potentials could be described by the non-trivial Hopf maps written below

$$h(\vec{r}),\ h^*(\vec{r}) : S^3 \to S^2 \tag{27}$$

These non-trivial Hopf maps can be classified in homotopy classes labeled by the value of the corresponding Hopf indexes, integer numbers, and the topological invariants[1,17]. The other names of the topological invariants are the topological charge, and the winding number (the degree of a continuous mapping, such as the Hopf maps). The topological charge is metric tensor-independent, it can be interpreted as energy[27].

The complex scalar potentials in the non-trivial Hopf maps (27) are time-independent. Analogous to the time-independent complex scalar fields, this problem could be solved by interpreting some of the quantities that appear in Hopf's theories as Cauchy's initial time values[24]. We consider that the two-dimensional spheres, S^2, as codomain in the Hopf maps, could be interpreted as two-dimensional spheres with constant curvature. In turn, we will interpret these two-dimensional spheres with constant curvature as de Sitter space.

VII. HOPF INVARIANT, HOPF INDEX, CHERN-SIMONS ACTION

The Hopf invariant, $\mathcal{H}$, can be expressed as[15,28,29]

$$\mathcal{H} = \int_{S^3} \varepsilon^{\mu\nu\rho}\ \Omega_\mu\ \partial_\nu \Omega_\rho\ d^3x \tag{28}$$

where Ω_μ can be interpreted as a connection one-form, a gauge potential, d^3x represents the volume element on S^3. In hydrodynamics, this partial derivative of connection one-form, $\partial_\nu\Omega_\rho = W_{\nu\rho}$, can be interpreted as the vorticity.

The Hopf invariant can be related to the Hopf index, H, written explicitly as[1]

$$\mathcal{H} = H\ \gamma^2 \tag{29}$$

where γ is the total strength of the field[1].

The concept of the Hopf invariant arises naturally from the geometry of the Hopf maps. It measures the degree (number) of linking of the preimages on S^3 mapped to a point on S^2 under the Hopf maps. The linking number tells us how many times one of these loops wraps around the other. If the linking (integer) number is zero, it means that there is no entanglement between two loops. These two loops can be separated or untangled without cutting. We could call these two separated loops the distant union of two unknots (the unknot is the knot) which is a perfectly fine link. It is because links do not actually need to be linked[30]. If the linking number is not zero, then there exists an entanglement in the Hopf maps, the (continuous) non-trivial maps.

The Hopf invariant is identical to the circulation in hydrodynamics[25] where Ω and $d\Omega$ in the Hopf invariant are identical to the velocity field and the vorticity in hydrodynamics, respectively. If we relate hydrodynamics (self-helicity) to a gauge theory, it can be interpreted naturally that the Hopf invariant has a deep relationship with the Chern-Simons action (the Chern-Simons integral)[15]. The Hopf invariant is just the winding number of Gauss mapping[15]. The Hopf invariant is an important topological invariant in describing the topological characteristics of the knot family. More precisely, the Hopf invariant or the Chern-Simons action is the total sum of all the linking and all the self-linking numbers of the knot family[15,16]. The linking and self-linking numbers by themselves have a topological structure.

VIII. NON-LINEAR AND LINEAR RICCI THEORIES

Analogous to a non-linear field theory in Maxwell's theory[1], the non-trivial Hopf maps (27) have a consequence that we could write a non-linear Ricci theory as

$$R_{\mu\nu} = \frac{1}{(1+h^*h)^2}(\partial_\mu h^* \partial_\nu h - \partial_\nu h^* \partial_\mu h) \tag{30}$$

The nonlinearity of eq.(30) is shown by the h^*h term in the denominator.

