QUANTUM CORRECTION TO THE SYNCHROTRON PHOTON SPECTRUM

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Abstract

We calculate the quantum corrections to synchrotron radiation by method developed by Schwinger and Tsai. The traditional calculation method working with the particle wave functions is here replaced by the mass operator approach avoiding the particle final states and the integration over photon angular distribution. The algorithm of the calculation of the quantum corrections to the synchrotron radiation consists in the evaluation of the forward Compton scattering process in the external magnetic field and then in application of the optical theorem to obtain the angular and frequency distribution.

1 Introduction

Around year 1947 Floyd Haber, a young staff member and technician in the laboratory of prof. Pollock, visually observed radiation of electrons moving circularly in the magnetic field of the chamber of an accelerator (Ternov, 1994). It occurred during adjustment of cyclic accelerator-synchrotron which accelerated electrons up to 100 MeV (Elder et al., 1948). The radiation was observed as a bright luminous patch on the background of the chamber of the synchrotron. It was clearly visible in the daylight. In this way the "electron light" was experimentally revealed for the first time as the radiation of relativistic electrons of large centripetal acceleration. The radiation was identified with the Ivanenko and Pomeranchuk radiation, or with the Schwinger radiation and later was called the synchrotron radiation since it was observed for the first time in synchrotron. The radiation was considered as the mysterious similarly to the Roentgen mysterious x-rays.

A number of theoretical studies on the emission of a relativistic accelerating electron had been carried out long before the cited experiment. The first steps in this line was treated by Liénard (1898). He used the Larmor formula and extended it to the high-velocity particles. He also received the total radiation of an electron following a circle of an circumference $2\pi R$.

In modern physics, Schwinger (1945, 1949) used the relativistic generalization of the Larmor formula to get the total synchrotron radiation. Schwinger also obtained the spectrum of the synchrotron radiation from the method which was based on the electron work on the electromagnetic field, $P = -\int(\mathbf{j}\cdot\mathbf{E})d\mathbf{x}$, where the intensity of electric field he expressed as the subtraction of the retarded and advanced electric field of a moving charge in a magnetic field, $\mathbf{E} = \frac{1}{2}(\mathbf{E}_{ret} - \mathbf{E}_{adv})$, (Schwinger, 1949).

Schott in 1907 was developing the classical theory of electromagnetic radiation of electron moving in the uniform magnetic field. His calculation was based on the Poynting vector. The goal of Schott was to explain the spectrum of radiation of atoms. Of course the theory of Schott was unsuccessful because only quantum theory is adequate to explain the emission spectrum of atoms. On the other hand the activity of Schott was not meaningless because he elaborated the theory of radiation of charged particles moving in the electromagnetic field. His theory appeared to be only of the academical interest for 40 years. Then, it was shown that the theory and specially his formula has deep physical meaning and applicability. His formula is at the present time the integral part of the every textbook on the electromagnetic field.

The classical derivation of the Schott formula is based on the Poynting vector $\mathbf{S}$ (Sokolov et al. 1966) $\mathbf{S} = (c/4\pi)(\mathbf{E}\times\mathbf{H})$, where $\mathbf{E}$ and $\mathbf{H}$ are intensities of the electromagnetic field of an electron moving in the constant magnetic field, where the magnetic field is in the direction of the axis z. In this case electron moves along the circle with radius R and the electromagnetic field is considered in the wave zone and in a point with the spherical coordinates r, θ, φ. In this case it is possible to show that the nonzero components of the radiated field are $-H_\theta = E_\varphi, H_\theta = E_\theta$ (Sokolov et al. 1966). They are calculated from the vector potential $\mathbf{A}$ which is expressed as the Fourier integral.

The circular classical trajectory of the electron is created by the Lorentz force $F = (e/c)(\mathbf{v}\times\mathbf{H})$. The trajectory is stationary when the radiative reaction is not considered. The radiative reaction causes the transformation of the circular trajectory to the spiral trajectory. In quantum mechanics, the trajectory is stationary when neglecting the interaction of an electron with the vacuum field. The interaction of an electron with the vacuum field, causes the electron jumps from the higher energetic level to the lower ones. In quantum electrodynamics description of the motion of electron in a homogeneous magnetic field, the stationarity of the trajectories is broken by including the mass operator into the wave equation. Then, it is possible from the mass operator to derive the power spectral formula (Schwinger, 1973). Different approach is involved in the Schwinger et al. article (1976).

