The Star Flare Method

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Abstract

We find a finite neighbourhood of the star flare $(10, 20)$.

Keywords: Flare, Square, Star, Wings, Epsilons, Dini's Theorem, Mean Value Theorem

1 Introduction

There is a special triangle with angles (10,20,150) in degrees which has a periodic path inside it. See Figure 1 below. Unfortunately this periodic path is not stable and breaks down in a neighbourhood of this triangle.

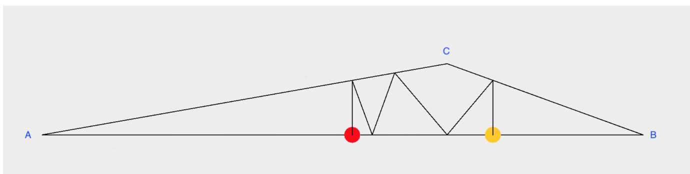

Figure 1: CNS (2,6) in (10,20) is unstable

In this paper we show how to find a small neighbourhood of the point (10,20) which we plot in a X-Y coordinate system. We then use six different kinds of periodic paths to completely cover this neighbourhood. See Figure 2 (drawn by Shehraj Singh).

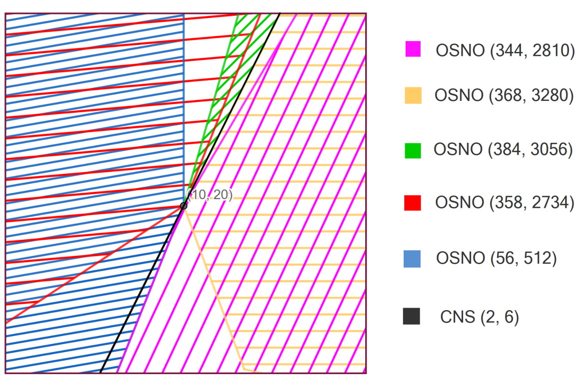

Figure 2: The 10-20 Star Flare Square

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Points such as (10,20) are very special and rare and are called **star flare** points. In fact this is the first and only star flare point found that has a **finite** neighbourhood which is not on the diagonal y=x or on the line x+y=90. This will make use of a computer. We will develop the theory and find a few other star flare points until we go to the big one, the (10,20) point.

2 Dini's Theorem

We use a version of Dini's Theorem which can be found in an Advanced Calculus text as in [1]. If a function of two variables $F(x, y)$ is continuous differentiable over a subset of the plane then $F(x, y) = 0$ can be a funny looking curve. See Figure 3.

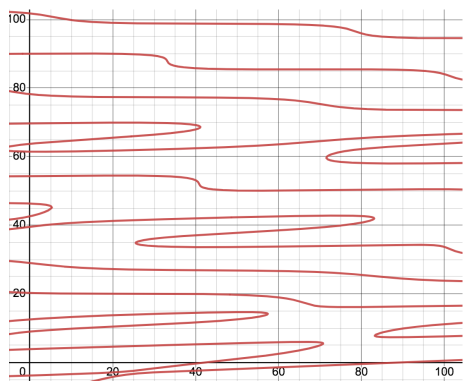

Figure 3: A funny looking curve $F(x, y) = 0$

In Figure 3 $F(x, y) = 10cos(x+15y) - 3cos(x+17y) + cos(x+19y) - 4cos(x+21y) - cos(x+23y) - 2cos(x+25y) + 2cos(x+27y) + 6cos(x+29y) - 2cos(x+35y) - 2cos(x+37y) + 6cos(3x-35y) = 0$

To prove the 10-20 Star Square Theorem we will use **functions of two variables** which are of the form $F(x, y) = \sum_{i=1}^{k} k_i cos(m_i x + n_i y)$ or $F(x, y) = \sum_{i=1}^{k} k_i sin(m_i x + n_i y)$ where k_i, m_i, n_i are integers and have a non-zero gradient ∇F at some point (not necessarily at every point). These functions are defined over the plane, double periodic and continuous differentiable and all their level curves $F(x, y) = k$ form an infinite sum of closed curves or an infinite sum of infinite curves going off to infinity in both directions. Each curve $F(x, y) = 0$ has two sides where ∇F points towards increasing k which is its **positive side** and $-\nabla F$ points towards decreasing k and forms its **negative side**. Note that if $\nabla F = 0$ at a particular point of the curve, then the curve intersects itself.

We first want to restrict the curve $F(x, y) = 0$ to a small enough **closed centered square** centered at (a, b) where $F(a, b) = 0$ with $\nabla F(a, b) \neq 0$ so that the curve inside this square is a **function of one variable** as either a function of x or a function of y or both. Recall from calculus that if the gradient $\nabla F(x, y)$ is different from zero then either its derivative is $y'(x, y) = -F_x/F_y$ or $x'(x, y) = -F_y/F_x$ or both. There are four cases.

1. If $F_x(x, y)$ and $F_y(x, y)$ are both positive or both negative throughout a given closed square then the curve $F(x, y) = 0$ is a function of x and a function of y in the square and is a strictly decreasing function of x and of y since $y'(x, y) < 0$ and $x'(x, y) < 0$ on the curve throughout.

2. If $F_x(x, y) > 0$ and $F_y(x, y) < 0$ or $F_x(x, y) < 0$ and $F_y(x, y) > 0$ throughout a given closed square then the curve $F(x, y) = 0$ is a function of x and a function of y in the square and is a strictly increasing function of x and of y since $y'(x, y) > 0$ and $x'(x, y) > 0$ on the curve throughout.

3. If $F_x(a,b) = 0$ and $F_y(x,y) > 0$ or $F_y(x,y) < 0$ throughout a given closed square then $F(x,y) = 0$ is a function of x in the square.

4. If $F_y(a,b) = 0$ and $F_x(x,y) > 0$ or $F_x(x,y) < 0$ throughout a given closed square then $F(x,y) = 0$ is a function of y in the square.

Note: Any of these four cases make the curve $F(x,y) = 0$ a continuous function in the given square and must hit the sides of the square in exactly two points. It is also worth noticing that $F(x,y) = 0$ has no self-intersection points in the square.

3 The Mean Value Theorem, Bounds and the Centered 2r-square

Let $F(x,y) = \sum_{i=1}^{k} k_i sin(m_i x + n_i y)$ where $\nabla F(x,y)$ is not identically zero and $k_i \neq 0$ for all i.

Let $F_x(x,y) = \sum_{i=1}^{k} m_i k_i \cos(m_i x + n_i y)$ where we let $\sum_{i=1}^{k} |m_i k_i| = |F_x|$.

Let $F_y(x,y) = \sum_{i=1}^{k} n_i k_i \cos(m_i x + n_i y)$ where we let $\sum_{i=1}^{k} |n_i k_i| = |F_y|$.

Let $|F_x| + |F_y|$ be called the **F-bounds** for F=$F(x,y)$.

Similar if F is a sum of cosines.

Theorem: A curve $F(x,y) = 0$ which is a sum of sines or a sum of cosines stays outside a **closed 2r-square** of side 2r centered at (a,b) if $F(a,b) > 0$ and $0 < r < F(a,b)/(|F_x| + |F_y|)$. Note **r is in radians** and r is called its **radius**. We will use this in later sections. Similarly if $F(a,b) < 0$ and $0 < r < |F(a,b)|/(|F_x| + |F_y|)$.

Proof: Let $F(a,b) > 0$ then let $0 < r < F(a,b)/(|F_x| + |F_y|)$ and (x,y) be anywhere on the 2r-square. Using the **Mean Value Theorem of two variables**, it follows that $F(a,b) - F(x,y) = F_x(c_1,c_2)(a-x) + F_y(c_1,c_2)(b-y) \leq |F_x||a-x| + |F_y||b-y| \leq r(|F_x| + |F_y|) < F(a,b)$ where (c_1,c_2) on the line between (x,y) and (a,b). This makes $F(x,y) > 0$ throughout this 2r-square and hence the curve $F(x,y) = 0$ misses this square.□

Important note: These two conditions above restricts r on the size of this 2r-square centered at (a,b) to make the curve $F(x,y) = 0$ miss this closed 2r-square entirely.

4 How to find the actual size of a closed centered 2r-square.

Because these curves of two variables $F(x,y) = 0$ is a sum of sines or a sum of cosines, we can actually find the size of a centered square to make the curve a function of x or a function of y or both centered at the point (a,b) where $\nabla F(a,b) \neq 0$.

As an example let $F(x,y) = \sum_{i=1}^{k} k_i sin(m_i x + n_i y)$ where the gradient of $F(x,y)$ is not identically zero and $k_i \neq 0$ for all i. Now let $\mathcal{C} : F(x,y) = 0$ be a curve with a point (a,b) on it where $F(a,b) = 0$ and we want to find a square with center at (a,b). Lets look at case 1 with both partial derivatives positive at (a,b).

Let $F_x(x,y) = \sum_{i=1}^{k} m_i k_i \cos(m_i x + n_i y)$ with absolute value $|F_x(x,y)| \leq \sum_{i=1}^{k} |m_i k_i| = |F_x|$. Suppose $F_x(a,b) = \sum_{i=1}^{k} m_i k_i \cos(m_i a + n_i b) > 0$. Caution $|F_x(x,y)|$ and $|F_x|$ are different definitions.

Let $F_y(x,y) = \sum_{i=1}^{k} n_i k_i \cos(m_i x + n_i y)$ with absolute value $|F_y(x,y)| \leq \sum_{i=1}^{k} |n_i k_i| = |F_y|$. Suppose $F_y(a,b) = \sum_{i=1}^{k} n_i k_i \cos(m_i a + n_i b) > 0$. Caution $|F_y(x,y)|$ and $|F_y|$ are different definitions.

We want to make (a, b) the center of a small enough square of side $2r_1 > 0$ in which $F_x(x, y)$ is positive everywhere in the closed square including its boundary.

Observe that $F_{xx}(x, y) = \sum_{i=1}^{k} -m_i^2 k_i \sin(m_i x + n_i y)$ with absolute value $|F_{xx}(x, y)| \le \sum_{i=1}^{k} |m_i^2 k_i| = |F_{xx}|$. and that $F_{xy}(x, y) = \sum_{i=1}^{k} -m_i n_i k_i \sin(m_i x + n_i y)$ with absolute value $|F_{xy}(x, y)| \le \sum_{i=1}^{k} |m_i n_i k_i| = |F_{xy}|$.

Let $|F_{xx}| + |F_{xy}|$ be called the F_x**-bounds** for F_x=$F_x(x, y)$.

Now observe that if (x, y) is an arbitrary point in this closed square and if we choose r_1 such that $0 < r_1 < F_x(a, b)/\sum_{i=1}^{k}(|m_i n_i k_i| + |m_i^2 k_i|) = F_x(a, b)/(|F_{xx}| + |F_{xy}|)$ and by the Mean Value Theorem there is a (c_1, c_2) on the line between (x, y) and (a, b) such that

$F_x(a, b) - F_x(x, y) = F_{xx}(c_1, c_2)(a - x) + F_{xy}(c_1, c_2)(b - y)$
$\le \sum_{i=1}^{k}(|m_i^2 k_i||a - x| + |m_i n_i k_i||b - y|) \le r_1 \sum_{i=1}^{k}(|m_i^2 k_i| + |m_i n_i k_i|) < F_x(a, b)$ which makes $F_x(x, y) > 0$.

This makes $F_x(x, y)$ positive throughout the square of side $2r_1$ and by Dini's Theorem the curve $F(x, y) = 0$ is a **function of y in the square**.

Similarly if $0 < r_2 < F_y(a, b)/\sum_{i=1}^{k}(|m_i n_i k_i| + |n_i^2 k_i|)$ where $F_y(a, b) > 0$ then this makes $F_y(x, y)$ positive throughout the square of side $2r_2$ and the curve $F(x, y) = 0$ is a **function of x in the square**. Note that $|F_{yx}(x, y)| = |F_{xy}(x, y)| \le \sum_{i=1}^{k} |m_i n_i k_i| = |F_{xy}| = |F_{yx}|$ and $|F_{yy}(x, y)| \le \sum_{i=1}^{k} |n_i^2 k_i| = |F_{yy}|$.

Let $|F_{yx}| + |F_{yy}|$ be called the F_y**-bounds** for F_y=$F_y(x, y)$.

Now if we choose $0 < r \le min(r_1, r_2)$ then the curve $F(x, y) = 0$ is a **function of both** x **and** y **in the square** of side $2r$.

Important note: These bound conditions restricts r on the size of this **2r-square** centered at (a,b) to make the curve $F(x, y) = 0$ a **function of both** x **and** y. More generally, below is a summary.

Fact I: If $F_x(a, b) > 0$ and $0 < r < F_x(a, b)/(|F_{xx}| + |F_{xy}|)$ then $F(x, y) = 0$ is a **function of y in a 2r-square** centered at (a,b). Similarly if $F_x(a, b) < 0$ and $0 < r < |F_x(a, b)|/(|F_{xx}| + |F_{xy}|)$.

Fact II: If $F_y(a, b) > 0$ and $0 < r < F_y(a, b)/(|F_{yx}| + |F_{yy}|)$ then $F(x, y) = 0$ is a **function of x in a 2r-square** centered at (a,b). Similarly if $F_y(a, b) < 0$ and $0 < r < |F_y(a, b)|/(|F_{yx}| + |F_{yy}|)$.

5 More Useful Facts about functions and slopes

If a curve $F(x, y) = 0$ has a slope y' or x' at (a,b) where its gradient is not zero, then we can treat the curve as a function in a small enough 2r-square.

Fact 1: If $F(a, b) = 0$ and $F_x(a, b) \neq 0$ then the curve $F(x, y) = 0$ is a function of y in a $2r$-square where $0 < r < |F_x(a, b)|/(|F_{xx}| + |F_{xy}|)$ since $F_x(x, y)$ keeps the same sign in the $2r$-square.

(Similarly If $F(a, b) = 0$ and $F_y(a, b) \neq 0$ then the curve $F(x, y) = 0$ is a function of x in a $2r$-square where $0 < r < |F_y(a, b)|/(|F_{yx}| + |F_{yy}|)$ since $F_y(x, y)$ keeps the same sign in the $2r$-square.)

Fact 2: If $F(a, b) = 0$ and its slope $y'(a, b) = -F_x(a, b)/F_y(a, b) \neq 0$ then its slope $y'(x, y)$ keeps the same sign in a 2r-square on points along the curve $F(x, y) = 0$ where $0 < r < min(|F_x(a, b)|/(|F_{xx}| + |F_{xy}|), |F_y(a, b)|/(|F_{yx}| + |F_{yy}|))$.

(Similarly if $F(a,b)=0$ and its slope $x'(a,b)=-F_y(a,b)/F_x(a,b)\neq 0$ then its slope $x'(x,y)$ keeps the same sign in a 2r-square on points along the curve $F(x,y)=0$ where $0<r<min(|F_x(a,b)|/(|F_{xx}|+|F_{xy}|),|F_y(a,b)|/(|F_{yx}|+|F_{yy}|))$.

Fact 3: If $y'(a,b)=0$ where $F_x(a,b)=0$ and $F_y(a,b)\neq 0$ then the curve $F(x,y)=0$ is a function of x where $0<r<|F_y(a,b)|/(|F_{yx}|+|F_{yy}|)$ since $F_y(x,y)$ keeps the same sign in a 2r-square.

(Similarly if If $x'(a,b)=0$ where $F_x(a,b)\neq 0$ and $F_y(a,b)=0$ then the curve $F(x,y)=0$ is a function of y where $0<r<|F_x(a,b)|/(|F_{xx}|+|F_{xy}|)$ since $F_x(x,y)$ keeps the same sign in a 2r-square.

These slope conditions further restricts r on the size of a 2r-square.

6 Trapping a Curve using Wings and Epsilon

We will show how to **trap a curve** $\mathcal{C}:F(x,y)=0$ with $F(a,b)=0$, $\nabla F(a,b)\neq 0$ in a 2r-square in a **wing** between two straight lines centered at (a,b) by examining it's derivative $m=y'(a,b)$ where $y'(a,b)=-F_x(a,b)/F_y(a,b)$ or its derivative $x'(a,b)$ where $x'(a,b)=-F_y(a,b)/F_x(a,b)$. There are four major cases with the idea to make the curve $\mathcal{C}:F(x,y)=0$ look like a straight line in a small enough square.

We first choose $0<r\leq \epsilon min(r_1,r_2)<min(r_1,r_2)$ in radians by multiplying by any **epsilon** $0<\epsilon<1$. Now take a closed square with side $2r$ centered at a fixed (a,b) on $\mathcal{C}:F(x,y)=0$ and any arbitrary point (x,y) on $\mathcal{C}$ inside the square. We then make use of the Mean Value Theorem in the following cases.

Note: These epsilon conditions using $0<\epsilon<1$ again further restricts the size of a 2r-square.

Case 1a: Where $y'(a,b)$ is positive with $F_x(a,b)<0$ and $F_y(a,b)>0$ and choose a square of side $2r$ centered at (a,b) where $0<r\leq \epsilon min(r_1,r_2)<min(r_1,r_2)$ and $0<\epsilon<1$. This makes $F(x,y)=0$ an increasing function of x and y in the square.

Here
$0<r_1\leq -F_x(a,b)/\sum_{i=1}^k(|m_in_iu_i|+|m_i^2u_i|)$
$0<r_2\leq F_y(a,b)/\sum_{i=1}^k(|m_in_iu_i|+|n_i^2u_i|)$

where
$0<r\leq \epsilon min(r_1,r_2)<min(r_1,r_2)\leq min(-F_x(a,b)/\sum_{i=1}^k(|m_in_iu_i|+|m_i^2u_i|),F_y(a,b)/\sum_{i=1}^k(|m_in_iu_i|+|n_i^2u_i|))$

And since $-F_x(a,b)>0$, we must have $-F_x(x,y)=-F_x(a,b)+F_{xx}(c_1,c_2)(a-x)+F_{xy}(c_1,c_2)(b-y)$ where (c_1,c_2) lies somewhere on the line between (x,y) and (a,b).
And since $F_y(a,b)>0$, we must have $F_y(x,y)=F_y(a,b)-F_{yx}(c_1',c_2')(a-x)-F_{yy}(c_1',c_2')(b-y)$ where (c_1',c_2') lies somewhere on the line between (x,y) and (a,b).

Hence
$0<-F_x(a,b)-r\sum_{i=1}^k(|m_in_iu_i|+|m_i^2u_i|)\leq -F_x(x,y)\leq -F_x(a,b)+r\sum_{i=1}^k(|m_in_iu_i|+|m_i^2u_i|)$ and
$0<F_y(a,b)-r\sum_{i=1}^k(|m_in_iu_i|+|n_i^2u_i|)\leq F_y(x,y)\leq F_y(a,b)+r\sum_{i=1}^k(|m_in_iu_i|+|n_i^2u_i|)$

This means
$0<-F_x(a,b)(1-\epsilon)\leq -F_x(x,y)\leq -F_x(a,b)(1+\epsilon)$ and
$0<F_y(a,b)(1-\epsilon)\leq F_y(x,y)\leq F_y(a,b)(1+\epsilon)$

and thus

$$0 < -F_x(a,b)(1-\epsilon)/F_y(a,b)(1+\epsilon) \leq -F_x(x,y)/F_y(x,y) \leq -F_x(a,b)(1+\epsilon)/F_y(a,b)(1-\epsilon)$$

Conclusion: Let (x,y) on the curve $\mathcal{C} : F(x,y) = 0$ then its derivative $y'(x,y)$ satisfies $0 < y'(a,b)(1-\epsilon)/(1+\epsilon) \leq y'(x,y) \leq y'(a,b)(1+\epsilon)/(1-\epsilon)$ or $0 < m(1-\epsilon)/(1+\epsilon) \leq y'(x,y) \leq m(1+\epsilon)/(1-\epsilon)$. It follows that the curve $\mathcal{C}$ is an increasing function of x and y that is trapped in a wing between two straight lines centered at (a,b) with these two positive slopes and inside the given 2r-square. Choosing ϵ closer to zero makes this wing "thinner".

Special Case: If $0 < r < min(r_1, r_2)$ setting $\epsilon = 1$, then the curve $\mathcal{C}$ is just an arbitrary increasing function of x and y in a 2r-square.

Case 1b: Where $y'(a,b)$ is positive with $F_x(a,b) > 0$ and $F_y(a,b) < 0$ and choose a square of side $2r$ centered at (a,b) where $0 < r \leq \epsilon min(r_1, r_2) < min(r_1, r_2)$ and $0 < \epsilon < 1$. This makes $F(x,y) = 0$ an increasing function of x and y in the square.

Here
$0 < r_1 \leq F_x(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |m_i^2 u_i|)$
$0 < r_2 \leq -F_y(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |n_i^2 u_i|)$

where
$0 < r \leq \epsilon min(r_1, r_2) < min(r_1, r_2) \leq min(F_x(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |m_i^2 u_i|), -F_y(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |n_i^2 u_i|))$

And since $F_x(a,b) > 0$, we must have $F_x(x,y) = F_x(a,b) - F_{xx}(c_1,c_2)(a-x) - F_{xy}(c_1,c_2)(b-y)$ where (c_1, c_2) lies somewhere on the line between (x,y) and (a,b).
And since $-F_y(a,b) > 0$, we must have $-F_y(x,y) = -F_y(a,b) + F_{yx}(c_1', c_2')(a-x) + F_{yy}(c_1', c_2')(b-y)$ where (c_1', c_2') lies somewhere on the line between (x,y) and (a,b).

Hence
$0 < F_x(a,b) - r\sum_{i=1}^{k}(|m_i n_i u_i| + |m_i^2 u_i|) \leq F_x(x,y) \leq F_x(a,b) + r\sum_{i=1}^{k}(|m_i n_i u_i| + |m_i^2 u_i|)$
$0 < -F_y(a,b) - r\sum_{i=1}^{k}(|m_i n_i u_i| + |n_i^2 u_i|) \leq -F_y(x,y) \leq -F_y(a,b) + r\sum_{i=1}^{k}(|m_i n_i u_i| + |n_i^2 u_i|)$

This means
$0 < F_x(a,b)(1-\epsilon) \leq F_x(x,y) \leq F_x(a,b)(1+\epsilon)$
$0 < -F_y(a,b)(1-\epsilon) \leq -F_y(x,y) \leq -F_y(a,b)(1+\epsilon)$

and thus
$0 < -F_x(a,b)(1-\epsilon)/F_y(a,b)(1+\epsilon) \leq -F_x(x,y)/F_y(x,y) \leq -F_x(a,b)(1+\epsilon)/F_y(a,b)(1-\epsilon)$

Conclusion: $0 < y'(a,b)(1-\epsilon)/(1+\epsilon) \leq y'(x,y) \leq y'(a,b)(1+\epsilon)/(1-\epsilon)$ and it follows that the curve $\mathcal{C}$ is an increasing function of x and y that is trapped in a wing between two straight lines centered at (a,b) with these two positive slopes and inside the given square. Choosing ϵ closer to zero makes this wing "thinner".

Special Case: If $0 < r < min(r_1, r_2)$ setting $\epsilon = 1$, then the curve $\mathcal{C}$ is just an arbitrary increasing function of x and y in a 2r-square.

Case 2a: Where $y'(a,b)$ is negative with $F_x(a,b) > 0$ and $F_y(a,b) > 0$ and choose a square of side $2r$ centered at (a,b) where $0 < r \leq \epsilon min(r_1, r_2) < min(r_1, r_2)$ and $0 < \epsilon < 1$. This makes $F(x,y) = 0$ an decreasing function of x and y in the square.

Here
$0 < r_1 \leq F_x(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |m_i^2 u_i|)$

$0 < r_2 \leq F_y(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |n_i^2 u_i|)$

where
$0 < r \leq \epsilon min(r_1, r_2) < min(r_1, r_2) \leq min(F_x(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |m_i^2 u_i|), F_y(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |n_i^2 u_i|))$

And since $F_x(a,b) > 0$, we must have $F_x(x,y) = F_x(a,b) - F_{xx}(c_1,c_2)(a-x) - F_{xy}(c_1,c_2)(b-y)$ where (c_1,c_2) lies somewhere on the line between (x,y) and (a,b).
And since $F_y(a,b) > 0$, we must have $F_y(x,y) = F_y(a,b) - F_{yx}(c_1',c_2')(a-x) - F_{yy}(c_1',c_2')(b-y)$ where (c_1',c_2') lies somewhere on the line between (x,y) and (a,b).

Hence
$0 < F_x(a,b) - r\sum_{i=1}^{k}(|m_i n_i u_i| + |m_i^2 u_i|) \leq F_x(x,y) \leq F_x(a,b) + r\sum_{i=1}^{k}(|m_i n_i u_i| + |m_i^2 u_i|)$
$0 < F_y(a,b) - r\sum_{i=1}^{k}(|m_i n_i u_i| + |n_i^2 u_i|) \leq F_y(x,y) \leq F_y(a,b) + r\sum_{i=1}^{k}(|m_i n_i u_i| + |n_i^2 u_i|)$

This means
$0 < F_x(a,b)(1-\epsilon) \leq F_x(x,y) \leq F_x(a,b)(1+\epsilon)$
$0 < F_y(a,b)(1-\epsilon) \leq F_y(x,y) \leq F_y(a,b)(1+\epsilon)$

and thus
$0 < F_x(a,b)(1-\epsilon)/F_y(a,b)(1+\epsilon) \leq F_x(x,y)/F_y(x,y) \leq F_x(a,b)(1+\epsilon)/F_y(a,b)(1-\epsilon)$

Conclusion: $y'(a,b)(1+\epsilon)/(1-\epsilon) \leq y'(x,y) \leq y'(a,b)(1-\epsilon)/(1+\epsilon) < 0$ and it follows that the curve $\mathcal{C}$ is a decreasing function of x and y that is trapped in a wing between two straight lines centered at (a,b) with these two negative slopes and inside the given square. Choosing ϵ closer to zero makes this wing "thinner".