In the case of a weak-field limit, the complex scalar potentials are very small, $|h^*h| << 1$, so eq.(30) reduces to a linear Ricci theory as written below

$$R_{\mu\nu} = \partial_\mu h^* \partial_\nu h - \partial_\nu h^* \partial_\mu h \tag{31}$$

If eq.(31) is written using the (real) vector potential (25), then we obtain

$$R_{\mu\nu} = \partial_\mu h_\nu - \partial_\nu h_\mu \tag{32}$$

This linear Ricci theory (32) is equivalent to the linearized Ricci curvature tensor[31,32]

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \partial_\nu \Gamma^\alpha{}_{\mu\alpha} \tag{33}$$

It means that the linearized Ricci curvature tensor (33) could be interpreted the same as the linear Ricci theory (32) where the vector potential, h_μ is equivalent to the Christoffel symbol, $\Gamma^\alpha{}_{\mu\alpha}$. The curvature, $R_{\mu\nu}$, in eqs.(32), (33), are equivalent. Both are the second rank tensor which is symmetric in μ and ν.

By using the vector potential (25), the linear Ricci theory (32) could be written as

$$R_{\mu\nu} = \partial_\mu(f\ \partial_\nu q) - \partial_\nu(f\ \partial_\mu q) \tag{34}$$

This is the linear Ricci theory written in terms of the Clebsch scalar variables. We see that the vector potential written using the Clebsch variables (34) is equivalent to the Levi-Civita connection (the Christoffel symbol) in eq.(33).

IX. GAUGE POTENTIAL AND GAUGE FIELDS

In the dreibein formalism of general relativity, we have the gauge fields (the dreibein and the spin connection). These gauge fields could be viewed identically to the gauge potential. In this case, the gauge potential can be written as[4,6,8]

$$A_\mu = e^a_\mu\ P_a + \omega^a_\mu\ J_a \tag{35}$$

where e^a_μ is a component of the vierbein (the translational part), $e^a_\mu\ P_a$ is the vierbein field or shortly the vierbein, ω^a_μ is a component of the spin connection (the rotational part), $\omega^a_\mu\ J_a$ is the spin connection field or the spin connection, P_a, J_a, are the generators of translation and rotation, respectively.

Analogous to (25), we could write the gauge potential as

$$A_\mu = f\ \partial_\mu q \tag{36}$$

By substituting eq.(36) into (35), we obtain

$$f\ \partial_\mu q = e^a_\mu P_a + \omega^a_\mu J_a \tag{37}$$

If we assume that the components of the vierbein and the spin connection could be written using the Clebsch variables, they could be written respectively, as

$$e^a_\mu = f_e\ \partial_\mu q^a_e \tag{38}$$

and

$$\omega^a_\mu = f_\omega\ \partial_\mu q^a_\omega \tag{39}$$

where f_e, f_ω, are the amplitude functions of the vierbein and the spin connection, respectively. Both of them relate to gravitational "strength" or scaling effects, i.e. they scale the contributions of $\partial_\mu q^a_e$ and $\partial_\mu q^a_\omega$. q^a_e, q^a_ω, are the phase of the vierbein and the spin connection, respectively. Their gradient encodes information about the direction or orientation of the gravitational field.

By substituting eqs.(38), (39), into (37), the gauge potential becomes

$$f\ \partial_\mu q = f_e\ \partial_\mu q^a_e\ P_a + f_\omega\ \partial_\mu q^a_\omega\ J_a \tag{40}$$

where the amplitude functions of the vierbein and the spin connection are constant due to the amplitude in an empty space-time is constant.

In the case of the Newtonian limit, eqs.(38), (39), (40), become

$$e^a_i = f_e\ \partial_i q^a_e \tag{41}$$

$$\omega^a_i = f_\omega\ \partial_i q^a_\omega \tag{42}$$

$$f\ \partial_i q = f_e\ \partial_i q^a_e\ P_a + f_\omega\ \partial_i q^a_\omega\ J_a \tag{43}$$

