The Geodesic Principle and the Nature of Passive Mass

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Abstract

The geodesic principle is a central feature of general relativity, as it expresses the geometric structure of the space-time manifold, but can also be interpreted as the effective influence of the metric field on the passive mass of a freely falling test particle. - The equation of motion is obtained from the stress-energy tensor of an isolated body by imposing the covariant conservation condition and evaluating its moments in the near-limit approximation. The reduced stress-energy tensor, defined in terms of the body's proper mass density, then enters the integral momentum balance, yielding a tensor equation that takes the form of the geodesic equation. Finally, a broader perspective on the weak equivalence principle is discussed.

I. Introduction

In A. Einstein and N. Rosen "The Particle Problem in the General Theory of Relativity" one can read: "One of the imperfections of the original relativistic theory of gravitation was that as a field theory it was not complete; it introduced the independent postulate that the law of motion of a particle is given by the equation of the geodesic." [1]. This postulate states: *Free massive point particles traverse timelike geodesics.* Einstein tried to remedy that shortcoming without success. "Over the last century numerous ostensible proofs claiming to have derived the geodesic principle from Einstein's field equations have been developed. (…) Grouping these results into three major families, which I refer to as (1) limit operation proofs, (2) 0th-order proofs, and (3) singularity proofs, (…) none of these strategies successfully demonstrates the geodesic principle, canonically interpreted as a dynamical law that massive bodies must actually follow geodesic paths in Einstein's theory." [2] "By reviewing the three major classes of proof, we have seen that would-be geodesic following bodies are forced either (i) to meet unrealistically restrictive special-case conditions, (ii) to have no matter-energy at all (i.e. vanish), (iii) to violate Einstein's field equations, or (iv) to be located on paths that don't just fail to be geodesic but fail to exist in the space-time manifold at all." [2] "Though the geodesic principle can be recovered as theorem in general relativity, it is not a consequence of Einstein's equation (or the conservation principle) alone. Other assumptions are needed to drive the theorems in question."[3]. Ignored, this issue still remains relevant. – *The following presents a proof of the geodesic principle* and its implications for the concept of passive mass. The proof is non-canonical insofar as it does not directly establish the limiting geodesic itself, but instead shows the solution trajectory converging sufficiently fast to it. This convergence depends on the diameter Ø of a spatial region encompassing both the body and the current point along the geodesic, and requires that the relevant bound grows at most on the order of O(Ø). The argument is formulated in terms of density moments rather than distributions; however, it still belongs to the limit operation proof family. Unlike the Geroch-Jang theorem, it does not rely on the "(strengthened) dominant energy condition" [3]; it assumes solely the physically natural requirement that the energy of a body in its locally inertial (LI) frame: $E_0 = mc^2$ is always positive. Additionally, the metric in a neighborhood of the geodesic, excluding the body's own gravitational contribution, is assumed to be sufficiently smooth, and the construction has to remain strictly compatible with the weak equivalence principle. The analysis focuses on the physically relevant case where the body's density is bounded ($m = O(d^3)$). It is demonstrated that even for $m = O(d)$ the gravitational field of the body remains effectively well separated from the external gravitational field. This separation, enabled by the sufficient rate of convergence, thus limits the test body problem, so that it plays only a marginal role in the overall solution. Whether convergence to a geodesic still holds when the mass is only bounded by $O(d^0)$, which corresponds to the canonical account [2], remains unresolved. – The below proof is carried out in 3 steps. In the first step: (1,2) an appropriate stationary (S)LI coordinate system is constructed. In the second step: (3,4) the approximation uncertainties and errors, and the deviation of the four-momentum derivative are estimated. Lastly, in the third step: (5) the geodesic principle is confirmed using the SE tensor and the geodesic equation is derived from the reduced SE tensor. - For simplicity, natural units are adopted throughout the following sections. Additionally, in summation notation, the corresponding indices are visibly crossed out when summing to provide a clearer overview.

II. The Physics Behind the Geodesic Principle

1) The locally gauged, stationary, locally (within some $\Delta\tau$) inertial coordinate system: $\underline{x}^{\hat{\mu}} : \mathcal{P} \mapsto x^{\hat{\mu}}(\mathcal{P})$

▷ a) A space-time coordinate system: $\underline{x}^{\mu}$ with its basis: $\boldsymbol{e}_\alpha$ and the metric: $g_{\alpha\beta}$. $\eta \equiv [\text{diag}(-1,1,1,1)]$

$$\tau \in \mathbb{R} \ ; \ \ \underline{x}^{\nu} : \ \forall x_{,}^{\nu}(\mathcal{P}(\tau)) \ \exists \Lambda_{\mu}^{\mu'}(x^{\nu} \to x_{,}^{\nu}) \ , \ \ g_{\alpha\beta} \equiv \boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta = \Lambda_{\alpha}^{\alpha'}\Lambda_{\beta}^{\beta'}\eta_{\alpha'\beta'} \ \ ; \tag{1.1}$$

▷ b) For *any* $\underline{x}^{\mu}$, the *stationary locally inertial* (SLI) coordinate system: $\underline{x}^{\hat{\mu}}$ is (implicitly) pre-defined

(1.10) $$\tau := x^{\hat{0}} \quad ; \quad x^{\hat{0}} := x^{\tilde{0}} \ \big| \ x^{\tilde{n}} = 0 \quad \to \quad \mathcal{P}(\tau) := \mathcal{P}(x^{\tilde{0}}, x^{\tilde{n}} = 0) \tag{1.2a,b,c}$$

$$\mathbf{e}_{\tilde{\alpha}} \equiv \mathbf{e}_{\tilde{\alpha}}(\tau) := \boldsymbol{e}_{\tilde{\alpha}}(x^{\tilde{0}}, \ x^{\tilde{n}} \to 0) \quad ; \quad \Lambda_{\tilde{\nu}}^{\mu} \equiv \Lambda_{\tilde{\nu}}^{\mu}(\tau) := \Lambda_{\tilde{\nu}}^{\mu}(x^{\tilde{0}}, \ x^{\tilde{n}} \to 0) \tag{1.3a,b}$$

(2.5a)
(2.5b) $$\boldsymbol{\Delta}\mathcal{P}(\tau) = \boldsymbol{e}_{\boldsymbol{\alpha}}(\mathcal{P}(\tau))\Lambda_{\tilde{0}}^{\boldsymbol{\alpha}}(\tau)\Delta\tau := \mathbf{e}_{\tilde{0}}(\tau)\Delta\tau \ \big| \ \Delta\tau \to 0 \quad \to \quad \frac{\mathrm{d}x_{,}^{\mu}(\mathcal{P}(\tau))}{\mathrm{d}\tau} = \Lambda_{\tilde{0}}^{\mu}(\tau) \tag{1.4a,b}$$

(1.6a)
(1.9a) $$x^{\mu} =: x_{,}^{\mu}(\mathcal{P}(\tau)) + \Lambda_{\tilde{\boldsymbol{n}}}^{\mu}(\tau)x^{\tilde{\boldsymbol{n}}} + 2^{-1}\Lambda_{\tilde{\boldsymbol{n}},\tilde{\boldsymbol{m}}}^{\mu}(\tau)x^{\tilde{\boldsymbol{n}}}x^{\tilde{\boldsymbol{m}}} \ \ \big| \ 2|x^{\hat{k}}| \leq \emptyset_0 : \text{"small enough"} \tag{1.5}$$

▷ c) Conditions for the SLI basis in the (infinitesimal) *neighborhood*: $x^{\tilde{n}} \to 0$; *of* any point: $\mathcal{P}(\tau)$ of the trajectory following the geodesic (a kind of situation like inside a freely moving non-rotating spaceship)

$$\Lambda_{\tilde{\nu}}^{\mu}(\tau) : \ \mathrm{g}_{\tilde{\alpha}\tilde{\beta}}(\tau) := g_{\tilde{\alpha}\tilde{\beta}}(x^{\tilde{0}}, \ x^{\tilde{n}} \to 0) \equiv \mathbf{e}_{\tilde{\alpha}} \cdot \mathbf{e}_{\tilde{\beta}} := \eta_{\tilde{\alpha}\tilde{\beta}} \quad \to \quad \mathbf{e}_{\tilde{0}} \cdot \mathbf{e}_{\tilde{0}} = -1 \tag{1.6a,b}$$

