ON THE PRINCIPLE OF STRUCTURAL DEPENDENCY AND APPLICATIONS

THEOPHILUS AGAMA

Abstract. This note formalizes and applies the *Principle of Structural Dependency*, which asserts that if the foundation of a mathematical structure B consists of another structure A, then A cannot exhibit a property distinct from B, while B may possess properties not shared by A. We verify this principle and apply it systematically to reconstruct concise proofs of several classical theorems, including Cantor's theorem, the Fundamental Theorem of Algebra, the Jordan Curve Theorem, the Monotone Convergence Theorem, and the Pythagorean Theorem. These reconstructions emphasize the structural underpinnings of these results, offering a novel perspective and demonstrating how foundational relationships can simplify complicated proofs.

1. Introduction

The study of mathematical structures often reveals deep interdependency between their properties. These relationships frequently allow for insights into proofs by leveraging the foundational composition of one structure within another. The *Principle of Structural Dependency* provides a systematic framework for analyzing and understanding such relationships.

This principle articulates that if a structure B is fundamentally built upon another structure A, then A cannot possess a property not shared by B. This dependency stems from the embedding of A within the foundation of B. However, B, as a more complex structure, may acquire properties beyond those of A. This insight provides a versatile tool for reasoning about the properties of structures and reconstructing proofs of classical mathematical results with clarity and brevity.

In this paper, we begin by rigorously formalizing the Principle of Structural Dependency and proving it within a mathematical framework. Following this, we apply the principle to several classical theorems, illustrating how it can simplify proofs and illuminate the relationships between foundational and derived structures. The focus is on an expository style to make the applications accessible while maintaining mathematical rigor.

2. The Principle of Structural Dependency

2.1. **Statement of the Principle.** Let A and B be mathematical structures. Suppose the foundation of B consists of A, which we denote by $A \subset \mathbb{F}(B)$. Then:

$$\mathbb{P}(A) \cap \mathbb{P}(B) = \mathbb{P}(A),$$

Date: December 2, 2024.
Key words and phrases. structural; dependency; principle.

where $\mathbb{P}(A)$ and $\mathbb{P}(B)$ represent the sets of properties of A and B, respectively. This means that A cannot exhibit a property distinct from B, while B may exhibit properties that extend beyond those of A.

2.2. **Proof of the Principle.** Since the foundation of B consists of A, we have $A \subset \mathbb{F}(B)$, where $\subset$ denotes an embedding of one structure into another. Furthermore, by the definition of foundational embedding, $\mathbb{F}(B) \subset B$. Combining these inclusions, we deduce $A \subset \mathbb{F}(B) \subset B$. Now, any property $p \in \mathbb{P}(A)$ must also belong to $\mathbb{P}(B)$ because A is embedded in B. This guarantees that $\mathbb{P}(A) \cap \mathbb{P}(B) = \mathbb{P}(A)$. Conversely, as B is a more comprehensive structure, it may exhibit properties that are not present in A, consistent with $\mathbb{P}(B) \supseteq \mathbb{P}(A)$. This asymmetry underscores the directional nature of the embedding, completing the proof.

3. Applications to Classical Theorems

The utility of the Principle of Structural Dependency lies in its ability to simplify and rewrite classical proofs by focusing on structural embeddings and property inheritance. Below, we apply the principle to reconstruct concise proofs of several well-known results, some of which can be found in [1].

3.1. **Cantor's Theorem. Statement:** The power set $\mathcal{P}(A)$ of a set A has a strictly greater cardinality than A.

Proof Reconstruction: The foundation of $\mathcal{P}(A)$ is A, as $\mathcal{P}(A)$ consists of subsets of A. Thus, $A \subseteq \mathbb{F}(\mathcal{P}(A)) \subset \mathcal{P}(A)$. Thus by the principle of *structural dependency*, we deduce that $|\mathcal{P}(A)| > |A|$.

3.2. **Fundamental Theorem of Algebra. Statement:** Every non-constant polynomial $p(z)$ with complex coefficients has at least one root in $\mathbb{C}$.