Special Case: If $0 < r < min(r_1, r_2)$ setting $\epsilon = 1$, then the curve $\mathcal{C}$ is just an arbitrary decreasing function of x and y in a 2r-square.

Case 2b: Where $y'(a,b)$ is negative with $F_x(a,b) < 0$ and $F_y(a,b) < 0$ and choose a square of side $2r$ centered at (a,b) where $0 < r \leq \epsilon min(r_1, r_2) < min(r_1, r_2)$ and $0 < \epsilon < 1$. This makes $F(x,y) = 0$ an decreasing function of x and y in the square.

Here
$0 < r_1 \leq -F_x(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |m_i^2 u_i|)$
$0 < r_2 \leq -F_y(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |n_i^2 u_i|)$

where
$0 < r \leq \epsilon min(r_1, r_2) < min(r_1, r_2) \leq min(-F_x(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |m_i^2 u_i|), -F_y(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |n_i^2 u_i|))$

And since $-F_x(a,b) > 0$, we must have $-F_x(x,y) = -F_x(a,b) + F_{xx}(c_1,c_2)(a-x) + F_{xy}(c_1,c_2)(b-y)$ where (c_1,c_2) lies somewhere on the line between (x,y) and (a,b).
And since $-F_y(a,b) > 0$, we must have $-F_y(x,y) = -F_y(a,b) + F_{yx}(c_1',c_2')(a-x) + F_{yy}(c_1',c_2')(b-y)$ where (c_1',c_2') lies somewhere on the line between (x,y) and (a,b).

Hence
$0 < -F_x(a,b) - r\sum_{i=1}^{k}(|m_i n_i u_i| + |m_i^2 u_i|) \leq -F_x(x,y) \leq -F_x(a,b) + r\sum_{i=1}^{k}(|m_i n_i u_i| + |m_i^2 u_i|)$
$0 < -F_y(a,b) - r\sum_{i=1}^{k}(|m_i n_i u_i| + |n_i^2 u_i|) \leq -F_y(x,y) \leq -F_y(a,b) + r\sum_{i=1}^{k}(|m_i n_i u_i| + |n_i^2 u_i|)$

This means
$0 < -F_x(a,b)(1-\epsilon) \leq -F_x(x,y) \leq -F_x(a,b)(1+\epsilon)$

$0 < -F_y(a,b)(1-\epsilon) \leq -F_y(x,y) \leq -F_y(a,b)(1+\epsilon)$

and thus
$0 < F_x(a,b)(1-\epsilon)/F_y(a,b)(1+\epsilon) \leq F_x(x,y)/F_y(x,y) \leq F_x(a,b)(1+\epsilon)/F_y(a,b)(1-\epsilon)$

Conclusion: $y'(a,b)(1+\epsilon)/(1-\epsilon) \leq y'(x,y) \leq y'(a,b)(1-\epsilon)/(1+\epsilon) < 0$ and it follows that the curve $\mathcal{C}$ is a decreasing function of x and y that is trapped in a wing between two straight lines centered at (a,b) with these two negative slopes and inside the given square. Choosing ϵ closer to zero makes this wing "thinner".

Special Case: If $0 < r < min(r_1, r_2)$ setting $\epsilon = 1$, then the curve $\mathcal{C}$ is just an arbitrary decreasing function of x and y in a 2r-square.

Case 3a: Where $y'(a,b) = 0$ with $F_x(a,b) = 0$ and $F_y(a,b) > 0$ and choose a square of side $2r$ centered at (a,b) where $0 < r \leq \epsilon min(r_1,r_2) < min(r_1,r_2)$ and $0 < \epsilon < 1$. This makes $F(x,y) = 0$ a function of x in the square.

Here
$0 < r_1 \leq F_y(a,b)/\sum_{i=1}^{k}(|m_i^2 u_i| + |m_i n_i u_i|)$ **Caution: We will use $F_y(a,b)$ here instead of $F_x(a,b)$**
$0 < r_2 \leq F_y(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |n_i^2 u_i|)$

where
$0 < r \leq \epsilon min(r_1,r_2) < min(r_1,r_2) \leq min(F_y(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |m_i^2 u_i|), F_y(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |n_i^2 u_i|))$

And since $F_x(a,b) = 0$, we must have $F_x(x,y) = -F_{xx}(c_1,c_2)(a-x) - F_{xy}(c_1,c_2)(b-y)$ where (c_1,c_2) lies somewhere on the line between (x,y) and (a,b).
And since $F_y(a,b) > 0$, we must have $F_y(x,y) = F_y(a,b) - F_{yx}(c_1',c_2')(a-x) - F_{yy}(c_1',c_2')(b-y)$ where (c_1',c_2') lies somewhere on the line between (x,y) and (a,b).

Hence
$-r\sum_{i=1}^{k}(|m_i^2 u_i| + |m_i n_i u_i|) \leq F_x(x,y) \leq r\sum_{i=1}^{k}(|m_i^2 u_i| + |m_i n_i u_i|)$
$0 < F_y(a,b) - r\sum_{i=1}^{k}(|m_i n_i u_i| + |n_i^2 u_i|) \leq F_y(x,y) \leq F_y(a,b) + r\sum_{i=1}^{k}(|m_i n_i u_i| + |n_i^2 u_i|)$

This means
$-\epsilon F_y(a,b) \leq F_x(x,y) \leq \epsilon F_y(a,b)$ and
$0 < F_y(a,b)(1-\epsilon) \leq F_y(x,y) \leq F_y(a,b)(1+\epsilon)$

and thus
$-\epsilon/(1-\epsilon) \leq -F_x(x,y)/F_y(x,y) \leq \epsilon/(1-\epsilon)$

Conclusion: $-\epsilon/(1-\epsilon) \leq y'(x,y) \leq \epsilon/(1-\epsilon)$ and it follows that the curve $\mathcal{C}$ is a function of x that is trapped in a wing between two straight lines centered at (a,b) with these two slopes $m = \pm\epsilon/(1-\epsilon)$ which include the horizontal line $y = b$ and inside the given square. Choosing ϵ closer to zero makes this wing "thinner".

Special Case: If $0 < r < min(r_1, r_2)$ setting $\epsilon = 1$, then the curve $\mathcal{C}$ is just an arbitrary function of x in a 2r-square.

Example: Let $F(x,y) = sin(x-4y) - sin(x+4y)$ where $F(45,45) = 0$, $F_x(45,45) = 0$, $F_y(45,45) > 0$ and thus $y'(45,45) = 0$ where $F_x(x,y) = cos(x-4y) - cos(x+4y)$ with F_x-bounds $= 10$ and $F_y(x,y) = -4cos(x-4y) - 4cos(x+4y)$ with F_y-bounds $= 40$.

Further $F_{xx}(x,y) = -sin(x-4y) + sin(x+4y)$ with $|F_{xx}| = 2$ and $F_{xy}(x,y) = 4sin(x-4y) + 4sin(x+4y)$ with $|F_{xy}| = 8$ and

$F_{yx}(x,y) = 4sin(x-4y) + 4sin(x+4y)$ with $|F_{yx}|= 8$ and $F_{yy}(x,y) = -16sin(x-4y) + 16sin(x+4y)$ with $|F_{yy}|= 32$.

Here let
$0 < r_1 \le F_y(45,45)/10$ Caution: We will use $F_y(45,45)$ here instead of $F_x(45,45)$
$0 < r_2 = 0.141421 \le F_y(45,45)/40$

where
$0 < r \le 0.141421\epsilon < \epsilon F_y(45,45)/40 < \epsilon F_y(45,45)/10$

Hence
$-10r \le F_x(x,y) \le 10r$
$0 < F_y(45,45) - 40r \le F_y(x,y) \le F_y(45,45) + 40r$

This means
$-\epsilon F_y(45,45) \le F_x(x,y) \le \epsilon F_y(45,45)$ and
$0 < F_y(45,45)(1-\epsilon) \le F_y(x,y) \le F_y(45,45)(1+\epsilon)$

and thus
$-\epsilon/(1-\epsilon) \le -F_x(x,y)/F_y(x,y) \le \epsilon/(1-\epsilon)$

Case 3b: Where $y'(a,b) = 0$ with $F_x(a,b) = 0$ and $F_y(a,b) < 0$ and choose a square of side $2r$ centered at (a,b) where $0 < r \le \epsilon min(r_1,r_2) < min(r_1,r_2)$ and $0 < \epsilon < 1$. This makes $F(x,y) = 0$ a function of x in the square.

Here
$0 < r_1 \le -F_y(a,b)/\sum_{i=1}^{k}(|m_i^2u_i|+|m_in_iu_i|)$ **Caution: We will use $F_y(a,b)$ here instead of $F_x(a,b)$**
$0 < r_2 \le -F_y(a,b)/\sum_{i=1}^{k}(|m_in_iu_i|+|n_i^2u_i|)$

where
$0 < r \le \epsilon min(r_1,r_2) < min(r_1,r_2) \le min(-F_y(a,b)/\sum_{i=1}^{k}(|m_in_iu_i|+|m_i^2u_i|), -F_y(a,b)/\sum_{i=1}^{k}(|m_in_iu_i|+|n_i^2u_i|))$

And since $F_x(a,b) = 0$, we must have $F_x(x,y) = -F_{xx}(c_1,c_2)(a-x) - F_{xy}(c_1,c_2)(b-y)$ where (c_1,c_2) lies somewhere on the line between (x,y) and (a,b).
And since $F_y(a,b) < 0$, we must have $F_y(x,y) = F_y(a,b) - F_{yx}(c_1',c_2')(a-x) - F_{yy}(c_1',c_2')(b-y)$ where (c_1',c_2') lies somewhere on the line between (x,y) and (a,b).

Hence
$-r\sum_{i=1}^{k}(|m_i^2u_i|+|m_in_iu_i|) \le F_x(x,y) \le r\sum_{i=1}^{k}(|m_i^2u_i|+|m_in_iu_i|)$ and
$F_y(a,b) - r\sum_{i=1}^{k}(|m_in_iu_i|+|n_i^2u_i|) \le F_y(x,y) \le F_y(a,b) + r\sum_{i=1}^{k}(|m_in_iu_i|+|n_i^2u_i|) < 0$

This means
$\epsilon F_y(a,b) \le F_x(x,y) \le -\epsilon F_y(a,b)$
$F_y(a,b)(1+\epsilon) \le F_y(x,y) \le F_y(a,b)(1-\epsilon) < 0$

and thus
$-\epsilon/(1-\epsilon) \le -F_x(x,y)/F_y(x,y) \le \epsilon/(1-\epsilon)$

Conclusion: $-\epsilon/(1-\epsilon) \le y'(x,y) \le \epsilon/(1-\epsilon)$ and it follows that the curve $\mathcal{C}$ is a function of x that is trapped in a wing between two straight lines centered at (a,b) with these two slopes $m = \pm\epsilon/(1-\epsilon)$ which include the horizontal line $y = b$ and inside the given square. Choosing ϵ closer to zero makes this wing "thinner".

Special Case: If $0 < r < min(r_1, r_2)$ setting $\epsilon = 1$, then the curve $\mathcal{C}$ is just an arbitrary function of x in a 2r-square.

Case 4a: If $x'(a, b) = 0$ where $F_x(a, b) > 0$ and $F_y(a, b) = 0$ choose a square of side $2r$ centered at (a, b) where $0 < r \leq \epsilon min(r_1, r_2) < min(r_1, r_2)$ and $0 < \epsilon < 1$. This makes $F(x, y) = 0$ a function of y in the square.

Here
$0 < r_1 \leq F_x(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |m_i^2 u_i|)$
$0 < r_2 \leq F_x(a,b)/\sum_{i=1}^{k}(|n_i^2 u_i| + |m_i n_i u_i|)$ **Caution: We will use $F_x(a, b)$ here instead of $F_y(a, b)$**

where
$0 < r \leq \epsilon min(r_1, r_2) < min(r_1, r_2) \leq min(F_x(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |m_i^2 u_i|), F_x(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |n_i^2 u_i|))$

And since $F_x(a, b) > 0$, we must have $F_x(x, y) = F_x(a, b) - F_{xx}(c_1', c_2')(a - x) - F_{xy}(c_1', c_2')(b - y)$ where (c_1', c_2') lies somewhere on the line between (x, y) and (a, b).
And since $F_y(a, b) = 0$, we must have $F_y(x, y) = -F_{yx}(c_1, c_2)(a - x) - F_{yy}(c_1, c_2)(b - y)$ where (c_1, c_2) lies somewhere on the line between (x, y) and (a, b).

Hence
$0 < F_x(a,b) - r\sum_{i=1}^{k}(|m_i n_i u_i| + |m_i^2 u_i|) \leq F_x(x,y) \leq F_x(a,b) + r\sum_{i=1}^{k}(|m_i n_i u_i| + |m_i^2 u_i|)$
$-r\sum_{i=1}^{k}(|n_i^2 u_i| + |m_i n_i u_i|) \leq F_y(x,y) \leq r\sum_{i=1}^{k}(|n_i^2 u_i| + |m_i n_i u_i|)$

This means

$0 < F_x(a,b)(1-\epsilon) < F_x(x,y) < F_x(a,b)(1+\epsilon)$
$-\epsilon F_x(a,b) \leq F_y(x,y) \leq \epsilon F_x(a,b)$

and thus
$-\epsilon/(1-\epsilon) \leq -F_y(x,y)/F_x(x,y) \leq \epsilon/(1-\epsilon)$

Conclusion: $-\epsilon/(1-\epsilon) \leq x'(x, y) \leq \epsilon/(1-\epsilon)$ and it follows that the curve $\mathcal{C}$ is a function of y that is trapped in a wing between two straight lines centered at (a, b) with these two slopes $m = \pm(1-\epsilon)/\epsilon$ which include the vertical line $x = a$ and inside the given square. Choosing ϵ closer to zero makes this wing "thinner".

Special Case: If $0 < r < min(r_1, r_2)$ setting $\epsilon = 1$, then the curve $\mathcal{C}$ is just an arbitrary function of y in a 2r-square.

Case 4b: If $x'(a, b) = 0$ where $F_x(a, b) < 0$ and $F_y(a, b) = 0$ choose a square of side $2r$ centered at (a, b) where $0 < r \leq \epsilon min(r_1, r_2) < min(r_1, r_2)$ and $0 < \epsilon < 1$. This makes $F(x, y) = 0$ a function of y in the square.

Here
$0 < r_1 \leq -F_x(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |m_i^2 u_i|)$
$0 < r_2 \leq -F_x(a,b)/\sum_{i=1}^{k}(|n_i^2 u_i| + |m_i n_i u_i|)$ **Caution: We will use $F_x(a, b)$ here instead of $F_y(a, b)$**

where
$0 < r \leq \epsilon min(r_1, r_2) < min(r_1, r_2) \leq min(-F_x(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |m_i^2 u_i|), -F_x(a,b)/\sum_{i=1}^{k}(|m_i n_i u_i| + |n_i^2 u_i|))$

And since $-F_x(a, b) > 0$, we must have $-F_x(x, y) = -F_x(a, b) + F_{xx}(c_1', c_2')(a - x) + F_{xy}(c_1', c_2')(b - y)$ where (c_1', c_2') lies somewhere on the line between (x, y) and (a, b).

And since $F_y(a,b)=0$, we must have $F_y(x,y) = -F_{yx}(c_1,c_2)(a-x) - F_{yy}(c_1,c_2)(b-y)$ where (c_1,c_2) lies somewhere on the line between (x,y) and (a,b).

Hence
$0 < -F_x(a,b) - r\sum_{i=1}^{k}(|m_in_iu_i| + |m_i^2u_i|) \le -F_x(x,y) \le -F_x(a,b) + r\sum_{i=1}^{k}(|m_in_iu_i| + |m_i^2u_i|)$
$-r\sum_{i=1}^{k}(|n_i^2u_i| + |m_in_iu_i|) \le F_y(x,y) \le r\sum_{i=1}^{k}(|n_i^2u_i| + |m_in_iu_i|)$

This means

$0 < -F_x(a,b)(1-\epsilon) < -F_x(x,y) < -F_x(a,b)(1+\epsilon)$
$\epsilon F_x(a,b) \le F_y(x,y) \le -\epsilon F_x(a,b)$

and thus
$-\epsilon/(1-\epsilon) \le -F_y(x,y)/F_x(x,y) \le \epsilon/(1-\epsilon)$

Conclusion: $-\epsilon/(1-\epsilon) \le x'(x,y) \le \epsilon/(1-\epsilon)$ and it follows that the curve $\mathcal{C}$ is a function of y inside a 2r-square that is trapped in a wing between two straight lines centered at (a,b) with these two slopes $m = \pm(1-\epsilon)/\epsilon$ and which include the vertical line $x=a$. Choosing ϵ closer to zero makes this wing "thinner".

Special Case: If $0 < r < min(r_1,r_2)$ setting $\epsilon = 1$, then the curve $\mathcal{C}$ is just an arbitrary function of y in a 2r-square.

Summary: There are four kinds of wings centered at (a,b) that traps a curve $\mathcal{C}$ where F(x,y)=0, F(a,b)=0, $\nabla F(a,b) \ne 0$ and $y'(a,b) = m$ with $m > 0$, $m < 0$, m=0, $m = \pm infinity$. We let $0 < \epsilon < 1$.

1. **Positive wing** between two straight lines centered at (a,b) with slopes $0 < m(1-\epsilon)/(1+\epsilon) \le m(1+\epsilon)/(1-\epsilon)$ where $y'(a,b) = m > 0$
2. **Negative wing** between two straight lines centered at (a,b) with slopes $m(1+\epsilon)/(1-\epsilon) \le m(1-\epsilon)/(1+\epsilon) < 0$ where $y'(a,b) = m < 0$
3. **Horizontal wing** between two straight lines centered at (a,b) with slopes $\pm\epsilon/(1-\epsilon)$ where $y'(a,b) = m = 0$
4. **Vertical wing** between two straight lines centered at (a,b) with slopes $\pm(1-\epsilon)/\epsilon$ where $y'(a,b) = m = \pm infinity$

Note: A wing that traps a curve $\mathcal{C}$ forms a **wedge** from two intersecting lines where we will orient them in a counter-clockwise order. We will label a wing so that it goes counter-clockwise from the **lower wing** line to the **upper wing** line.

7 Separate Wings

Given two curves intersecting at (a,b) and $0 < \epsilon < 1$, we want to create **two separate wings** that force the two curves F with slope m_1 and G with slope m_2 to intersect once only at (a,b) in a 2r-square. We can do this if the slopes m_1, m_2 of the two curves at (a,b) are different. As an example suppose $0 < m_2 < m_1$ where we start by using $0 < r \le \epsilon$ min($|F_x(a,b)|/F_x$-bounds, $|F_y(a,b)|/F_y$-bounds, $|G_x(a,b)|/G_x$-bounds, $|G_y(a,b)|/G_y$-bounds) to make F and G functions.

Now we will choose an ϵ from these facts below to separate the two wings and keep the two curves F with slope m_1 and G with slope m_2 at (a,b) and inside separate wings.

Fact 1: If $0 < m_2 < m_1$, then we can create two separate wings by choosing $0 < \epsilon < \frac{\sqrt{m}_1 - \sqrt{m}_2}{\sqrt{m}_1 + \sqrt{m}_2}$
Proof:
We choose $0 < \epsilon < 1$ to make the two wings separate if the upper wing of G with $(\frac{1+\epsilon}{1-\epsilon})m_2 < (\frac{1-\epsilon}{1+\epsilon})m_1$ from

the lower wing of F $\iff (\frac{1+\epsilon}{1-\epsilon}) < \sqrt{\frac{m_1}{m_2}} \iff 0 < \epsilon < \frac{\sqrt{m}_1 - \sqrt{m}_2}{\sqrt{m}_1 + \sqrt{m}_2}$
□

Fact 2: If $0 < m_2 < m_1$, then we can also create two separate wings by choosing $0 < \epsilon \leq \frac{m_1 - m_2}{2(m_1+m_2)} < 1/2$ since $0 < \frac{m_1-m_2}{2(m_1+m_2)} < \frac{\sqrt{m}_1 - \sqrt{m}_2}{\sqrt{m}_1 + \sqrt{m}_2}$
Proof:
$0 < \frac{m_1-m_2}{2(m_1+m_2)} < \frac{\sqrt{m}_1 - \sqrt{m}_2}{\sqrt{m}_1 + \sqrt{m}_2} \iff \frac{\sqrt{m}_1 + \sqrt{m}_2}{2(m_1+m_2)} < \frac{1}{\sqrt{m}_1 + \sqrt{m}_2} \iff (\sqrt{m}_1 + \sqrt{m}_2)^2 < 2(m_1 + m_2) \iff 0 < (\sqrt{m}_1 - \sqrt{m}_2)^2$ which is true.
□

Note: This second fact forces $0 < \epsilon < 1/2$.

More generally allowing negatives if m_1 and m_2 have different negative slopes, then we use $0 < \epsilon < \frac{||m_1|-|m_2||}{2(|m_1|+|m_2|)}$

Special cases:
1. If the two curves are two distinct straight lines, then the wing of each line is itself and we can let $\epsilon = 1$.
2. If one curve is straight with slope m_1 and the second curve is not straight with slope m_2 and suppose that $0 < m_2 < m_1$ then the two wings are separate if $(\frac{1+\epsilon}{1-\epsilon})m_2 < m_1 \iff (\frac{1+\epsilon}{1-\epsilon}) < \frac{m_1}{m_2} \iff 0 < \epsilon < \frac{m_1-m_2}{m_1+m_2}$.
3. If the two curves have slopes of opposite sign, then the the two wings are automatically separate and we can let $\epsilon = 1$.

Note: It is nicer to use rationals instead of irrationals.

8 Summary of the List of Square Rules

1. A curve $F(x, y) = 0$ will miss a 2r-square centered at (a,b) if $F(a, b) > 0$ and $0 < r < F(a, b)/(|F_x| + |F_y|)$ since this makes $F(x, y) > 0$ throughout this 2r-square.

2. A curve $F(x, y) = 0$ is a **function of y** in a 2r-square centered at (a,b) if F(a,b)=0, $F_x(a, b) > 0$ and $0 < r < F_x(a, b)/(|F_{xx}| + |F_{xy}|)$ since this makes $F_x(x, y) > 0$ throughout this 2r-square.
A curve $F(x, y) = 0$ is a **function of x** in a 2r-square centered at (a,b) if F(a,b)=0, $F_y(a, b) > 0$ and $0 < r < F_y(a, b)/(|F_{yx}| + |F_{yy}|)$ since this makes $F_y(x, y) > 0$ throughout this 2r-square.

3. A curve $F(x, y) = 0$ is a **function of x and y** in a 2r-square centered at (a,b) if F(a,b)=0, $F_x(a, b) \neq 0$,and $F_y(a, b) \neq 0$ and $0 < r < min(|F_x(a, b)|/(|F_{xx}| + |F_{xy}|), |F_y(a, b)(|F_{yx}| + |F_{xy}|)$
a. The curve $F(x, y) = 0$ is an **increasing function of x and y** in a 2r-square if $F_x(a, b)$ and $F_y(a, b)$ have different signs since $y'(x, y) > 0$ and $x'(x, y) > 0$ throughout this 2r-square.
b. The curve $F(x, y) = 0$ is a **decreasing function of x and y** in a 2r-square if $F_x(a, b)$ and $F_y(a, b)$ have the same signs since $y'(x, y) < 0$ and $x'(x, y) < 0$ throughout this 2r-square.

4. A curve $F(x, y) = 0$ is **trapped in a wing** between two intersecting lines with slopes $m(1-\epsilon)/(1+\epsilon)$ and $m(1+\epsilon)/(1-\epsilon)$ at (a,b) in a 2r-square centered at (a,b) if $F(a, b) = 0$, $F_x(a, b) \neq 0$, $F_y(a, b) \neq 0$, $m = -F_x(a, b)/F_y(a, b)$, $0 < \epsilon < 1$ and $0 < r < \epsilon min(|F_x(a, b)|/(|F_{xx}| + |F_{xy}|), |F_y(a, b)|/(|F_{yx}| + |F_{xy}|)$ where m is between $m(1-\epsilon)/(1+\epsilon)$ and $m(1+\epsilon)/(1-\epsilon)$.

5. Two curves $F(x, y) = 0$ with slope m_1 and $G(x, y) = 0$ with slope m_2 at (a,b) with $m_1 \neq m_2$ are **trapped by separate wings** if we use $\epsilon = .5||m_1| - |m_2||/(|m_1| + |m_2|)$.

There are special case of these rules which we will mention them when they are needed.

9 MPFI and Big Decimal

We use the **MPFI library** to guarantee that one of our equations have a value at a point that lies in an interval and lies in between its two endpoints.

As an example let $F(x,y) = 63sin(x - 63y) - 110sin(x - 55y) + 153sin(x - 51y) + 47sin(x - 47y)$ at the point (10,20) where its value $F(10,20)$ lies in the interval (175.43443533058970, 175.43443533058974). This means $F(10,20) = 175.4344353305897...$ and is correct to 13 decimals. This is more decimals than we need and in most cases, we will just use $F(10,20) = 175.434435...$ which is correct to 6 decimals.

We also use the **Big Decimal** library on rational numbers to get a precise number of decimals.

As an example let 507005/146819=3.453265... which is correct to 6 decimals and 3.45326558551... is correct to 11 decimals.

10 The 2005-2006 period

This comes from the early work of Tokarsky and Marinov about periodic paths in triangles.