(1.6b)
(2.5b) $$\mathbf{e}_{\tau}(\tau) := \mathbf{e}_{\tilde{0}}(\tau) \quad : \quad \frac{\mathrm{d}\mathbf{e}_{\tau}}{\mathrm{d}\tau} \equiv \frac{\partial \mathbf{e}_{\tilde{0}}}{\partial x^{\tilde{0}}} = 0 \quad \to \quad \Gamma^{\tilde{\gamma}}{}_{\tilde{0}\tilde{0}}(\tau) := \Gamma^{\tilde{\gamma}}{}_{\tilde{0}\tilde{0}}(x^{\tilde{0}}, \ x^{\tilde{n}} \to 0) = 0 \tag{1.7a,b}$$

(1.2b)
(1.6a) $$\frac{\mathrm{d}\Lambda_{\tilde{\nu}}^{\mu}}{\mathrm{d}\tau} \equiv \frac{\partial\Lambda_{\tilde{\nu}}^{\mu}}{\partial x^{\tilde{0}}} \equiv \frac{\partial\Lambda_{\tilde{0}}^{\mu}}{\partial x^{\tilde{\nu}}} \ : \ \frac{\partial \mathbf{e}_{\tilde{\alpha}}}{\partial x^{\tilde{0}}} \equiv \frac{\partial \mathbf{e}_{\tilde{0}}}{\partial x^{\tilde{\alpha}}} = 0 \quad \to \quad \Gamma^{\tilde{\gamma}}{}_{\tilde{\alpha}\tilde{0}}(\tau) := \Gamma^{\tilde{\gamma}}{}_{\tilde{\alpha}\tilde{0}}(x^{\tilde{0}}, \ x^{\tilde{n}} \to 0) = 0 \tag{1.8a,b}$$

$$\to \quad \Lambda_{\tilde{m},\tilde{n}}^{\mu} \equiv \Lambda_{\tilde{n},\tilde{m}}^{\mu} := \frac{\partial\Lambda_{\tilde{n}}^{\mu}}{\partial x^{\tilde{m}}} \ : \ \frac{\partial \mathbf{e}_{\tilde{n}}}{\partial x^{\tilde{m}}} \equiv \frac{\partial \mathbf{e}_{\tilde{m}}}{\partial x^{\tilde{n}}} = 0 \quad \to \quad \Gamma^{\tilde{\gamma}}{}_{\tilde{\alpha}\tilde{\beta}}(\tau) := \Gamma^{\tilde{\gamma}}{}_{\tilde{\alpha}\tilde{\beta}}(x^{\tilde{0}}, \ x^{\tilde{n}} \to 0) = 0 \tag{1.9a,b}$$

▷ d) The SLI gauge transformation of the $\underline{x}^{\tilde{\mu}}$ SLI coordinates and the SLI Lorentz gauge as its example

$$2|x^{\hat{k}}| \leq \emptyset_0 \quad \Rightarrow \quad x^{\hat{\mu}} := x^{\tilde{\mu}} + \hat{\xi}^{\tilde{\mu}} \ \ \big| \ \ \hat{\xi}^{\tilde{\mu}}(\tau, 0) = 0 \ , \ \hat{\xi}^{\tilde{\mu}}{}_{,\tilde{\nu}}(\tau, 0) = 0 \ , \ \hat{\xi}^{\tilde{\mu}}{}_{,\tilde{\nu},\tilde{\kappa}}(\tau, 0) = 0 \tag{1.10a,b}$$

[6] $$\hat{\xi}^{\tilde{\mu}}(\tau, x^{\tilde{n}}) : \ \bar{h}^{\hat{\alpha}\hat{\nu}}{}_{,\hat{\nu}} = 0 \quad \leftarrow \quad \bar{h}^{\hat{\alpha}\hat{\beta}} := h^{\hat{\alpha}\hat{\beta}} - 2^{-1}\,\eta^{\hat{\alpha}\hat{\beta}}\eta^{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}h_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} \quad \leftarrow \quad h_{\hat{\alpha}\hat{\beta}} := \Delta g_{\hat{\alpha}\hat{\beta}} \tag{1.11a,b,c}$$

2) General definitions in the context of the body's stress-energy (SE) tensor field: $T^{\alpha\beta}(x^{\nu}) := T^{\beta\alpha}(x^{\nu})$

▷ a) A closed spatial region: $\underline{V}(\tau) : \in \underline{\boldsymbol{V}}$ with minimal diameter: Ø containing the whole body *and* $\mathcal{P}(\tau)$

$$\underline{V} := \underline{V}(\tau) : \ \underline{V} \cup \partial\underline{V} = \underline{V} \ , \ \ \mathcal{P}(x^{\hat{\nu}}) \in \underline{V}(\tau) \ \ \Rightarrow \ \ \mathcal{P}(x^{\hat{\nu}}) = \mathcal{P}(\tau, x^{\hat{n}}) \ \ ; \tag{2.1}$$

$$\underline{V}(\tau) : \ \mathcal{P}(\tau, x^{\hat{n}}) \in (\partial\underline{V} \cup \overline{\underline{V}}) \ \ \Rightarrow \ \ T^{\hat{\alpha}\hat{\beta}}(\tau, x^{\hat{n}}) = 0 \tag{2.2}$$

The body diameter:
(2.5a) $$\mathrm{d} := \mathrm{d}(\tau) :\leq \emptyset := \emptyset(\tau) := \emptyset\big(\underline{V}(\tau)\big) :\leq \frac{1}{2}\emptyset_0 \quad ; \quad \emptyset(\tau_0) := \mathrm{d}(\tau_0) \tag{2.3a,b}$$

▷ b) The notation of the proper 3D area integral (taken over the $\mathrm{d}\tau$-slab of the space-time domain: $\underline{\boldsymbol{V}}$)

(1.2a) $$< f > := \int_{\underline{V}(\tau)} f \ |\mathrm{d}\underline{\boldsymbol{V}}| \ := \lim_{\Delta\tau \to 0} \frac{1}{\Delta\tau}\int_{x^{\hat{0}}}^{x^{\hat{0}}+\Delta\tau} \int_{\underline{V}(\tau)} f \ \sqrt{-g} \ |\mathrm{d}x^{\hat{1}}\mathrm{d}x^{\hat{2}}\mathrm{d}x^{\hat{3}}\Delta\tau| \tag{2.4a,b}$$

▷ c) Synchronizing (initial) condition for $\underline{x}^{\hat{\mu}} \mid x^{\hat{0}} = \tau_0$, which codetermine the matrix: $\Lambda_{\tilde{\nu}}^{\mu}$ at $\tau = \tau_0$

(5.2)
(1.4,5)
$$\begin{cases} \mathcal{P}(\tau_0) \ : \ < x^{\hat{n}}T^{\hat{0}\hat{0}}(\tau_0, x^{\hat{n}}) > \ = \ 0 \quad \leftarrow \quad \text{1st moments of } T^{\hat{0}\hat{0}} \ \leftarrow \ \text{position: } x^{\hat{n}} & (2.5\text{a}) \\ \mathbf{e}_{\tilde{0}}(\tau_0) : \ \ < T^{\hat{n}\hat{0}}(\tau_0, x^{\hat{n}}) > \ = 0 \quad \leftarrow \quad \text{0th moment of } T^{\hat{n}\hat{0}} \ \leftarrow \ \text{velocity} & (2.5\text{b}) \end{cases}$$

If the SLI coordinate system satisfies this condition at $\tau = \tau_0$, it represents the (locally inertial momentarily comoving) *proper frame* (of reference) on $\underline{\boldsymbol{V}}(\tau \to \tau_0)$; and, as long as $2\emptyset \leq \emptyset_0$ holds, the *locally inertial comoving frame* (of reference). - The coordinate-invariant parameter: τ is known as proper time.

