ON MAPS BETWEEN PROBLEM AND SOLUTION SPACES

THEOPHILUS AGAMA

ABSTRACT. In this paper, we study the map between problem and solution spaces. We introduce and develop a functional analysis on the topology of problems and their solution spaces. We introduce the notion of an *isotope* and corresponding *isotope* problem spaces and examine various notions compatible with this space.

1. **Background**

In [2],[1], [3] and [4] the theory of problems and their solution spaces with their topology was introduced and studied in great detail.
Let X denotes a solution (resp. answer) to problem Y (resp. question). Then we call the collection of all problems to be solved to provide a solution X to the problem Y the problem space induced by providing a solution X to problem Y. We denote this space by $\mathcal{P}_Y(X)$. If K is any subspace of the space $\mathcal{P}_Y(X)$, then we denote this relation by $K \subseteq \mathcal{P}_Y(X)$. If the space K is a subspace of the space $\mathcal{P}_Y(X)$ with $K \neq \mathcal{P}_Y(X)$, then we write $K \subset \mathcal{P}_Y(X)$. We say that problem V is a sub-problem of problem Y if providing a solution to problem Y furnishes a solution to problem V. If V is a subproblem of the problem Y, then we write $V \leq Y$. If V is a subproblem of the problem Y and $V \neq Y$, then we write $V < Y$ and call V a proper sub-problem of Y.
Similarly, Let X denotes a solution (resp. answer) to problem Y (resp. question). Then we call the collection of all solutions to problems obtained as a result of providing the solution X to the problem Y the solution space induced by providing the solution X to problem Y. We denote this space by $\mathcal{S}_Y(X)$. If K is any subspace of the space $\mathcal{S}_Y(X)$, then we denote this relation by $K \subset \mathcal{S}_Y(X)$. We make the assignment $T \in \mathcal{S}_Y(X)$ if solution T is also a solution in this space.
Let $\mathcal{P}_Y(X)$ be the problem space induced by providing the solution X to problem Y. Then we call the number of problems in the space (size) the **complexity** of the space and denote by $\mathbb{C}[\mathcal{P}_Y(X)]$ the complexity of the space. We make the assignment $Z \in \mathcal{P}_Y(X)$ if problem Z is also a problem in this space. Similarly, let $\mathcal{S}_Y(X)$ be the solution space induced by providing the solution X to problem Y. Then we call the number of solutions in the space (size) the **index** of the space and denote by $\mathbb{I}[\mathcal{S}_Y(X)]$ the index of this space.
Let $\mathcal{P}_X(Y)$ and $\mathcal{S}_X(Y)$ denotes the problem and solutions spaces, respectively, induced by providing solution X to problem Y. We say the problem space $\mathcal{P}_X(Y)$ is *compact* if and only if there exists a finite number of problem spaces

$$\mathcal{P}_{U_1}(V_1), \mathcal{P}_{U_2}(V_2), \ldots, \mathcal{P}_{U_k}(V_k)$$

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such that

$$\mathcal{P}_X(Y) \subset \mathcal{P}_{U_1}(V_1) \cup \mathcal{P}_{U_2}(V_2) \cup \cdots \cup \mathcal{P}_{U_k}(V_k).$$

Similarly, we say the solution space $\mathcal{S}_X(Y)$ is *compact* if and only if there exists a finite number of solution spaces $\mathcal{S}_{U_1}(V_1), \mathcal{S}_{U_2}(V_2), \ldots, \mathcal{S}_{U_k}(V_k)$ such that

$$\mathcal{S}_X(Y) \subset \mathcal{S}_{U_1}(V_1) \cup \mathcal{S}_{U_2}(V_2) \cup \cdots \cup \mathcal{S}_{U_k}(V_k).$$

2. Maps between problem and solution spaces

In this section, we study the analysis of map between between problem spaces and solution spaces. We examine how the notion of *boundedness* and *compactness* are preserved under the map.