Proof Reconstruction: The foundation of the complex polynomial $p(z)$ is $\mathbb{C}$, as $p(z)$ is constructed over the field of complex numbers. Hence, $\mathbb{F}(p(z)) \supseteq \mathbb{C}$. - $\mathbb{C}$, being algebraically closed, transfers this property to $p(z)$, ensuring that $p(z)$ cannot lack roots within $\mathbb{C}$. By the principle of *structural dependency*, it follows that $p(z)$ must share the algebraic closure property, guaranteeing at least one root.

3.3. **Jordan Curve Theorem. Statement:** A simple closed curve C in $\mathbb{R}^2$ divides the plane into two disjoint regions: an interior and an exterior.

Proof Reconstruction: C is embedded in $\mathbb{R}^2$, with $C \subset \mathbb{F}(\mathbb{R}^2) \subset \mathbb{R}^2$. By the principle of *structural dependence*, it follows that $\mathbb{P}(C) \subset \mathbb{P}(\mathbb{R}^2)$. The claim follows from this embedding of properties.

3.4. **Monotone Convergence Theorem.** Statement: Let (a_n) be a bounded monotone sequence of real numbers. Then (a_n) converges to a limit in $\mathbb{R}$.

Proof Reconstruction:

The sequence (a_n) is a subset of $\mathbb{R}$ and its foundation is $\mathbb{R}$, which means $\mathbb{R} = \mathbb{F}((a_n))$. That is, each term in the sequence (a_n) can be constructed from $\mathbb{R}$. The completeness property of $\mathbb{R}$ guarantees that every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound (supremum). The sequence (a_n), being bounded and monotone, inherits this completeness property from its foundation $\mathbb{F}((a_n)) = \mathbb{R}$. Thus, the supremum of (a_n) exists in $\mathbb{R}$, and it is the limit of the sequence. By the *structural dependency principle*, we know that the sequence (a_n) cannot possess properties distinct from those of its foundation, $\mathbb{R}$. Hence, the boundedness and monotonicity of (a_n) ensure that it inherits the completeness property of $\mathbb{R}$. The convergence of (a_n) is therefore guaranteed by the structure of $\mathbb{R}$.

3.5. **Pythagorean Theorem.** Statement: In a right triangle with side lengths a, b, and hypotenuse c, the relation $c^2 = a^2 + b^2$ holds.

Proof Reconstruction:

The triangle is a geometric object embedded in the Euclidean plane $\mathbb{R}^2$. Therefore, the foundation of the triangle is $\mathbb{R}^2$, implying $\mathbb{R}^2 = \mathbb{F}(\text{Triangle})$. The Euclidean plane $\mathbb{R}^2$ is equipped with the standard Euclidean metric, which measures the distance between two points (x_1, y_1) and (x_2, y_2) as $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. The right triangle inherits this metric as a property of its foundation. In a right triangle, the hypotenuse c is the distance between two points, which can be computed using the Euclidean metric. The two sides a and b are also distances measured along the axes. By the property inherited from the topology of the Euclidean metric, the distance c satisfies $c^2 = a^2 + b^2$, where a^2 and b^2 are the squares of the lengths of the perpendicular sides. Applying the *structural dependency* principle, the triangle cannot have any property distinct from those of its foundation $\mathbb{R}^2$. The Euclidean metric, as a property of $\mathbb{R}^2$, applies to the triangle, guaranteeing the Pythagorean relation.

4. Conclusion

The Principle of Structural Dependency provides a powerful framework for understanding the interplay between mathematical structures and their properties. By focusing on foundational embeddings, this principle not only elucidates the relationships between structures but also simplifies proofs of classical theorems. The reconstructions presented here illustrate its versatility and potential for broad application across various fields of mathematics. Future investigations could explore its implications in topology, analysis, and algebra, extending its reach and utility.

References

1. Rudin, Walter and others *Principles of mathematical analysis*, McGraw-hill New York, vol. 3, 1964.

Department of Mathematics, African institute for mathematical sciences, Ghana, Cape-coast