I: Code Sequences We introduced code sequences rather than side sequences. The code is quite simple to understand. You just count the number of times a periodic path goes across an angle in a triangle. If the number is even, then the next angle is the same angle as the previous angle. If the number is odd, then the next angle is different from the previous two angles.

As an example say the path code is 5 2 3 ... which goes 5 times across the y angle and 2 times across the x angle then since 2 is even then the path goes across the y angle again. If we continue this code and since 3 is odd, the path must next go across the z angle.

It is much easier to use our notation then this long **side sequence** which gives the sequence of sides hit.
123232323232323232323231313131313131313131313131313132323232323232323232323131313131313131313131313131313132
32323232323232323231313131313131313131313131313 which becomes 1 21 30 20 28 20 29 in code sequences using 7 **code numbers**.

We will always write a code sequence in its lowest value for example as above and not as 1 29 20 28 20 30 21 viewing it as an integer in a base 30 number system. Note that a code sequence is viewed on a circle forwards or backwards.

II: Stable Regions We classified three types of stable regions using code numbers.

The code is **stable** if it has an even number of code numbers viewed on a circle and the sum of the **top angles** = the sum of the **bottom angles** as in 1x+3z+3y=3y+1x+3z.

```
x   z   z   y
1 1 2 2 1 1 3 3
  y   y   x   z
```

1. OSO is stable, has an odd number of code numbers and has a periodic path which hits every side at an acute angle. Example 1 21 30 20 28 20 29 is of type OSO(7,149) where 7=the number of code numbers and 149=the sum of the code numbers=the number of integers in a corresponding side sequence.
Note: A OSO code sequence as for example 1 3 3 is of type OSO(3,7) and must be doubled to see that it is a stable code as in 1x+3z+3y=3y+1x+3z

x   z   y

1 3 3 1 3 3
y x z

2. CS is stable, has an even number of code numbers and has a periodic path which hits two sides perpendicularly. Example 1 1 1 1 2 1 1 1 1 2 is of type CS(10,12) where 10=the number of code numbers and 12=the sum of the code numbers.

3. OSNO is stable, has an even number of code numbers and has a periodic path which doesn't hit any side perpendicularly. Example 1 1 2 2 1 1 3 3 is of type OSNO(8,14) where 8=the number of code numbers and 14=the sum of the code numbers.

A **stable region** is an open nonempty set in the Cartesian plane.

III: Unstable Regions We classified two types of unstable regions using code numbers.

1. CNS is not stable, has an even number of code numbers and has a periodic path which hits two sides perpendicularly. Example 2 2 is of type CNS(2,4) where 2=the number of code numbers and 4=the sum of the code numbers.

2. ONS is not stable, has an even number of code numbers and has a periodic path which doesn't hit any side perpendicularly. Example 1 1 1 1 3 3 is of type ONS(6,10) where 6=the number of code numbers and 10=the sum of the code numbers.

An **unstable region** is an open line segment and non-empty.

A CNS code is of the form as in this example CNS(6,14) 1 3 ② 3 1 ④ where two code numbers are even as in the 2 and 4 and all the code numbers between the 2 and 4 are in opposite order 1 3 and 3 1. We circled the 2 and the 4 where the path hits a side perpendicularly. If we use side sequences the two sides are hit perpendicularly at side 1 which are circled 12323①323213①3.

Note: An ONS(6,10) code as below creates a **linear equation** 1z+1y+3x=1x+1z+3y which makes 2x=2y and hence the open line segment region is an open segment of the line y=x. Similarly for the CNS codes we can create its corresponding **linear equation**.

z y x
1 1 1 1 3 3
x z y

IV: All Equations The all equations which belong to a valid code sequence are finite sums $F(x, y)$ of sines or a sum of cosines and the intersection of all (x,y) over the all equations F where $F(x, y) > 0$ form a non-empty region. It creates an open region with positive area or an open line segment region of positive length. These all equations are not saved in our database. Rather it is faster and easier to generate them one by one, use them one by one in a calculation and then delete them. If you do want to see the all equations belonging to any code, you can find and create them under the Info button in the star jar [7].

An **open region R** belongs to a code sequence if it is the finite intersection over all its equations $F(x, y)$ where $F(x, y) > 0$ which is an open region with positive area.

$$R = \bigcap_{allequationsF} F(x, y) > 0$$

.

A **corner C of an open region R at the point (a,b)** is the non-empty finite intersection only over all its equations $F(x, y)$ where $F(a, b) = 0$ and where $F(x, y) > 0$.

$$C = \bigcap_{allFwhereF(a,b)=0} F(x, y) > 0$$

.

Note: A corner of a region comes from two successive boundary sides.

The **boundary** of a closed region is a clockwise or counterclockwise finite list of the curves $F(x, y) = 0$.

An **open line segment** with its corresponding linear equation is the finite intersection over all its equations F where $F(x, y) > 0$ which is an open line segment with positive length.

V: MRR Equations This is a shorter and preliminary list by refining the all equations.

11 The 90x90 Big Square and its Subdivisions

The **90x90 big square** is a square with coordinates (0,0),(90,0),(90,90),(0,90) using degrees. Each side has length $2r_1$=90=90/1 where r_1=45 is called its **radius**. We then keep subdividing the big square into 4 equal squares of side length $2r_1 = 90/2^k$ degrees with $0 \leq k$ as far as needed.

A **subdivided** $2r_1$**-square of side** $2r_1$ **belonging to a code sequence** and its all equations F is a **closed square** which is inside its corresponding open region. (s_1, t_1) is its center and every all equation F satisfies $F(s_1, t_1) > 0$ and $0 < r_1 < F(s_1, t_1)/(|F_x| + |F_y|)$. It is important to realize that a closed $2r_1$-square can never reach the boundary of its corresponding open region.

There are three more types of **subdivided squares** used.

1. A **stable square** in which every point inside it including the boundary has the same type of periodic path for that triangle which is given by its stable code sequence.
2. A **triple square** in which the square intersects a non-stable code sequence which is a finite open straight line segment which intersects the sides of the square. The non-stable then separates the square into two different stable code regions.
3.A **star flare square** in which there are multiple stable and multiple non-stable regions which intersect at a single point. We call a point (s_1, t_1) in the plane a **flare** if no open stable region contains that point and thus needs a family of open regions, open intervals and rational points to cover a neighbourhood of it. It is an **infinite flare** if it needs an infinite family, it is a **finite flare** otherwise.

In radians, the big square is a square with coordinates $(0, 0), (\pi/2, 0), (\pi/2, \pi/2)$ and $(0, \pi/2)$ where each side has length $2r_1$ radians where $2r_1 = \pi/2$. We now subdivide it into 4 equal squares of length $2r_1 = \pi/4$. In general we keep subdividing into 4 equal squares and get a $2r_1$-square where $2r_1 = \pi/2^k$ radians with $1 \leq k$.

Before we have been using a **centered** 2r-square where it is centered at (a,b). Caution: This 2r-square need not be one of the subdivided squares. We want to create a **subdivided** $2r_1$-square which is a subset of the 2r-square and contains the point (a,b) which in general is not its center.

Important Fact: A subdivided square must have side length $\pi/2^k$ in lowest terms and its coordinates of the square are $(s\pi/2^k, t\pi/2^k), ((s+1)\pi/2^k, t\pi/2^k), ((s+1)\pi/2^k, (t+1)\pi/2^k), (s\pi/2^k, (t+1)\pi/2^k)$. Its center is then $((2s+1)\pi/2^{k+1}, (2t+1)\pi/2^{k+1})$.

Given a centered 2r-square centered at (a,b), below is the method of finding a subdivided $2r_1$-square with $r_1 \leq \pi/4$ in radians or $r_1 \leq 45$ in degrees inside the 2r-square and includes (a,b).

Method in degrees with a,b in degrees:
1. Find the smallest integer k so that $90/2^k < r$ iff $90/r < 2^k$ iff $ln(90/r)/ln2 < k$, $0 \leq k$.
2. This means the subdivided $2r_1$-square has radius $r_1 = 90/2^{k+1}$.
To find its center (s_1, t_1) of the $2r_1$-square

3. Find the largest integer s so that $90s/2^k < a$ which means $s_1 = (2s+1)90/2^{k+1}$
4. Find the largest integer t so that $90t/2^k < b$ which means $t_1 = (2t+1)90/2^{k+1}$

Method in radians with a,b in radians:
1. Find the smallest integer k so that $\pi/2^k < r$ iff $\pi/r < 2^k$ iff $ln(\pi/r)/ln2 < k$, $1 \leq k$.
2. This means the subdivided $2r_1$-square has radius $r_1 = \pi/2^{k+1}$.

To find its center (s_1, t_1) of the $2r_1$-square
3. Find the largest integer s so that $s\pi/2^k < a$ which means $s_1 = (2s+1)\pi/2^{k+1}$
4. Find the largest integer t so that $t\pi/2^k < b$ which means $t_1 = (2t+1)\pi/2^{k+1}$

Example in radians: In a centered 2r-square centered at $(\pi/18, \pi/9)$ with r=6.187503E-10 radians
1. We need smallest integer k so that $ln(\pi/r)/ln2 < k$ which makes k=33.
2. Hence we use r_1=1.828647E-10< $\pi/2^{34}$ to form a subdivided $2r_1$-square in radians.
3. Here $a = \pi/18$ and we need the largest s with $s\pi/2^{33} < \pi/18$ iff $s < 2^{33}/18$ which means s=477218588 and $s_1 = \pi(2s+1)/2^{34} = \pi 954437177/2^{34}$=0.174532... rad
4. Here $b = \pi/9$ and we need the largest t with $t\pi/2^{33} < \pi/9$ iff $t < 2^{33}/9$ which means t=954437176 and $t_1 = \pi(2t+1)/2^{34} = \pi 1908874351/2^{34}$=0.349065... rad

Thus the center of the subdivided $2r_1$-square is $(\pi(2s+1)/2^{34}, \pi(2t+1)/2^{34}) = (0.174532..., 0.349065...)$.

Example in degrees: In a centered 2r-square centered at (36,54), r=0.0328296 degrees, then
1. We need the smallest integer k so that $ln(90/0.0328296)/ln2 < k$ which makes k=12.
2. Hence we will use $r_1 = 0.010986328125 = 90/2^{13}$ to form a subdivided $2r_1$-square in degrees.
3. Here a=36 and we need the largest integer s so that $90s/2^{12} < 36$ iff $s < 2^{13}/5$ which means s=1638 and $s_1 = (2s+1)90/2^{13} = 36.002197265625$
4. Here b=54 and we need the largest integer t so that $90t/2^{12} < 54$ iff $t < 6(2^{11})/5$ which means t=2457 and $t_1 = (2t+1)90/2^{13} = 53.997802734375$

Thus the center of the subdivided $2r_1$-square is $((2s+1)90/2^{13}, (2t+1)90/2^{13})$=(36.002197265625,53.997802734375).

12 Two Examples of a Centered Square Subdivided

Example one: Take the equation $F(x, y) = -cos(x+y) = 0$ which is a periodic set of lines $x + y = 90 + 180k$ for any integer k. Now choose the point (36,54) which is on the line $x + y = 90$. We now want to make a small enough square centered at (36,54) and of side $2r$ where the square only intersects this single line $x + y = 90$ and hence treats this curve as a function of both x and y.

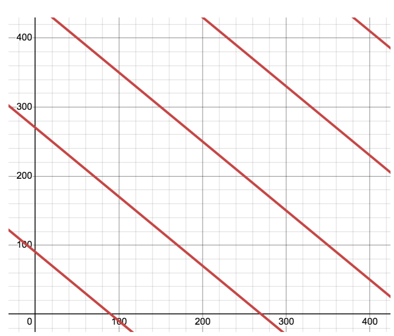

Figure 4: -cos(x+y)=0

We can do this by first calculating the two partials $F_x = sin(x + y)$ and $F_y = sin(x + y)$ and making sure that both of these partials at (x, y) are totally positive throughout the 2r-square. This will happen if we choose $0 < r < sin(90)/2 = 1/2$ radians.

Using the first partial $F_x(x,y) = sin(x+y)$ where $F_x(36,54) = 1 > 0$. By the Mean Value Theorem $F_x(36,54) - F_x(x,y) = 1 - F_x(x,y) = F_{xx}(c_1,c_2)(36-x) + F_{xy}(c_1,c_2)(54-y) = cos(c_1+c_2)(36-x) + cos(c_1+c_2)(54-y) \leq 2r < 1$ when $0 < r < 1/2$ radians. Hence $0 < F_x(x,y)$ in this 2r-square. Similarly for the second partial.

This 2r-square is not a subdivision of the Big Square. However we can eventually find a subdivision of the Big Square which contains (36,54) and is inside the centered square. It's radius is $r_1 < \pi/16$ radians or $< 90/8$ degrees.

Example two.

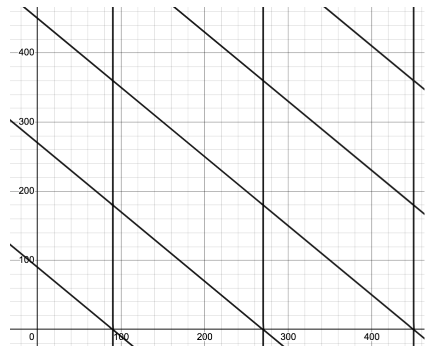

Figure 5: cos y+cos(2x+y)=0

Let $F(x,y) = cosy + cos(2x+y) = 0$ which is a double set of straight lines. One set is $x + y = 90 + 180k$ with slope -1 and the other set is vertical lines $x = 90 + 180k$. Again choose the point (36,54) which is on the line $x + y = 90$.

$F_x(x,y) = -2sin(2x+y) = -2sin54 = -1.618033...$ at (36,54). Thus $F_x(x,y)$ is negative throughout a 2r-square where $0 < r = 0.269672 < |F_x(36,54)|/6$ and makes F a function of y.
$F_y(x,y) = -sin(y) - sin(2x+y) = -2sin54 = -1.618033...$ at (36,54). Thus $F_y(x,y)$ is negative throughout a 2r-square where if $0 < r = 0.404508 < |F_y(36,54)|/4$ and makes F a function of x.

It follows that if r=0.269672 then the curve $F(x,y) = 0$ is a decreasing function of x and y in a 2r-square centered at (36,54). This means that $x + y = 90$ is the only portion of this curve inside this 2r-square. It is an exercise to find a subdivision of the big square containing (36,54) and inside the given centered 2r-square.

13 Corners of regions

Notation Reviewed and Terminology
Let $F(x,y) = \sum_{i=1}^{k} k_i cos(m_i x + n_i y)$ or $F(x,y) = \sum_{i=1}^{k} k_i sin(m_i x + n_i y)$

Let $|F_x| = \sum_{i=1}^{k} |m_i k_i|$ and $|F_y| = \sum_{i=1}^{k} |n_i k_i|$

Let $|F_x| + |F_y|$ be called the F-bounds for F=$F(x,y)$.

Let $|F_{xx}| = \sum_{i=1}^{k} |m_i^2 k_i|$, $|F_{xy}| = |F_{yx}| = \sum_{i=1}^{k} |m_i n_i k_i|$ and $|F_{yy}| = \sum_{i=1}^{k} |n_i^2 k_i|$

Let $|F_{xx}| + |F_{xy}|$ be called the F_x-bounds for F_x=$F_x(x,y)$.

Let $|F_{yx}| + |F_{yy}|$ be called the F_y-bounds for F_y=$F_y(x, y)$.

Note: For each code and its all equations F corresponding to $F(x, y)$ of two variables and the point (a,b), we will list all the

i) equations or curves $F(x, y) = 0$ where $F(a, b) > 0$ and for convenience we will call them the **positive equations**.

Observe that in a 2r-square with $0 < r < F(a, b)/(|F_x| + |F_y|)$ all the positive equations will miss this 2r-square centered at (a,b).

ii) equations or curves $F(x, y) = 0$ where $F(a, b) = 0$ and for convenience we will call them the **zero equations**.

Observe that in a 2r-square where $0 < r < min(|F_x(a, b)|/(|F_{xx}| + |F_{xy}|), |F_y(a, b)|/(|F_{yx}| + |F_{yy}|))$, a zero curve F is a function of x, of y or of both. As a bonus this will keep the signs of F_x and F_y the same in this 2r-square as long as their value at (a,b) are non-zero. This means that the slopes of the curve at every point on the curve $F(x, y) = 0$ are non zero and have the same sign.

Corners
1. An open corner at (a,b) of an open stable region R is the finite intersection of all its zero equations F where $0 < F(x, y)$ with a non zero area.
2. A closed corner at (a,b) of a closed stable region $\overline{R}$ is the finite intersection of all its zero equations F where $0 \leq F(x, y)$ with a non zero area.
3. We will list the slopes of the zero equations in numerical order $m_1, m_2, ..., m_n$ and view them on a circle. A given corner will have two successive slopes for a fixed i where m_i, m_{i+1} or be at the beginning and end of the list m_n, m_1. The slopes can be finite or infinite.
4. We make a 2r-square centered at (a,b) small enough so that the all its zero equations F are functions of x or of y or of both.
5. We make a 2r-square centered at (a,b) small enough so that the all its zero equations F intersect only at (a,b) using wings as needed.

Comments: In a corner there are two zero sides with one called the **bottom corner side** F(x,y)=0 and the other called the **top corner side** G(x,y)=0 with slopes which are viewed counter clockwise from the bottom to the top where their slopes are different. We make a square small enough so that the corner is **separated** in a 2r-square. This means the two corner sides intersect only at the center of the square. In general the 2r-square has four regions and the corner is the region with both sides positive where $F(x, y) > 0$ and $G(x, y) > 0$. The other three regions are - +, + - or - -. We do allow a degenerate corner where the corner is straight as in an unstable code.

Corner Sides Method (where the gradients of F and G are non zero at (a,b).)

1. Make the corner's two sides F(x,y)=0 and G(x,y)=0 become functions in a 2r-square .
2. Separate the corner's two sides in a 2r-square by using an appropriate small enough $0 < \epsilon < 1$.
3. Make two wings to separate the two corner sides (if they are not straight lines). This forms an actual straight corner inside the original corner in a 2r-square with the same given code. Note: This is from the upper wing line of the bottom side to the lower wing line of the top side.
4. Separate any other non-corner zero at the corner from each of the corner's two sides by wings.

14 The Star Flare Square

To make a star flare square of radius r_1, we do the following using a computer and its database [7].

1. Find a point (a, b) which is not found inside any stable region.
2. Find a neighbourhood of a finite set of stable and unstable regions which totally surround the given point (a, b) and arrange them counterclockwise. The stable regions form corners at (a, b). The unstable regions point at or go through (a, b). Successive regions must overlap or abut.
3. Find a centered 2r-square at (a,b) which is a subset of this neighbourhood.
4. For each stable region and its code, its all trig sums $F(x, y)$ must be positive or zero at (a, b). Note there are no trig sums $F(x, y)$ which are negative at (a, b).

a) For each positive equation F where $F(a, b) > 0$ we find the minimum of r where $0 < r < F(a, b)/(|F_x| + |F_y|)$. This means all these curves $F(x, y) = 0$ miss this 2r-square.

b) For each zero equation F where $F(a, b) = 0$ we find the minimum of r where $0 < r < min(|F_x(a, b)|/(|F_{xx}| + |F_{xy}|), (|F_y(a, b)|/(|F_{yx}| + |F_{yy}|))$. This means all these zero curves $F(x, y) = 0$ are functions of x or of y or of both in a 2r-square.

c) Every zero equation F at (a,b) is trapped in a wing using epsilon in a small enough 2r-square as in section 6. Note: If $F(x, y) = 0$ is a straight line, then the wing is itself.

d) Every corner at (a,b) of a stable region is separated by two wings from its bottom side to its top side counterclockwise. Using a 2r-square and epsilon, this guarantees that the counterclockwise angle between the upper wing line of the bottom side to the lower wing line of the top side is inside this stable region.

e) Every non-corner zero equation at (a,b) of this stable region is separated from its corner in a 2r-square using epsilon.

5. For each unstable region and its code, its all trig sums $F(x, y)$ must be positive or zero at (a, b). Note: There are no trig sums $F(x, y)$ which are negative at (a, b). Note again: An unstable region is a straight open line segment.

a) For each positive equation F where $F(a, b) > 0$ we find the minimum of r where $0 < r < F(a, b)/(|F_x| + |F_y|)$. This means all these curves $F(x, y) = 0$ miss this 2r-square.

b) If $F(a, b) = 0$ then the line segment ends at that point. In this case we need to show that the point (a, b) has a periodic path. For example if (a, b) is a rational point.

6. Find a covering from the given neighbourhood which separates the bottom side of a corner of a successive region to the top side of a corner of the previous region using wings and epsilon as necessary in a 2r-square. This means two adjacent regions overlap or abut.
7. Find a subdivided $2r_1$-square which contains (a,b) and is completely inside the given centered 2r-square. This creates the (a,b) subdivided Star Flare square.

Since the 10-20 star flare square involves thousands and thousands of equations, we put them in a database as needed and you can access them there. To be comfortable that the equations and values in the database are correct we will give some other examples of star squares. We will start with the 36-54 star where we can do all the calculations by hand if needed.

Here is some more detail **in a 2r-square** centered at (a,b).

A positive equation misses the 2r-square

If $F(a, b) > 0$ then the curve F(x,y)=0 misses the 2r-square where $0 < r < F(a, b)/(|F_x| + |F_y|)$

A zero equation is a function in a 2r-square

If $F(a, b) = 0$ then the curve F(x,y)=0 is a function if

$F_x(a, b) > 0$ then F(x,y)=0 is a function of y in a 2r-square where $0 < r < F_x(a, b)/(|F_{xx}| + |F_{xy}|)$
$F_y(a, b) > 0$ then F(x,y)=0 is a function of x in a 2r-square where $0 < r < F_y(a, b)/(|F_{yx}| + |F_{yy}|)$
$F_x(a, b) < 0$ then F(x,y)=0 is a function of y in a 2r-square where $0 < r < |F_x(a, b)|/(|F_{xx}| + |F_{xy}|)$
$F_y(a, b) < 0$ then F(x,y)=0 is a function of x in a 2r-square where $0 < r < |F_y(a, b)|/(|F_{yx}| + |F_{yy}|)$

Two zero corner equations use wings and epsilon to separate them

Given a corner of a stable region which is the intersection point (a, b) of two curves $F(x, y) = 0$ and $G(x, y) = 0$. We need $F(x, y) > 0$ and $G(x, y) > 0$ to form the corner. We put the corner inside a square of side 2r centered at (a,b) so that the two curves intersect only at the center. We say that the two curves are **separated** at the center which creates four regions in the square. The **corner** at (a,b) is the only one where $F(x, y) > 0$ and $G(x, y) > 0$.

If $F(a, b) = 0$ and $G(a, b) = 0$ then the two curves F(x,y)=0 with slope m1 and G(x,y)=0 with slope m2 where $\epsilon = ||m2| - |m1||/2(|m2| + |m1|)$ intersect only at (a,b) in a 2r-square centered at (a,b). They are separated by two wings where $0 < m1 < m2$ or $m2 < m1 < 0$ where

$0 < r < \epsilon min(|F_x(a, b)|/(|F_{xx}| + |F_{xy}|), |F_y(a, b)|/(|F_{yx}| + |F_{yy}|), |G_x(a, b)|/(|G_{xx}| + |G_{xy}|), |G_y(a, b)|/(|G_{yx}| + |G_{yy}|))$

Two zero equations are separated if one is straight and the other is not

If $F(a, b) = 0$ and $G(a, b) = 0$ then the two curves (one straight and the other curved) F(x,y)=0 with slope m1 and G(x,y)=0 with slope m2 where $\epsilon = ||m2| - |m1||/(|m2| + |m1|)$ intersect only at (a,b) in a 2r-square centered at (a,b). They are separated by one wing where $0 < m1 < m2$ or $m2 < m1 < 0$ if

$0 < r < \epsilon min(|F_x(a, b)|/(|F_{xx}| + |F_{xy}|), |F_y(a, b)|/(|F_{yx}| + |F_{yy}|), |G_x(a, b)|/(|G_{xx}| + |G_{xy}|), |G_y(a, b)|/(|G_{yx}| + |G_{yy}|))$

Two zero equations are separated if they are both straight

If $F(a, b) = 0$ and $G(a, b) = 0$ then the two curves (both straight) F(x,y)=0 with slope $m1 \neq 0$ and G(x,y)=0 with slope $m2 \neq 0$ letting $\epsilon = 1$ intersect only at (a,b) in a 2r-square centered at (a,b). They are separated by themselves and no wings are needed where $m_1 \neq m_2$ if

$0 < r < min(|F_x(a, b)|/(|F_{xx}| + |F_{xy}|), |F_y(a, b)|/(|F_{yx}| + |F_{yy}|), |G_x(a, b)|/(|G_{xx}| + |G_{xy}|), |G_y(a, b)|/(|G_{yx}| + |G_{yy}|))$

Note: There are special cases and degenerate cases as they arrive.

15 The 36-54 Star Flare Square

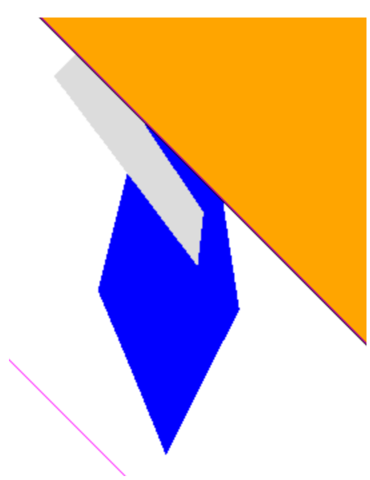

Figure 6: The 36-54 Star Flare

We will give a proof that the **36-54 square** is a finite star flare of the point (36,54) using three stables and one unstable. Observe that the point (36,54) is not inside any stable region by [2] and that (36,54) is not the center of a subdivided square. Keep in mind that a subdivided square is closed and is completely inside an open neighbourhood of these three stable regions and one unstable region.