Definition 2.1. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces. We say f is *continuous* if and only if for any subspace $\mathcal{P}_R(U) \subseteq \mathcal{P}_S(T)$ with complexity $\mathbb{C}[\mathcal{P}_R(U)] \geq k$ there exists a subspace $\mathcal{P}_W(Z) \subseteq \mathcal{P}_X(Y)$ with complexity $\mathcal{C}[\mathcal{P}_W(Z)] \geq k$ such that $f(\mathcal{P}_W(R)) \subseteq \mathcal{P}_R(U)$. Similarly, we say the map $f : \mathcal{S}_X(Y) \longrightarrow \mathcal{S}_S(T)$ between problem spaces is *continuous* if and only if for any subspace $\mathcal{S}_R(U) \subseteq \mathcal{S}_S(T)$ with index $\mathbb{I}[\mathcal{S}_R(U)] \geq k$ there exists a subspace $\mathcal{S}_W(Z) \subseteq \mathcal{S}_X(Y)$ with index $\mathbb{I}[\mathcal{S}_W(Z)] \geq k$ such that $f(\mathcal{S}_W(R)) \subseteq \mathcal{S}_R(U)$.

Definition 2.2. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces. We say f is *bounded* if $f(\mathcal{P}_U(T))$ is a finite subset of problems in $\mathcal{P}_S(T)$ for each bounded $\mathcal{P}_U(T) \subset \mathcal{P}_X(Y)$.

Definition 2.3. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces. We say f is *compact* if and only if $f(\mathcal{P}_X(Y))$ is *compact*.

We expose the fact that *compactness* of a map between problem spaces can be inherited from the compactness of the space on which it acts.

Theorem 2.4 (Stability theorem)**.** *Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces. If $\mathcal{P}_X(Y)$ is compact, then f is compact.*

Proof. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces and suppose the space $\mathcal{P}_X(Y)$ is *compact*. Then there exists a finite number of problems spaces $\mathcal{P}_{K_1}(L_1), \cdots, \mathcal{P}_{K_n}(L_n)$ such that

$$\mathcal{P}_X(Y) \subset \mathcal{P}_{K_1}(L_1) \cup \cdots \cup \mathcal{P}_{K_n}(L_n).$$

We observe that $f(\mathcal{P}_X(Y) \cap \mathcal{P}_{K_1}(L_1)) \subseteq f(\mathcal{P}_{K_1}(L_1)))$. Using this relation, we can put

$$f(\mathcal{P}_X(Y)) \subseteq \bigcup_{j=1}^{n} f(\mathcal{P}_X(Y) \cap \mathcal{P}_{K_j}(L_j)) \subseteq \bigcup_{j=1}^{n} f(\mathcal{P}_{K_j}(L_j)).$$

This proves that the range $f(\mathcal{P}_X(Y))$ is *compact* and hence f is also compact. □

3. Isotope and Isotope problem and solution spaces

In this section we study the notion of an *isotope* of problem and solution spaces.

Definition 3.1. Let V and U be any two problems. We say V and U are *compatible* if there exists a problem space $\mathcal{P}_X(Y)$ such that $V, U \in \mathcal{P}_X(Y)$. We denote this compatibility by $V \diamond U$ or $U \diamond V$. Similarly, we say two solutions R, S to some (possibly) distinct problems are compatible if there exists a solution space $\mathcal{S}_X(Y)$ such that $R, S \in \mathcal{S}_X(Y)$. We denote this compatibility by $R \diamond S$ or $S \diamond R$.

Definition 3.2. Let U and V be compatible problems. We say V and U admits a *merger* in the space $\mathcal{P}_X(Y)$ if there exists a problem $S \in \mathcal{P}_X(Y)$ such that $V < S$ and $U < S$ and V, U are the only maximal subproblem of S. In notation, we write $V \bowtie U = S \in \mathcal{P}_X(Y)$ or $U \bowtie V = S \in \mathcal{P}_X(Y)$. Similarly, let R and T be compatible solutions. We say R and T admits a *merger* in the space $\mathcal{S}_X(Y)$ if there exists a solution $W \in \mathcal{S}_X(Y)$ such that $R < W$ and $T < W$ and R, T are the only maximal sub-solutions of W. In notation, we write $R \bowtie T = W \in \mathcal{S}_X(Y)$ or $R \bowtie T = W \in \mathcal{P}_X(Y)$

We now launch the notion of an *isotope*.

