The Golden Ratio's Siblings

Bassam Abdul-Baki

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Abstract

On real numbers whose fractional part remain constant after squaring.

Golden Ratio

The golden ratio[1], denoted by the Greek letter phi $(\varphi \; or \; \phi)$, is the value of the ratio of two quantities a, b, with a > b > 0, such that $\frac{a}{b} = \frac{a+b}{a} = \varphi = 1.61803398874989 \ldots$.

One of the consequences of the golden ratio is that $\varphi^2 = \varphi + 1$. Thus, squaring the golden ratio does not change the fractional part of the number, only the integer part.

Analysis

Let $i, j \in \mathbb{N}$ and $r \in \mathcal{R}$.

We need to find all real numbers such that when you square them, there fractional parts stay the same. A similar exercise can be done for higher integer powers, non-integer powers, or even complex numbers.

$$(i + r)^2 = j + r$$

$$i^2 + 2ir + r^2 = j + r$$

$$r^2 + r(2i - 1) + i^2 - j = 0$$

$$\begin{aligned} r &= \frac{-(2i-1) \pm \sqrt{(2i-1)^2 - 4(i^2 - j)}}{2} \\ &= \frac{-2i + 1 \pm \sqrt{4(j-i) + 1}}{2} \end{aligned}$$

Since r has to be real, for each i + j = n, we have at most $\left\lfloor \frac{n+1}{2} \right\rfloor$ possible solutions (ignoring complex numbers).

For all j > i ≥ 0, 0 ≤ i + j ≤ n, we get the following non-integers when excluding integers:

Note: The last few decimal places may not be exact.

Complete Table (up to n = 13)

i	j	r-	r+	left constant	right constant
0	1	-0.618033988749895	1.618033988749890	1	1
0	3	-1.302775637731990	2.302775637731990	2	2
1	2	-1.618033988749890	0.618033988749895	1	1
0	4	-1.561552812808830	2.561552812808830	3	3
0	5	-1.791287847477920	2.791287847477920	4	4
1	4	-2.302775637731990	1.302775637731990	2	2
2	3	-2.618033988749890	-0.381966011250105	1	5
1	5	-2.561552812808830	1.561552812808830	3	3
0	7	-2.192582403567250	3.192582403567250	6	6
1	6	-2.791287847477920	1.791287847477920	4	4
2	5	-3.302775637731990	0.302775637731995	2	2
3	4	-3.618033988749890	-1.381966011250110	1	5
0	8	-2.372281323269010	3.372281323269010	7	7
2	6	-3.561552812808830	0.561552812808830	3	3
0	9	-2.541381265149110	3.541381265149110	8	8
1	8	-3.192582403567250	2.192582403567250	6	6
2	7	-3.791287847477920	0.791287847477920	4	4
3	6	-4.302775637731990	-0.697224362268005	2	9
4	5	-4.618033988749890	-2.381966011250110	1	5
0	10	-2.701562118716420	3.701562118716420	10	10
1	9	-3.372281323269010	2.372281323269010	7	7
3	7	-4.561552812808830	-0.438447187191170	3	11
0	11	-2.854101966249680	3.854101966249680	12	12
1	10	-3.541381265149110	2.541381265149110	8	8
2	9	-4.192582403567250	1.192582403567250	6	6
3	8	-4.791287847477920	-0.208712152522080	4	13
4	7	-5.302775637731990	-1.697224362268010	2	9
5	6	-5.618033988749890	-3.381966011250110	1	5
1	11	-3.701562118716420	2.701562118716420	10	10
2	10	-4.372281323269010	1.372281323269010	7	7
4	8	-5.561552812808830	-1.438447187191170	3	11
0	13	-3.140054944640260	4.140054944640260	14	14
1	12	-3.854101966249680	2.854101966249680	12	12
2	11	-4.541381265149110	1.541381265149110	8	8
3	10	-5.192582403567250	0.192582403567252	6	6
4	9	-5.791287847477920	-1.208712152522080	4	13
5	8	-6.302775637731990	-2.697224362268010	2	9
6	7	-6.618033988749890	-4.381966011250110	1	5

Summary Table (up to n = 71)

Every i = 0 obviously introduces a new constant r when the discriminant is not a perfect square.

Negative constants occur when $j < i^2$, with r = 0 for $j = i^2$. However, the fractional parts of the negative right constants are just the complement of the fractional parts of the left constants (e.g., from the previous table, the first unique constant = $\varphi = 1.618033988749890$ and the fifth unique constant = $-2 + \varphi = -0.381966011250105$).