X. THE NEWTONIAN KNOT

By substituting eqs.(41), (42), into eq.(21), and by assuming that f_e, f_ω, are constants (we could take as 1), we obtain

$$\begin{aligned} I_{\rm CS} = \int_M \varepsilon^{ijk}\ \partial_i q_{ea} \{&(\partial_j\partial_k - \partial_k\partial_j) q^a_\omega \\ &- \frac{\Lambda}{3}\ \varepsilon^a{}_{bc}\ \partial_j q^b_e\ \partial_k q^c_e\Big\}\, d^{2+1}x \end{aligned} \tag{44}$$

The action, $I_{\rm CS}$ (44), could be interpreted as the Newtonian knot. The Levi-Civita symbols have a role as structure constants that couple the interaction between the gauge fields. The amplitude functions, f_e, f_ω, have a role as scale factors. We see that the Chern-Simons action (44) is identical to the Hopf invariant (28). This Newtonian knot could be interpreted as an integer number. That is what we mean by a set of curvature components obeying the topological quantum condition.

XI. THE FIELD EQUTIONS AND SOLUTIONS

The field equations or the equations of motion can be derived by applying the least action principle to an Abelian Chern-Simons action (44). We obtain the field equations below

$$\varepsilon^{ijk}\ \partial_k e_{ia} = 0 \tag{45}$$

and

$$\varepsilon^{ijk}\left(\partial_j \omega_k^a - \partial_k \omega_j^a\right) = \frac{\Lambda}{3}\ \varepsilon^{ijk}\ \varepsilon^a{}_{bc}\ e_j^b\ e_k^c \tag{46}$$

Analogous to hydrodynamics, eq.(45) is identical to the curl-free velocity, $\Omega = 0$, where the vorticity $\Omega = \varepsilon^{ij}\ \partial_i v_j$[25], v_j is the velocity. We see from (45) the dreibein is identical to the velocity, and the vorticity is identical to the curvature. Eq.(45) imposes a constraint on the components of the dreibein.

The consequence of the curl-free vector field, such as the velocity or the dreibein, is the vector field could be written as the gradient of a scalar function[26]. It implies that the dreibein could be written as

$$e_{ia} = f_e\ \partial_i q_{ea} \tag{47}$$

where f_e is an amplitude function of the dreibein and q_{ea} is the dreibein phase. Mathematically, to ensure that the curl-free vector field can be replaced by the gradient of a scalar function, we should take the scalar function f_e as a constant, so that its derivative vanishes. Physically, this corresponds to the fact that f_e takes a constant value in an empty space-time. In such a space-time, we can assume that the dreibein phase is a linear scalar function.

The linear scalar function of the dreibein phase could be written in two spatial dimensions of the polar coordinate as

$$q_{ea} = k_{er}\ r\ \delta_{ar} + k_{e\theta}\ \theta\ \delta_{a\theta} \tag{48}$$

where k_{er}, $k_{e\theta}$, are constants (as tuning parameters), δ_{ar} is the Kronecker delta. To give the non-zero result, we take $a = r, \theta$, and for simplicity in this article, $k_{er} = k_{e\theta} = 1$, then we obtain

$$q_{ea} = r + \theta \tag{49}$$

By substituting eq.(49) into (47) and for simplicity in this article we take $f_e = 1$, we obtain

$$e_{ia} = \partial_i (r + \theta) \tag{50}$$

Assume that r and θ are linear functions, we can write both as

$$r = a_i x^i = x^i, \quad \theta = a_i x^i = x^i \tag{51}$$

where a_i are constants and we set $a_i = 1$, $x^i = r, \theta$. In (2+1)-dimensional space-time, $i = 1, 2$. In polar coordinates, we can write $x^1 = r$ and $x^2 = \theta$.

By substituting (51) into (50), we obtain

$$e_{ia} = \partial_i (x^i + x^i) = \partial_r r + \partial_\theta \theta = 2 \tag{52}$$

By substituting (52) into (45), we obtain

$$\varepsilon^{ijk} \partial_k (2) = 0 \tag{53}$$

This shows that the second derivative of a linear scalar function gives a zero result. It means that there exists no curvature.