We need to do the following steps where we start with a centered square centered at (36,54). Only the first two regions overlap and the rest abut. You can see the four regions in Billiards Everything and you can see the subdivided square in Billiards Covers [6] or the Star Flare jar [7], all of which you can find in sourceforge.net.

There are five codes going counterclockwise around the point (36,54) using three stables and one unstable used twice. These four regions cover an centered 2r-square where r=**0.000572984** radians and the center of this square is (36,54).

CS(28,48) 1 1 1 1 2 1 3 2 3 1 2 1 1 1 1 3 2 3 1 1 2 2 2 1 1 3 2 3 grey
CS(14,28) 1 1 2 4 2 1 1 3 3 1 2 1 3 3 blue
CNS(4,6) 1 2 1 2 lower black
OSO(3,3) 1 1 1 orange
CNS(4,6) 1 2 1 2 upper black

For every F the **positive equations** where $F(x,y) = 0$ and $F(36,54) > 0$ must satisfy $0 < r < F(36,54)/(|F_x| + |F_y|)$ for every F. This means none of these positive curves $F(x,y) = 0$ intersect the 36-54 square of side 2r.

For every F the **zero equations** where $F(x,y) = 0$ and $F(36,54) = 0$ must satisfy $0 < r < min(|F_x(36,54)|/(|F_{xx}| + |F_{xy}|), (|F_y(36,54)|/(|F_{yx}| + |F_{yy}|))$. This means these zero curves $F(x,y) = 0$ are functions of x or of y or of both in this 2r-square.

The **All equations** belonging to a code sequence comprise the positive and zero equations F above with its F-bounds and can be found in the star jar using the info button [7]. Caution: These do not show if it is a positive or zero equation because that depends on the point (a,b). To see if an equation is positive or zero for example at (36,54) you can put it into the gradient button.

1. **CS(28,48)-** This stable code has 151 positive equations and 3 zero equations at the point 36-54 for a total of 154 all equations. This is the grey portion of the square.

a. **Positive equations:** Of these there are 151 positive equations and we will just give the equation that leads to the smallest r=0.00467737 and we will only store that r in the database.

F-bounds=48, F(x,y)=cos(x-3y)+cos(x-y)+cos(x+y)+cos(3x+y)-cos(3x+3y)-2cos(5x+y)-cos(5x+3y)+cos(7x+3y) where $0 < r = 0.00467737 < F(36,54)/(|F_x| + |F_y|) = |F(36,54)|/48$

We will use this minimum **r=0.00467737** for section a.

b. **Corner Zero equations:** We need to separate the top and bottom curved sides of this corner in a 2r-square. Here the corner is the intersection of two zero equations of which the bottom is straight and the top is curved at the point (36,54).

i) **The bottom corner curve** is $F(x,y) = cos(2y) + cos(2x - 2y) + cos(4x) - cos(4x + 2y) = 0$ which is two infinite families of curves, one of which is straight and the other is not and (36,54) is on only one straight line $x+y = 90$ and where $y' = -1$. Note $F(x,y)$ factors as $-2cos(x+y)(-cos(x-y)-cos(3x-y)+cos(3x+y))$. You can find these factors using the Triples button in the star jar.

$$F_x(x,y)|_{(36,54)} = -2sin(2x - 2y) - 4sin(4x) + 4sin(4x + 2y)|_{(36,54)} < -4.979796 < 0$$
$$F_y(x,y)|_{(36,54)} = -2sin(2y) + 2sin(2x - 2y) + 2sin(4x + 2y)|_{(36,54)} < -4.979796 < 0$$

Here are the second derivatives

$F_{xx}(x, y) = -4cos(2x - 2y) - 16cos(4x) + 16cos(4x + 2y)$ with $|F_{xx}| = 36$
$F_{xy}(x, y) = 4cos(2x - 2y) + 8cos(4x + 2y)$ with $|F_{xy}| = 12$
where F_x-bounds=48

We let $0 < r < 4.979796/48$ which makes $F_x(x, y)$ negative throughout this 2r-square.

$F_{yx}(x, y) = 4cos(2x - 2y) + 8cos(4x + 2y)$ with $|F_{yx}| = 12$
$F_{yy}(x, y) = -4cos(2y) - 4cos(2x - 2y) + 4cos(4x + 2y)$ with $|F_{yy}| = 12$
where Fybounds=24

We let $0 < r < 4.979796/24$ which makes $F_y(x, y)$ negative throughout this 2r-square.

Thus the bottom line curve is a decreasing function of both x and y in the 2r square and let the smallest **r=0.103745** $< 4.979796/48$

ii) **The top corner curve** is $G(x, y) = cos(0) + cos(2y) - cos(4y) + cos(2x) - cos(2x + 4y) + 2cos(4x) - 2cos(4x + 2y) - cos(6x + 2y) + cos(6x + 4y) = 0$ which is an infinite family of curves, none of which is straight and (36,54) is on the top curve and where $y' = -1.505091... < -1.505091 < -1$.

$G_x(x, y)|_{(36,54)} = -2sin(2x) + 2sin(2x + 4y) - 8sin(4x) + 8sin(4x + 2y) + 6sin(6x + 2y) - 6sin(6x + 4y)|_{(36,54)} < -25.348010 < 0$
$G_y(x, y)|_{(36,54)} = -2sin(2y) + 4sin(4y) + 4sin(2x + 4y) + 4sin(4x + 2y) + 2sin(6x + 2y) - 4sin(6x + 4y)|_{(36,54)} < -16.841502 < 0$

Here are the second derivatives

$G_{xx}(x, y) = -4cos(2x) + 4cos(2x + 4y) - 32cos(4x) + 32cos(4x + 2y) + 36cos(6x + 2y) - 36cos(6x + 4y)$ with $|G_{xx}| = 144$
$G_{xy}(x, y) = 8cos(2x + 4y) + 16cos(4x + 2y) + 12cos(6x + 2y) - 24cos(6x + 4y)$ with $|G_{xy}| = 60$
where Gxbounds=204

We let $0 < r = 0.124254 < 25.348010/204$ which makes $G_x(x, y)$ negative throughout this 2r-square.

$G_{yx}(x, y) = 8cos(2x + 4y) + 16cos(4x + 2y) + 12cos(6x + 2y) - 24cos(6x + 4y)$ with $|G_{yx}| = 60$
$G_{yy}(x, y) = -4cos(2y) + 16cos(4y) + 16cos(2x + 4y) + 8cos(4x + 2y) + 4cos(6x + 2y) - 16cos(6x + 4y)$ with $|G_{yy}| = 64$
where Gybounds=124

We let $0 < r = 0.135818 < 16.841502/124$ which makes $G_y(x, y)$ negative throughout this 2r-square.

Thus the top curve is a decreasing function of both x and y in the 2r square and let the smallest **r=0.124254**

The smallest r so far of i) and ii) is **r=0.103745** to make both sides of the corner into functions in this 2r-square.

iii) **Corner wings:** We separate these two corner sides by using two wings centered at (36,54) so that the top corner $G(x, y) = 0$ doesn't intersect the bottom corner $F(x, y) = 0$ in the 2r-square other than at the point (36,54).

We separate the top side corner from the bottom side corner at (36,54)
Top slope $m1 = -1.505091...$ and Bottom slope m2=-1.0.
Let $r' = \min(0.124254, 0.103745) = 0.103745$ This is the min r from i) and ii)

Let epsilon = $0.100813 \leq .5(|m1| - |m2|)/(|m1| + |m2|)$
Then we use $0 < r = 0.0104589 < \epsilon r'$ to separate the corner using wings.

Special Case: Since the bottom side is a straight line, we could also separate the two corner sides using one wing instead of using two wings.

We separate the top side corner from the bottom side corner at (36,54)
Top slope m1=-1.505091... and Bottom slope m2=-1.0.
Let r' = min(0.124254,0.103745) = 0.103745 This is the min r from i) and ii)
Let epsilon = $0.201625 \leq (|m1| - |m2|)/(|m1| + |m2|)$
Then we can use $0 < r = 0.0209175 < \epsilon r'$ to separate the corner using wings.

Note: We will still use the smaller **r= 0.0104589** to separate the corner since both work.

c. **Non Corner Zero equations:** There is only one zero non corner curve and we need to separate it from the corner in a 2r-square.

This zero curve is
F(x,y)= cos(y)+cos(2x-3y)-cos(2x-y)+cos(2x+y)-cos(2x+3y)+cos(4x-y)-cos(4x+3y)-cos(6x+y)+cos(6x+3y)=0
with slope -1.466931... where its

F_x-bounds=176, $F_x(x,y)$=-2sin(2x-3y)+2sin(2x-y)-2sin(2x+y)+2sin(2x+3y)-4sin(4x-y)+4sin(4x+3y)+6sin(6x+y)-6sin(6x+3y), $F_x(36,54) = -15.708203...$
Fybounds=97, $F_y(x,y)$=-sin(y)+3sin(2x-3y)-sin(2x-y)-sin(2x+y)+3sin(2x+3y)+sin(4x-y)+3sin(4x+3y)+sin(6x+y)-3sin(6x+3y), $F_y(36,54) = -10.708203...$

Let $0 < r = 0.0892511 < min(|F_x(36,54)|/176, |F_y(36,54)|/97)$ to make both derivatives negative in a 2r-square.

We will use this minimum **r=0.0892511** to make this F a function.

Zero wings: We need to put a wing around this **zero curve** F=0 so that its wing separates and only intersects the corner at (36,54) in a 2r-square.

We separate the top side corner from this zero curve at (36,54) using wings.
Top slope m1=-1.505091... and zero curve slope m2=-1.466931...
Let r' = min(0.12425,0.0892511) = 0.0892511
Let epsilon = $0.00641991 \leq .5(|m1| - |m2|)/(|m1| + |m2|)$
Let $0 < r = 0.000572984 < \epsilon r'$

We separate the bottom side corner from this zero curve at (36,54) using wings.
Slopes m3=-1.0 and zero curve slope m2=-1.466931...
Let r' = min(0.103745, 0.0892511) = 0.0892511
Let epsilon = $0.0946381 \leq .5(|m3| - |m2|)/(|m3| + |m2|)$
Let $0 < r = 0.00844656 < \epsilon r'$

We will use this minimum **r=0.000572984** to separate F from both sides of the corner.

The minimum r for CS(28,48) is **r=0.000572984**

2. **CS(14,28)-** This stable code has 49 positive equations and 3 zero equations at the point (36,54) for a total of 52 all equations. This is the blue portion of the square.

a) **Positive equations** This code has 49 positive equations F(x,y)=0 where $F(36,54) > 0$ and we will only list the equation that gives the smallest r and we store only that r in the database.

F-bounds= 20, F(x,y)=cos(2y)-cos(2x-2y)+cos(2x)+cos(4x-2y)-cos(4x+2y)=0 where $0 < r = 0.0154508 < |F(36,54)|/20$

We will use this minimum $0 <$ **r=0.0154508** $< F(36,54)/(|F_x| + |F_y|)$ for all the positive equations to miss the 2r-square.

b) **Corner Zero equations:** Note: This blue corner at (36,54) has two straight sides and every point in this corner has a periodic path of this code type CS(14,28). Thus corner wings and epsilons are not needed to separate the corner.

i) The **bottom corner** is F=-cos(4x-y)=0 with slope=4

$F_x$bounds=20, $F_x(x,y) = 4sin(4x - y)$, $F_x(36,54) = 4$
$F_y bounds$=5, $F_y(x,y) = -sin(4x - y)$, $F_y(36,54) = -1$

Let $0 < r < min(4/20, 1/5) = 0.2$ which makes F an increasing straight line function cos(4x-y)=0.

ii) The **top corner** is G= cos(y)+cos(2x+y)=0 with slope=-1 and G factors to 2cos(x+y)cos(x).

$G_x$bounds=6, $G_x(x,y) = -2sin(2x + y)$, $G_x(36,54) = -2sin54$
$G_y bounds$=4, $G_y(x,y) = -sin(y) - sin(2x + y)$, $G_y(36,54) = -2sin54$

Let $0 < r = 0.269672 < min(2sin(54)/6, 2sin(54)/4)$ which makes G a decreasing straight line function cos(x+y)=0.

We will use this minimum $0 < r <$ **0.2** to make both sides of the corner into straight line functions in a 2r-square.

c) **Non Corner Zero equations:** There is only one and we need to separate it from the corner.

This is the zero equation F=sin(3x-2y)=0 with a straight line slope=1.5

$F_x$bounds=15, $F_x(x,y) = 3cos(3x - 2y)$, $F_x(36,54) = 3$
$F_y bounds$=10, $F_y(x,y) = -2cos(3x - 2y)$, $F_y(36,54) = -2$

Let $0 < r < min(3/15, 2/10) = 0.2$

We will use this minimum $0 < r <$ **0.2** to make F a straight line function sin(3x-2y)=0.

Note: Since this zero equation and the corner sides are all straight line functions with different slopes, F is automatically separate from the corner.

The minimum r for CS(14,28) is **r= 0.0154508**

3. **OSO(3,3)** This stable OSO(3,3) 1 1 1 has three positive equations and one zero equation at the point (36,54). Note: There is only one zero equation here since (36,54) is a **straight degenerate corner**.

a) **Positive equations** These 3 positive equations where $F(x,y) > 0$ at (36,54) are listed below.

F-bounds=0, F=cos(0) where $r = infinity$
F-bounds=1, F=$cos(y)$ where $0 < r = 0.587785 < cos(54)/1$

F-bounds=1, F=$cos(x)$ where $0 < r = 0.809016 < cos(36)/1$

We will use this minimum $0 <$ **r=0.587785** $< F(36,54)/(|F_x| + |F_y|)$ for all the positive equations and which is stored in the database.

Note: These three curves $F(x,y) = 0$ will all miss the smallest 2r-square using r=0.587785 In particular this means the curves y=90 and x=90 will miss this 2r-square.

b) **Corner Zero equations:** This corner $F(x,y) = -cos(x+y) = 0$ is degenerate and separated into a **bottom lower corner side** and a **top upper corner side** at the point (36,54) viewed counter clockwise.

This code has only 1 zero equation from the all equations where $F(x,y) = -cos(x+y) = 0$ and $F(36,54) = -cos(90) = 0$ which is a point on a set of straight lines $x + y = 90 + 180k$. In particular this means that the curve x+y=90 intersects this centered 2r-square at its center.

$F_x$bounds=2, F_x=sin(x+y) where $0 < r < .5 = sin(90)/2$
$F_y$bounds=2, F_y=sin(x+y) where $0 < r < .5 = sin(90)/2$

We will use this minimum r where $0 < r < min(.5, .5)$=**.5** to make F a straight line function.

Note that the slope $dy/dx = -F_x/F_y = -sin(x+y)/sin(x+y) = -1$ and that $F_x = F_y = sin(x+y) = 1$ at (36,54) and we need $0 < r < sin(90)/2 = .50$ to keep a 2r-square centered at (36,54) so that F has only one line through it.

c) **Non Corner Zero equations:** There are none.

The minimum r for OSO(3,3) is $0 < r <$ **0.5**

Note: As a very special case if (x,y) is in the open triangular region with coordinates (0,90), (90,90), (90,0) or alternately $0 < x < 90, 0 < y < 90$ and $90 < x + y$ then every point in this region satisfies that $cos0 > 0$, $cosx > 0$, $cosy > 0$ and $-cos(x+y) > 0$. This is the orange portion of the given square.

4. **CNS(4,6)** Below are all the unstable equations F for CNS(4,6) 1 2 1 2 which is a straight line segment x+y=90 and where $F(36,54) > 0$ and then r satisfies $0 < r < F(36,54)/(|F_x| + |F_y|)$ for every F. Note: There are exactly 5 positive equations and no zero equations at (36,54).

F-bounds 0 F=cos(0), where r=infinity
F-bounds 1 F=cos(x), where r = 0.809016...
F-bounds 3 F=-cos(2x+y), where $r = 0.195928 < cos(126)/3$
F-bounds 4 F= -cos(2x+2y), where r=.25
F-bounds 3 F=sin(2x+y), where r= 0.269672...

The minimum r for CNS(4,6) is **r=0.195928** and all these positive equations miss this 2r-square centered at (36,54).

5. **Covering:** We now put the regions above counterclockwise as in the Figure 6 so that they overlap or abut to fully cover and surround a 2r-square and use wings and epsilons as necessary. Here it turns out that no wings and no epsilons are needed. Bot=Bottom corner side, Top=Top cprner side.

Bot CS(14,28) overlaps Top CS(28,48):
- Bot CS(14,28) increasing straight line has slope m_1=4 and r_1=.2
- Top CS(28,48) decreasing curved line has slope m_2=-1.505091... and r_2=0.124254...
- It turns out that no wings are needed since the slopes are opposite sign.

r = 0.124254 <min(r_1,r_2)

Bot CNS(4,6) abuts Top CS(14,28):
- Bot CNS(4,6) straight line has slope -1.0 and r_1=0.195928...
- Top CS(14,28) straight line has slope m=-1 r=0.269672...

r = 0.195928 < min(r_1,r_2)

Bot OSO(3,3) abuts Top CNS(4,6):
- Bot OSO(3,3) straight line has slope -1.0 and r_1=0.5
- Top CNS(4,6) straight line has slope -1.0 and r_2=0.195928...

r = 0.195928 <min(r_1,r_2)

Bot CNS(4,6) abuts Top OSO(3,3):
- Bot CNS(4,6) straight line has slope -1.0 and r_1=0.195928...
- Top OSO(3,3) straight line has slope -1.0 and r_2=0.5

r = 0.195928 < min(r_1,r_2)

Bot CS(28,48) abuts Top CNS(4,6):
- Bot CS(28,48) straight line has slope -1.0 and r_1=0.103745...
- Top CNS(4,6) straight line has slope -1.0 and r_2=0.195928...

r = 0.103745 <min(r_1,r_2)

The minimum r for this covering is **0.103745**

The **overall minimum r** for a centered 2r-square is **r=0.000572984**.

6. **The 2r-squares:** Finally we form the two types of squares.

The **centered 2r-square** centered at (36,54) uses this overall minimum **r=0.000572984** from all the sections above and every point in this closed square has a periodic path of at least one of the 5 given code types.

The **subdivided $2r_1$-square** which contains (36,54) and is inside the centered square has side $2r_1$ where $r_1 = \pi/2^{14}$ radians or $r_1 = 90/2^{13}$ degrees by using subdivisions of the big square. Its center is at $(3277\pi/2^{14}, 4915\pi/2^{14})$ = (0.628356..., 0.942439...) radians = $(589860/2^{14}, 884700/2^{14})$ = (36.002197..., 53.997802...) degrees.

This is the **subdivided 36-54 Star Flare square** which you can find in the star jar [7] and also in Billiards Covers [6].

16 The 45-45 Star Flare Square

There are twelve codes counterclockwise surrounding the point (45,45) with six stables and 6 nonstables as in the Figure 7 below with the minimum r of the positive equations listed to the right for each code. We store each of these r in the database.

OSO(3,3) 1 1 1 orange r=0.707106
CNS(4,6) 1 2 1 2 black r= 0.235702
CS(20,36) 1 1 2 2 2 2 2 1 1 3 2 2 1 1 4 1 1 2 2 3 yellow r=0.021427
CNS(10,20) 1 2 1 2 1 2 1 3 4 3 black r=0.0128564
CS(28,52) 1 1 2 2 2 2 2 2 2 2 2 1 1 3 2 2 2 2 1 1 4 1 1 2 2 2 2 3 blue **r=0.0108785**
CNS(6,14) 1 2 1 3 4 3 black r=0.0261891
OSO(3,7) 1 3 3 magenta r=0.101015
CNS(6,14) 1 2 1 3 4 3 black r=0.0261891

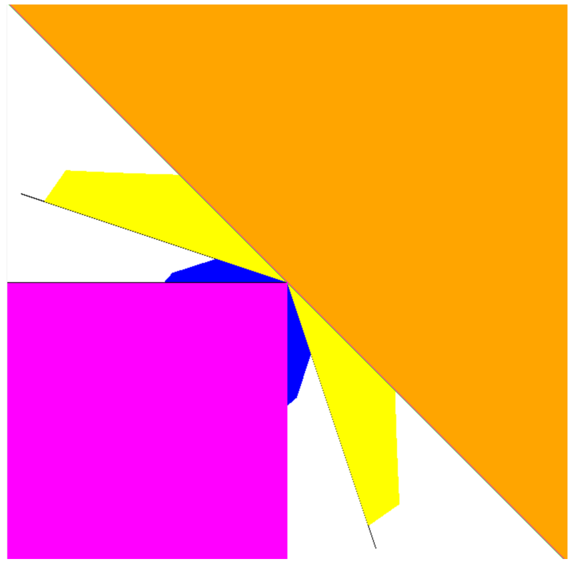

Figure 7: The 45-45 Star Flare

CS(28,52) 1 1 2 2 2 2 2 2 2 2 2 1 1 3 2 2 2 2 1 1 4 1 1 2 2 2 2 3 blue **r= 0.0108785**
CNS(10,20) 1 2 1 2 1 2 1 3 4 3 black r=0.0128564
CS(20,36) 1 1 2 2 2 2 2 1 1 3 2 2 1 1 4 1 1 2 2 3 yellow r=0.021427
CNS(4,6) 1 2 1 2 black r= 0.235702

Note: The minimum **r=0.0108785** from all the positive equations and from the stable and unstable codes is listed above.

Special case: All these regions are symmetric in the diagonal y=x. This means that if the point (x,y) is in a centered 2r-square then so is the point (y,x). In particular the point (45,45) is the center of a centered 2r-square and it has the periodic path 1 2 1 2 inside it.

For every F the **positive equations** $F(x, y) = 0$ where $F(45, 45) > 0$ must satisfy $0 < r < F(45, 45)/(|F_x|+|F_y|) = F(45, 45)/Fbounds$ for every F. This means none of these positive curves $F(x, y) = 0$ intersect the 45-45 square of side 2r.

For every F the **zero equations** $F(x, y) = 0$ where $F(45, 45) = 0$ must satisfy $0 < r < min(|F_x(45, 45)|/(|F_{xx}|+|F_{xy}|), (|F_y(45, 45)|/(|F_{yx}|+|F_{yy}|)) = min(|F_x(45, 45)|/F_x bounds, |F_y(45, 45)|/F_y bounds)$. This means these zero curves $F(x, y) = 0$ become functions in this 2r-square.

The Unstables

We have 3 unstable codes and their corresponding open regions.

CNS(4,6) 1 2 1 2 which is the open line segment y=90-x from (0,90) to (90,0).
CNS(10,20) 1 2 1 2 1 2 1 3 4 3 which is the upper open line segment from y=60-x/3 from (36,48) to (45,45).
CNS(6,14) 1 2 1 3 4 3 which is the upper open line segment y=45 from (22.5,45) to (45,45).

Again note that for the second two CNS, we only need to look at the upper side of the diagonal as the lower side is automatic by symmetry.

The positive CNS equations:

We will only list that positive CNS equation F which gives the smallest r where $0 < r < F(45,45)/(|F_x|+|F_y|)$.

i. CNS(4,6) 1 2 1 2 We use F-bounds=3, F(x,y)= -cos(2x+y), where $r = 0.235702 < cos(45)/3$
Note: This has 5 positive equations (and no zero equations) for a total of 5 all equations.

Important note: This CNS region is an oblique open straight line segment y=90-x from (0,90) to (90,0).

ii. CNS(6,14) 1 2 1 3 4 3 We use F-bounds=27, F(x,y)=cos(3y)-cos(5y)-cos(2x-5y)+cos(2x-3y)-cos(2x+5y) where $r = 0.0261891 < cos(45)/27$
Note: This has 25 positive equations (and 2 zero equations) for a total of 27 all equations.

iii. CNS(10,20) 1 2 1 2 1 2 1 3 4 3 We use F-bounds=55, F(x,y)=cos(y)-cos(5y)+cos(7y)+cos(2x+y)-cos(2x+3y)-cos(2x+5y)+cos(2x+7y)-cos(4x+3y)+cos(4x+7y) where **r=0.0128564** $< cos(45)/55$
Note: This has 41 positive equations (and 3 zero equations) for a total of 44 all equations.

Conclusion: The smallest r from the positive CNS equations is **r=0.0128564** and all these equations miss this 2r-square.

The zero CNS equations

There are five zero equations F at (45,45) which are amongst only two CNS codes and where $0 < r < min(|F_x(45,45)|/(|F_{xx}| + |F_{xy}|), (|F_y(45,45)|/(|F_{yx}| + |F_{yy}|))$ to make F a function in a 2r-square. It turns out that all five zero equations are straight lines that go through (45,45). Each zero is separate from the corresponding CNS region which is an open straight line segment ending at (45,45). This means no wings or epsilon are needed.

a) zero CNS(6,14) equations: We will list the 2 zeros F for CNS(6,14) 1 2 1 3 4 3 Note: This CNS region is a horizontal open straight line segment y=45 from (22.5,45) to (45,45) with slope m=0.

i. F(x,y)= cos(x)+cos(x+2y) which is a double periodic set of straight lines and dy/dx = -1 and only one straight line goes through (45,45) in this 2r-square.