3) Locally inertial coordinate system: vector integration and the spatial integral of SE tensor divergence

▷ a) The flat basis approximation and factoring of the local coordinate basis out of the spatial integral
The norm: $|x^{\bar{n}}|$ is defined as the *spatial* distance, which is the shortest length from $\mathcal{P}(\tau, x^{\bar{n}})$ to $\mathcal{P}(\tau, 0^{\bar{n}})$.

$$\mathbf{e}_\alpha := \boldsymbol{e}_\alpha\big(x^\nu \to x_,{}^\nu(\mathcal{P}_0)\big) \quad \to \quad \frac{\partial \mathbf{e}_\alpha}{\partial x^\beta} \equiv \Gamma^{\gamma}{}_{\alpha\beta}\mathbf{e}_\gamma \equiv \Gamma^{\gamma}{}_{\beta\alpha}\mathbf{e}_\gamma := \Gamma^{\gamma}{}_{\alpha\beta}\big(x^\nu \to x_,{}^\nu(\mathcal{P}_0)\big)\, \mathbf{e}_\gamma \qquad (3.1a,b)$$

$$\boldsymbol{e}_\alpha = \mathbf{e}_\alpha + \Gamma^{\mu}{}_{\alpha\nu}\mathbf{e}_\mu x^\nu + \tilde{O}\left(2^{-1}\left|\Gamma^{\mu}{}_{\alpha\kappa,\gamma} + \Gamma^{\beta}{}_{\alpha\kappa}\Gamma^{\mu}{}_{\beta\gamma}\right| \mathbf{e}_\mu |x^\kappa x^\gamma|\right) \qquad (3.2)$$

$$\Phi \;:\; \underline{x}^\mu \to \underline{x}^{\bar{\mu}}(\mathcal{P}_0) \;:\; x^{\bar{\mu}}(\mathcal{P} = \mathcal{P}_0) = 0 \;,\; \Gamma^{\bar{\gamma}}{}_{\bar{\alpha}\bar{\beta}} = 0 \;,\; g_{\bar{\mu}\bar{\nu};\bar{\beta}} \equiv 0 \quad \to \quad g_{\bar{\mu}\bar{\nu},\bar{\beta}} = 0 \qquad (3.3a,b)$$

$$\boldsymbol{e}_{\bar{\alpha}} = \mathbf{e}_{\bar{\alpha}} + \tilde{O}\big(2^{-2}\big|(\Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\kappa},\bar{\gamma}} + \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\gamma},\bar{\kappa}}) - R^{\bar{\mu}}{}_{\bar{\alpha}\bar{\kappa}\bar{\gamma}}\big||x^{\bar{\kappa}}||x^{\bar{\gamma}}|\mathbf{e}_{\bar{\mu}}\big) = \mathbf{e}_{\bar{\alpha}} + \tilde{O}^{\bar{\mu}}{}_{\bar{\alpha}}(|x^{\bar{0}}|^2 + |x^{\bar{n}}|^2)\mathbf{e}_{\bar{\mu}} \qquad (3.4)$$

$$\big(x^{\bar{0}} \leftarrow 0\big) \;\to\; \boldsymbol{e}_{\bar{\alpha}} = \mathbf{e}_{\bar{\alpha}} + \tilde{O}^{\bar{\mu}}{}_{\bar{\alpha}}(|x^{\bar{n}}|^2)\mathbf{e}_{\bar{\mu}} \qquad (3.5)$$

$$Y^\nu \!:\in \mathbb{R}^4 \;\to\; < Y^{\bar{\alpha}}\boldsymbol{e}_{\bar{\alpha}} > = < Y^{\bar{\alpha}}\mathbf{e}_{\bar{\alpha}} > + < Y^{\bar{\alpha}}\, \tilde{\mathbf{O}}_{\bar{\alpha}}(|x^{\bar{n}}|^2) > = < Y^{\bar{\alpha}} > \mathbf{e}_{\bar{\alpha}} + \|Y^{\bar{\alpha}}\|\, \tilde{\mathbf{O}}_{\bar{\alpha}}(\emptyset^2) \qquad (3.6)$$

(1.9b)
(3.3a)
$$\forall \tau, \quad \underline{\underline{x^{\hat{\mu}} - x_,{}^{\hat{\mu}}(\mathcal{P}(\tau,))}} \subset \underline{x^{\bar{\mu}}(\mathcal{P}(\tau,))} \quad \Big| \quad 2|x^{\hat{k}}| \le \emptyset_0 \qquad (3.7)$$

This means that (3.6) is also valid for $\underline{x}^{\hat{\mu}}$. For the rest the LI coordinates components can be separated:

(3.6)
$$\forall \hat{\alpha} \qquad < Y^{\hat{\alpha}}\boldsymbol{e}_{\hat{\alpha}} > = \Big(< Y^{\hat{\alpha}} > + \tilde{O}^{\hat{\alpha}}\big(\|Y^{\hat{\mu}}\|\emptyset^2\big)\Big)\, \mathbf{e}_{\hat{\alpha}} \qquad (3.8)$$

The uncertainty: $\tilde{O}^{\hat{\alpha}}(\emptyset^2)$ is the cost of making vector integration taken over a curved domain justifiable.

▷ b) The total neutrality of spatial divergence for the body's SE tensor field in the LI comoving frame

(2.2)
$$T^{\hat{\alpha}\hat{\beta}}(\mathcal{P} \in \underline{S} := \partial\underline{V}) \equiv 0 \;\to\; < T^{\hat{\mu}\hat{n}}{}_{,\hat{n}}(\tau, x^{\hat{k}}) > + \;\tilde{O}^{\hat{\mu}}(m\emptyset^2) = \oint_{\underline{S}} T^{\hat{\mu}\hat{n}}(\tau, x^{\hat{k}})\, n_{\hat{n}} |d\underline{\boldsymbol{S}}| \equiv 0 \qquad (3.9)$$

4) Finding the $O(m\emptyset^2)$ approximation of the four-momentum temporal partial derivative for a small body

▷ a) The test body problem: the cross effect of gravity fields, convergence of the upper bound estimate
Based on the equation for weak gravitational field and its source given by a hypothetical scalable massive body with a density limit (no singularities), which on closer inspection, can be treated as the superposition of the infinitesimally small, dispersed point masses with their $d\bar{h}_{\hat{0}\hat{0}} \approx 4d\rho/r$ [6] partial fields; by negligible external reflections, the following basic estimates of the metric deviation and its derivatives can be obtained in the originally (without the body's active mass as the field source) LI comoving frame.

[6]
(1.11)
$$\eta^{\hat{\mu}\hat{\nu}}\, \bar{h}^{\hat{\alpha}\hat{\beta}}{}_{,\hat{\mu},\hat{\nu}} \equiv \eta^{\hat{\mu}\hat{\nu}} \left(h^{\hat{\alpha}\hat{\beta}} - \frac{1}{2}\,\eta^{\hat{\alpha}\hat{\beta}} h\right)_{,\hat{\mu},\hat{\nu}} = -16\pi T^{\hat{\alpha}\hat{\beta}} \quad \leftarrow \quad h = h^{\hat{\alpha}}{}_{\hat{\alpha}} = \eta^{\hat{\alpha}\hat{\beta}} h_{\hat{\alpha}\hat{\beta}} \qquad (4.1a,b)$$

(2.3)
(5.1)
$$T^{\hat{\alpha}\hat{\beta}} := O^{\hat{\alpha}\hat{\beta}}(d^0) \in \mathbb{R}^{4\times 4} \;;\; m := \left\|T^{\hat{\alpha}\hat{\beta}}\right\| \quad \to \quad m = 8m_0\emptyset_0^{-3}d(\tau_0)^3 = \tilde{O}(d^3) \qquad (4.2a..c)$$
⁋(4.3)

$$g_{\hat{\alpha}\hat{\beta}} := \eta_{\hat{\alpha}\hat{\beta}} + \Delta g_{(ex)\hat{\alpha}\hat{\beta}} + \Delta g_{(in)\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}} + h_{(ex)\hat{\alpha}\hat{\beta}} + h_{(in)\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}} + O_{\hat{\alpha}\hat{\beta}}(\emptyset^2) + O_{\hat{\alpha}\hat{\beta}}(md^{-1})$$

$$h_{(ex)\hat{\alpha}\hat{\beta}} = O_{...}(\emptyset^2) \;\to\; h_{(ex)\hat{\alpha}\hat{\beta},\hat{\gamma}} = O_{...}(\emptyset) \;\to\; h_{(ex)\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon}} = O_{...}(\emptyset^0) \;\to\; h_{(ex)\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon},\hat{\kappa}} = O_{...}(\emptyset^0) \qquad (4.4a..d)$$

$$h_{(in)\hat{\alpha}\hat{\beta}} = O_{...}(md^{-1}) \to h_{(in)\hat{\alpha}\hat{\beta},\hat{\gamma}} = O_{...}(md^{-2}) \to h_{(in)\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon}} = O_{...}(md^{-3}) \to h_{(in)\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon},\hat{\kappa}} = O_{...}(md^{-4}) \qquad (4.5a..d)$$

These fields: of the body (internal) and of the externally imposed curvature; are superimposed, resulting in two partial Christoffel symbol fields and creating additionally the cross-term: Δ_X in whose range the separation between the two fields is no longer definite. The convergence order of this term and its derivatives can be determined. As Newton's first law applies, the internal field itself in an inertial system can have no effect on the overall motion; hence, it is arbitrarily omitted, but the cross-term remains valid.