Let us analyze the second field equations, especially the right-hand side of eq.(46). Analogous to (47)-(52), we obtain

$$e_j^b = 2 \tag{54}$$

and

$$e_k^c = 2 \tag{55}$$

Let us consider the term of eq.(46) below

$$\varepsilon^{ijk}\ \varepsilon^a{}_{bc} = \varepsilon^{ijk}\ \eta^{ad}\ \varepsilon_{dbc} \tag{56}$$

where η^{ad} is the Minkowski metric written using the Lorentz indices. From eq.(56), we could write

$$\begin{aligned} \varepsilon^{ijk}\ \varepsilon_{dbc} = \delta_d^i\ \delta_b^j\ \delta_c^k - \delta_d^i\ \delta_c^j\ \delta_b^k + \delta_b^i\ \delta_c^j\ \delta_d^k - \delta_b^i\ \delta_d^j\ \delta_c^k \\ + \delta_c^i\ \delta_d^j\ \delta_b^k - \delta_c^i\ \delta_b^j\ \delta_d^k \end{aligned} \tag{57}$$

where δ_d^i is the Kronecker delta.

If we contract eq.(57) with η^{ad}, this will replace d with a, giving

$$\begin{aligned} \varepsilon^{ijk}\ \eta^{ad}\ \varepsilon_{dbc} = \delta_a^i\ \delta_b^j\ \delta_c^k - \delta_a^i\ \delta_c^j\ \delta_b^k + \delta_b^i\ \delta_c^j\ \delta_a^k - \delta_b^i\ \delta_a^j\ \delta_c^k \\ + \delta_c^i\ \delta_a^j\ \delta_b^k - \delta_c^i\ \delta_b^j\ \delta_a^k \end{aligned} \tag{58}$$

If we assume that $i = a$, $j = b$, $k = c$, eq.(58) or (56) becomes

$$\varepsilon^{ijk}\ \varepsilon^a{}_{bc} = 1 \tag{59}$$

By substituting eqs.(54), (55), (59), into the term of the right-hand side of eq.(46), we obtain

$$\varepsilon^{ijk}\ \varepsilon^a{}_{bc}\ e_j^b\ e_k^c = (1)(2)(2) = 4 \tag{60}$$

By substituting eq.(60) into eq.(46), we obtain

$$\varepsilon^{ijk}\left(\partial_j \omega_k^a - \partial_k \omega_j^a\right) = \frac{4}{3}\ \Lambda \tag{61}$$

Let us analyze the left-hand side of the eq.(61). We assume that the solution of eq.(61) is analogous to the solution of eq.(45), so we propose a similar ansatz field as the solution for the spin connection written below

$$\omega_j^a = f_\omega\ \partial_j q_\omega^a \tag{62}$$

where f_ω, q_ω^a, are the amplitude function and the phase of the spin connection, respectively. In empty space-time, such as the amplitude function of the dreibein, f_ω takes a constant value.

If we set $f_\omega = 1$, then eq.(62) becomes

$$\omega_j^a = \partial_j q_\omega^a \tag{63}$$

The same procedures apply to ω_k^a, then we obtain

$$\omega_k^a = \partial_k q_\omega^a \tag{64}$$

By substituting eqs.(63), (64), into eq.(61), we obtain

$$\varepsilon^{ijk} \left(\partial_j \partial_k - \partial_k \partial_j\right) q_\omega^a = \frac{4}{3}\,\Lambda \tag{65}$$

In commutation notation, eq.(65) can be written as

$$\varepsilon^{ijk}\,[\partial_j, \partial_k] q_\omega^a = \frac{4}{3}\,\Lambda \tag{66}$$

where

$$[\partial_j, \partial_k] = f_{jkl}\,\partial_l \tag{67}$$

f_{jkl} are the non-zero antisymmetric structure coefficients (the structure constants), the non-zero constants. This non-zero commutation relation in eqs.(66)-(67) arises because space has a non-commutative structure (an intrinsic curvature). A (constant) curvature which is due to the non-zero cosmological constant shows this non-commutative structure of space.