F_x-bounds=4, $F_x(45,45)/4 = -0.353553...$
F_y-bounds=6 , $F_y(45,45)/6 = -0.235702...$

We use r=0.235702 which makes this F a function and a straight line in this 2r-square and misses the CNS(6,14) open line segment region and goes through its closed endpoint (45,45).

ii. F(x,y)= -cos(2x+4y) which is a periodic set of straight lines and dy/dx = -0.5 and only one straight line goes through (45,45) in this 2r-square.

F_x-bounds=12, $F_x(45,45)/12 = -1/6$
F_y-bounds=24, $F_y(45,45)/24 = -1/6$

We use r=1/6 which makes this F a function and a straight line in this 2r-square and misses the CNS(6,14) open line segment region and goes through its closed endpoint (45,45).

Conclusion: The smallest r in a) is **r=1/6**. Note these two zeros are separate from the CNS(6,14) region in this 2r-square as their slopes are different.

b) zero CNS(10,20) equations: We will list the 3 zeros for CNS(10,20) 1 2 1 2 1 2 1 3 4 3 Note: This CNS region is an oblique open straight line segment y=60-x/3 from (36,48) to (45,45) with slope m=-1/3.

i. F(x,y)= -sin(4x+3y)-sin(4x+5y) which is a double periodic set of straight lines and dy/dx = -1.

F_x-bounds=64, $F_x(45,45)/64 = -0.0883883...$
F_y-bounds=66, $F_y(45,45)/66 = -0.0857099...$

We use $r = 0.0857099 < 8cos(45)/66$ which makes this F a function and a straight line in this 2r-square and misses the CNS(10,20) open line segment region at the closed endpoint (45,45).

ii. F(x,y)= cos(2y)-cos(4x+2y)+cos(4x+6y) which is a periodic set of curved lines and dy/dx = -0.8.

F_x-bounds=64, $F_x(45,45)/64 = -0.125$
F_y-bounds=76, $F_y(45,45)/76 = -10/76 = -0.131578...$

We use r=0.125 which makes this F a function and a curved line in this 2r-square and misses the CNS(10,20) open line segment region at the closed endpoint (45,45).

iii. F(x,y)= cos(4x+6y) whiich is a periodic set of straight lines and dy/dx = -2/3.

F_x-bounds=40, $F_x(45,45)/40 = -0.1$
F_y-bounds=60, $F_y(45,45)/60 = -0.1$

We use r=0.1 which makes this F a function and a straight line in this 2r-square and misses the CNS(10,20) region at the closed endpoint (45,45).

Conclusion: The smallest r in b) is **r=0.0857099**. Note these three zeros are separate from the CNS(10,20) region in this 2r-square as their slopes are all different.

Final Conclusion: The smallest r from all the unstables positive and zero equations is **r=0.0128564** $< cos(45)/55$.

The Stables

We have four stable codes with the minimum positive r listed to the right for each stable code and the number of its all equations.

OSO(3,3) 1 1 1 orange r=0.707106 with 4 all equations.
CS(20,36) 1 1 2 2 2 2 2 1 1 3 2 2 1 1 4 1 1 2 2 3 yellow r=0.0214274 with 84 all equations.
CS(28,52) 1 1 2 2 2 2 2 2 2 2 2 1 1 3 2 2 2 2 1 1 4 1 1 2 2 2 2 3 blue r=0.0108785 with 174 all equations.
OSO(3,7) 1 3 3 magenta r=0.101015 with 25 all equations.

CS(28,52) 1 1 2 2 2 2 2 2 2 2 2 1 1 3 2 2 2 2 1 1 4 1 1 2 2 2 2 3

This code has 174 all equations of which 134 are positive equations and 15 zero sines, 25 zero cosines equations . We will list the smallest r of each.

i. **Smallest Positive equation r:** This code has 134 positive equations F(x,y)=0 where $F(45,45) > 0$ all of which miss a 2r-square. We will only list the F that gives the smallest r.

F(x,y)=cos(x-6y)-cos(x-4y)-cos(x-2y)+2cos(x)-cos(x+2y)+cos(x+4y)-cos(3x-4y)+cos(3x-2y)-cos(3x)+cos(3x+2y)-cos(5x-6y)+cos(5x-4y)=0 and $r = 0.0108785 \leq F(45,45)/65 = 0.0108785... = F(45,45)/(|F_x| + |F_y|)$.

This is the equation that we use for the minimum **r=0.0108785** and we will store r in the database.

ii. **Corner Zero equations:** This blue corner at (45,45) has two straight lines and every point in this corner has a periodic path of this code type CS(28,52). Thus corner wings or epsilon are not needed to separate this corner.

Bottom side: F(x,y)=-sin(x-5y)+sin(x+3y)+sin(3x+y)=sin(x+3y)(cos(0)+2cos(2x-2y))=0 with slope=-1/3

$r = 0.136363 < F_y(45,45)/(|F_{xx}|+|F_{xy}|) = 9/22 = min(|F_x(45,45)|/F_x bounds, |F_y(45,45)|/F_y bounds).$

Top side: G(x,y)=sin(x-6y)+sin(x-4y)-sin(x+2y)-sin(x+4y)+sin(3x-6y)-sin(3x+2y)=cos(2y)(sin(x-4y)+sin(x-2y)-sin(x)-sin(x+2y)+sin(3x-4y)-sin(3x))=0 with slope=0

$r = 0.111648 < G_y(45,45)/(|G_{yx}|+|G_{yy}|) = 16.970562.../152 = min(|G_y(45,45)|/G_x bounds, |G_y(45,45)|/G_y bounds).$

Note: Since $G_x(45,45) = 0$ we use $r < G_y(45,45)/(|G_{xx}|+|G_{xy}|) = G_y(45,45)/62$ instead of $G_x(45,45)/(|G_{xx}|+|G_{xy}|)$. See section 6.

We will use this smallest r for this corner which is **r=0.111648**.

iii. **Smallest Non Corner Zero equation r:** This code has zero equations F(x,y)=0 where $F(45,45) = 0$ and we will only list the one that gives the smallest r which is not one of the corner sides. There are 38 non corner zero equations at (45,45).

F(x,y)=-cos(2y)+cos(6y)-cos(2x-4y)+cos(2x+4y)-cos(4x-6y)+cos(4x+2y)+cos(6x-4y)=0 a periodic set of curves with slope=.214285...=6/28.
$F_x$bounds=148, $F_y$bounds=200, $F_x(45,45) = -6$, $F_y(45,45) = 28$

$r = 0.0405405 < min(|F_x(45,45)|/F_x bounds, |F_y(45,45)|/F_y bounds) = min(6/148, 28/200) = 6/148$

Caution: This zero F gives the smallest r=0.0405405 by itself to make it a function but F does not necessarily give the smallest r that is separate from both sides of the corner. Below is the F that does both.

F(x,y)=cos(2y)+cos(2x-4y)+cos(2x)-cos(2x+4y)-cos(4x+2y)=0 with slope -0.5
$F_x$bounds=52, $F_y$bounds=64, $F_x(45,45) = -6$, $F_y(45,45) = -12$

$r = 0.115384 < min(|F_x(45,45)|/F_x bounds, |F_y(45,45)|/F_y bounds) = min(6/52, 12/64) = 6/52$
This r makes this a function in this 2r-square.

We now make r smaller by making F separate from the corner by using wings and epsilon.

Bottom side corner with slope -1/3 compared with F with slope -0.5 makes $\epsilon = 0.1$
r = min(0.136363...,0.115384...) = 0.115384..., ϵr = 0.0115384...

Top side corner with slope 0 compared with F with slope -0.5 makes $\epsilon = 0.5$
r = min(0.111648..., 0.115384...) = 0.111648..., ϵr = 0.0558242...

This zero F separates from the corner and gives this smallest **r=0.0115384**

Conclusion: The minimum r for this section is $r <$ **0.0108785** which comes from the positive equations.

OSO(3,3) 1 1 1

b) **OSO(3,3)** 1 1 1 at the point (45,45) which has 3 positive and 1 zero equation for a total of 4 all equations.

i. **Positive equations:** This code has 3 positive all equations F listed below with minimum $0 < r < 0.707106$ and is found in the database. These three curves miss this 2r-square.

F-bounds 0, F=cos(0) where $r = infinity$
F-bounds 1, F=$cos(y)$ where $0 < r < 0.707106 < cos(45)/1$
F-bounds 1, F=$cos(x)$ where $0 < r < 0.707106 < cos(45)/1$

We will use this minimum r where **r=0.707106**

ii. **Corner zero equations** This degenerate corner at (45,45) has 1 zero All equations F with minimum $0 < r < 0.5$ to make F a function.

$F = -cos(x + y) = 0$ where $F(45, 45) = -cos(90) = 0$ which is a point on a set of straight lines $x + y = 90 + 180k$ with slope m=-1.

$F_x$bounds=2, F_x=sin(x+y) where $r < .5 = sin(90)/2$
$F_y$bounds=2, F_y=sin(x+y) where $r < .5 = sin(90)/2$

We will use this minimum r where $r <$ **0.5**

Conclusion: The minimum r for this section is $r <$ **0.5** which comes from the corner zero equation.

OSO(3,7) 1 3 3

This code has 3 cos zeros and 22 positive equations for a total of 25 all equations.

i. **Positive equations:** This code has 22 positive equations F(x,y)=0 where $F(45, 45) > 0$ and we will only list the two(tied) that gives the smallest r. These 22 equations miss this 2r-square.

F-bounds 7 F=-sin(y)+sin(3y)+sin(2x-y) where $0 < r = 0.101015 < sin(45)/7$
F-bounds 7, F=-sin(x-2y)-sin(x)+sin(3x) where $0 < r = 0.101015 < sin(45)/7$

This is the equation that gives the minimum **r=0.101015**. You can check the others using the Info and Gradient buttons [7].

ii. **Corner zero equations** Note: This magenta corner at (45,45) has two straight sides and every point in this corner has a periodic path of this code type OSO(3,7). Thus corner wings and epsilons are not needed.

Bottom side: F(x,y)=cos(x-2y)+cos(x+2y)=0 where F(45,45)=0 which is a point on a double set of straight lines with slope m=0.
$r = 0.235702 < min(|F_y(45, 45)|/6, |F_y(45, 45)|/12)$ Note we use F_y twice since $F_x(45, 45) = 0$ and $F_y(45, 45) < 0$. See section 6.

Top side: G(x,y)=cos(2x-y)+cos(2x+y)=0 where G(45,45)=0 which is a point on a double set of straight lines with slope infinity.
$r = 0.235702 < min(|G_x(45, 45|)/12, |G_x(45, 45|)/6)$ Note we use G_x twice since $G_y(45, 45) = 0$ and $G_x(45, 45) < 0$. See section 6.

We will use this minimum r where **r=0.235702** to make both sides of the corner functions and separate.

iii. **Non corner Smallest Zero equation:** This code has 3 zero equations F(x,y)=0 at (45,45) and we will only list the one that gives the smallest r.

F(x,y)=cos(x+y) with slope -1.

$F_x(45.0, 45.0) = -1$ with $F_x bounds$ 2
$F_y(45.0, 45.0) = -1$ with $F_y bounds$ 2
$r < 0.5 = min(|F_x(45, 45)|/F_x bounds, |F_y(45, 45)|/F_y bounds$ = min(0.5, 0.5) = 0.5

We will use this minimum r where $r <$ **0.5**

Conclusion: The minimum r for this section is $r =$ **0.101015** from the positive equations.

CS(20,36) 1 1 2 2 2 2 2 1 1 3 2 2 1 1 4 1 1 2 2 3

This code has 84 all equations with 65 positive equations and 19 zero equations.

The yellow corner has straight sides which have slope $m_1 = -1.0$ and slope $m_2 = -1/3$.

i. **Smallest Positive equation:** This code has 65 positive equations F(x,y)=0 where $F(x, y) > 0$ at (45,45) and we will only list the F that gives the smallest r.

F-bounds 33, F(x,y)=-cos(x-2y)+2cos(x)-cos(x+2y)+cos(x+4y)-cos(3x-4y)+cos(3x-2y)-cos(3x)+cos(3x+2y) where $r = 0.0214274 < |F(45, 45)|/33$

This is the equation F that gives the minimum **r=0.0214274**. You can check the others using the Info or Gradient buttons [7]. All these positive equations miss this 2r-square centered at (45,45).

ii. **Corner zero equations:** This yellow corner at (45,45) has two straight lines and every point in this corner has a periodic path of this code type CS(20,36). Thus wings are not needed to separate the corner sides. We can make the two sides separate by making them functions in a 2r-square.

Bottom side: F(x,y)=cos(2y)+cos(2x)=0 where F(45,45)=0 which is a point on a double family of straight lines with slope -1
$r < 0.5 = min(|F_x(45, 45)|/4, |F_y(45, 45)|/4) = min(2/4, 2/4)$

Top side: F(x,y)=sin(x-4y)-sin(x+2y)-sin(x+4y)-sin(3x+2y)=0 where F(45,45)=0 which is a point on a triple family of straight lines with slope -1/3.
$r = 0.101015 < min(|Fx(45, 45)|/28, |Fy(45, 45)|/56) = min(-4(cos225)/28, -12(cos225)/56)$

We will use this minimum r for ii where **r=0.101015** to make both sides of the corner functions and separate.

iii. **Non corner Smallest Zero equation:** This code has 19 zeros (6 sin zeros, 13 cos zeros) at the point (45,45). Two of these form the corner and we will only give the non corner zero G that gives the smallest r that is a function and separate from the corner sides.

Caution: This zero F(x,y)=-2cos(2y)-cos(2x-4y)+cos(2x)+cos(2x+4y)+cos(4x-2y)+cos(4x+2y)=0 with slope=.125 is the smallest r=0.0263157 which makes all these zero equations as functions in this 2r-square. The key is to find that zero G that is separate from the corner sides and also gives the smallest r. Here it is.

G(x,y)=cos(2y)+cos(2x)-cos(4x+2y)=0 with slope -1.5 where $r_2 = 0.214285 < min(G_x(45, 45)/28, G_y(45, 45)/16)$ makes G a function. Note: This gives a larger r than the F above

We now separate G from the the two corner sides in ii. using wings and epsilons.

Bottom side with slope -1 and r_1=0.5, G with slope -1.5 and r_2=0.214285 where r_3=min(r_1,r_2)=0.214285 and epsilon=0.1 which makes $r = \epsilon r_3 = 0.0214285$

Top side with slope -1/3 and r_1=0.101015, G with slope -1.5 and r_2=0.214285 where r_3=min(r_1,r_2)=0.101015 and epsilon=0.318181 which makes $r = \epsilon r_3 = 0.0321412$

We will use this minimum r for iii. where **r=0.0214285** Note: this G yields a smaller r than from F.

Conclusion: The minimum r for CS(20,36) is **r=0.0214274** which comes from the positive equations.

Covering

The last step is to put the regions counterclockwise so that they overlap or abut to fully cover the final 2r-square. Here it turns out that no wings and no epsilons are needed as all corner sides are straight lines.

Bot upper CNS(4,6) = Top upper OSO(3,3): orange
Bot CNS(4,6) has slope -1.0 and r_1=0.235702
Top OSO(3,3) has slope -1.0 and r_2=0.5
r = min(r_1,r_2) = 0.235702

Bot upper CS(20,36) yellow = Top upper CNS(4,6):
-Bot CS(20,36) has slope -1.0 and r_1=0.5
-Top CNS(4,6) has slope -1.0 and r_2=0.235702
r = min(r_1,r_2)= 0.235702

Bot upper CNS(10,20) = Top upper CS(20,36): yellow
-Bot CNS(10,20) has slope -1/3 and r_1=0.047140...
-Top CS(20,36) has slope -1/3 and r_2=0.101015...
r = min(r_1,r_2) = 0.0471404

Bot upper CS(28,52) blue = Top upper CNS(10,20):
-Bot CS(28,52) has slope -1/3 and r_1=0.136363
-Top CNS(10,20) has slope -1/3 and r_2=0.0471404
r = min(r_1,r_2) = 0.0471404

Bot upper CNS(6,14) = Top upper CS(28,52): blue
-Bot CNS(6,14) has slope -0.0 and r_1=0.235702
-Top CS(28,52) has slope -0.0 and r_2=0.111648
r = min(r_1,r_2) = 0.111648

Bot upper OSO(3,7) magenta = Top upper CNS(6,14):
-Bot OSO(3,7) has slope zero and r_1=0.235702
-Top CNS(6,14) has slope -0.0 and r_2=0.235702
r = min(r_1,r_2) = 0.235702

Bot lower CNS(6,14) = Top lower OSO(3,7): magenta
-Bot CNS(6,14) has slope infinity and r_1=0.235702
-Top OSO(3,7) has slope infinity and r_2=0.235702
r = min(r_1,r_2) = 0.235702

Bot lower CS(28,52) blue = Top lower CNS(6,14):
-Bot CS(28,52) has slope infinity and r_1=0.111648

-Top CNS(6,14) has slope infinity and r_2=0.235702
r = min(r_1,r_2) = 0.111648

Bot lower CNS(10,20) = Top lower CS(28,52): blue
-Bot CNS(10,20) has slope -3 and r_1=0.0471404
-Top CS(28,52) has slope -3 and r_2=0.136363
r = min(r_1,r_2) = 0.0471404

Bot lower CS(20,36) yellow = Top lower CNS(10,20):
-Bot CS(20,36) has slope -3 and r_1=0.101015
-Top CNS(10,20) has slope -3 and r_2=0.0471404
r = min(r_1,r_2) = 0.0471404

Bot lower CNS(4,6) = Top lower CS(20,36): yellow
-Bot CNS(4,6) has slope -1 and r_1=0.235702
-Top CS(20,36) has slope -1 and r_2=0.5
r = min(r_1,r_2) = 0.235702

Bot lower OSO(3,3) orange = Top lower CNS(4,6):
-Bot OSO(3,3) has slope -1.0 and r_1=0.5
-Top CNS(4,6) has slope -1.0 and r_2=0.235702
r = min(r_1,r_2) = 0.235702

The minimum r for this covering is **0.0471404**.

The overall minimum r for a centered 2r-square is **r=.0108785** from all sections.

The 2r-squares

The **centered 2r-square** centered at (45,45) where **r=0.0108785** has a periodic path of one of the 12 given code types. We will call it a centered 45-45 Star Flare square.

The **subdivided** $2r_1$**-square** which contains (45,45) and is inside the centered square has side $2r_1$ where $r_1 = \pi/2^{10}$ radians or $r_1 = 90/2^9$ degrees using subdivisions of the big square. We will call it the subdivided 45-45 Star Flare square which is the one we use in the star jar. Its center is at $(255\pi/2^{10}, 255\pi/2^{10})$ = (0.782330..., 0.782330...) radians = $(45900/2^{10}, 45900/2^{10}) = (44.82421875, 44.82421875)$ degrees.

Note: (45,45) is the top right corner of this subdivided square.

17 The 10-20 Flare Square

In the search of all periodic paths in triangles, flares appear to be a key cornerstone. We call a point (a, b) in the plane a **flare** if no known open stable region contains that point and thus needs a family of regions to cover a neighbourhood of it. It is an **infinite flare** if it needs an infinite family, it is a **finite flare** otherwise. In this paper we will show that a centered **10-20 square** is a finite flare of the point (10,20) using five stables and one unstable region. Note: (10,20) is the center of a centered 10-20 square. We will later put it inside a subdivided square in which (10,20) is not its center.

There are six codes counterclockwise surrounding the point (10,20) with five stables and one nonstable as in the Figure.

Here is the list of code sequences.

Figure 8: The centered 10-20 Star Flare Square

OSNO (56, 512) 1 7 12 6 12 6 14 8 16 7 1 17 8 16 8 14 6 12 6 12 7 1 15 8 16 6 8 4 8 4 10 6 12 6 14 8 16 8 17 1 7 16 8 14 6 12 6 10 4 8 4 8 6 16 8 15 blue

OSNO (344, 2810) 1 7 12 6 12 4 4 2 6 4 8 4 10 6 12 6 14 8 16 6 8 4 10 6 12 6 14 8 16 7 1 17 8 15 1 7 12 6 13 1 8 1 15 8 17 1 7 15 1 8 1 13 6 12 7 1 15 8 17 1 7 16 8 14 6 12 6 10 4 8 6 16 8 15 1 7 12 6 13 1 8 1 15 7 1 17 8 15 1 7 12 6 12 4 4 2 6 4 8 4 10 6 12 6 14 8 16 6 8 4 10 6 12 6 14 8 16 7 1 17 8 15 1 7 12 6 13 1 8 1 15 8 17 1 7 15 1 8 1 13 6 12 7 1 15 8 17 1 7 16 8 14 6 12 6 10 4 8 6 16 8 15 1 7 12 6 13 1 8 1 15 7 1 17 8 15 1 7 12 6 12 4 4 2 6 4 8 4 10 6 12 6 14 8 16 6 8 4 10 6 12 6 14 8 16 7 1 17 8 15 1 7 12 6 13 1 8 1 15 8 17 1 7 15 1 8 1 13 6 12 7 1 15 8 17 1 7 16 8 14 6 12 6 10 4 8 6 16 8 15 1 7 12 6 13 1 8 1 15 7 1 17 8 15 1 8 1 13 6 12 7 1 15 8 17 1 7 16 8 14 6 12 6 10 4 8 6 16 8 14 6 12 6 10 4 8 4 6 2 4 4 12 6 10 4 8 4 6 2 4 2 2 2 8 4 8 6 16 8 14 6 12 6 10 4 8 6 16 8 15 1 7 12 6 13 1 8 1 15 7 1 17 8 15 lime

OSNO (368, 3280) 1 7 12 6 12 6 14 8 16 7 1 17 8 16 8 14 6 12 6 12 7 1 15 8 16 6 8 4 8 4 10 6 12 6 13 1 8 1 15 8 17 1 7 16 8 15 1 7 12 6 12 6 14 8 16 7 1 17 8 16 8 14 6 12 6 12 7 1 15 8 16 6 8 4 8 4 10 6 12 6 13 1 8 1 15 8 17 1 7 16 8 15 1 7 12 6 12 6 14 8 16 7 1 17 8 16 8 14 6 12 6 12 7 1 15 8 16 6 8 4 8 4 10 6 12 6 13 1 8 1 15 8 17 1 7 16 8 15 1 7 12 6 12 6 14 8 16 7 1 17 8 16 8 14 6 12 6 12 7 1 15 8 16 7 1 17 8 15 1 8 1 13 6 12 6 10 4 8 4 8 6 16 8 15 1 7 12 6 12 6 14 8 16 8 17 1 7 16 8 14 6 12 6 12 7 1 15 8 16 7 1 17 8 15 1 8 1 13 6 12 6 10 4 8 4 8 6 16 8 15 1 7 12 6 12 6 14 8 16 8 17 1 7 16 8 14 6 12 6 12 7 1 15 8 16 6 8 4 8 4 10 6 12 6 13 1 8 1 15 8 17 1 7 16 8 15 1 7 12 6 12 6 14 8 16 8 17 1 7 16 8 14 6 12 6 12 7 1 15 8 16 7 1 17 8 15 1 8 1 13 6 12 6 10 4 8 4 8 6 16 8 15 1 7 12 6 12 6 14 8 16 8 17 1 7 16 8 14 6 12 6 12 7 1 15 8 16 7 1 17 8 15 1 8 1 13 6 12 6 10 4 8 4 8 6 16 8 15 magenta

CNS (2,6) 2 4 black

OSNO (384, 3056) 1 7 12 4 6 4 10 6 14 6 10 6 14 7 1 16 1 8 1 13 7 1 16 1 7 14 6 10 6 15 1 7 14 7 1 17 8 14 7 1 15 6 10 6 14 7 1 16 1 7 13 1 8 1 16 1 7 14 6 10 6 14 6 10 4 6 4 12 7 1 16 1 7 14 6 10 6 15 1 7 14 7 1 17 8 14 7 1 15 6 10 6 14 7 1 16 1 7 13 1 8 1 16 1 7 14 6 10 6 14 6 10 4 6 4 12 7 1 16 1 7 14 6 10 6 15 1 7 14 7 1 17 8 14 7 1 15 6 10 6 14 7 1 16 1 7 13 1 8 1 16 1 7 14 6 10 6 14 6 10 4 6 4 12 7 1 16 1 7 14 6 10 6 15 1 7 14 7 1 17 8 14 7 1 15 6 10 6 14 7 1 16 1 7 12 4 6 4 10 6 14 6 10 6 14 7 1 16 1 8 1 13 7 1 16 1 7 14 6 10 6 15 1 7 14 8 17 1 7 14 7 1 15 6 10 6 14 7 1 16 1 7 12 4 6 4 10 6 14 6 10 6 14 7 1 16 1 8 1 13 7 1 16 1 7 14 6 10 6 15 1 7 14 8 17 1 7 14 7 1 15 6 10 6 14 7 1 16 1 7 13 1 8 1 16 1 7 14 6 10 6 14 6 10 4 6 4 12 7 1 16 1 7 14 6 10 6 15 1 7 14 8 17 1 7 14 7 1 15 6 10 6 14 7 1 16 1 7 12 4 6 4 10 6 14 6 10 6 14 7 1 16 1 8 1 13 7 1 16 1 7 14 6 10 6 15 1 7 14 8 17 1 7 14 7 1 15 6 10 6 14 7 1 16 yellow

OSNO (358, 2734) 1 7 12 4 4 2 6 4 10 6 12 4 6 4 8 2 2 2 6 4 10 6 12 4 4 2 6 4 10 6 13 1 8 1 15 6 8 4 10 6 14 8 17 1 7 15 1 7 12 6 14 7 1 17 8 14 6 12 7 1 15 7 1 17 8 14 6 10 4 8 6 15 1 7 12 6 14 8 17 1 7 15 1 7 12 4 4 2 6 4 10 6 13 1 8 1 15 6 8 4 10 6 14 8 17 1 7 15 1 7 12 6 14 8 17 1 7 14 6 12 7 1 15 7 1 17 8 14 6 10 4 8 6 15 1 8 1 13 6 10 4 6 2 4 4 12 6 10 4 6 2 2 2 8 6 16 8 14 6 10 4 8 6 15 1 7 12 6 14 8 17 1 7 15 1 7 12 4 4 2 6 4 10 6 13 1 8 1 15 6 8 4 10 6 14 8 17 1 7 15 1 7 12 6 14 8 17 1 7 14 6 12 7 1 15 7 1 17 8 14 6 10 4 8 6 15 1 8 1 13 6 10 4 6 2 4 4 12 6 10 4 6 2 2 2 8 6 16 8 14 6 10 4 8 6 15 1 7 12 6 14 8 17 1 7

15 1 7 12 4 4 2 6 4 10 6 13 1 8 1 15 6 8 4 10 6 14 8 17 1 7 15 1 7 12 6 14 8 17 1 7 14 6 12 7 1 15 7 1 17 8 14 6 10 4 8 6 15 1 8 1 13 6 10 4 6 2 4 4 12 6 10 4 6 2 2 2 8 6 16 8 14 6 10 4 8 6 15 1 7 12 6 14 8 17 1 7 15 orange

For every F the **positive equations** $F(x, y) = 0$ where $F(10, 20) > 0$ must satisfy $0 < r < F(10, 20)/(|F_x| + |F_y|) = F(10, 20)/Fbounds$ for every F. This means none of these positive curves $F(x, y) = 0$ intersect the 10-20 square of side 2r.