We calculate the quantum corrections to synchrotron radiation by method developed by Schwinger and Tsai. The traditional calculation method working with the particle wave functions is here replaced by the mass operator approach avoiding the particle final states and the integration over photon angular distribution. The algorithm of the calculation of the quantum corrections to the synchrotron radiation consists in the evaluation of the forward Compton scattering process in the external magnetic field and then in application of the optical theorem to obtain the angular and frequency distribution.

We answer the question whether the quantum effects are significant correction to the power spectrum of radiation through the modification of the coefficient attached to the speed of the charged particle. The quantum correction, if significant, might have impact on the construction of the next generation of synchrotrons and on the theory of non thermal radiation from the magnetic stars.

The traditional calculation method of Latal and Erber (2003) working with the particle wave functions is here replaced by the mass operator approach avoiding the particle final states and the integration over photon angular distribution.

2 The quantum corrections to synchrotron radiation

The algorithm of the calculation of the quantum corrections to the synchrotron radiation consists in the evaluation of the forward Compton scattering process in the external magnetic field and then in application of the optical theorem to obtain the angular and frequency distribution $P(\omega, t)$. The used calculation method has the following advantages.

1. It is used the Green function and only the electron final states are summed from the beginning.

2. The proper time method enables to solve the problem in the coordinate representation. The resulting quantum expansion is similar to its classical counterpart, so that the quantum modification can be easily identified.

3. $P(\omega, t)$ can be evaluated exactly in a one-parameter integral.

We will see that there is no indication of a significant second-order quantum correction. Since the sensitive quantum correction do not depend on the spin, we can consider the process with the spin-0 particle. We choose the homogenous magnetic field $\mathbf{H}$ to be along the $+z$-axis and particles to be moving in the xy plane.

We use the forward Compton scattering to compute the spectral density of radiation $P(\omega, t)$. Here, the scattering amplitude describes the scattering of an outgoing real photon with momentum k^μ and polarization $e_\lambda^{\mu *}$, $\lambda = 1, 2$ by a charged particle of energy E. The radiative power $P(\omega, t)$ is related through the optical theorem to the imaginary part of the forward scattering.

The action for the forward Compton scattering is of the form [Schwinger, 1973; 1989]:

$$W = \frac{1}{2}\int (dx)(dx')\varphi(x)(2\Pi - k)_\mu e\hat{q}\mathcal{A}^\mu_\lambda(x)\Delta^A_+(x-x')e\hat{q}\mathcal{A}^{\nu *}_\lambda(x')(2\Pi - k)_\nu\varphi(x') =$$

$$= -\frac{1}{2}\int (dx)(dx')\varphi(x)\mathcal{M}_\lambda(x,x')\varphi(x'), \qquad (1)$$

where $\hat{q}$ is so called the charge matrix (with $q_{11} = 0, q_{12} = -i, q_{21} = i, q_{22} = 0$) (Schwinger, 1969; 1970) and

$$\mathcal{A}^\mu_\lambda(x) = (d\omega_k)^{1/2}e^{ikx}\varepsilon^\mu_\lambda, \qquad (2)$$

$$\mathcal{M}_\lambda(x,x') = -e^2 d\omega_k(2\Pi - k)_\mu\varepsilon^\mu_\lambda e^{ikx}\Delta^A_+(x,x')e^{-ikx}(2\Pi - k)_\nu\varepsilon^{\nu *}_\lambda \quad =$$

$$\varepsilon^\mu_\lambda\mathcal{M}_{\mu\nu}(x,x')\varepsilon^{\nu *}_\lambda, \qquad (3)$$

and field $\varphi(x)$ concerns spin-0 particles of the Klein-Gordon equation

$$(m^2 + \Pi^2)\varphi(x) = 0. \qquad (4)$$

The corresponding Green function equation to eq. (4) is

$$(m^2 + \Pi^2)\Delta^A_+ = \delta(x - x'). \qquad (5)$$

Furthermore $d\omega_k$ is the invariant phase space measure for the photon

$$d\omega_k = \frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{2\omega} = \frac{\omega d\omega d\Omega}{16\pi^3}; \quad \omega = |k^0| \equiv |\mathbf{k}| \qquad (6)$$

and Π^μ is defined by

$$\Pi^\mu = \frac{1}{i}\partial^\mu - e\hat{q}A^\mu, \qquad (7)$$

where A^μ is the vector potential for the external magnetic field.

For the unpolarized photons we may sum over the polarization using relations

$$\sum \varepsilon^\mu_\lambda\varepsilon^{\nu *}_\lambda = g^{\mu\nu} - \frac{(k^\mu\bar{k}^\nu - k^\nu\bar{k}^\mu)}{k\bar{k}} \qquad (8)$$

with

$$k^\mu = \omega(1, \mathbf{n}); \qquad \bar{k}^\mu = \omega(1, -\mathbf{n}), \qquad (9)$$

where $\mathbf{n}$ is a unit vector along the direction of propagation.