Analogous to the case of the dreibein above, we assume that the spin connection phase is a linear scalar function of space written below

$$q_\omega^a = k_{\omega r}\,r\,\delta_r^a + k_{\omega\theta}\,\theta\,\delta_\theta^a = r + \theta \tag{68}$$

where $k_{\omega r}$, $k_{\omega\theta}$, are constants and we set $k_{\omega r} = k_{\omega\theta} = 1$, $a = r, \theta$, and

$$r = a_l x^l = x^l, \quad \theta = a_l x^l = x^l \tag{69}$$

where a_l are constants and we set $a_l = 1$. By substituting eq.(69) into eq.(68), we obtain

$$q_\omega^a = x^l + x^l = 2x^l \tag{70}$$

By substituting eqs.(70), (67), into (66), we obtain

$$\varepsilon^{ijk}\,f_{jkl} = \frac{2}{3}\,\Lambda \tag{71}$$

where ε^{ijk} is fully antisymmetric in its indices, f_{jkl} are antisymmetric in the first two indices, j and k. We see that the non-zero contraction result of $\varepsilon^{ijk}\,f_{jkl}$ (71) is guaranteed by the non-zero commutation relation (67) in turn by the non-zero cosmological constant.

If we treat the contraction $\varepsilon^{ijk}\,f_{jkl}$ as a non-zero scalar quantity, λ, then eq.(71) becomes

$$\lambda = \frac{2}{3}\,\Lambda \tag{72}$$

Equation (72) shows that the contraction result λ is directly proportional to the cosmological constant Λ, with a proportionality factor of $2/3$. This factor acts as a scaling parameter that relates the structure constants to the cosmological constant. The positive sign in eq.(72) indicates that the contribution of the structure constants to the curvature aligns constructively with the effect introduced by the cosmological constant, rather than opposing it.

XII. DISCUSSION AND CONCLUSION

It has been realized that the role of topology has become more and more important in recent days and the future of physics. But to understand topology is complicated enough because topology is inherently related to nonlinearity. It has been widely believed that topological objects can not exist in linear theories, such as an Abelian Chern-Simons action in the topological quantum field theory. But this belief can no longer be maintained. The discovery of the electromagnetic knot in vacuum Maxwell's theory more than thirty years ago has shown that the topological object could exist in the linear theory.

We adopt the idea[1] of the electromagnetic knot and apply it to gravity. This is because electromagnetism and gravity are similar. The electromagnetic or Maxwell's theory is a gauge theory and gravity theory (the general theory of relativity) could be treated as a gauge theory. Maxwell's theory is an Abelian $U(1)$ local gauge theory of internal space and general relativity, a non-linear theory, is a gauge theory of translation in (3+1)-dimensional (external) space-time. The gauge potential and the field strength tensor in electromagnetism are identical to the connection and the curvature in gravity theory, respectively.

In (2+1) empty space-time, the curvature can be non-zero due to a non-zero cosmological constant. This (constant) curvature causes the gravitational fields to interact with themselves, as shown by the term $-\frac{\Lambda}{3}\,\varepsilon_{abc}\,e_i^a\,e_j^b\,e_k^c$ (14) and its equivalent form in a gauge theory (17), giving rise to non-trivial topological objects, such as gravitational knots. The gravitational field can be visualized as a string to form a closed loop, a knot. These knots are localized in space-time because it is related to the interaction between gravitational fields or curvature that occur locally.

We propose that the curvature i.e. the Ricci theory has a set of curvature components. We consider this set of curvature components analogous to the linearized (small) metric perturbations, i.e. a set of curvature components could consist of the scalar potentials. It is because, the linearized metric perturbations take a role as potentials, i.e. the linearized metric perturbations are the scalar potentials. A set of curvature components could be complex, such as in the case of the electromagnetic knot, a set of subset fields could consist of the complex scalar fields. It has a consequence that the complex scalar potentials could be interpreted equivalently as the complex scalar fields. In other words, a set of curvature components could be interpreted equivalently as the complex scalar fields. It means that, roughly speaking, the curved space-time could be treated as the complex fields. What does it imply?