[6]
(4.3)⁋
$$\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} := \Gamma_{(in)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} + \Gamma_{(ex)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} + \Delta_X{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} \quad \leftarrow \quad \Gamma^{\mu}{}_{\alpha\beta} \equiv 2^{-1} g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}) \qquad (4.6a,b)$$
⁋(4.7)

$$\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} \approx \frac{\{\eta_{\hat{\mu}\hat{\nu}} - h_{(in)\hat{\mu}\hat{\nu}} - h_{(ex)\hat{\mu}\hat{\nu}}\}^{\hat{\mu}\hat{\nu}}}{2}\Big(h_{(in)\hat{\nu}\hat{\alpha},\hat{\beta}} + h_{(in)\hat{\nu}\hat{\beta},\hat{\alpha}} - h_{(in)\hat{\alpha}\hat{\beta},\hat{\nu}} + h_{(ex)\hat{\nu}\hat{\alpha},\hat{\beta}} + h_{(ex)\hat{\nu}\hat{\beta},\hat{\alpha}} - h_{(ex)\hat{\alpha}\hat{\beta},\hat{\nu}}\Big)$$

$$\Delta_{X}{}^{\hat\mu}{}_{\hat\alpha\hat\beta} \approx \frac{\{h_{(in)\hat\mu\hat\vartheta}\}^{\hat\mu\hat\vartheta}}{2}\left(h_{(ex)\hat\alpha\hat\beta,\hat\vartheta} - h_{(ex)\hat\vartheta\hat\alpha,\hat\beta} - h_{(ex)\hat\vartheta\hat\beta,\hat\alpha}\right) + \frac{\{h_{(ex)\hat\mu\hat\vartheta}\}^{\hat\mu\hat\vartheta}}{2}\left(h_{(in)\hat\alpha\hat\beta,\hat\vartheta} - h_{(in)\hat\vartheta\hat\alpha,\hat\beta} - h_{(in)\hat\vartheta\hat\beta,\hat\alpha}\right)$$

(4.4,5,6) (2.3b) $$\left(\Gamma_{(in)}{}^{\hat\mu}{}_{\hat\alpha\hat\beta} \leftarrow 0\ ,\ \tau \to \tau_0\right) \Rightarrow \Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta} = \Gamma_{(ex)}{}^{\hat\mu}{}_{\hat\alpha\hat\beta} + \Delta_{X}{}^{\hat\mu}{}_{\hat\alpha\hat\beta} = \Gamma_{(ex)}{}^{\hat\mu}{}_{\hat\alpha\hat\beta} + O^{\hat\mu}{}_{\hat\alpha\hat\beta}(\mathrm{md}^0)$$ ☝(4.8) (4.9a,b)

$$\rightarrow \quad \Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta,\hat\gamma} = \Gamma_{(ex)}{}^{\hat\mu}{}_{\hat\alpha\hat\beta,\hat\gamma} + O^{\hat\mu}{}_{\hat\alpha\hat\beta\hat\gamma}(\mathrm{md}^{-1}) \quad \rightarrow \quad \Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta,\hat\gamma,\hat\varepsilon} = \Gamma_{(ex)}{}^{\hat\mu}{}_{\hat\alpha\hat\beta,\hat\gamma,\hat\varepsilon} + O^{\hat\mu}{}_{\hat\alpha\hat\beta\hat\gamma\hat\varepsilon}(\mathrm{md}^{-2}) \quad (4.9c,d)$$

Based on (4.9), the deviation caused by the cross-effect in the equation (4.19) converges one order faster than the approximation error. Moreover, it is worth noting that even for m = O(d), the solutions (5.8,21) retain convergence at the O(mØ) rate. Therefore, the cross-term: Δ_X is neglected from this point onward.

▷ b) An approximate factoring of the Christoffel symbol out of the spatial integral over the volume: $\underline{V}$

$$g_{\hat\alpha\hat\beta} := \eta_{\hat\alpha\hat\beta} + h_{(ex)\hat\alpha\hat\beta} \;\rightarrow\; \Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta} = \Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta} + \Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta,\hat n}x^{\hat n} + O\left(2^{-1}\left|\Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta,\hat k,\hat l}\right| |x^{\hat k}x^{\hat l}|\right) = \Gamma_{(ex)}{}^{\hat\mu}{}_{\hat\alpha\hat\beta} \quad (4.10a,b)$$

(4.6b)(1.6a) $$\Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta}(\tau) = 2^{-1}g^{\hat\mu\hat\vartheta}\left(g_{\hat\vartheta\hat\alpha,\hat\beta} + g_{\hat\vartheta\hat\beta,\hat\alpha} - g_{\hat\alpha\hat\beta,\hat\vartheta}\right) = \Gamma_{(ex)}{}^{\hat\mu}{}_{\hat\alpha\hat\beta}\left(x^{\hat 0},\ x^{\hat n} \to 0\right) \quad (4.11)$$

(4.2) $$< \Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta}T^{\hat\gamma\hat\nu} > = \Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta} < T^{\hat\gamma\hat\nu} > + \Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta,\hat n} < x^{\hat n}T^{\hat\gamma\hat\nu} > + O^{\hat\mu}{}_{\hat\alpha\hat\beta}\left(\|T^{\hat\gamma\hat\nu}\|Ø^2\right) \quad (4.12)$$

With (1.9), an estimate of an upper bound for deviation of the temporal partial derivative (4.19) results.

(4.12) (1.9b) $$< \Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta}T^{\hat\gamma\hat\nu} > = O\left(\left|\Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta,\hat n}\right| \|T^{\hat\gamma\hat\nu}\|Ø\right) + O^{\hat\mu}{}_{\hat\alpha\hat\beta}\left(\|T^{\hat\gamma\hat\nu}\|Ø^2\right) = O^{\hat\mu}{}_{\hat\alpha\hat\beta}\left(\|T^{\hat\gamma\hat\nu}\|Ø\right) \quad (4.13)$$

(4.2) $$< \Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta}T^{\hat\alpha\hat\beta} > = O^{\hat\mu}(mØ) \quad ; \quad < \Gamma^{\hat\beta}{}_{\hat\alpha\hat\beta}T^{\hat\mu\hat\alpha} > = O^{\hat\mu}(mØ) \quad (4.14a,b)$$

▷ c) The temporal partial derivative of the four-momentum, obtained from the conservation condition:

$$T^{\mu\beta}{}_{;\beta} \equiv T^{\mu\beta}{}_{,\beta} + \Gamma^{\mu}{}_{\alpha\beta}T^{\alpha\beta} + \Gamma^{\beta}{}_{\alpha\beta}T^{\mu\alpha} := 0 \quad (4.15)$$

$$T^{\mu\beta}{}_{,\beta} = -\Gamma^{\mu}{}_{\alpha\beta}T^{\alpha\beta} - \Gamma^{\beta}{}_{\alpha\beta}T^{\mu\alpha} \quad (4.16)$$

(3.8) $$< T^{\hat\mu\hat\beta}{}_{,\hat\beta} > = -< \Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta}T^{\hat\alpha\hat\beta} > - < \Gamma^{\hat\beta}{}_{\hat\alpha\hat\beta}T^{\hat\mu\hat\alpha} > + \tilde O^{\hat\mu}(mØ^2) \quad (4.17)$$

(3.9) $$< T^{\hat\mu\hat 0}{}_{,\hat 0} > = -< \Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta}T^{\hat\alpha\hat\beta} > - < \Gamma^{\hat\beta}{}_{\hat\alpha\hat\beta}T^{\hat\mu\hat\alpha} > + \tilde O^{\hat\mu}(mØ^2) \quad (4.18)$$

(4.12,14) (1.9b) $$< T^{\hat\mu\hat 0}{}_{,\hat 0} > = -\Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta,\hat n} < x^{\hat n}T^{\hat\alpha\hat\beta} > - \Gamma^{\hat\beta}{}_{\hat\alpha\hat\beta,\hat n} < x^{\hat n}T^{\hat\mu\hat\alpha} > + O^{\hat\mu}(mØ^2) = O^{\hat\mu}(mØ) \quad (4.19)$$

5) The freely falling small body and its, founded on the conservation conditions, near geodesic solutions

Because $\mathcal{P}(\tau_0)$ can be any given point on the geodesic, in the ongoing section it's assumed that the body is currently situated in the *proper* or at least in the LI *comoving* frame of reference, the behavior of the body in the vicinity of the spatial coordinate origin: $\mathcal{P}(\tau)$ is analyzed, and if the result follows the geodesic in the limit case (for $d \to 0$), it must also follow it inside the $Ø_0$ tube for $d > 0$ over a time period.

