Thue-Morse Sequence

Edgar Valdebenito

November 27, 2024

ABSTRACT:
In this note we give some formulas related to Prouhet-Thue-Morse constant.

I. Introduction

The Thue-Morse sequence was discovered in 1851 by Prouhet with interest applied in number theory, also during the 20th century, it’s rediscovered by Axel Thue and by Marston Morse who applied it to the Combinatorics of words and to differential geometry respectively.

The Prouhet-Thue-Morse sequence is defined as

$$\mathrm{t} = (t_n)_{n\geq 0} = 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ \ 1\ 0\ 1\ 1\ \ 0\ 1\ 0\ 0\ \ 1\ 0\ 1\ 1\ \ldots \tag{1}$$

where

$$t_0 = 0\,,\ \ t_{2n} = t_n\,,\ \ t_{2n+1} = 1 - t_n\ ,\ \ n \geq 0 \tag{2}$$

for details see [1],[2],[3],[4].

II. Pi formulas

Recall that

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \ldots\right) \tag{3}$$

we have

$$\frac{\pi}{12} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(2 - \sqrt{3}\right)^{2n+3}}{2n+3} \sum_{k=0}^{n} (-1)^{t_{k+1}} + \int_0^{2-\sqrt{3}} \prod_{n=1}^{\infty} \left(1 - x^{2^{n+1}}\right) dx \tag{4}$$

$$\frac{\pi}{6} = \frac{2}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n\, t_n\, 3^{-n}}{2n+1} + \int_0^{1/\sqrt{3}} \left(1 + x^2\right) \prod_{n=1}^{\infty} \left(1 - x^{2^{n+1}}\right) dx \tag{5}$$

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \int_0^{1/\sqrt{3}} \left(\left(1 - x^2\right) T\left(x^2\right) - \left(1 - x^2\right)^2 T\left(x^4\right)\right) dx \tag{6}$$

where

$$T(x) = \sum_{n=0}^{\infty} x^n \, t_n \tag{7}$$

III. Prouhet-Thue-Morse Constant

The Prouhet-Thue-Morse constant is the number τ whose binary expansion is the Prouhet-Thue-Morse sequence. That is,

$$\tau = \sum_{n=0}^{\infty} \frac{t_n}{2^{n+1}} = 0.4124540336\ldots \tag{8}$$

where t_n is the Prouhet-Thue-Morse sequence.

Some formulas for τ:

$$\tau = \frac{1}{3} + \frac{1}{4} \sum_{n=0}^{\infty} \frac{t_n}{2^{2n}} \tag{9}$$

$$\tau = \frac{1}{3} + \frac{1}{15} + \frac{3}{16} \sum_{n=0}^{\infty} \frac{t_n}{2^{4n}} \tag{10}$$

$$\tau = \frac{2}{3} - \frac{1}{2} \cdot \sum_{n=0}^{\infty} 2^{-n} \left(1 - 2^{-n}\right) t_n \tag{11}$$

$$\tau = \frac{1}{3} + \frac{1}{4} \cdot \sum_{n=0}^{\infty} 2^{-n} \left(1 - 2^{-n-1}\right) \left(2\, t_n - t_{n+1}\right) \tag{12}$$

$$\tau = \frac{1 + \sqrt{2}}{2} \cdot \sum_{n=0}^{\infty} \left(\sqrt{2}\right)^{-n} \left(1 - \left(\sqrt{2}\right)^{-n-1}\right) \left(\sqrt{2}\, t_n - t_{n+1}\right) \tag{13}$$

$$\tau = \sum_{n=0}^{\infty} 2^{-n-2} \sum_{k=0}^{n} t_k \tag{14}$$

$$\tau = \frac{3}{8} \cdot \sum_{n=1}^{\infty} 2^{-2n} \cdot \sum_{k=1}^{n} 2^k \, t_k \tag{15}$$

$$\tau = \frac{3}{8} + \frac{1}{4} \cdot \sum_{n=0}^{\infty} 2^{-2^{n+1}} \prod_{k=0}^{n} \left(1 - 2^{-2^k}\right) \tag{16}$$

$$\tau = \frac{1}{4} \prod_{n=1}^{\infty} \left(1 + \frac{t_{n+1}}{\sum_{k=1}^{n} 2^{n-k+1} \, t_k}\right) \tag{17}$$

$$4\,\tau^2 = \sum_{n=0}^{\infty} \frac{t_n}{2^{2n}} + \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} t_k \, t_{n-k+1} \tag{18}$$

$$\tau = \frac{3}{4} \sum_{n=0}^{\infty} (-1)^n \, 2^{-n} \sum_{k=0}^{n} (-1)^k \, t_k \tag{19}$$

$$\tau = \frac{1}{3} + \frac{3}{8} \sum_{n=0}^{\infty} (-1)^n \, 2^{-n} \sum_{k=0}^{n} (-1)^k \, 2^{-k} \, t_k \tag{20}$$

$$\tau = \frac{1}{3} + \frac{3}{8}\sum_{n=0}^{\infty}(-1)^n\, 2^{-n}\sum_{k=0}^{\lfloor n/2 \rfloor} t_k \tag{21}$$

$$\tau = \frac{1}{3} + \frac{1}{8}\cdot\sum_{n=0}^{\infty} 2^{-n}\cdot\sum_{k=0}^{n} 2^{-k}\, t_k \tag{22}$$

$$\tau = \frac{1}{3} + \frac{1}{8}\cdot\sum_{n=0}^{\infty} 2^{-n}\cdot\sum_{k=0}^{\lfloor n/2 \rfloor} t_k \tag{23}$$