The scalar potentials, such as the scalar fields, could be described using wave language. Both could be denoted by the amplitude times the exponential of iq, where q is the phase, and i is the imaginary number. The related (real) vector potential can be written using the Clebsch (scalar) variables, f, and q (25). We chose the real part of the related vector potential because we still do not know what the consequence of the imaginary part formulation in physics.

The Clebsch variables are not uniquely defined, but many different choices are possible for them. In this way, the vector potential can be understood simply. These Clebsch variables are related to any divergenceless vector field, i.e. the divergence of any vector field gives the zero result. Examples of a divergenceless vector field are the vorticity in hydrodynamics, i.e. $\vec{\nabla}\cdot\vec{W}=0$, so $\vec{W}=\vec{\nabla}\times\vec{v}$ where $\vec{v}$ is the velocity field vector, and in electromagnetism $\vec{\nabla}\cdot\vec{B}=0$, $\vec{B}=\vec{\nabla}\times\vec{A}$ where $\vec{B}$ is the magnetic field, $\vec{A}$ is the potential.

The condition $\vec{\nabla}\cdot\vec{W}=0$ implies that vorticity is solenoidal, meaning it is sourceless or has no sinks. The vorticity originates from the curl of the velocity field. This means that the "source" of vorticity is the rotational motion of the velocity field rather than a scalar charge such as in electromagnetism. We can observe the non-zero vorticity phenomenon in the rotational flows of fluids, e.g. a whirlpool. We see from eqs.(21), (44), the vorticity is identical to the curvature, and the velocity field is identical to the gauge fields (the dreibein).

In the case of the weak-field limit in (2+1)-dimensional empty space-time with a small negative cosmological constant, a non-linear Ricci curvature tensor (15) is reduced to a linearized Ricci curvature tensor (16). This small constant curvature accommodates our model in the limit of the infinite radius where the space-time is isotropic. What we mean by a linearized Ricci curvature tensor is, in terms of the spin connection, the Ricci curvature tensor is linear. We see that a linearized Ricci curvature tensor in a gauge theory (19) is precisely equivalent to a linearized Ricci curvature tensor in gravity theory (16).

The main difference between a non-Abelian and an Abelian gravity (a gauge theory) is that the curvature term, $\varepsilon^{a}{}_{bc}\,\omega^{b}_{j}\,\omega^{c}_{k}$, in gravity (15) or a gauge theory (18) is no longer exist in a linearized curvature in gravity (16) or a gauge theory (19). In the case of the weak-field limit, the multiplication between the weak fields gives a very small result that we can assume to be ignored. It means there is no interaction between the spin connections in an Abelian gravity or an Abelian gauge theory. In terms of the spin connection, an Abelian Chern-Simons action (21) is a linear equation.

The dreibein formalism of general relativity makes general relativity similar to a gauge theory. For this reason, we need to reformulate the gauge potential related to the gauge (vector) fields, i.e. in terms of the dreibein and the spin connection, as written in eq.(35). Analogous to the vector potential (25), the gauge potential could be written using the Clebsch variables (36). The gauge potential is not a total derivative, otherwise, it would be a pure gauge[24]. A pure gauge in this context means that the field configuration does not produce any observable curvature or field strength. Since the gauge potential is not a total derivative then it is not a pure gauge and therefore represents a physical, non-trivial field configuration.