▷ a) The proper- (rest) mass: m, four-position: $x^{\hat\alpha}$, four-velocity: $U^{\hat\alpha}$, and the rest (minimum) energy: E_0

(3.8) (2.3b) $$m(\tau = \tau_0) := < T^{\hat 0\hat 0}(\tau_0, x^{\hat n}) > + \tilde O(\mathrm{md}^2) \quad (5.1)$$

(1.2a) (2.3a) $$x^{\hat 0}(\tau) := \tau \quad ; \quad x^{\hat n}(\tau) := < T^{\hat 0\hat 0}(\tau, x^{\hat n}) >^{-1} < x^{\hat n}T^{\hat 0\hat 0}(\tau, x^{\hat n}) > + \tilde O^{\hat n}(Ø^2 d) \quad (5.2a,b)$$

(2.3b,5a) (1.2a) $$x^{\hat 0}(\tau_0) = \tau_0 \quad ; \quad x^{\hat n}(\tau_0) = \tilde O^{\hat n}(d^3) \quad (5.3a,b)$$

$$U^{\hat\nu} := \frac{\partial x^{\hat\nu}}{\partial\tau} \equiv \frac{\partial x^{\hat\nu}}{\partial x^{\hat 0}} \;\Bigg|\; |U^{\hat n}| \ll 1 \;\leftarrow\; \frac{\partial^2 O^{\hat\alpha}(d^l, \tau \to \tau_0)}{\partial\tau^2} = O^{\hat\alpha}(d^l) \;\Leftarrow\; \frac{O^{\hat\alpha}(d^l, \tau \to \tau_0)}{d^l} \in C^2 \quad (5.4a..c)$$

(2.5b) (2.3b) $$E_0 := E(\tau_0) = \min\left(m(\tau_0)/\sqrt{1 - U^{\hat n}(\tau_0)U_{\hat n}(\tau_0)}\right) \Rightarrow U^{\hat n}(\tau_0) = \tilde O^{\hat n}(d^3) \quad (5.5a,b)$$

(5.2a,4) $$U^{\hat 0}(\tau_0) = 1\ ,\ U^{\hat n}(\tau_0) = \tilde O^{\hat n}(d^3) \quad \rightarrow \quad U^{\hat 0}{}_{,\hat 0}(\tau_0) = 0 \ ; \ U^{\hat\mu}{}_{,\hat k}(\tau_0) \equiv 0 \quad (5.6a,b,c)$$

▷ b) The solution based on the stress-energy tensor, in the locally inertial comoving frame of reference
The four momentum: $p^{\bar\mu}$ can be defined as the 'T4-momentum': $< T^{\bar\mu 0} >$ or as the four-velocity based 'U4-momentum': $mU^{\bar\mu}$. Assuming they are equivalent and the mass does not change, then for $|U^{\hat n}| \ll 1$

(1.2a) (4.19) (4.12)℘ $$\frac{\mathrm{d}p^{\hat\mu}}{\mathrm{d}\tau} = -\Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta,\hat n} < x^{\hat n} T^{\hat\alpha\hat\beta} > - \Gamma^{\hat\beta}{}_{\hat\alpha\hat\beta,\hat n} < x^{\hat n} T^{\hat\mu\hat\alpha} > + O^{\hat\mu}(m\emptyset^2) = O^{\hat\mu}(m\emptyset) \qquad (5.7)\ ℘(5.8)$$

$$\frac{\mathrm{d}p^{\hat\mu}}{\mathrm{d}\tau} = -\Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta} < T^{\hat\alpha\hat\beta} > - \Gamma^{\hat\beta}{}_{\hat\alpha\hat\beta} < T^{\hat\mu\hat\alpha} > - \Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta,\hat n} < x^{\hat n} T^{\hat\alpha\hat\beta} > - \Gamma^{\hat\beta}{}_{\hat\alpha\hat\beta,\hat n} < x^{\hat n} T^{\hat\mu\hat\alpha} > + O^{\hat\mu}(m\emptyset^2)$$

(5.1,4) $$p^{\hat\mu} \approx mU^{\hat\mu} \quad \Rightarrow \quad \frac{\mathrm{d}U^{\hat\mu}}{\mathrm{d}\tau} \approx \frac{\mathrm{d}(p^{\hat\mu}/m)}{\mathrm{d}\tau} = \frac{1}{m}\left(\frac{\mathrm{d}p^{\hat\mu}}{\mathrm{d}\tau} - \frac{p^{\hat\mu}}{m}\frac{\mathrm{d}m}{\mathrm{d}\tau}\right) = O^{\hat\mu}(\emptyset) \qquad (5.9a)\ (5.9b)$$

Since the origin of $\underline{x}^{\hat n}$ follows the geodesic, this implication shows that in the limit the body follows the geodesic as well. The tidal forces in (5.7) also depend on the SLI gauge and can influence the trajectory. The critical physical issue is that (5.8) is not coordinate-invariant, and the essential point of this is that the body must freely levitate in the domain: $\underline{V}$ close to the spatial origin, where gravity nearly vanishes. That effect is possible owing to the LI comoving frame, which ensures the local translational symmetries (*t-xyz*) inside $\underline{V}$ in $\underline{x}^{\hat\mu}$. However, these symmetries are not perfect because the derivative of the metric: $g_{\hat\mu\hat\nu,\hat k} = O_{\hat\mu\hat\nu\hat k}(\emptyset) \neq 0$ on $\underline{V}$, and this constitutes the limiting factor for the convergence of the solution.

▷ c) Body's (proper) mass density, a body-energy (BE) tensor (field) and its *local integral* divergence
The state of the body's overall motion is defined by its four-velocity; therefore, to find the tensor equation of motion the SE tensor constituent that incorporates only this four-velocity should be used which, for a small body, is the BE tensor $\boldsymbol{T}_M$ (5.11a) reflecting the convective flux of the matter's mass. However, as simultaneity in the space-time domain: $\underline{\boldsymbol{V}}$ is relative, $\boldsymbol{T}_M$ is ambiguous depending on the SLI $\hat\xi^0$ gauge in the $O(d^2)$ range. Consequently, in order to evade the problem resulting from (5.8), instead of the SE the BE tensor is applied so that gravitation acts exclusively upon this stress-free constituent. Owing to the energy conservation from the τ-translation symmetry (5.14a) and since for $\tau \to \tau_0$ the first moments of $\boldsymbol{T}_M$ are nullified because of the condition (2.5a), the LHS of (5.14b) disappears creating for the $\boldsymbol{T}_M$ field an integral covariant conservation condition that has an $\tilde O(md^2)$ convergence order in the proper frame.
- *The following equations are studied for* $\tau \to \tau_0$, where the offset: $x^{\hat n}$ (5.2b) is negligible due to (2.5a).

(3.3) (3.7) $$\underline{x}^{\bar\nu}(\mathcal{P}(\tau_0)):\ g_{\bar 0\bar n} = 0\ ,\ x^{\bar 0} = x^{\hat 0} - \tau_0 \quad \to \quad \Lambda^{\hat 0}_{\bar\nu} = \delta^0_{\bar\nu} \quad ; \quad \bar\rho(\tau_0) := T^{\bar 0\bar 0}\ ,\ \bar U^{\bar\nu}(\tau_0) := \delta^{\bar\nu}_0 \qquad (5.10\ a..c)$$

(5.6a,18c) (1.9b) $$T_M^{\hat\mu\hat\nu} := \bar\rho\, \bar U^{\hat\mu} \bar U^{\hat\nu} \quad \leftarrow \quad \bar U^{\hat\mu} = \Lambda^{\hat\mu}_{\bar\kappa}\delta^{\bar\kappa}_0 \approx \left(U^{\hat\mu} - \tilde O^{\hat m}(d^3)\right) + \Lambda^{\hat m}_{\bar 0,\hat k,\hat l} x^{\hat k} x^{\hat l} \quad \Big|\tau = \tau_0 \qquad (5.11\ a,b)$$

$$T_S^{\mu\nu} := T^{\mu\nu} - T_M^{\mu\nu} \quad \to \quad T_M^{\bar\mu\bar\nu} T_{S\bar\mu\bar\nu} = 0 \quad \to \quad T_M^{\mu\nu} T_{S\mu\nu} \equiv 0 \quad \to \quad T_M = -\bar\rho\ \ ,\ \ T_S =: 3\bar p \qquad (5.12\ a..d)$$