The synchrotron radiation is determined only by the imaginary part of $\mathcal{M}_\lambda$. We will compute the matrix element of $\mathcal{M}_\lambda$ for a particle state, so that we replace Π^2 by $-m^2$, wherever Π^2 appears to the left or right hand side of $\mathcal{M}_\lambda$. It may be easily seen that Im $\mathcal{M}_\lambda$ is gauge invariant because

$$k^\mu(\text{Im } \mathcal{M}_{\mu\nu}) \equiv \text{Im } \left\{-e^2 d\omega_k(2k\Pi - k^2)e^{ikx}\Delta_+^A(x,x')e^{-ikx'}(2\Pi - k)_\nu\right\} =$$

$$\text{Im } \left\{e^2 d\omega_k\left[(-k+\Pi)^2 - \Pi^2\right]e^{ikx}\Delta_+^A(x,x')e^{-ikx'}(2\Pi - k)_\nu\right\} \to$$

$$\to \text{Im } \left\{e^2 d\omega_k(2\Pi - k)_\nu\right\} = 0 \tag{10}$$

since the diagonal matrix element of $(2\Pi - k)_\nu$ is a real number.

In the derivation of the gauge invariance we have used

$$(m^2 + \Pi^2)\varphi(x) = 0 \tag{11}$$

$$(m^2 + \Pi^2)\Delta_+^A(x,x') = \delta(x,x') \tag{12}$$

and

$$e^{-ikx}\left[(-k+\Pi)^2 + m^2\right]e^{ikx}\Delta_+^A(x,x') = (m^2+\Pi^2)\Delta_+^A(x,x') = \delta(x,x'). \tag{13}$$

After the polarization summation, we can write

$$\mathcal{M}(x,x') \equiv \sum_{\lambda=1}^{2}\mathcal{M}_\lambda(x,x') \to -4e^2 d\omega_k \Pi^\mu e^{ikx}\Delta_+^A(x,x')e^{-ikx'}\Pi_\mu. \tag{14}$$

It is well known that in terms of matrix notation in the coordinate representation, the Green function can be expressed as

$$\Delta_+^A(x,x') = \langle x|\frac{1}{m^2+\Pi^2 - i\varepsilon}|x'\rangle = \langle x|i\int_0^\infty ds\ e^{-is\mathcal{H}}|x'\rangle, \tag{15}$$

where

$$\mathcal{H} = m^2 + \Pi^2 \tag{16}$$

so that

$$\mathcal{M}(x,x') = \langle x|\mathcal{M}|x'\rangle, \tag{17}$$

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$$\mathcal{M}(x,x') = \langle x| - 4ie^2 d\omega_k\int_0^\infty ds\ \Pi^\mu e^{ikx}e^{-i\mathcal{H}}e^{-ikx}\Pi_\mu|x'\rangle. \tag{18}$$

In the standard proper-time formulation of quantum mechanics, the time evolution of an operator Δ is defined by equation

$$A(s) = e^{is\mathcal{H}} A e^{-is\mathcal{H}}. \tag{19}$$

Using

$$e^{A+B} = e^A e^B e^{\frac{1}{2}[A,B]} \tag{20}$$

for $[A, B]$ commuting with both A and B we have

$$e^{-is\mathcal{H}} = e^{-\frac{1}{2}is\mathcal{H}} e^{-\frac{1}{2}is\mathcal{H}} \tag{21}$$

and for Π it is:

$$\mathcal{M} \to -4ie^2 d\omega_k \int_0^\infty ds\ \Pi^\mu(s/2) e^{ikx(s/2)} e^{-ikx(-s/2)} \Pi_\mu(-s/2). \tag{22}$$

To involve the Compton scattering, we must supplement to the original spin-0-action

$$\int (dx)[K(x)\varphi(x) - \frac{1}{2}\varphi(x)(\Pi^2 + m^2)\varphi(x)] \tag{23}$$

the term concerning the Compton scattering. The resulting equation of motion is then

$$(\Pi^2 + m^2 + \mathcal{M})\varphi(x) \equiv (\Pi_\perp^2 - E'^2 + m^2 + \mathcal{M})\varphi(x) = 0, \tag{24}$$

where

$$\Pi_\perp^2 = \Pi_x^2 + \Pi_y^2 \tag{25}$$

and

$$E'^2 \approx (E^2 + \mathcal{M}')^{1/2} \approx E + \frac{1}{2E}\mathcal{M}', \tag{26}$$

where E' is the new energy eigenvalue of the system and $\mathcal{M}'$ is the diagonal matrix element of E for particle state of energy E. The total decay rate is then identified to be

$$\gamma = -2\text{Im}\ E' \approx -\frac{1}{E}\text{Im}\ \mathcal{M}', \tag{27}$$

which follows from the time dependent of the wave function $\exp\{-iE't\}$ and from the definition of the decay rate.