Definition 3.3. Let $\mathcal{P}_X(Y)$ and $\mathcal{S}_X(Y)$ be the problem space and the corresponding solution space, induced by assigning solution Y to problem X. We denote an *isotope* on $\mathcal{P}_X(Y)$ as the map $\mathrm{Iso} : \mathcal{P}_X(Y) \longrightarrow \mathbb{R}$ such that

(i) $\mathrm{Iso}(V) \geq 0$ for each $V \in \mathcal{P}_X(Y)$ and
(ii) $\mathrm{Iso}(V \bowtie U) \leq \mathrm{Iso}(V) + \mathrm{Iso}(U)$ provided $U, V \in \mathcal{P}_X(Y)$ admits a merger.

A similar axiom also holds for solution spaces.

The notion of an *isotope* may not be viewed as an abstract notion. For example, if we consider a problem $V \in \mathcal{P}_X(Y)$ with solution $U \in \mathcal{S}_X(Y)$ and the induced problem space $\mathcal{P}_V(U) \subset \mathcal{P}_X(Y)$, then we can associate a number to problem V to be

$$(\mathbb{C}[\mathcal{P}_V(U)])^{\frac{1}{\mathbb{C}[\mathcal{P}_V(U)]}-1}$$

where $\mathbb{C}[\mathcal{P}_V(U)]$ as usual denotes the complexity of the space. Similarly for a solution U in the solution space $\mathcal{S}_X(Y)$, we can assign a number to the solution U to be

$$(\mathbb{I}[\mathcal{S}_V(U)])^{\frac{1}{\mathbb{I}[\mathcal{P}_V(U)]}-1}$$

where $\mathbb{I}[\mathcal{P}_V(U)]$ as usual denotes the index of the space. One could verify that these two maps satisfy the axioms of an *isotope*. In particular, an isotope is a pseudo semi-norm.

Definition 3.4. Let $\mathcal{P}_X(Y)$ and $\mathcal{S}_X(Y)$ be a problem and a corresponding solution space whose topology admits an *isotope*. A problem (resp. solution) space equipped with an isotope is an isotope problem (resp. isotope solution) space. We denote these spaces with $(\mathcal{P}_X(Y), \mathrm{Iso}(\cdot))$ and $(\mathcal{S}_X(Y), \mathrm{Iso}(\cdot))$, respectively.

Definition 3.5. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between isotope problem spaces. We put the isotope of f, denoted $\mathrm{Iso}(f)$, to be

$$\mathrm{Iso}(f) := \sup_{\substack{V \in \mathcal{P}_X(Y) \\ \mathrm{Iso}(V) \neq 0}} \frac{\mathrm{Iso}(f(V))}{\mathrm{Iso}(V)}.$$

We say f is bounded if $\mathrm{Iso}(f) < \infty$. A similar characterization also holds for solution spaces.

Proposition 3.1. *Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces. Then $\mathrm{Iso}(f) < \infty$ if and only if there exists an absolute constant $c > 0$ such that $\mathrm{Iso}(f(V)) \leq c\, \mathrm{Iso}(V)$ for all $V \in \mathcal{P}_X(Y)$.*

Proof. Suppose $\text{Iso}(f) < \infty$ then by definition 3.5 there exists an absolute constant $c > 0$ such that $\frac{\text{Iso}(f(V))}{\text{Iso}(V)} \leq c$ for all $V \in \mathcal{P}_X(Y)$. It implies immediately that $\text{Iso}(f(V)) \leq c\,\text{Iso}(V)$ for all $V \in \mathcal{P}_X(Y)$. Conversely, suppose $\text{Iso}(f(V)) \leq c\,\text{Iso}(V)$ for all $V \in \mathcal{P}_X(Y)$ then

$$\text{Iso}(f) := \sup_{\substack{V \in \mathcal{P}_X(Y) \\ \text{Iso}(V) \neq 0}} \frac{\text{Iso}(f(V))}{\text{Iso}(V)} < \infty.$$

□

3.1. **Bounded isotope problem spaces.** In this section, we introduce and study the notion of a *bounded* isotope problem and solution spaces.

Definition 3.6. Let $\mathcal{P}_X(Y)$ be an isotope problem space induced by providing solution Y to problem X. We say the space $\mathcal{P}_X(Y)$ is bounded if $\text{Iso}(V) < \infty$ for all $V \in \mathcal{P}_X(Y)$.