(3.8,9) (1.6a,9b)(3.3b)℘ $$< T_M^{\hat\mu\hat\beta}{}_{;\hat\beta} > = < T_M^{\hat\mu\hat 0}{}_{,\hat 0} > + < \Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta} T_M^{\hat\alpha\hat\beta} > + < \Gamma^{\hat\beta}{}_{\hat\alpha\hat\beta} T_M^{\hat\mu\hat\alpha} > + \tilde O^{\hat\mu}(md^2) \qquad (5.13)$$

(1.2a) (2.5a) $$g_{\hat\mu\hat\nu,\hat 0} = \tilde O_{\hat\mu\hat\nu}(d^2) \quad \to \quad < T_M^{\hat\mu\hat 0}{}_{,\hat 0} > + < \Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta} T_M^{\hat\alpha\hat\beta} > + < \Gamma^{\hat\beta}{}_{\hat\alpha\hat\beta} T_M^{\hat\mu\hat\alpha} > = \tilde O^{\hat\mu}(md^2) \qquad (5.14a,b)$$

▷ d) The coordinate-invariant solution founded on the BE tensor in the (LI) proper frame of reference
As shown above, (5.14) corresponds directly to (4.18); and, since the body's four-position (5.2) along the world-line is identically defined for both the BE tensor and the SE tensor, the subsequent proof of the geodesic solution for the BE tensor also confirms in the limiting case the result (5.9b) for the SE tensor.

(2.4)(5.14a) (4.12)(2.5a) $$< T_M^{\hat\mu\hat 0} >_{,\hat 0} = -\Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta} < T_M^{\hat\alpha\hat\beta} > - \Gamma^{\hat\beta}{}_{\hat\alpha\hat\beta} < T_M^{\hat\mu\hat\alpha} > + O^{\hat\mu}(md^2) = \tilde O^{\hat\mu}(md^2) \qquad (5.15)$$

(5.4b)℘ (5.11) $$\left(< \bar\rho > U^{\hat 0} U^{\hat\mu}\right)_{,\hat 0} = - < \bar\rho > \Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta} U^{\hat\alpha} U^{\hat\beta} - < \bar\rho > \Gamma^{\hat\beta}{}_{\hat\alpha\hat\beta} U^{\hat\mu} U^{\hat\alpha} + O^{\hat\mu}(md^2) \qquad (5.16)$$

(5.1) $$\left(mU^{\hat 0}U^{\hat\mu}\right)_{,\hat 0} = \left(mU^{\hat 0}\right)_{,\hat 0} U^{\hat\mu} + \left(mU^{\hat 0}\right) U^{\hat\mu}{}_{,\hat 0} = -m\Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta} U^{\hat\alpha} U^{\hat\beta} - m\Gamma^{\hat\beta}{}_{\hat\alpha\hat\beta} U^{\hat\alpha} U^{\hat\mu} + O^{\hat\mu}(md^2) \qquad (5.17)$$

Even though here $\Gamma^{\hat\alpha}{}_{\hat\nu\hat\kappa} = 0$, it is the $\Gamma^{\hat\alpha}{}_{\hat\nu\hat\kappa}$ that carries the key information about the origin of this zero. Because $U^{\hat 0}$ is a constant value and $U^{\hat n} \to 0$, (5.17) can be decomposed into the system of two equations:

(5.6a,b) $$\begin{cases} (mU^{\hat 0}) U^{\hat\mu}{}_{,\hat 0} = -m\Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta} U^{\hat\alpha} U^{\hat\beta} + O^{\hat m}(md^2) \quad \leftarrow \quad \left(\hat m := \hat\mu = 0 \ \Rightarrow\ Y^{\hat m} := 0\right) & (5.18a)\ (5.18c) \\ \left(mU^{\hat 0}\right)_{,\hat 0} U^{\hat\mu} = m_{,\hat 0} U^{\hat 0} U^{\hat\mu} = -m\Gamma^{\hat\beta}{}_{\hat\alpha\hat\beta} U^{\hat\alpha} U^{\hat\mu} + O(md^2) U^{\hat\mu} & (5.18b) \end{cases}$$

(1.2a) (5.3b) (5.6c)

$$\tau \to \tau_0 \begin{cases} m\frac{\eth U^{\hat{\mu}}}{\eth\tau} = -m\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}U^{\hat{\alpha}}U^{\hat{\beta}} + O^{\hat{\mu}}(md^2) \ - m\Gamma^{\mu}{}_{\alpha\beta,n}U^{\alpha}U^{\beta}\tilde{O}^{n}(d^3) & (5.19a) \\ \left(mU^{\hat{\nu}}\right)_{,\hat{\nu}} + m\Gamma^{\hat{\nu}}{}_{\hat{\alpha}\hat{\nu}}U^{\hat{\alpha}} \equiv \left(mU^{\hat{\nu}}\right)_{;\hat{\nu}} = \left(mU^{\hat{0}}\right)_{,\hat{0}} = O(md^2) & (5.19b) \end{cases}$$

The equation (5.19a) turns out to be coordinate-invariant in (5.20a) since it is the four-momentum form of the geodesic equation. The second one: (5.19b) is coordinate-invariant as well, and is equivalent to (5.18b) because the total mass flux of an isolated object in the proper frame does not depend on spatial coordinates. That continuity equation solved in the proper frame, holds in all coordinate systems (5.20b).

(1.2a) (5.6b)

$$\forall\tau \begin{cases} m\frac{\eth U^{\mu}}{\eth\tau} = -m\Gamma^{\mu}{}_{\alpha\beta}U^{\alpha}U^{\beta} + O^{\mu}(md^2) \ \to \ \frac{DU^{\mu}}{\eth\tau} = O^{\mu}(d^2) & \textbf{(5.20a)}\ (5.20c) \\ \frac{\eth m}{\eth\tau} = O(md^2) & \textbf{(5.20b)} \end{cases}$$

▷ e) The limit case turns out to be the (rest) mass conservation law and the standard geodesic equation:

[6] (5.4)

$$\lambda := a\tau + b\ ,\quad d \to 0 \qquad \Rightarrow \qquad \frac{\eth^2 x^{\mu}}{\eth\lambda^2} = -\Gamma^{\mu}{}_{\alpha\beta}\frac{\eth x^{\alpha}}{\eth\lambda}\frac{\eth x^{\beta}}{\eth\lambda} \qquad \blacksquare \qquad (5.21)$$

III. Summary

The behavior of a massive body situated in a gravitational field, free of other influences and with *negligible* radiation, has been analyzed here. The body is characterized by its stress–energy (SE) tensor field, which, along with the geodesic curve it follows, determines the space-time domain: $\underline{\boldsymbol{V}}$ in which it exists. Based on general coordinates, a stationary locally inertial (SLI) coordinate system has been constructed within this region. The SLI coordinates allow one to describe the motion of the body suitably within a finite time interval. When their spatial origin coincides with the comoving body at a given τ, they define the proper frame there. This configuration is achieved by selecting proper initial conditions for the SLI coordinate system itself. The proper frame remains locally inertial (LI) in the neighborhood of its origin. The spatial origin of the SLI system follows a geodesic, forming a kind of geodesic tunnel in space-time; however, this itself does not affect the tensorial solution. The requested SLI gauge is intended to suppress fictitious tidal forces, particularly in flat regions; nevertheless, it may be replaced, without affecting the convergence-order estimations, by any gauge satisfying the SLI boundary conditions at the spatial origin. – Furthermore, the local integral divergence of the SE tensor over the spatial domain plays a significant role within this framework. If the body is sufficiently small and the local coordinate basis is adequately flattened, four distinct conservation equations for energy and momentum arise within the LI comoving frame. These follow from the zero divergence of the SE tensor due to translational symmetries, which in curved space-time are only locally valid through approximation. It is therefore essential to estimate the convergence rate of the associated uncertainties, deviations, and errors. To obtain upper-bound estimates, big-O notation, which represents *a member* of a class of functions with explicitly specified convergence, has been employed. Since the SE tensor field has a limited spatial extent, its local integral divergence reduces to the temporal derivative. This reduction is crucial, since it enables the derivation of the body's equation of motion from the zero divergence of the SE tensor. To ensure that the test-body approximation remains non-critical, cross-effects of gravitational fields has been shown to be negligible up to m=O(d). – Based on the SE tensor, the geodesic trajectory of the body is determined only in the SLI coordinates. This reinforces the concept that the SE tensor by itself is only conditionally suitable to be a basis for the geodesic equation. To address this limitation, the SE tensor has been reduced by arbitrarily setting all its subcomponents, except energy density, to zero in the *orthochronous proper* frame. This method defines the BE tensor that depends only on the body's mass density, while excluding contributions from its stress-density field. The latter is expressed as the body-stress (BS) tensor: $\boldsymbol{T}_S$ (5.12a) comprising spatial stress but also internal momentum density integrating to zero in the proper frame due to the condition (2.5b). Exploiting the local time symmetry of the SLI system then directly gives rise to the integral conservation condition for the body-energy component of the SE tensor. By replacing the SE tensor with its BE tensor, the dependence on the BS tensor is eliminated, reflecting this condition. The corresponding momentum-balance equation now becomes solvable once it is split into a system of two equations: the first equation governing the acceleration of the body and the second equation determining the mass change of the body. Ultimately, these equations in the boundary case are represented by the tensor equations that correspond to the geodesic equation and the mass conservation law for freely falling bodies in a general space-time.