$$\tau = \sum_{n=0}^{\infty} 2^{-1-2^n} + \frac{1}{2}\cdot\sum_{n=1}^{\infty}\sum_{k=2^n+1}^{2^{n+1}-1} 2^{-k}\, t_k \tag{24}$$

$$\tau = \sum_{n=0}^{\infty} 3^{-n-1}\sum_{k=0}^{n}\binom{n}{k} t_k \tag{25}$$

$$\tau = \frac{1}{3} + \sum_{n=0}^{\infty} 5^{-n-1}\sum_{k=0}^{n}\binom{n}{k} t_k \tag{26}$$

$$\tau = \frac{1}{2}\cdot\sum_{n=0}^{\infty} 3^{-n-1}\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k} t_m \tag{27}$$

$$\tau = \frac{6}{15} + \sum_{n=1}^{\infty}\frac{1}{2^{2^{n+2}}-1}\prod_{k=1}^{n}\left(2^{2^k}-1\right) \tag{28}$$

$$\tau = a(m) + b(m)\sum_{n=0}^{\infty} t_n\, 2^{-2^m n} \quad , \quad m = 0,\ 1,\ 2,\ 3,\ \ldots \tag{29}$$

where

$$a(m+1) = a(m) + \frac{2^{2^m}}{2^{2^{m+1}}-1}\, b(m) \quad , \quad a(0) = 0 \tag{30}$$

$$b(m+1) = b(m)\,\frac{2^{2^m}-1}{2^{2^m}} \quad , \quad b(0) = 1/2 \tag{31}$$

$$a(m) = \left\{0,\ \frac{1}{3},\ \frac{2}{5},\ \frac{7}{17},\ \frac{106}{257},\ \frac{27\,031}{65\,537},\ \frac{1\,771\,476\,586}{4\,294\,967\,297},\ \ldots\right\} \tag{32}$$

$$b(m) = \left\{\frac{1}{2},\ \frac{1}{4},\ \frac{3}{16},\ \frac{45}{256},\ \frac{11\,475}{65\,536},\ \frac{752\,014\,125}{4\,294\,967\,296},\ \frac{3\,229\,876\,072\,253\,041\,875}{18\,446\,744\,073\,709\,551\,616},\ \ldots\right\} \tag{33}$$

$$\tau = \frac{1}{2}\cdot\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} 2^{-c(k)\, 2^n} \tag{34}$$

where

$$c(k) = \{1,\ 7,\ 11,\ 13,\ 19,\ 21,\ 25,\ 31,\ 35,\ 37,\ 41,\ 47,\ 49,\ 55,\ 59,\ 61,\ 67,\ 69,\ 73,\ 79,\ 81,\ \ldots\} \tag{35}$$

Remarks:

$$\{2\, t_n - t_{n+1},\ n \geq 0\} = \{-1,\ 1,\ 2,\ -1,\ 2,\ 0,\ -1,\ 1,\ 2,\ 0,\ -1,\ \ldots\} \tag{36}$$

$$\left\{\sum_{k=0}^{n} t_k\ ,\ n \geq 0\right\} = \{0,\ 1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 5,\ 5,\ 5,\ \ldots\} \tag{37}$$

$$\left\{\sum_{k=1}^{n} 2^k\, t_k\,,\ n \ge 0\right\} = \{0, 2, 6, 6, 22, 22, 22, 150, 406, 406, 406, \ ...\} \tag{38}$$

$$\left\{\sum_{k=1}^{n} 2^{n-k+1}\, t_k\,,\ n \ge 0\right\} = \{0, 2, 6, 12, 26, 52, 104, 210, 422, 844, 1688, \ ...\} \tag{39}$$

$$\left\{\sum_{k=0}^{\lfloor n/2 \rfloor} t_k\, t_{n-k+1}\,,\ n \ge 0\right\} = \{0, 0, 1, 0, 1, 1, 0, 1, 2, 1, 1, \ ...\} \tag{40}$$

$$\left\{\sum_{k=0}^{\lfloor n/2 \rfloor} t_k\,,\ n \ge 0\right\} = \{0, 0, 1, 1, 2, 2, 2, 2, 3, 3, 3, \ ...\} \tag{41}$$

$$\left\{\sum_{k=0}^{n} (-1)^k\, t_k\,,\ n \ge 0\right\} = \{0, -1, 0, 0, 1, 1, 1, 0, 1, 1, 1, \ ...\} \tag{42}$$

$$\left\{\sum_{k=0}^{n} \binom{n}{k} t_k\,,\ n \ge 0\right\} = \{0, 1, 3, 6, 11, 20, 36, 64, 115, 216, 430, \ ...\} \tag{43}$$

IV. References

[1] J.P. Allouche and J. Shallit, The ubiquitous Prouhet-Thue-Morse sequence. Journal of Funny Physics, Vol.35,1997.

[2] J. Shallit, The Ubiquitous Thue Morse sequence. https://cs.uwaterloo.ca/shallit/Talks/green3.pdf, 2010.

[3] J.P. Allouche, A. Arnold, J. Berstel, S. Brlek, W. Jockusch, S. Plouffe, and B.E. Sagan, A relative of the Thue-Morse sequence, Discrete Math. 139, 1995.

[4] M. Dekking, Transcendance du nombre de Thue-Morse, C. R. Acad. Sci. Paris Sér. I 285, 1977.

[5] J.-Y. Yao, Généralisations de la suite de Thue-Morse, Ann. Sci. Math. Québec 21, 1997.

[6] L. Tóth, Transcendental Infinite Products Associated with the ±1 Thue-Morse Sequence. J. Integer Seq. 23, 2020.

[7] OEIS: Sequences: A010060,A048707,A000069,A001969.