Analogous to the gauge potential, we assume that the gauge fields could be written using the Clebsch variables, (38), (39). So, the relation between the gauge potential and the gauge fields could be written using the Clebsch variables (40). We could interpret the first term on the right-hand side (40), $f_e\,\partial_i q^a_e\,P_a$, as the rate of translation, f_e being an amplitude (scaling) factor that scales this translation rate. The second term on the right-hand side (40), $f_\omega\,\partial_i q^a_\omega\,J_a$, shows the rate of rotation, f_ω being a scaling factor that scales this rotational rate.

Expressing the gauge potential and the gauge fields in terms of the Clebsch variables simplifies the formulation. The Clebsch variables, by showing explicitly the amplitude function and the phase, enable the separation of the underlying physical dynamics (the amplitude function, the phase) and certain properties of the gauge potential and the gauge fields, such as topological structures. Separating the gauge potential and the gauge fields into their amplitude function and phase makes the topological features (related to non-zero vorticity) inherent in the gauge potential and the gauge fields more apparent.

The problems in the higher dimension can often be more complex than those in the lower dimension. By mapping onto the lower dimensional space, such as in the non-trivial Hopf maps, the problem becomes simpler, without losing the information about the non-trivial topological properties of space. If we relate the non-trivial Hopf maps to physics, we could interpret the Hopf maps to represent the properties of a set of curvature components, consisting of the complex scalar potentials. In the infinite radius, the value of the complex scalar potentials is weak. The complex scalar potentials have isotropic (well-defined) properties in the infinite radius.

We show that a set of curvature components satisfies the non-trivial Hopf maps (27). We assume that the time-independent problems of a set of curvature components, such as in the case of the electromagnetic knot, could be solved by interpreting some of the quantities that appear in Hopf's theories as Cauchy's initial time values.

There exists (one) dimensional reduction in the non-trivial Hopf maps. Physically, we could relate this dimensional reduction to the isotropic (well-defined) property of the complex scalar potentials, in turn, the gauge potential and the gauge fields. The isotropic (well-defined) property of the complex scalar potentials could be interpreted as an empty (a vacuum) space-time where space-time is homogeneous (isotropic). A space-time without a source but with a small negative cosmological constant can be viewed to be an empty space-time with a constant negative curvature (anti-de Sitter space-time).

The non-zero constant negative curvature resulting from a small negative cosmological constant (44) is reflected in the non-commutativity between ∂_ν and ∂_ρ. This non-zero curvature is linked to the non-trivial (which can not be smoothly or continuously deformed into the trivial configurations, such as a point) topological configurations of the gauge fields. In particular, the non-zero vorticity term $(\partial_j\partial_k - \partial_k\partial_j)q^a_\omega$ (44), can indeed contribute to a topological invariant, such as the winding number (an integer number). These non-trivial configurations are supporting evidence for the existence of the weak gravitational knot. We could say that the phenomenon of the weak gravity knot is related to the presence of the local vortex in space-time. The gravitation field could be imagined as a line (a field line)[33]. The winding number counts how often this field line winds around. The winding number is related to the energy of the field configuration.

Theoretically, the empirical or the observational evidence to support the existence of the weak gravity knot in (2+1)-dimensional empty space-time is guaranteed by the formal equivalence between the weak gravity knot and the electromagnetic knot formulations for which the electromagnetic knot solutions had been known to exist[1,11].

Experimentally, probably, we could observe the existence of the weak gravity knot by observing the gravitational wave (the ripples of space-time) as it passes through space-time influenced by the weak gravity knot. The presence of topological structures, such as the weak gravity knot, might influence the propagation of these ripples, detectable through their specific polarization modes. The weak gravity knot could change the properties of the gravitational waves. These changes would appear as additional or modified polarization modes (beyond the standard $+$ and $\times$) in the detected gravitational waves.

The existence of the Newtonian knot in empty (2+1)-dimensional space-time with the small negative cosmological constant (anti-de Sitter space-time) does not support the wide belief[5,35] that there exists no Newtonian limit in (2+1)-dimensional space-time. The Newtonian knot in empty (2+1)-dimensional space-time with a small positive cosmological constant is discussed separately[34].

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