IV. Conclusions and Discussion

For a small body diameter: d there are two verifiable possibilities: The effective gravitational tidal forces: *A*) *can* influence the trajectory of the freely falling body at the O(d) deviation level. *B*) *cannot* because the internal stress tensor field of the body vanishes or, on average, doesn't interact with the gravity field.
– It has been proven that even if no coordinate-invariant equation of motion based on the SE tensor has been found, free 'point' bodies associated with their SE tensors traverse timelike geodesics. Essentially, the covariant conservation condition of the SE tensor is sufficient to determine this, provided the body is isolated from all conventional forces and, according to the local symmetries in the LI comoving frame of the SLI system, is additionally apparently isolated from gravity. In this case, possibility "*A*" arises if the distribution of any component of the SE tensor does not correspond to the mass density of the body.
– Simultaneously this statement can be extended analogously to a solution that involves the body-energy (BE) tensor, which is the stress-free constituent of the SE tensor, however in this case the direct coordinate-invariant solution is the geodesic equation; yet should "*A*" be true, a body's trajectory error at the O(d) level may occur. Thereby, the BE tensor provides a mechanism through which gravity influences the massive body, while maintaining the conservation of the body's mass. Applying the SE tensor instead of the BE tensor in a tensor equation of motion may lead to disregard for the weak equivalence principle, but given strong reasons to believe that this principle remains valid, the solution on the BE tensor basis is preferable since it eliminates the theoretical dependence on the stress tensor of the matter of the body.
- As indicated by that, the following thesis may be proposed, namely: The total gravitational influence on a sufficiently small freely falling massive body arises solely from the interaction with its BE tensor.
- When formulated in this manner, the above thesis in the near-limit case leads not only to the geodesic principle and the weak equivalence principle, but also to the mass conservation law for an isolated body. It further implies that rest mass and passive gravitational mass must always be equal. Besides, this approach has the added advantage of not relying on a quasi-mathematical axiom, but instead actually being grounded in the local integral conservation condition for the BE tensor field within the SLI proper frame.

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For now, there is an opportunity to substantiate the weak equivalence principle for (*not only stable*) extended objects in a (locally) uniform gravitational field. Since space-time is flat there, curvature-related integration uncertainty vanishes and so the derivation of the geodesic principle is exact for such objects. In this case, owing to the translational symmetries, internal forces within the object are unable to accelerate the object as a whole, so these symmetries reveal their masking effect on the stress tensor (5.12a). Hence, only the velocity of the object determines its world-line path in this setup. Yet, it is important to consider that the position of such an object is defined in its (locally) inertial *proper* frame, and is unambiguous only in this integration context. Thus, the outcome is always the trajectory of its *proper* position.
- Therefore, under these circumstances a short-term solution, expressed as a specific geodesic trajectory within a flat space-time domain, complies with the criterion defined by the weak equivalence principle.

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Since the geodesic principle has not been derived here from the geometrical approach but from the conservation condition, it is natural not to restrict oneself to freely falling bodies alone and external influences can also be considered. In the case of the Lorentz force, it introduces the tensor equation of motion:

$$(5.20a) \qquad m\frac{\eth U^{\mu}}{\eth\tau} = -m\Gamma^{\mu}{}_{\alpha\beta}U^{\alpha}U^{\beta} + qF^{\mu}{}_{\alpha}U^{\alpha} \equiv m^{\alpha\beta}\Gamma^{\mu}{}_{\alpha\beta} + qU^{\alpha}F^{\mu}{}_{\alpha} \Big|\ d \to 0 \qquad (C.1)$$

Within this equation, the mass: m is not simply a result of 'adjusting' the 'geometrical' geodesic equation to the Lorentz force, but was already there in (5.20a). It is also clear that m on the RHS expresses the passive mass, while m on the LHS the inertial mass having the same value. Secondly, it is notable here that the inertial mass 'hides' in the body-bound coordinates: $\eth U^{\mu}/\eth\tau = 0$, and the passive mass 'hides' in the free-falling coordinates where $\Gamma^{\mu}{}_{\alpha\beta} = 0$; as in the famous free-falling elevator thought experiment.
– The above implies that the influence of gravity on an extended object can be decomposed into interactions of its parts with the metric field, *incorporating* (internal) influences that act externally on each part. If one reduces the maximal diameter: $d_{\hat{p}}$ of all the parts at $t = t_0$, each part's SE tensor field (expectation value) becomes smoother. Approximating this field by a polynomial, one can easily confirm that possibility "*A*" disappears for each such a subobject, because now the field's first moments (5.7) decrease at $O(d_{\hat{p}}^2)$ convergence order. In this case, the mechanism universally masking the stress tensor in flat space is likewise capable of masking this tensor of any subobject in curved space to that order of convergence. Furthermore, if that maximum diameter is stepwise reduced, the superposition of their BE tensor fields,

at least statistically, approaches an energy (E) tensor field: $\boldsymbol{T}_{\mathrm{E}} = \rho \boldsymbol{U} \otimes \boldsymbol{U}$ at the object's description level. In the specific case of the object made of pressureless (dust) matter, the E and SE tensor fields are equal.
- In the same way that the E tensor, the stress (S) tensor, which is always orthogonal to the energy tensor, emerges in the proper frame from spatial stress, but also from momentum density whose value but not necessarily derivative vanishes there. It isn't stress density there that produces stress-force density, but its spatial divergence which results in coordinate-invariant stress-force density principally independent of gravity much like the Lorentz force. Yet, the stress tensor as such can *indirectly* depend on gravitation.

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What is known is that physical objects can also contain massless particles such as photons, which can still contribute to the body's mass. However, for a free photon there is no physical proper frame, thus its trajectory must be parametrized using an affine parameter: $\lambda := \tau$ from a LI proper frame of an observer. By adopting the proper frame of a suitable observer as the reference, the (null-norm) four-velocity of the photon: $\eth x^{\mu'}/\eth\tau'$ is defined through the propagation vector: $\omega' \mathrm{U}^{\mu'} := \mathrm{K}^{\mu'}$, so that $\mathrm{U}^{\mu'} = \{1,1,0,0\}^{\mu'}$ for a (locally) flat wave. After substituting $\mu' \leftarrow \hat{\mu}$ and introducing $\underline{x}^{\hat{\mu}}$ as null-geodesic 'tunnel' LI (instead of SLI) coordinates; and by defining the energy density as $\rho := (A\omega')^2$, an analogous derivation of the geodesic principle becomes possible. Although in this case the four-velocity and the associated energy density are inevitably ambiguous under Lorentz boosts and hence cannot be regarded as proper, their E and SE tensor fields: $T^{\mu\nu} = T_{\mathrm{E}}^{\mu\nu} := \rho \mathrm{U}^{\mu}\mathrm{U}^{\nu}$ are well-defined. This conclusion is physically consistent, as it accords with observations of light deflection, which reliably also confirm that the geodesic principle applies universally and without exceptions, including to null-geodesic trajectories of massless particles.
- The E tensor field of the photon can be superposed with such fields of other photons, yielding the total SE tensor field. To produce the E tensor at points where the superposition has its local proper frame, the stress components of the SE tensor has to be removed; elsewhere, just as in the flat wave case above, the stress is correctly taken to be zero. So far, the analysis applies to the particular situation in which massless particles propagate at the speed of light in vacuum. In the general case, where photons also propagate through the object's material medium, the photons become coupled excitations whose propagation reflects an electromagnetic response of the material. Nonetheless, conservation principles, which constrain the SE and E tensor fields, must indeed still be upheld because both tensors are universal for matter and light, ensuring consistency of gravitational interaction with the object itself across all physical regimes.