For every F the **zero equations** $F(x, y) = 0$ where $F(10, 20) = 0$ must satisfy $0 < r < min(|F_x(10, 20)|/(|F_{xx}| + |F_{xy}|), (|F_y(10, 20)|/(|F_{yx}| + |F_{yy}|)) = min(|F_x(10, 20)|/F_x bounds, |F_y(10, 20)|/F_y bounds)$. This means these zero curves $F(x, y) = 0$ become functions in this 2r-square.

Note: These equations are way too long to be of use to be printed off. If you want to see them, you can find them by using the code above and putting it in the Info button from the star jar [7].

18 The 10-20 Positive Equations

Here are the minimum r of the positive equations listed to the right for each code. We store each of these minimum r in the star database for each code. We will use exponential notation since there are so many extra decimal zeros.

OSNO (56, 512) blue r =8.454055E-6
OSNO (344, 2810) lime **r= 5.253499E-8**
OSNO (368, 3280) magenta r= 1.103488E-6
CNS(2,6) black r= 8.55050E-2
OSNO (384, 3056) yellow r= 5.676630E-7
OSNO (358, 2734) orange r= 3.448833E-7

Note: The minimum r from all the positive equations from the stable and unstable codes is **r=5.253499E-8**.

19 The (10,20) Corners

A corner of a region is formed using two successive boundary curves where we put them in a counter clockwise orientation from the **bottom** curve to the **top** curve. We will now use wings and epsilon to separate the corners in a square as required. In this section we use rational bounds for the slopes by using integer bounds for the partial derivatives. This makes bigger bounds for the slopes and yields a smaller radius r.

OSNO (56, 512) corner

1. OSNO (56, 512) 1 7 12 6 12 6 ... This is the blue portion of the square and has 6394 all equations.

This region forms a corner between two curves $F(x, y) = 0$ and $G(x, y) = 0$ with slopes m_1=infinity and m_2=2.518671... intersecting at (10,20). We will use a 2r-square where r is rational and that the two corner sides only meet inside this square at only one point namely (10,20).

Note: Figure 9 and the following was done in [8] which allows a 20,000 sum of sines or cosines.

a) The bottom side of the corner is $F(x, y) = 0$ which is a vertical line and $F(x, y) = -sin(4y) + sin(8y)$...
Note: $F(x, y)$ factors to $cos(9x)(sin(x - 12y) + sin(x - 10y)...)$.

We use mpfi to find that

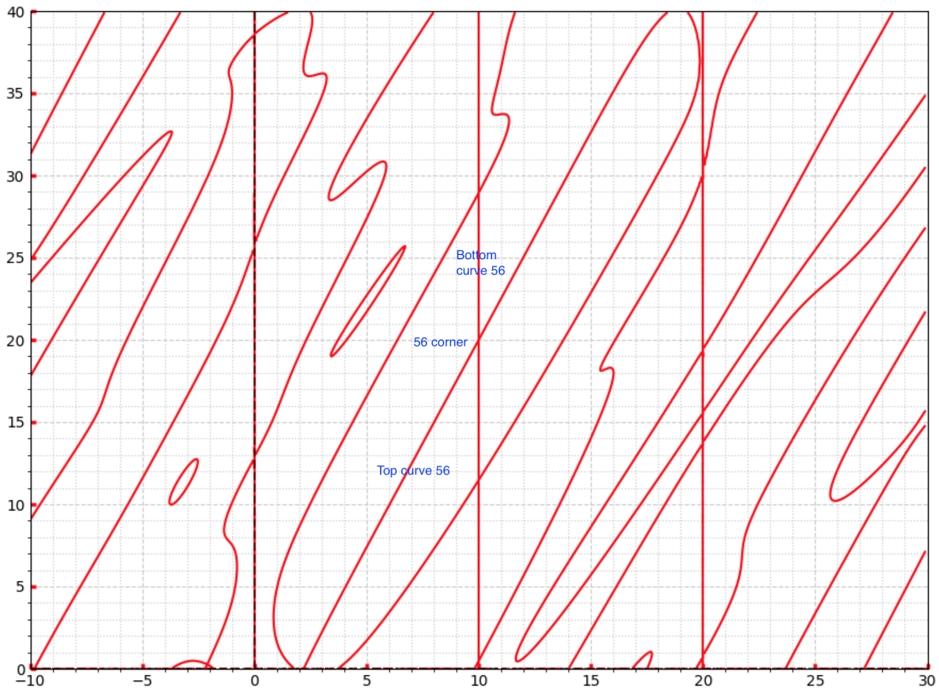

Figure 9: 56 red corner

$-9408 < F_x(10, 20) < -9407$

$F_y(10, 20) = 0$ since $cos(9x)$ is a factor of $F(x, y)$.

Hence $y'(10, 20) = m_1 = infinity$

Then

$F_{xx}(x, y)$ has bound $|F_{xx}| = 1192744$

$F_{xy}(x, y)$ has bound $|F_{xy}| = 428472$ and the sum of these bounds is 1621216

We choose $0 < r_1 = 0.00580243 < 9407/1621216$ which makes $F_x(x, y)$ negative and makes $F(x, y) = 0$ a function of y in a $2r_1$-square.

b) The top side of the corner is $G(x, y) = 0$ where $G(x, y) = cos(x - 13y) - cos(x - 9y)....$ and let $y'(10, 20) = m_2 = 2.5186715...$ which need not be a rational so we will find rational bounds for the slope m_2.

We use mpfi to find that

$-23104 < G_x(10, 20) < -23103$

$9172 < G_y(10, 20) < 9173$

Hence let $m_2^* = 23103/9173 = 2.518587... < y'(10, 20) = m_2 < 23104/9172 = m_2^{**} = 2.518970...$

$G_{xx}(x, y)$ has bound $|G_{xx}| = 3049794$

$G_{xy}(x, y)$ has bound $|G_{xy}| = 1192234$ and the sum of bounds $=4242028$

We choose $r_2 = 23103/4242028 = 0.00544621...$ which makes $G_x(x, y)$ negative and makes $G(x, y) = 0$ a function of y within a $2r_2$-square.

$G_{yx}(x, y)$ has bound $|G_{yx}| = 1192234$

$G_{yy}(x, y)$ has value bound $|G_{yy}| = 536114$ and the sum of bounds is 1728348

We choose $r_3 = 9172/1728348 = 0.00530680...$ which makes $G_y(x, y)$ positive and makes $G(x, y) = 0$ a function of x within a $2r_3$-square.

Now take $min(r_2, r_3)$ which is r_3 and makes $G(x,y) = 0$ an increasing function of both x and y within this $2r_3$-square.

c) The smallest r from a) and b) is $min(r_1, r_3) = r_3$=**9172/1728348=0.00530680...** which makes G an increasing function of both x and y and F a vertical line in a $2r_3$-square. This means F and G and the corner intersects only at the point (10,20) in this square and thus we don't need to introduce any wings.

OSNO (344, 2810) corner

2. OSNO (344, 2810) = 1 7 12 6 12 4 ... This is the lime portion of the square.

This region forms a corner between two curves $F(x,y) = 0$ and $G(x,y) = 0$ with slopes m_1=1.724918... and m_2=2.493881... at the point (10,20). We will use a 2r-square where r is rational and that the corner only meets inside this square at one point namely (10,20).

a) The top side of this corner is $F(x,y) = 0$ which is curved and $F(x,y) = cos(x - 35y) - 2cos(x - 31y)$.... and let $y'(10,20) = m_1 = 1.724918...$ which need not be a rational so we will find rational bounds for m_1.

We use mpfi to find that
$824817 < F_x(10,20) < 824818$
$-478178 < F_y(10,20) < -478177$
Hence let $m_1^* = 824817/478178 = 1.724916... < y'(10,20) = m_1 < 824818/478177 = m_1^{**} = 1.724921...$

$F_{xx}(x,y)$ has bound $|F_{xx}| = 377676498$
$F_{xy}(x,y)$ has bound $|F_{xy}| = 199337742$ and the sum of bounds = 577014240
We use $0 < r_1 = 0.00142945 < 824817/577014240$ which makes $F_x(x,y)$ positive and makes $F(x,y) = 0$ a function of y within a $2r_1$-square.

$F_{yx}(x,y)$ has bound $|F_{yx}| = 199337742$
$F_{yy}(x,y)$ has bound $|F_{yy}| = 107543778$ and the sum of bounds is 306881520
We use $0 < r_2 = 0.00155818 < 478177/306881520$ which makes $F_y(x,y)$ negative and makes $F(x,y) = 0$ a function of x within a $2r_2$-square.

Now $min(r_1, r_2) = r_1$ and makes $F(x,y) = 0$ an increasing function of both x and y in a $2r_1$-square.

b) The bottom side of this corner is $G(x,y) = 0$ and $G(x,y) = -2cos(x - 35y) + 3cos(x - 31y)$... and let $y'(10,20) = m_2 = 2.493881...$ which need not be a rational so we will find rational bounds for m_2.

We use mpfi to find that
$664208 < G_x(10,20) < 664209$
$-266336 < G_y(10,20) < -266335$

Hence let $m_2^* = 664208/266336 = 2.493872... < y'(10,20) = m_2 < 664209/266335 = m_2^{**} = 2.493885...$

$G_{xx}(x,y)$ has bound $|G_{xx}| = 317510180$
$G_{xy}(x,y)$ has bound $|G_{xy}| = 154922020$ and the sum of bounds is 472432200
We use $0 < r_3 = 0.00140593 < 664208/472432200$ which makes $G_x(x,y)$ positive and makes $G(x,y) = 0$ a function of y within a $2r_3$-square.

$G_{yx}(x,y)$ has bound $|G_{xx}| = 154922020$
$G_{yy}(x,y)$ has bound $|G_{xy}| = 78187428$ and the sum of bounds is 233109448

We use $0 < r_4 = 0.00114253 < 266335/233109448$ which makes $G_y(x,y)$ negative and makes $G(x,y) = 0$ a function of x within a $2r_4$-square.

Now $min(r_3, r_4) = r_4$ and makes $G(x,y) = 0$ an increasing function of both x and y in a $2r_4$-square.

c) The smallest r from a) and b) is $r_4 =$ **0.00114253** which makes F an increasing function of both x and y and G an increasing function of both x and y in a $2r_4$-square. We need to introduce wings to separate the two curves at (10,20) by letting $\epsilon = 0.0911335 < .5(|m_2^*| - |m_1^{**}|)/(|m_2^{**}| + |m_1^{**}|)$.

We finally choose $0 < r$=**0.000104122**$< \epsilon r_4$ to use a 2r-square.

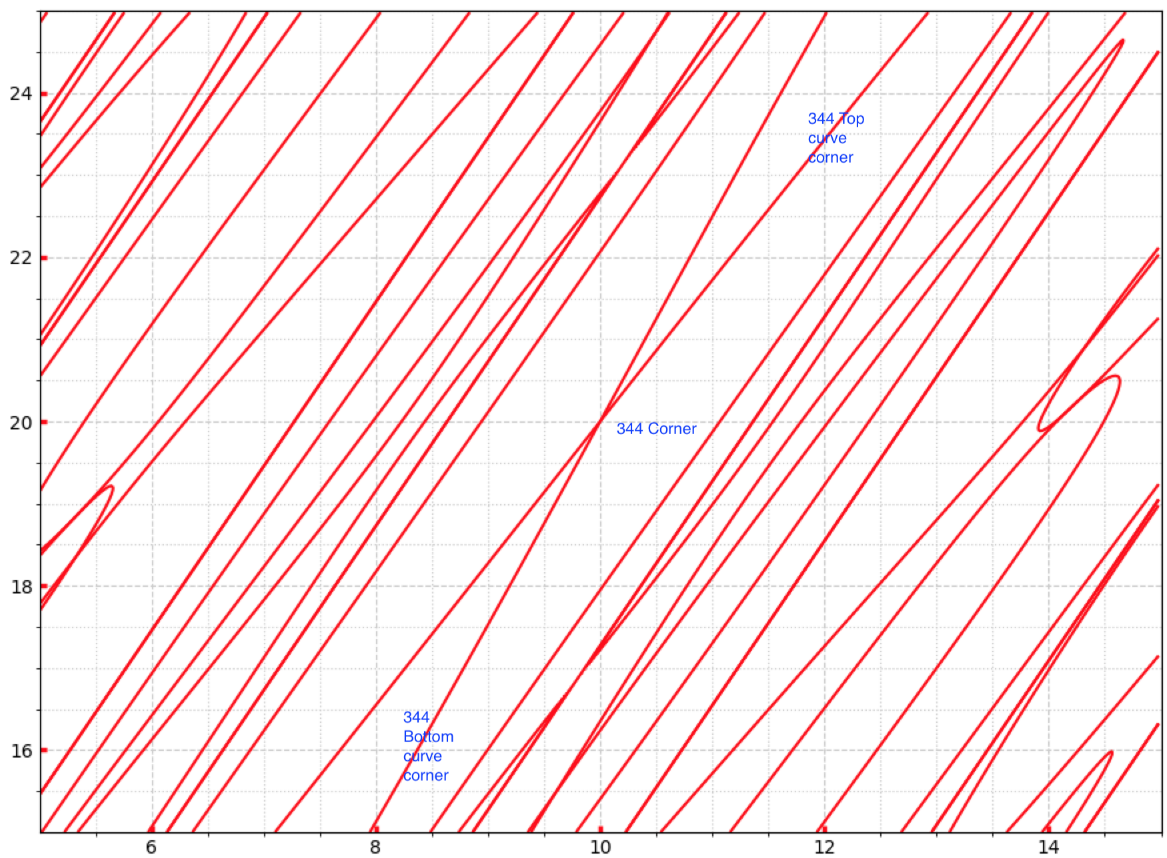

Figure 10: 344 red corner

OSNO (368, 3280) corner

3. OSNO (368, 3280) 1 7 12 6 12 6 ...

This region forms a corner between two curves $F(x, y) = 0$ and $G(x, y) = 0$ with slopes m_1=-2.662816... and m_2=2 at the point (10,20). We want to find a 2r-square where r is rational and that the corner only meets inside this square at (10,20).

a) The bottom side of the corner is $F(x, y) = 0$ which is curved and $F(x, y) = -6cos(x-37y)+2cos(x-35y)$.... and let $y'(10, 20) = m_1 = -2.662816...$ which need not be a rational so we will find rational bounds for m_1.

We use mpfi to find that
$151416 < F_x(10, 20) < 151417$
$56863 < F_y(10, 20) < 56864$
Hence let $m_1^* = -151417/56863 = -2.662838... < y'(10, 20) = m_1 < -151416/56864 = m_1^{**} = -2.662775...$

$F_{xx}(x, y)$ has bound $|F_{xx}| = 128067138$
$F_{xy}(x, y)$ has bound $|F_{xy}| = 31642206$ and the sum of bounds $= 159709344$
We choose $0 < r_1 = 0.000948072 < 151416/159709344$ which makes $F_x(x, y)$ positive since $r_1 < |F_x(10, 20)|/(|F_{xx}|+|F_{xy}|)$ and makes $F(x, y) = 0$ a function of y within a $2r_1$-square.

$F_{yx}(x, y)$ has bound $|F_{yx}| = 31642206$
$F_{yy}(x, y)$ has bound $|F_{yy}| = 10453314$ and the sum of bounds is 42095520
We choose $0 < r_2 = 0.00135080 < 56863/42095520$ which makes $F_y(x, y)$ positive since $r_2 < |F_y(10, 20)|/(|F_{yx}|+|F_{yy}|)$ and makes $F(x, y) = 0$ a function of x within a $2r_2$-square.

Now take $min(r_1, r_2)$ which is r_1 and makes $F(x, y) = 0$ an decreasing function of both x and y within this $2r_1$-square.

b) The top side of the corner is $G(x, y) = 0$ which is a straight line and $G(x, y) = -7sin(2y)-7sin(4y)$... Note: $G(x, y)$ factors to $sin(2x-y)h(x, y)$ and let $m_2 = y'(10, 20) = 2$ since $y'(10, 20) = -G_x(10, 20)/G_y(10, 20) = 2$ where $G_x(10, 20) = 2h(10, 20)$ and $G_y(10, 20) = -h(10, 20)$

We use mpfi to find that
$333094 < G_x(10, 20) < 333095$
$-166548 < G_y(10, 20) < -166547$
Note: We don't need to find bounds for m_2 since m_2 is exactly 2.

$G_{xx}(x, y)$ has bound $|G_{xx}| = 92289280$
$G_{xy}(x, y)$ has bound $|G_{xy}| = 25093856$ and the sum of bounds is 117383136
We choose $0 < r_3 = 0.00283766 < 333094/117383136$ which makes $G_x(x, y)$ positive since $r_3 < G_x(10, 20)/(|G_{xx}|+|G_{xy}|)$ and makes $G(x, y) = 0$ a function of y within a $2r_3$-square.

$G_{yx}(x, y)$ has bound $|G_{yx}| = 25093856$
$G_{yy}(x, y)$ has bound $|G_{yy}| = 8344560$ and the sum of bounds is 33438416
We choose $0 < r_4 = 0.00498070 < 166547/33438416$ which makes $G_y(x, y)$ negative since $r_4 < |G_y(10, 20)|/(|G_{yx}|+|G_{yy}|)$and makes $G(x, y) = 0$ a function of x within a $2r_4$-square.

Now take $min(r_3, r_4)$ which is r_3 and makes $G(x, y) = 0$ an increasing function of both x and y within this $2r_3$-square.

c) The smallest r from a) and b) is $min(r_1, r_2, r_3, r_4) = r_1$=**0.000948072** which makes F a decreasing function of both x and y and G an increasing function of both x and y in a $2r_1$-square. This means F and G and the corner intersects only at the point (10,20) in this square and thus we don't need to introduce any wings or we just let $\epsilon = 1$.

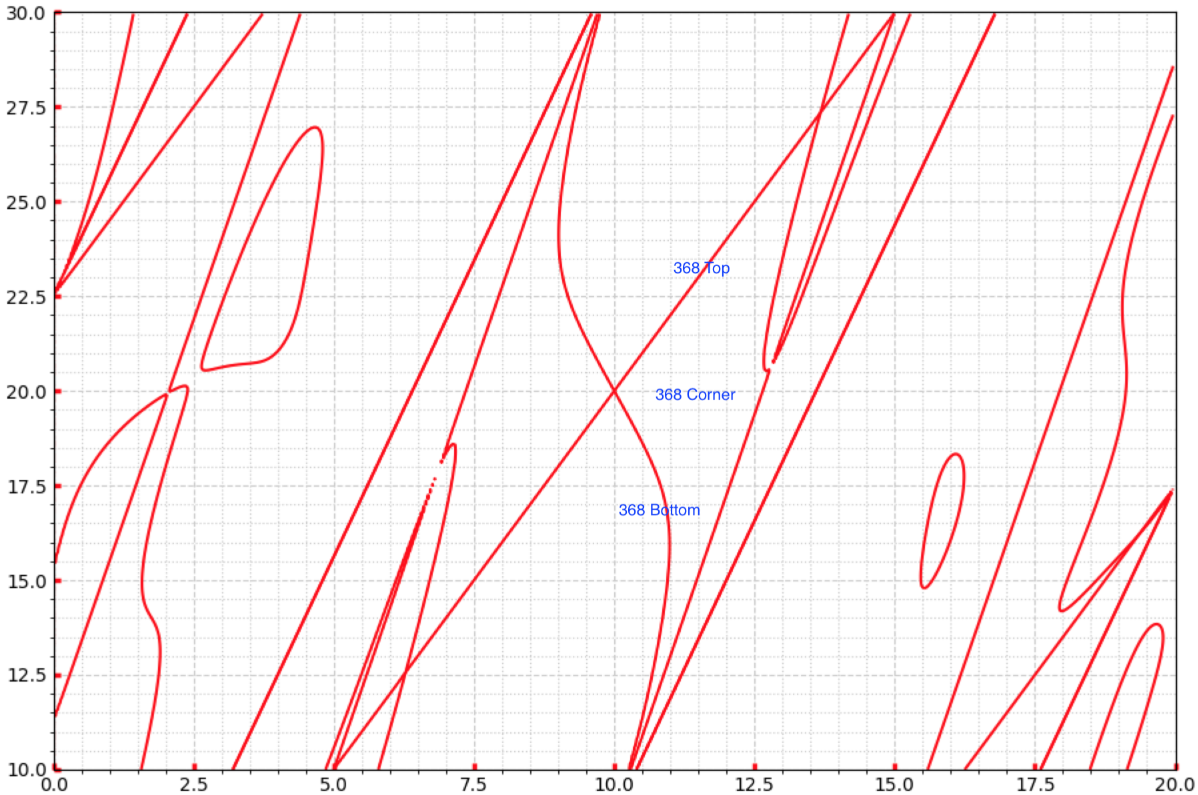

Figure 11: 368 red corner

OSNO (384,3056) corner

4. OSNO(384,3056) = 1 7 12 4 6 4....

This region has a corner between $F(x, y)$=0 and $G(x, y)$=0 with slopes m_1=2 and m_2=3.529402... We will use a 2r-square where r is rational and that the two corner sides only meet in this square at the point namely (10,20).

a) The bottom side of the corner is $F(x, y) = 0$ which is a straight line and $F(x, y) = -4sin(2y) - 3sin(4y)$...
Note: $F(x, y)$ factors to $sin(2x-y)h(x, y)$ and $m_1 = y'(10, 20)$=2 since $y'(10, 20) = -F_x(10, 20)/F_y(10, 20)$=2 where $F_x(10, 20) = 2h(10, 20)$ and $F_y(10, 20) = -h(10, 20)$

We use mpfi to find that
$-808256 < F_x(10, 20) =< -808255$
$404127 < F_y(10, 20) < 404128$
Note: We don't need to find bounds for m_1 since m_1 is exactly 2.

$F_{xx}(x, y)$ has bound $|F_{xx}| = 112671136$
$F_{xy}(x, y)$ has bound $|F_{xy}| = 71919576$ and the sum of bounds is 184590712

We choose $0 < r_1 = 0.00437863 < 808255/184590712$ which makes $F_x(x, y)$ negative since $r_1 < |F_x(10, 20)|/(|F_{xx}| + |F_{xy}|)$ and makes $F(x, y) = 0$ a function of y within a $2r_1$-square.

$F_{yx}(x, y)$ with bound $|F_{yx}| = 71919576$
$F_{yy}(x, y)$ with bound $|F_{yy}| = 151631208$ and the sum of bounds is 123550784
We choose $0 < r_2 = 0.00327093 < 404127/123550784$ which makes $F_y(x, y)$ positive since $r_2 < F_y(10, 20)/(|F_{yx}(x, y)| + |F_{yy}(x, y)|)$ and makes $F(x, y) = 0$ a function of x within this $2r_2$-square.

Now take $min(r_1, r_2)$ which is r_2 and makes $F(x, y) = 0$ a function of both x and y within this $2r_2$-square.

b) The top side of the corner is $G(x, y) = 0$ which is curved and $G(x, y) = cos(x - 63y) - 2cos(x - 55y)$.... and let $y'(10, 20) = m_2 = 3.529402...$ which need not be a rational so we will find rational bounds for m_2.

We use mpfi to find that
$527551 < G_x(10, 20) < 527552$
$-149474 < G_y(10, 20) < -149473$
Hence let $m_2^* = 527551/149474 = 3.529383... < y'(10, 20) = m_1 < 527552/149473 = m_2^{**}$=3.529413...

$G_{xx}(x, y)$ has bound $|G_{xx}| = 125308990$
$G_{xy}(x, y)$ has value bound $|G_{xy}| = 77722106$ and the sum of bounds = 203031096
Now choose $0 < r_3 = 0.00259837 < 527551/203031096$ which makes $G_x(x, y)$ positive since $r_3 < G_x(10, 20)/(|G_{xx}| + |G_{xy}|)$ and makes $G(x, y) = 0$ a function of y within a $2r_3$-square.

$G_{yx}(x, y)$ has bound $|G_{yx}| = 77722106$
$G_{yy}(x, y)$ has bound $|G_{yy}| = 57804670$ and the sum of bounds is 135526776
Now choose $0 < r_4 = 0.00110290 < 149473/135526776$ which makes $G_y(x, y)$ negative since $r_4 < |G_y(10, 20)|/(|G_{yx}| + |G_{yy}|)$ and makes $G(x, y) = 0$ a function of x within a $2r_4$-square.