Remark 3.7. We now show that a bounded map between problem spaces maps bounded subspaces to a bounded set of problems.

Proposition 3.2. *Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between isotope problem spaces. Suppose $\mathcal{P}_K(L) \subset \mathcal{P}_X(Y)$ be a bounded sub-problem space. If* $\text{Iso}(f) < \infty$, *then $f(\mathcal{P}_K(L))$ is bounded in $\mathcal{P}_S(T)$.*

Proof. Consider the map $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ such that $\text{Iso}(f) < \infty$. Then there exists an absolute constant $c > 0$ such that Iso(f(V)) $\leq c$ $\text{Iso}(V)$ for all $V \in \mathcal{P}_X(Y)$. The requirement that $\mathcal{P}_K(L)$ is bounded implies that $\text{Iso}(V) < \infty$ for all $V \in \mathcal{P}_K(L)$. This implies that $\text{Iso}(f(V)) \leq d$ for all $V \in \mathcal{P}_K(L)$. This proves that $f(\mathcal{P}_K(L))$ is bounded in $\mathcal{P}_S(T)$. □

A similar characterization could be made and proofs can be constructed by replacing the problem spaces $\mathcal{P}_K(L)$ with the corresponding induced solution spaces $\mathcal{S}_K(L)$.

3.2. **Continuous maps between isotope problem and solution spaces.** In this section, we introduce the notion of *continuity* of a map between isotope problem spaces.

Definition 3.8. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between isotope problem spaces. We say f is *continuous* if for any $\epsilon > 0$ there exists some $\delta > 0$ such that with $\text{Iso}(V) < \delta$ then $\text{Iso}(f(V)) < \epsilon$ for $V \in \mathcal{P}_X(Y)$.

We expose the relationship that exists between *continuity* and *boundedness* of maps between problem space. In fact, we show that these two seemingly disparate notions are equivalent in problem theory.

Theorem 3.9. *Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between isotope problem spaces. Then* $\text{Iso}(f) < \infty$ *if and only if f is continuous.*

Proof. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between isotope problem spaces. Suppose that $\text{Iso}(f) < \infty$, then there exists an absolute constant $c > 0$ such that $\text{Iso}(f(V)) \leq c$ $\text{Iso}(V)$ for all $V \in \mathcal{P}_X(Y)$. Let $\epsilon > 0$ and choose $\delta := \frac{\epsilon}{c}$ so that with $\text{Iso}(V) < \delta$ then $\text{Iso}(f(V)) \leq c$ $\text{Iso}(V) < c\delta = \epsilon$. This proves that f is continuous. Conversely, suppose that f is continuous and assume that f is not bounded. Then for each $n \geq 1$ there exists a sequence $\{V_n\} \subset \mathcal{P}_X(Y)$ such that $\mathrm{Iso}(f(V_n)) > n\ \mathrm{Iso}(V_n)$ for all $n \geq N_o > 0$. Put $\frac{1}{n} < \mathrm{Iso}(V_n) < 1 - \frac{1}{n}$, then (by continuity) we get $1 < n\ \mathrm{Iso}(V_n) < \mathrm{Iso}(f(V_n)) < 1$, which is absurd. □

4. **Further remarks**

In this work, we introduced a novel theoretical framework centred on abstract problem and solution spaces, alongside the innovative concept of isotopes and isotope spaces. The theory extends classical notions such as boundedness, continuity, and compactness into a more generalized setting tailored for these spaces. We established that boundedness and continuity are equivalent within the scope of this framework and proposed a new interpretation of compactness based on finite coverings, distinct from classical sequential compactness. Additionally, we defined the notion of isotopes, which, while analogous to operator norms, possess unique properties that reflect the underlying structure of the problem space. This foundational exposition is designed to offer a fresh perspective on abstract mappings and their behaviour, opening potential avenues for applications in both theoretical and applied mathematics. As the theory is still in its formative stages, further exploration is required to develop its sequential compactness, refine the behaviour of mappings on intersections, and examine its potential connections to existing mathematical disciplines.

References

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Department of Mathematics, African institute for mathematical sciences, Ghana, Cape-coast