In addition, $r^- = 1 - r^+ = \frac{-j}{r^+}$ or ${r^+}^2 = r^+ + j$. Thus, ∀ j, ∃ an r ∋ r^2 = r + j; if, however, i = 0 and j = k(k+1), then r is an integer.

One final point of interest is that $\lfloor r^+ \rfloor$ appears as often as $2\lfloor r^+ \rfloor - 1$.

Eliminating all rows but the first positive constant number for uniqueness, we get the following results:

#	i	j	r⁻	r⁺
0	0	1	-0.618033988749895	1.618033988749890
1	0	3	-1.302775637731990	2.302775637731990
2	0	4	-1.561552812808830	2.561552812808830
3	0	5	-1.791287847477920	2.791287847477920
4	0	7	-2.192582403567250	3.192582403567250
5	0	8	-2.372281323269010	3.372281323269010
6	0	9	-2.541381265149110	3.541381265149110
7	0	10	-2.701562118716420	3.701562118716420
8	0	11	-2.854101966249680	3.854101966249680
9	0	13	-3.140054944640260	4.140054944640260
10	0	14	-3.274917217635370	4.274917217635370
11	0	15	-3.405124837953330	4.405124837953330
12	0	16	-3.531128874149270	4.531128874149270
13	0	17	-3.653311931459040	4.653311931459040
14	0	18	-3.772001872658770	4.772001872658770
15	0	19	-3.887482193696060	4.887482193696060
16	0	21	-4.109772228646440	5.109772228646440
17	0	22	-4.216990566028300	5.216990566028300
18	0	23	-4.321825380496480	5.321825380496480
19	0	24	-4.424428900898050	5.424428900898050
20	0	25	-4.524937810560440	5.524937810560440
21	0	26	-4.623475382979800	5.623475382979800
22	0	27	-4.720153254455280	5.720153254455280
23	0	28	-4.815072906367320	5.815072906367320
24	0	29	-4.908326913195980	5.908326913195980
25	0	31	-5.090169943749470	6.090169943749470
26	0	32	-5.178908345800270	6.178908345800270
27	0	33	-5.266281297335400	6.266281297335400
28	0	34	-5.352349955359810	6.352349955359810
29	0	35	-5.437171043518960	6.437171043518960
30	0	36	-5.520797289396150	6.520797289396150
31	0	37	-5.603277807866850	6.603277807866850
32	0	38	-5.684658438426490	6.684658438426490
33	0	39	-5.764982043070830	6.764982043070830
34	0	40	-5.844288770224760	6.844288770224760
35	0	41	-5.922616289332560	6.922616289332560

#	i	j	r^-	r^+
36	0	43	-6.076473218982950	7.076473218982950
37	0	44	-6.152067347825040	7.152067347825040
38	0	45	-6.226812023536850	7.226812023536850
39	0	46	-6.300735254367720	7.300735254367720
40	0	47	-6.373863542433760	7.373863542433760
41	0	48	-6.446221994724900	7.446221994724900
42	0	49	-6.517834423809100	7.517834423809100
43	0	50	-6.588723439378910	7.588723439378910
44	0	51	-6.658910531638180	7.658910531638180
45	0	52	-6.728416147400480	7.728416147400480
46	0	53	-6.797259759663210	7.797259759663210
47	0	54	-6.865459931328120	7.865459931328120
48	0	55	-6.933034373659250	7.933034373659250
49	0	57	-7.066372975210780	8.066372975210780
50	0	58	-7.132168761236870	8.132168761236870
51	0	59	-7.197402159170330	8.197402159170330
52	0	60	-7.262087348130010	8.262087348130010
53	0	61	-7.326237921249260	8.326237921249260
54	0	62	-7.389866919029750	8.389866919029750
55	0	63	-7.452986860293430	8.452986860293430
56	0	64	-7.515609770940700	8.515609770940700
57	0	65	-7.577747210701760	8.577747210701760
58	0	66	-7.639410298049850	8.639410298049850
59	0	67	-7.700609733428360	8.700609733428360
60	0	68	-7.761355820929150	8.761355820929150
61	0	69	-7.821658488546620	8.821658488546620
62	0	70	-7.881527307120110	8.881527307120110
63	0	71	-7.940971508067070	8.940971508067070

References

[1] Wikipedia, The Free Encyclopedia (2024), https://en.wikipedia.org/wiki/Golden_ratio, ***Golden ratio***