Now take $min(r_3, r_4)$ which is r_4 and makes $G(x, y) = 0$ an increasing function of both x and y within this $2r_4$-square.

c) The smallest r from a) and b) is r_4=**0.00110290** which makes F and G functions of both x and y and a corner in a $2r_4$-square. We now want to make the square smaller so that the two corner sides intersect only at the point (10,20) in this square and thus we introduce wings and epsilon to separate the corner sides. Since the first curve F is straight in this square, we need to only make one wing for the second curve G and let $\epsilon = 15293/55295 = 0.276571... \leq (|m_2^*| - |m_1|)/(|m_2^{**}| + |m_1|)$=$(m_2^* - 2)/(m_2^{**} + 2)$. This means the two curves F and G centered at (10,20) form a corner and otherwise miss each other in a $2r_5$-square where we choose $0 < r_5$=**0.000305030**$\leq \epsilon r_4$.

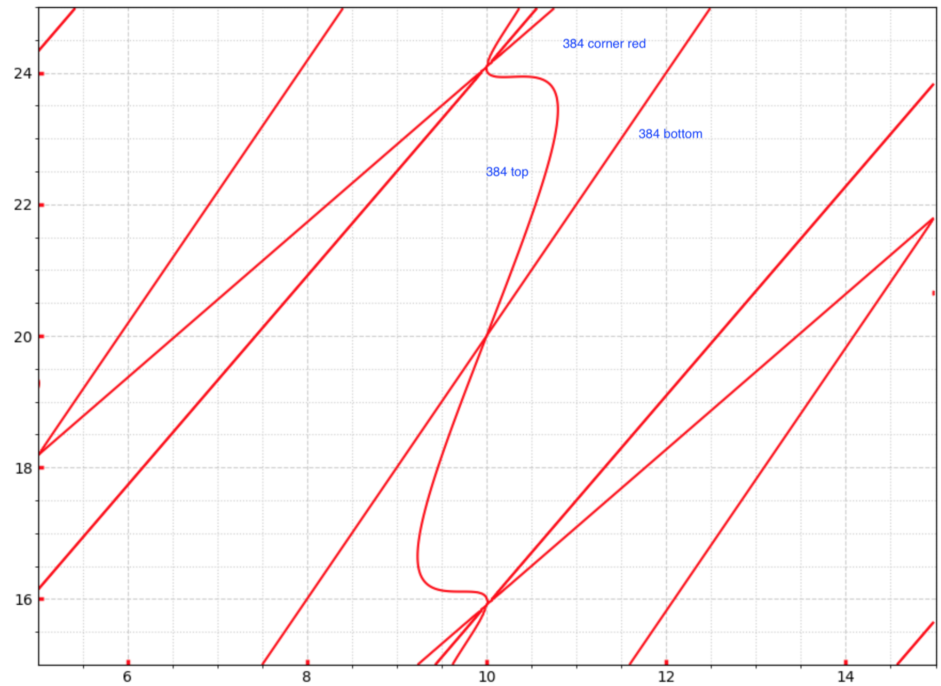

Figure 12: 384 red corner

OSNO (358, 2734) corner

5. OSNO (358, 2734) 1 7 12 4 4 2 ...

This region has a corner between $F(x, y) = 0$ and $G(x, y) = 0$ with slopes m_1=0.661351... and m_2=2.533682... at (10,20). We want to find a 2r-square where r is rational and that the two corner sides only meet in this square at the point namely (10,20). Both sides have positive slopes.

a) The top side of this corner is $F(x, y) = 0$ which is curved and $F(x, y) = -cos(x - 33y) + cos(x - 31y)$.... and let $y'(10, 20) = m_1$=0.661351... which need not be a rational so we will find rational bounds for m_1.

We use mpfi to find that

$-47747 < F_x(10, 20) < -47746$

$72194 < F_y(10, 20) < 72195$

Hence let $m_1^* = 47746/72195 = 0.661347... < y'(10, 20) = m_1 < 47747/72194 = m_1^{**} = 0.661370...$

$F_{xx}(x, y)$ has bound $|F_{xx}| = 52948434$

$F_{xy}(x, y)$ has bound $|F_{xy}| = 22832366$ and the sum of bounds = 75780800

We choose $0 < r_1 = 0.000630054 < 47746/75780800$ which makes $F_x(x, y)$ negative since $r_1 < |F_x(10, 20)|/(|F_{xx}|+|F_{xy}|)$ and makes $F(x, y) = 0$ a function of y within a $2r_1$-square.

$F_{yx}(x, y)$ has bound $|F_{yx}| = 22832366$

$F_{yy}(x, y)$ has bound $|F_{yy}| = 11361234$ and the sum of bounds is 34193600

We choose $0 < r_2 = 0.00211133 < 72194/34193600$ which makes $F_y(x, y)$ positive since $r_2 < F_y(10, 20)/(|F_{yx}|+|F_{yy}|)$ and makes $F(x, y) = 0$ a function of x within a $2r_2$-square.

Now take $min(r_1, r_2)$ which is r_1 and makes $F(x, y) = 0$ an increasing function of both x and y within this $2r_1$-square.

b) The bottom side of this corner is $G(x,y) = 0$ which is curved and $G(x,y) = -cos(x-27y)+cos(x-25y)$... and let $y'(10,20) = m_2$=2.533682... which need not be a rational so we will find rational bounds for it.

We use mpfi to find that

$-623773 < G_x(10,20) < -623772$

$246191 < G_y(10,20) < 246192$

Hence let $m_2^* = 623772/246192 = 2.533681... < y'(10,20) = m_2 < 623773/246191 = m_2^{**} = 2.533696...$

$G_{xx}(x,y)$ has bound $|G_{xx}| = 216045584$

$G_{xy}(x,y)$ has bound $|G_{xy}| = 87058008$ and the sum of bounds is 303103592

We choose $0 < r_3 = 0.00205794 < 623772/303103592$ which makes $G_x(x,y)$ negative since $r_3 < |G_x(10,20)|/(|G_{xx}|+|G_{xy}|)$ and makes $G(x,y) = 0$ a function of y within a $2r_3$-square.

$G_{yx}(x,y)$ has bound $|G_{xx}| = 87058008$

$G_{yy}(x,y)$ has bound $|G_{xy}| = 36835376$ and the sum of bounds is 123893384

We choose $r_4 = 0.00198711 < 246191/123893384$ which makes $G_y(x,y)$ positive since $r_4 < G_y(10,20)/(|G_{xy}|+|G_{yy}|)$ and makes $G(x,y) = 0$ a function of x within a $2r_4$-square.

Now take $min(r_3, r_4)$ which is r_4 and makes $G(x,y) = 0$ an increasing function of both x and y within this $2r_4$-square.

c) The smallest r from a) and b) is r_1=**0.000630054** which makes F an increasing function of both x and y and G an increasing function of both x and y in a $2r_1$-square. We need to introduce wings and epsilon to separate the two curves. We let $\epsilon = 0.293000 < .5(|m_2^*|-|m_1^{**}|)/(|m_2^{**}|+|m_1^{**}|) \leq .5(|m_2|-|m_1|)/(|m_2|+|m_1|)$ and we let $0 < r$=**0.000184605**$< \epsilon r_1$.

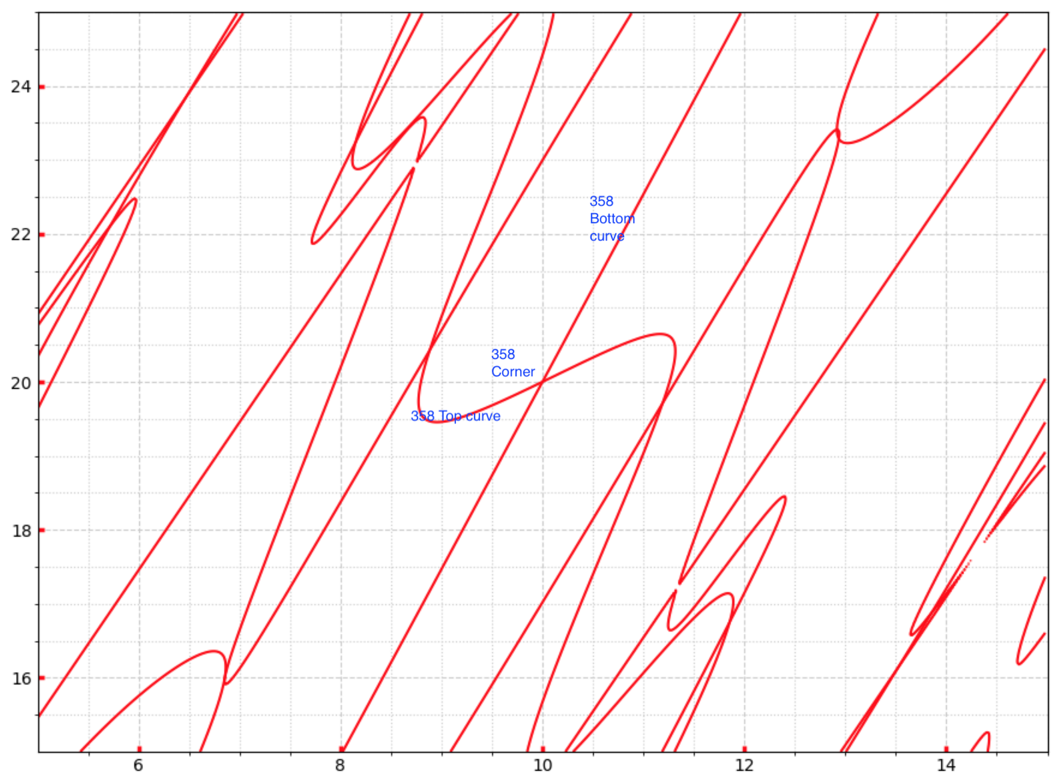

Figure 13: 358 red corner

Conclusion: These are the r's we use to separate the sides of each corner.

56 r=0.00530680
344 r=0.000104122
368 r=0.000948072
384 r=0.000305030
358 r=0.000184605

Thus we use this minimum r=**0.000104122** which separates each corner in this 2r-square.
Caution: We have to do much more work to prove that the corners actually cover a square.

20 The five smallest zero curve combinations at (10,20)

In this section we use decimal bounds for the slopes. This makes smaller bounds for the slopes and yields a larger radius r. To distinquish the three types of zero curves, we will use these new notations.

The zero curves are the curves $Z(x,y) = 0$ that go through the point (10,20) and are not the bottom $B(x,y) = 0$ or top $T(x,y) = 0$ zero curves of a corner. We now make the 2r-square small enough so that all the zero curves that go through a particular corner only intersects at (10,20) and nowhere else. For each corner's zero curves we will use the best combination of a zero curve $Z(x,y) = 0$ with either the bottom or with the top curve and which gives the smallest r. Here are the calculations that we use.

$0 < r' <$min$(|T_x(10,20)|/(|T_{xx}| + |T_{xy}|), |T_y(10,20)|/(|T_{yx}| + |T_{yy}|)$ to make T a function of x and y.
or
$0 < r' <$min$(|B_x(10,20)|/(|B_{xx}| + |B_{xy}|), |B_y(10,20)|/(|B_{yx}| + |B_{yy}|)$ to make B a function of x and y.

$0 < r'' <$min$(|Z_x(10,20)|/(|Z_{xx}| + |Z_{xy}|), |Z_y(10,20)|/(|Z_{yx}| + |Z_{yy}|)$ to make Z a function of x and y

then $r''' = min(r', r'')$ makes these curves functions of x and y in a $2r'''$-square.

We let $0 < \epsilon \leq .5||m_2| - |m_1||/(|m_2| + |m_1|)$ using appropriate slopes.

Finally we use $0 < r \leq \epsilon r'''$ to separate two curves in two wings in a 2r-square..

1. The OSNO (56, 512) has seven zero curves and its two corner curves that go through (10,20). Here is the combination that gives the smallest r. See the Figure.

We use mpfi to find

-that the top curve T(x,y) has slope m_1 between m_1^*=2.518587 and m_1^{**}=2.518971
-that the best zero curve Z(x,y) has slope m_2 between m_2^*=2.720395 and m_2^{**}=2.720396
-that for the top curve T(x,y), we use $r' = 0.00530680$ to make T a function of x and y.
-that for the zero curve Z(x,y), we use $r'' = 0.00400207$ to make Z a function of x and y.
-that r''' = min(0.00530730,0.00400207) = 0.00400207 to make both curves functions of x and y.
-that we let $\epsilon = 0.0192518$
-that we let r=**0.0000770470**$\leq \epsilon r'''$ which separates these curves in two wings in a 2r-square.

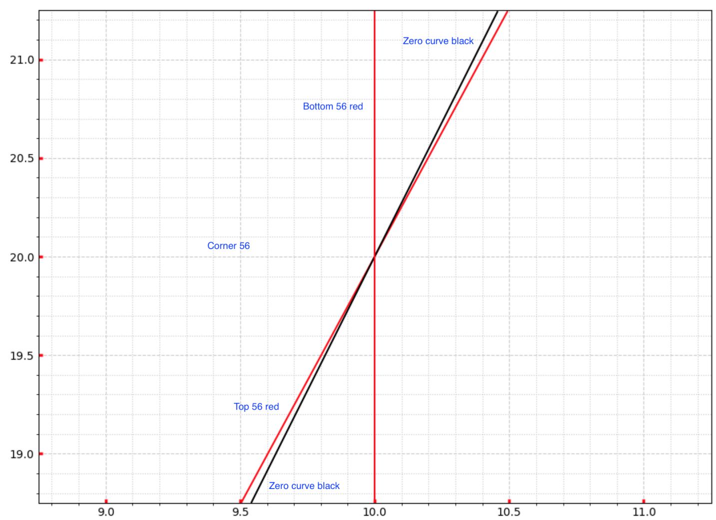

Figure 14: 56 red corner and black zero curve

2. The OSNO (344, 2810) has ten zero curves and its two corner curves that go through (10,20). Here is the combination that gives the smallest r. See the Figure.

We use mpfi to find

-that the top curve T(x,y) has slope m_1 between m_1^*=1.724918 and m_1^{**}=1.724919

-that the best zero curve Z(x,y) has slope m_2 between m_2^*=1.864274 and m_2^{**}=1.864275

-that for the top curve T(x,y), we use r' =0.00142945 to make T a function of x and y.

-that for the zero curve Z(x,y), we use $r'' = 0.00165523$ to make Z a function of x and y.

-that $r''' = \min(0.00142945, 0.00165523) = 0.00142945$ to make both curves functions of x and y.

-that we let $\epsilon = 0.0194131$

-that we let $r =$ **00002775005** $\leq \epsilon r'''$ which separates these curves in two wings in a 2r-square.

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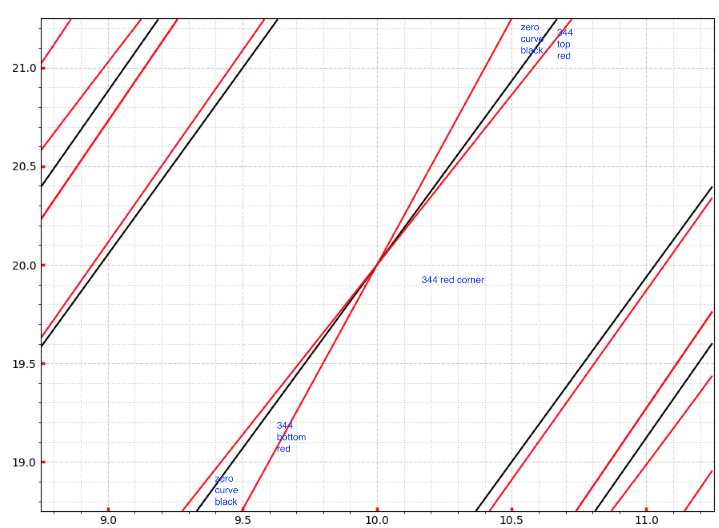

Figure 15: 344 red corner and black zero curve

3. The OSNO (368, 3280) has 94 zeros and here is the combination that gives the smallest r. See the Figure.

We use mpfi to find
-that the bottom curve B(x,y) has slope m_1 between m_1^*=-2.662817 and m_1^{**}=-2.662816
-that the best zero curve Z(x,y) has slope m_2 between m_2^*=-3.117212 and m_2^{**}=-3.117211
-that for the bottom curve B(x,y), we use r' =0.000948073 to make B a function of x and y.
-that for the zero curve Z(x,y), we use r'' = 0.00100252 to make Z a function of x and y
-that r''' = min(0.000948073,0.00100252) =0.000948073 to make both curves functions of x and y.
-that we let ϵ = 0.0393072
-that we let r = **0.0000372660** $\leq \epsilon r'''$ which separates these curves in two wings in a 2r-square.

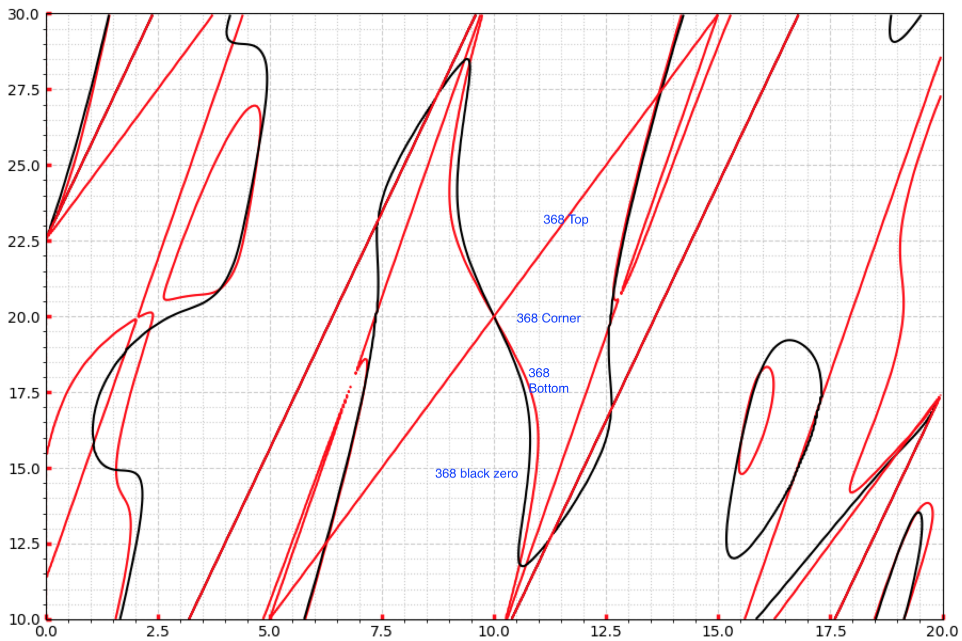

Figure 16: 368 red corner and black zero curve

4. OSNO (384, 3056) 1 7 12 4 6... This corner has 94 zero curves, 2 negative infinities, and 2 sides of the corner one of which is the straight line y=2x. Below in the Figure is the zero curve that gives the overall smallest radius r that we will use.

Important note: It is not important to find the largest r that will work. What we want is any subdivided square that that has a periodic path at every point inside the square. This particular black zero curve has slope = .000000507005... that is almost horizontal at (10,20).

We use mpfi to find
-that the bottom curve B(x,y) has slope $m_2 = 2$
-that the best zero curve Z(x,y) has slope m_1 between m_1^*=5.070057E-7 and m_1^{**}=5.070058E-7
-that for the bottom curve B(x,y), we use r' =0.00327094 to make B a function of x and y.
-that for the zero curve Z(x,y), we use r'' = 1.237501E-9 to make Z a function of x and y
-that r''' = min(0.00327094,1.237501E-9) = 1.237501E-9 to make both curves functions of x and y.
-that we let ϵ = 0.49999974
-that we let r = **0.000000000618750** $\leq \epsilon r'''$ which separates these curves in two wings in a 2r-square.

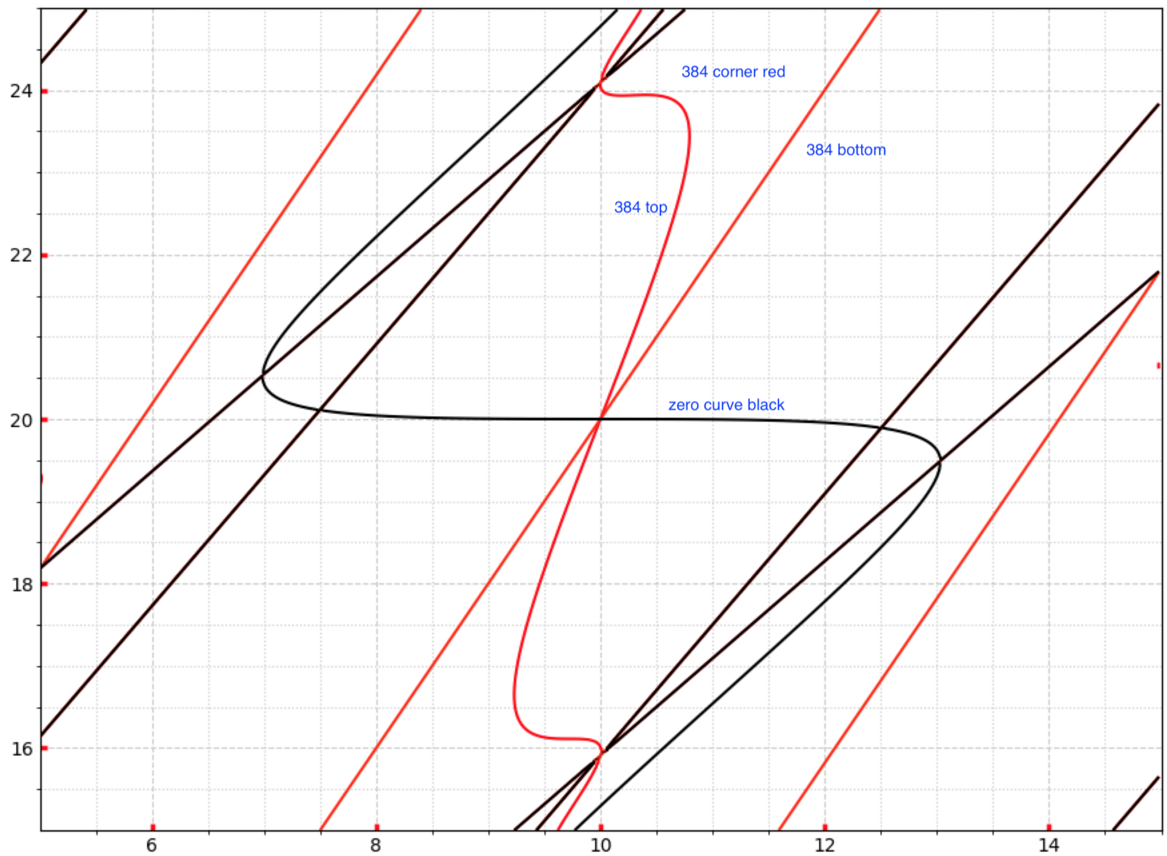

Figure 17: 384 red corner and black zero curve

Note: If we wanted to we could use $\epsilon = 0.999999$ since B(x,y) is a straight line and $r = \mathbf{0.00000000123749} \leq \epsilon r'''$. Never the less we will still use the smaller r.

5. The OSNO (358, 2734) has ten zero curves and its two corner curves that go through (10,20) Here is the combination that gives the smallest r. See the Figure.

We use mpfi to find

-that the bottom curve B(x,y) has slope m_2 between m_2^*=2.533682 and m_1^{**}=2.533683

-that the best zero curve Z(x,y) has slope m_1 between m_1^*=2.327233 and m_1^{**}=2.327234

-that for the bottom curve B(x,y), we use r' =0.00198712 to make B a function of x and y.

-that for the zero curve Z(x,y), we use $r'' = 0.00233097$ to make Z a function of x and y

-that $r''' = \min(0.00198712, 0.00233097) = 0.00198712$ to make both curves functions of x and y.

-that we let $\epsilon = 0.0212355$

-that we let $r = \mathbf{0.0000421974} \leq \epsilon r'''$ which separates these curves in two wings in a 2r-square.