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Since the object need not be fully homogeneous, the gravitational interaction is not straightforward. The object may be formed by multiple constituent objects, each of which contributes differently to the overall gravitational response. Without prior knowledge of the internal structure, these contributions cannot be resolved individually; nevertheless, the direct influence exerted on the object's SE tensor field can still be stated conclusively. Likewise, the same rule applies recursively to any subordinate constituent object. Therefore, the thesis formulated above generalizes naturally to extended objects as: The direct effect of gravity on the SE tensor of a physical object results only from the interaction with its constituent E tensor.
- This postulate forms the basis for incorporating gravity in the equations of motion for extended objects across all scales, including the particle range where the E tensor may be derived from the SE tensor of the wavefunction. Moreover, it can be treated as the most general form of the weak equivalence principle.

V. References

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VI. Symbols

A ; a, b	effective amplitude scalar field; constant real values
C^{n}	n·th differentiability class. If $f \in \mathrm{C}^{\mathrm{n}}$ then $\text{đ}^{\mathrm{n}} f / \text{đ}\tau^{\mathrm{n}}$ is bounded on a compact interval
d, d_{P} ; $\mathrm{d}_{\hat{\mathrm{P}}}$	diameter: of a body, of an object part (≡ subobject); max. diameter $\mathrm{d}_{\hat{\mathrm{P}}} := \max(\mathrm{d}_{\mathrm{P}k})$
D , đ , ∂, ∂_α	differential operators : absolute , total , partial , partial gradient
Δ , **Δ** ; Δ_{X}	difference, difference vector ; external-internal gravitational cross-term (4.6b)
δ^μ_ν	Kronecker delta (a selector): $\delta^\mu_\nu := 0 \mid \mu \neq \nu$, $\delta^\mu_\nu := 1 \mid \mu = \nu$
$\boldsymbol{e}_\alpha$, $\boldsymbol{e}_{\mathrm{\alpha}}$	coordinate basis, basis vector of the coordinate basis
$\mathbf{e}_\alpha$, $\mathbf{e}_{\mathrm{\alpha}}$	coordinate basis, basis vecto·r of the coordinate basis – near the (spatial) origin
$\eta_{\alpha\beta}, \eta^{\alpha\beta}; F_{\mu\nu}$	Minkowski metric (matrix value): $\equiv [\mathrm{diag}(-1,1,1,1)]_{\alpha\beta}$; electromagnetic tensor
Ø , $Ø_0$; Φ	diameter of: the spatial domain: $\underline{V}$, the validity domain: $\underline{V}_0$; coordinate transformation
$g_{\alpha\beta}, \mathrm{g}_{\alpha\beta}, g$	metric field, metric near the (spatial) origin , metric's matrix determinant
$\Gamma^\alpha_{\ \mu\nu}$, $\mathrm{\Gamma}^\alpha_{\ \mu\nu}$	Christoffel symbol (field), Christoffel symbol near the (spatial) origin
$h_{\alpha\beta}, h^{\alpha\beta}; \bar{h}_{\alpha\beta}$	perturbation of the Minkowski metric $h^{\alpha\beta} := \eta^{\alpha\mu}\eta^{\beta\nu}h_{\mu\nu}$; its trace reverse
$\Lambda^{\nu'}_\nu, \mathrm{\Lambda}^{\nu'}_\nu$; λ	(base) transformation matrix ($Y^\nu \mapsto Y^{\nu'}$), - near the (spatial) origin ; affine parameter
m , $\mathrm{m}^{\mu\nu}$	mass (5.1), (body's) mass tensor: $\mathrm{m}^{\hat\mu\hat\nu} := -\mathrm{m}\mathrm{U}^{\hat\mu}\mathrm{U}^{\hat\nu} \quad \rightarrow \quad \mathrm{m} \equiv \mathrm{m}^\mu_{\ \mu}$
n^n, n_n	unit normal vector: to a surface, - covariant form
$\tilde{O}$, $\tilde{\mathrm{O}}$	big-*O* symbol (converges to 0) as pure integration uncertainty, - τ dependent only
O , O ; ω	big-*O* symbol (converges to 0) , same τ dependent only ; photon's angular frequency
$\mathcal{P}, \mathcal{P}(x^\nu)$	(event in or) point of the space-time, - at certain coordinates
$\mathcal{P}_0 , \mathcal{P}(\tau)$	point of the space-time: at the origin of a coordinate system, on the trajectory
$\bar{p}$; p^μ; q	(body's) pressure-stress field (5.12d) ; body's four-momentum ; electric charge
$R^\alpha_{\ \mu\nu\sigma}, \mathrm{R}^\alpha_{\ \mu\nu\sigma}$	Riemann curvature: tensor field, - near the (spatial) origin
ρ, $\bar{\rho}$	energy density, (body's) mass-energy density (5.10c)
$\underline{S}$; $\vert\text{đ}\underline{S}\vert, \vert\text{đ}\underline{\boldsymbol{S}}\vert$	shell surface; proper element: of spatial 2D-area, of space-time layer: $\sqrt{\vert g\vert}\,\vert\text{đ}x^1\text{đ}x^2\vert$
$T^{\mu\nu}$; $T^{\mu\nu}_{\mathrm{S}}$	stress-energy tensor ; body-stress (BS)(5.12a) *or* stress (S) tensor: $T^{\mu\nu}_{\mathrm{S}} := T^{\mu\nu} - T^{\mu\nu}_{\mathrm{E}}$
$T^{\mu\nu}_{\mathrm{E}}, T^{\mu\nu}_{\mathrm{M}}$	$T^{\mu\nu}_{\mathrm{E}} := \rho U^\mu U^\nu$ energy density (E), body-energy (BE) (5.11a) – tensor fields
t ; τ , τ_0	time coordinate: x^0 ; proper time, - initial value (for the proper *and* comoving frame)
$U^\mu, \bar{U}^\mu$; U^μ	medium's mean: 4-velocity, mass density 4-velocity – tensor fields; proper 4-velocity
$\underline{V}(\tau)$, $\underline{V}_{\mathrm{P}}$	spatial domain of: min. Ø containing the whole body *and* $\mathcal{P}(\tau)$ (2.1,2) , the subobject
$\overline{\underline{V}}$, $\underline{\boldsymbol{V}}$	all space but $\underline{V}$, minimal space-time domain containing the whole $\underline{V}$ for all valid τ
$\text{đ}\underline{V}, \partial\underline{V}; \vert\text{đ}\underline{\boldsymbol{V}}\vert$	coordinate volume element, volume boundary ; proper space-time volume element
$x^\alpha, \mathrm{x}^\alpha ; x^n$	coordinates, a coordinate – of a point (or an event) in space-time ; spatial coordinates
$\underline{x}^\alpha$, $\underline{\underline{x}}^\alpha$	(all) coordinates = coordinate system, (all) coordinate systems (their full set)
$x^{\bar\alpha}, x^{\tilde\alpha}, x^{\hat\alpha}$	coordinates: (locally) inertial, SLI coordinates, SLI gauged coordinates
x^α	position of a body (5,2), which can also be a position of a sufficiently small subobject
ξ^μ , $\hat{\xi}^\mu$	gauge transformation shift, SLI gauge transformation shift (1.10)
$\boldsymbol{Y}, Y^\alpha, Y_\alpha$	abstract tensor: in boldface = a coordinateless tensor , as (contra)vector , as covector
⊗ ; ⊂	outer product ; "is a proper subset of"
⇐ , $A \mid B$	reverse implication , reverse implication as A "holds for" B
← ; ↦	substitution *or* side information ; "element is mapped to element"
→ ; ⇒	"it follows that" *or* "infinitesimally near" *or* "set is mapped to set" ; implication
$< f >$; [...]	a proper spatial 3D-area integral of f over $\underline{V}$ in $\underline{\boldsymbol{V}}$ (2.4a) ; (reformatting to a) matrix
$\vert\ldots\vert$; $\Vert T^{\mu\nu}\Vert$	abs. value *or* (vector's) norm *or* matrix determinant ; integral norm of a tensor over $\underline{V}$
{...}; $\{\ldots_{\mu\nu}\}^{\mu\nu}$	a set or collection; reformatting … to a contravariant second-rank tensor