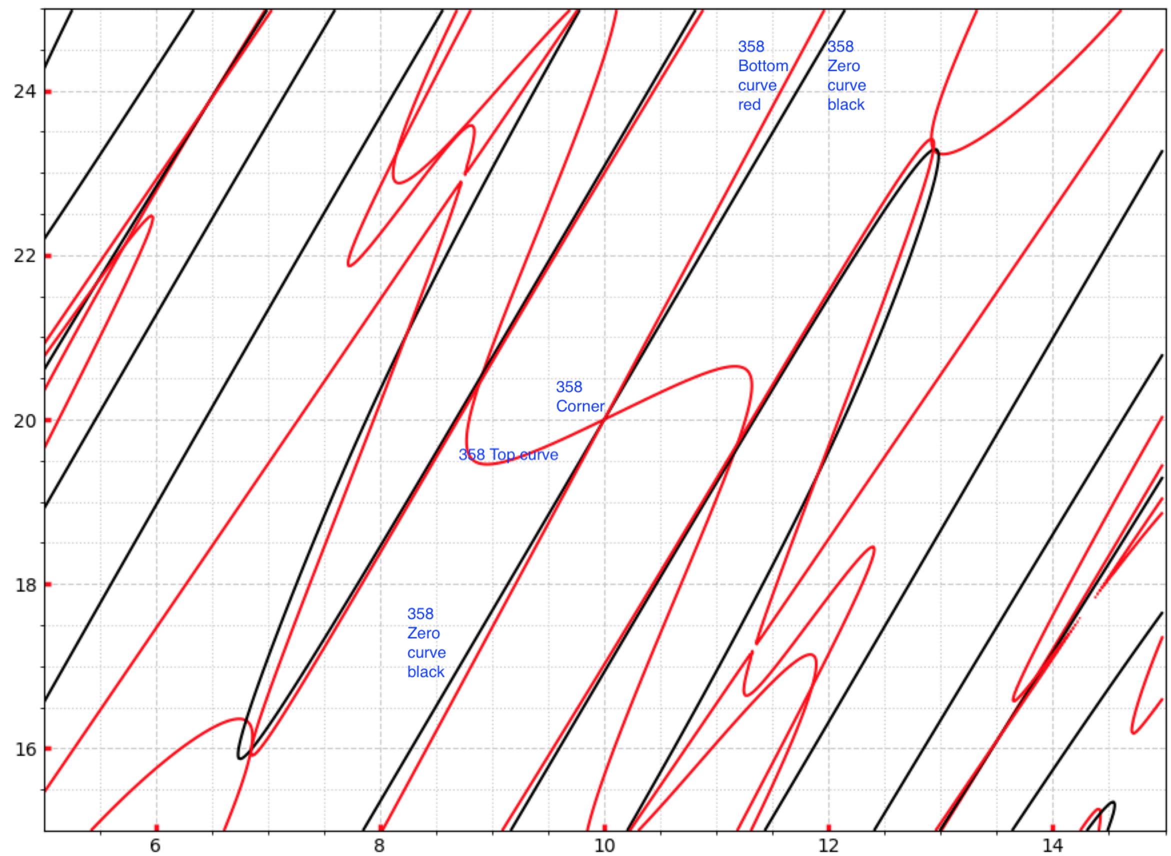

Figure 18: 358 red corner and black zero curve

The smallest r from the 10-20 zero combinations is r = **6.187503E-10**

21 Special Curves at (10,20)

These special zero curves go throught (10,20) and have slopes of the form infinity, -infinity or the constant 2. They are either a corner zero side or a non-corner zero curve of the (10,20) corner.

There are seven of these at (10,20), 1 from the OSNO 56, 3 from the OSNO 368 and 3 from the OSNO 384. The 5 infinite slopes are straight vertical lines and the other two are straight lines with slope 2.

1. The OSNO (56,512) has a bottom corner curve side 0=F(x,y)=-sin(4y)+sin(8y)+sin(10y)... which includes a vertical line x=10 through (10,20) and its slope is infinity since it factors to cos(9x)(sin(x-12y)+sin(x-10y)-3sin(x-6y)...). We want to find a 2r-square so that F(x,y)=0 is a function of y and only vertical in this square.

Since $-9408 < F_x(10,20) < -9407$ with $|F_{xx}|$=1192744 and $|F_{xy}|$=428472 and sum of bounds = 1621216 then if $0 < r = 0.00580243 < F_x(10,20)/(|F_{xx}| + |F_{xy}|)$ then $F_x < 0$ throughout this square and $F(x,y) = 0$ is a function of y which makes the vertical line x=10 alone in this 2r-square.

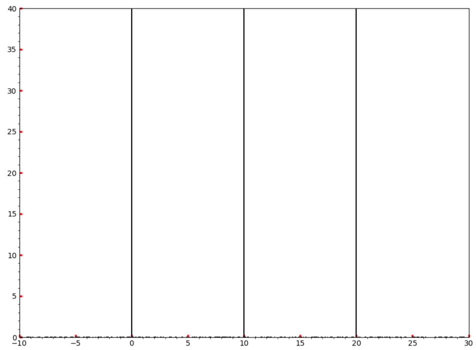

Figure 19: 56 vertical line at x=10

2. The OSNO(368,3280) has a non-corner zero curve $F(x,y) = -6cos(x-37y) + 2cos(x-35y) + 2cos(x-33y)... = 0$ which has a vertical line at (10,20) and its slope is -infinity since it factors to $cos(9x)(-46cos(y) - 12cos(3y) - 13cos(5y)...)$.

$317508 < F_x(10,20) < 317509$ with $|F_{xx}|$=111632010 and $|F_{xy}|$=26207746 and sum of bounds =137839756

then if $0 < r = 0.00230345 < F_x(10,20)/(|F_{xx}| + |F_{xy}|)$ then $F_x > 0$ in this square and $F(x,y) = 0$ is a function of y which makes the vertical line x=10 alone in this 2r-square.

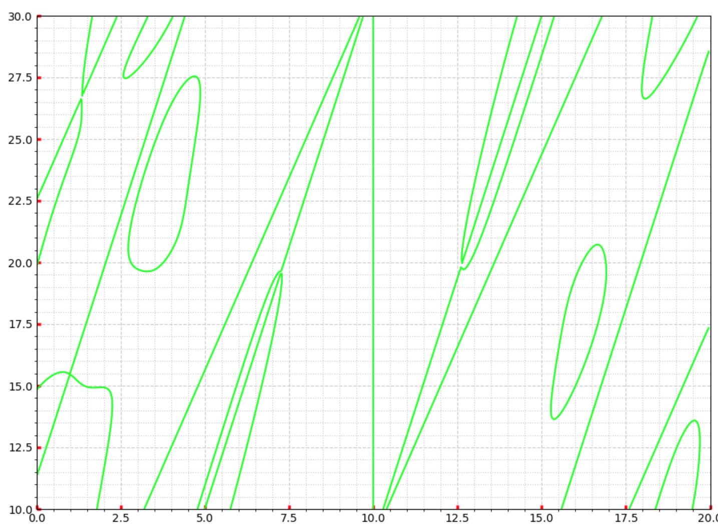

Figure 20: 368 vertical line 1 at x=10

3. The OSNO(368,3280) has a non-corner zero curve $F(x,y) = -6cos(x-37y) + 2cos(x-35y) + 2cos(x-33y)... = 0$ which has a vertical line at (10,20) and its slope is -infinity since it factors to $cos(9x)(-43cos(y) - 14cos(3y) - 12cos(5y)...)$

$317508 < F_x(10,20) < 317509$ with $|F_{xx}|$=112309926 and $|F_{xy}|$=26643218 and sum of bounds =138953144

then if $0 < r = 0.00228500 < F_x(10,20)/(|F_{xx}| + |F_{xy}|)$ then $F_x > 0$ in this square and $F(x,y) = 0$ is a function of y which makes the vertical line x=10 alone in this 2r-square.

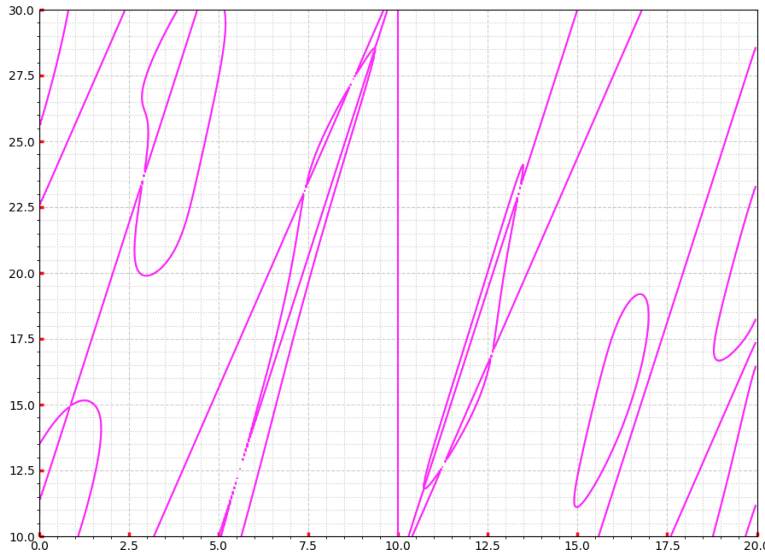

Figure 21: 368 vertical line 2 at x=10

4. The OSNO(368,3280) has a top corner curve side $F(x,y) = -7sin(2y) - 7sin(4y) + 18sin(8y)$....=0 which has a straight line through (10,20) and its slope is 2 since it factors to $sin(2x - y)(205cos(y) + 75cos(3y) - 57cos(5y)...)$ We want to find a 2r-square so that F(x,y)=0 is a straight line y=2x alone in this square.

333094< $F_x(10.0, 20.0)$ <333095 with $|F_{xx}|$=92289280, and $|F_{xy}|$=25093856 and sum of bounds=117383136
$-166548 < F_y(10.0, 20.0) < -166547$ with $|F_{yx}|$=25093856, and $|F_{yy}|$=8344560 and sum of bounds=33438416

then if $0 < r_1 = 0.00283766 < F_x(10, 20)/(|F_{xx}| + |F_{xy}|)$ then $F_x > 0$ in this square and $F(x, y) = 0$ is a function of y which makes the straight line y=2x alone in this square.
then if $0 < r_2 = 0.00498070 < |F_y(10, 20)|/(|F_{yx}| + |F_{yy}|)$ then $F_y < 0$ in this square and $F(x, y) = 0$ is a function of x which makes the straight line y=2x alone in this square.

This makes $r = min(r_1, r_2)$=**0.00283766**

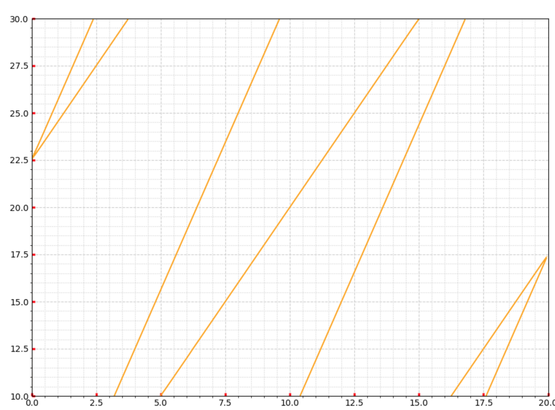

Figure 22: 368 m=2 at x=10

5. The OSNO(384,3956) has a non-corner zero curve $F(x,y) = cos(x-63y) - 2cos(x-55y) + 4cos(x-51y)... =$ 0 which has a vertical line at (10,20) and its slope is -infinity since it factors to $cos(9x)h_1(x,y)$

$280158 < F_x(10,20) < 280159$ with $|F_{xx}|$=74461876 and $|F_{xy}|$=46189908 and sum of bounds =120651784

then if $0 < r = 0.00232204 < F_x(10,20)/(|F_{xx}| + |F_{xy}|)$ then $F_x > 0$ in this square and $F(x,y) = 0$ is a function of y which makes the vertical line x=10 alone in this 2r-square.

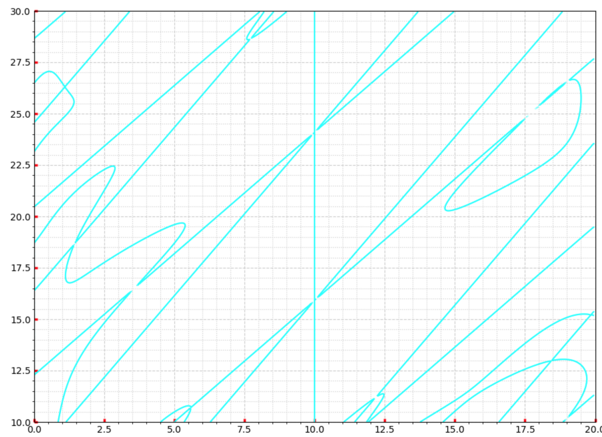

Figure 23: 384 vertical line 1 at x=10

6. The OSNO(384,3956) has a different non-corner zero curve $G(x,y) = cos(x-63y) - 2cos(x-55y) + 4cos(x-51y)... = 0$ which has a vertical line at (10,20) and its slope is -infinity since it factors to $cos(9x)h_2(x,y)$.

$280158 < G_x(10,20) < 280159$, $|G_{xx}|$=81076838, $|G_{xy}|$=49508498 and sum of bounds=130585336

then if $0 < r = 0.00214540 < G_x(10,20)/(|G_{xx}| + |G_{xy}|)$ then $G_x > 0$ in this square and $G(x,y) = 0$ is a function of y which makes the vertical line x=10 alone in this 2r-square.

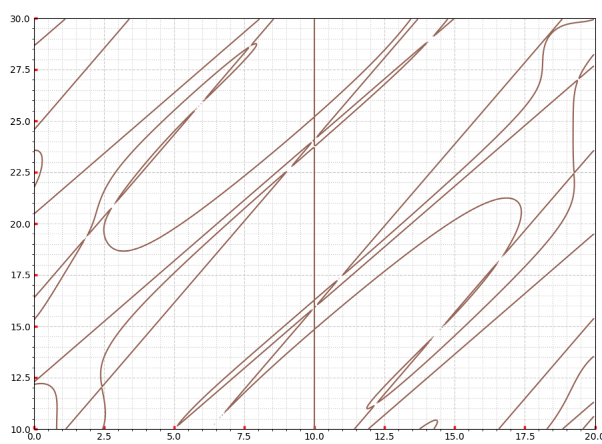

Figure 24: 384 vertical line 2 at x=10

7. The OSNO(384,3956) has a bottom corner curve side $F(x,y) = -4sin(2y) - 3sin(4y) - 9sin(8y)...$ =0 which has a straight line through (10,20) and its slope is 2 since it factors to $sin(2x-y)h(x,y)$. We want to

find a 2r-square so that F(x,y)=0 is a straight line y=2x alone in this square.

-808256< $F_x(10.0, 20.0)$ <-808255 with $|F_{xx}|$=112671136, and $|F_{xy}|$=71919576 and sum of bounds=184590712
404127 < $F_y(10.0, 20.0)$ < 404128 with $|F_{yx}|$=71919576, and $|F_{yy}|$=51631208 and sum of bounds=123550784

then if $0 < r_1 = 0.00437863 < |F_x(10, 20)|/(|F_{xx}| + |F_{xy}|)$ then $F_x < 0$ in this square and $F(x, y) = 0$ is a function of y which makes the straight line y=2x alone in this square.
then if $0 < r_2 = 0.00327093 < F_y(10, 20)/(|F_{yx}| + |F_{yy}|)$ then $F_y > 0$ in this square and $F(x, y) = 0$ is a function of x which makes the straight line y=2x alone in this square.

This makes $r = min(r_1, r_2)$=**0.00327093**

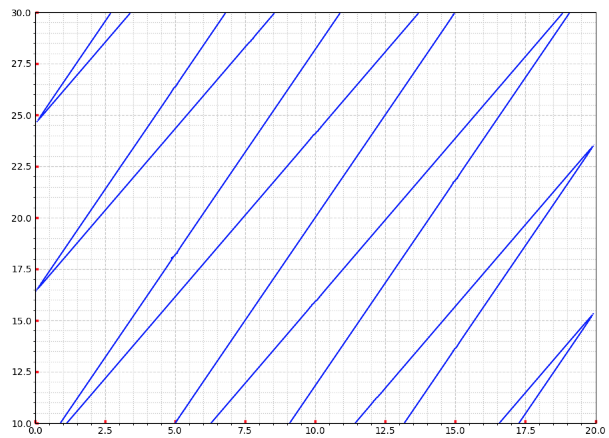

Figure 25: 384 slope m=2

The smallest r from these special curves is r=**0.00228500**

22 Covering of the regions at 10-20

The five stables and one unstable either overlap each other or abut each other to cover a 2r-square. Here we separate successive sides of the corners by using wings if needed.

We look at the region counterclockwise in this order below and we look at the two curved sides of the corners. We will call them the bottom or bot curve followed by the top curve counterclockwise.
Bot OSNO (56, 512) and Top OSNO (56, 512) Blue
Bot OSNO (344, 2810) and Top OSNO (344, 2810) Lime
Bot OSNO (368, 3280) and Top OSNO (368, 3280) Magenta
Bot CNS (2,6) and Top CNS (2,6) Black
Bot OSNO (384, 3056) and Top OSNO (384, 3056) Yellow
Bot OSNO (358, 2734) and Top OSNO (358, 2734) Orange

These are the curves that are the sides of the corners with their slopes at (10,20) and makes them a function in each given centered 2r-square.
Bot 56 r=0.00580256 slope = infinity and Top 56 r=0.00530730 slope = 2.5186715...
Bot 344 r=0.00114253 slope = 2.493881... and Top 344 r=0.00142945 slope = 1.724918...
Bot 368 r=0.000948073 slope = -2.662816... and Top 368 r=0.00283766 slope = 2.0
Bot 384 r=0.00327094 slope = 2.0 and Top 384 r=0.00110290 slope = 3.529402...

Bot 358 r=0.00198712 slope = 2.533682... and Top 358 r=0.000630055 slope = 0.661351...
The smallest r to make all corner sides as functions is r=**0.000630055**

To separate two curved equations and its slopes we use $\epsilon = .5||m_2| - |m_1||/(|m_2| + |m_1|)$
To separate one curved and one straight line equation, we use $\epsilon = ||m_2| - |m_1||/(|m_2| + |m_1|)$
To separate one curved and a vertical line equation, we don't need any epsilon or let $\epsilon = 1$.
To separate two straight line equations with different slopes, we don't need any epsilon or let $\epsilon = 1$.
To separate two curved equations with opposite signs slopes, we don't need any epsilon or let $\epsilon = 1$.

Bot OSNO(344,2810) overlaps Top OSNO(56,512):
- Bot OSNO(344,2810) has slope 2.493881... and r_1=0.00114253 an increasing function of x,y
- Top OSNO(56,512) has slope 2.5186715... and r_2=0.00530730 an increasing function of x,y

Epsilon = 0.00247279...

r = 2.825251E-6 < $\epsilon min(r_1, r_2)$

Bot OSNO(368,3280) overlaps Top OSNO(344,2810):
- Bot OSNO(368,3280) has slope -2.662816... and r_1=9.480732E-4, a decreasing function of x,y
- Top OSNO(344,2810) has slope 1.724918... and r_2=0.00142945, an increasing function of x,y

Epsilon not needed as both slopes have opposite signs.

r = min(r_1,r_2)= 9.480732E-4

Bot CNS(2,6) line 2x-y=0 abuts Top OSNO(368,3280) line sin(2x-y)=0:
- Bot CNS(2,6) has slope 2.0 and r_1=0.0855050, a straight increasing line
- Top OSNO(368,3280) has slope 2.0 and r_2=0.00283766, a straight increasing line

Epsilon not needed as both curves are straight in this 2r-square.

r = min(r_1,r_2) = 0.00283766

Bot OSNO(384,3056) line sin(2x-y)=0 abuts Top CNS(2,6) line 2x-y=0:
- Bot OSNO(384,3056) has slope 2.0 and r_1=0.00327094, a straight increasing line
- Top CNS(2,6) has slope 2.0 and r_2=0.0855050, a straight increasing line

Epsilon not needed as both curves are straight in this 2r-square.

r = min(r_1,r_2) = 0.00327094

Bot OSNO(358,2734) overlaps Top OSNO(384,3056):
- Bot OSNO(358,2734) has slope 2.533682... and r_1=0.00198712, an increasing function of x,y
- Top OSNO(384,3056) has slope 3.529402... and r_2=0.00110290, an increasing function of x,y

Epsilon = 0.0821132...

r = 9.0563249E-5< $\epsilon min(r_1, r_2)$

Bot OSNO(56,512) vertical line cos(9x)=0 overlaps Top OSNO(358,2734) curve:
- Bot OSNO(56,512) has slope Infinity and r_1=0.00580256, a straight vertical line
- Top OSNO(358,2734) has slope 0.661351... and r_2=6.300553 an increasing function of x,y

Epsilon not needed as OSNO(56,512) has a vertical line and Top OSNO(358,2734) is a function.

r = min(r_1,r_2) = 6.300553E-4

The smallest r from the 10-20 covering is r = **2.825251E-6**

23 10-20 Final Conclusion

Now using these sections in order, we get the following minimum positive r's

Positive Equations: r=5.253499E-8

Corners: r=1.04122E-4
Zero Combos: r=**6.187503E-10**
Special Curves: r=2.28500E-3
Covering: r=2.825251E-6

and in total we use $0 < r <$**6.187503E-10** for the centered star square.

Every point of a **centered 2r-square** centered at (10,20) where **r=6.187503E-10** has a periodic path from at least one of the 6 given code types. We will call it a **centered 10-20 Star Flare square**.

Thus the (10,20) flare square is covered by five overlapping stables and one unstable to form a (10,20) centered square with side 2r where $0 < r <$ 6.187503E-10 radians. It will follow that we will finish by putting a subdivided square of radius $r_1 = \pi/2^{34}$ = 1.828647...E-10 inside the centered square. The center of the subdivided square has center $(954437177\pi/2^{34}, 1908874353\pi/2^{34})$.

The overall smallest r_1 above is $0 < r_1 <$ **1.828647...E-10** to completely cover a (10,20) subdivided neighbourhood. Thus

The **subdivided** $2r_1$**-square** which contains (10,20) and is inside the centered square has side $2r_1$ where $r_1 = \pi/2^{34}$ radians or $r_1 = 90/2^{33}$ degrees by using subdivisions of the big square. Its center is at $(954437177\pi/2^{34}, 1908874353\pi/2^{34})$ in radians. We call it **the subdivided 10-20 Star Flare square** which is the one that comes from the Star jar.

Note: Any smaller r will work and the one used in Billiards Covers [6] uses a much smaller r.

24 Summary of subdivided squares

1. A (36,54) neighbourhood covering of a subdivided $2r_1$-square with center (36.002197265625,53.997802734375) uses 4 codes where $r_1 = 90/2^{13}$ using k=13.
2. A (45,45) neighbourhood covering of a subdivided $2r_1$-square with center (44.82421875, 44.82421875) uses 7 codes where $r_1 = 90/2^{9}$ using k=9.
3. A (10,20) neighbourhood covering of a subdivided $2r_1$-square with center (10.00000000116415...,19.99999999185092...) uses 6 codes where $r_1 = 90/2^{33}$ using k=33.

Any smaller r will cover these neighbourhoods. It is not necessary to find the largest r.

25 Conjecture

Conjecture: Prove that the point (15,30) is a star flare. It is also unknown if it is an infinite or finite star flare.

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References

[1] Patrick M. Fitzpatrick "Advanced Calculus, second edition" *pp. 442–444*

[2] Laurent Fousse, Guillaume Hanrot, Vincent Lefèvre, Patrick Pélissier, Paul Zimmermann, "MPFR: A Multiple-Precision Binary Floating-Point Library With Correct Rounding", *ACM Transactions on Mathematical Software* 33, no. 2 (June 2007): Article 13.

[3] Revol, N. and Rouilier, F., The MPFI Library, 2001, http://perso.ens-lyon.fr/nathalie.revol/software.html

[4] Torbjörn Granlund et al., GNU Multiple Precision Arithmetic Library, https://gmplib.org.

[5] Billiards Everything, https://sourceforge.net/projects/billiards-everything/

[6] Billiards Covers, https://sourceforge.net/projects/billiards-covers/

[7] Billiards Star Flares, https://sourceforge.net/projects/billiards-stars/

[8] The Tokarsky-Xichen Grapher, https://github.com/gwtokarsky/gwtokarsky.github.io/raw/refs/heads/main/graphing

[9] The Great Periodic Path Hunt https://gwtokarsky.github.io/

[10] Howard Masur, "Closed Trajectories for Quadratic Differentials with an Application to Billiards", *Duke Mathematical Journal* 53, no. 2 (1986): 307–314.

[11] Carlo Boldrighini, Michael Keane and Federico Marchetti, "Billiards in Polygons" *The Annals of Probability* 6, no. 4 (1978): 532–540.

[12] Michael D. Boshernitzan "Billiards and Rational Periodic Directions in Polygons" *The American Mathematical Monthly, Vol. 99, No. 6 (Jun. - Jul., 1992), pp. 522-529*

[13] M. Boshernitzan, G. Galperin, T. Kruger and S. Troubetzkoy, "Some Remarks on Periodic Billiard Orbits in Rational Polygons" *preprint SUNY Stony Brook* (1994)

[14] G. Galperin and D. Zvonkine, "Periodic Billiard Trajectories in Right Triangles and Right-Angled Tetrahedra" *Regular and Chaotic Dynamics* 8, no. 1 (2003): 29–44.

[15] Lorenz Halbeisen and Norbert Hungerbuhler, "On periodic billiard trajectories in obtuse triangles" *SIAM review* 42, no. 4 (2000): 657–670.

[16] Fred Holt, "Periodic Reflecting Paths in Right Triangles", *Geometriae Dedicata* 46, no. 1 (April 1993): 73–90.

[17] W. Patrick Hooper, "Periodic billiard paths in right triangles are unstable" *Geometriae Dedicata* 125, no. 1 (March 2007): 39–46.

[18] W. Patrick Hooper, Richard Evan Schwartz, "Introduction to McBilliards" *Geometriae Dedicata* 125, no. 1 (Sept. 2004): 39–46.

[19] W. Patrick Hooper, Richard Evan Schwartz, "Billiards in Nearly Isosceles Triangles", Journal of Modern Dynamics 3, no. 2 (2009): 159–231.

[20] Richard Evan Schwartz, "Obtuse Triangular Billiards II: One Hundred Degrees Worth of Periodic Trajectories", *Experimental Mathematics* 18, no. 2 (2009): 137–171.

[21] Ya. B. Vorobets, G. A. Galperin, and A. M. Stepin, "Periodic billiard trajectories in polygons generating mechanisms" *Russ. Math. Surv.* 47, no. 5 (1992): 5–80.

[22] Clayton C. Ward, "Calculation of Fuchsian groups associated to billiards in a rational triangle" *Ergodic Theory Dynam. Systems* 18, no. 4 (1998): 1019–1042.