$$e^{-\gamma t} \equiv |e^{-iE't}|^2 = e^{-2(\text{Im}\ E')t}. \tag{28}$$

We define the relation between γ and the spectral and angular distribution of the radiative power $P(\omega, \Omega)$ by equation

$$\gamma = \frac{d\omega}{\omega} d\Omega P(\omega, \Omega). \tag{29}$$

The comparison of eq. (29) with eq. (27) then gives

$$P(\omega, \Omega) = -\frac{\omega}{E}(\mathrm{Im}\ \mathcal{M}' d\omega d\Omega) \quad =$$

$$-\frac{\omega}{E}\mathrm{Im}\ \left(-4ie^2 d\omega_k \int_0^\infty ds\ \Pi^\mu(s/2) e^{ikx(s/2)} e^{-ikx(-s/2)} \Pi_\mu(-s/2)\right) d\omega d\Omega. \tag{30}$$

Then, using

$$d\omega_k = \frac{\omega d\omega d\Omega}{16\pi^3} \tag{31}$$

we get

$$P(\omega, \Omega) = \frac{e^2\omega^2}{4\pi^3} \frac{1}{E} \mathrm{Re}\ \left\langle \Pi(s/2) e^{ikx(s/2)} e^{-ikx(-s/2)} \Pi(-s/2) \right\rangle, \tag{32}$$

where we have introduced $\langle \cdots \rangle$ for diagonal matrix element of $\mathcal{M}$. Let us remark that the clasical formula for $P(\omega, \Omega)$ can be achieved by transformation $\Pi^0 \to E$, $\mathbf{\Pi} = E\mathbf{v}$, $s = \tau/2E$.

To evaluate $P(\omega, \Omega)$, we use eq. (19), or, its equivalent transcription called Heisenberg equation

$$\frac{d}{ds} A(s) = \frac{1}{i}[A(s), \mathcal{H}] \tag{33}$$

The equation (33) generates equations for $x(s)$ and $\Pi(s)$ in the form

$$\frac{d}{ds} x(s) = 2\Pi(s) \tag{34}$$

$$\frac{d}{ds} \Pi(s) = 2e\hat{q}F\Pi \tag{35}$$

with the corresponding solution

$$x(s) = x + \left[\frac{e^{2e\hat{q}Fs} - 1}{e\hat{q}F}\right] \Pi \tag{36}$$

with

$$\Pi(s) = e^{2e\hat{q}Fs}\Pi. \tag{37}$$

It is no problem to get the commutation relations

$$[x_\mu, \Pi_\nu] = ig_{\mu\nu} \tag{38}$$

$$[\Pi_\mu, \Pi_\nu] = ie\hat{q}F_{\mu\nu} \tag{39}$$

$$\left[ikx\left(\frac{s}{2}\right), -ikx\left(-\frac{s}{2}\right)\right] = -\frac{i}{eH}(\sin 2z - 2z)k_\perp^2. \tag{40}$$

Using the last relation and identity

$$e^a e^b = e^{a+b} e^{\frac{1}{2}[a,b]} \tag{41}$$

for both a, b commuting with $[a, b]$ we get

$$e^{ikx(s/2)} e^{-ikx(-s/2)} \quad =$$

$$e^{-2is\omega E} \exp\left\{\frac{2i}{eH} \sin z (k\Pi)_\perp\right\} \exp\left\{-\frac{i\omega^2}{eH}(\sin z \cos z - z)\sin^2\Theta\right\}, \tag{42}$$

where Θ is the angle between propagation direction of the photon and $+z$ direction and

$$(k\Pi)_\perp \equiv k_x \Pi_x + k_y \Pi_y \tag{43}$$

$$z = seH. \tag{44}$$

The further identity

$$be^a = e^a(b + [b, a]) \tag{45}$$

for a, b commuting with $[a, b]$ can be used to obtain the result

$$P(\omega, \Omega) = \frac{e^2\omega^2}{4\pi^3} \frac{1}{E} \quad \times$$

$$\mathrm{Re} \int_0^\infty \left\langle e^{-2is\omega E} \exp\left\{i\frac{\xi}{\omega E}(k\Pi)_\perp\right\} \exp\left\{-i\frac{\omega^2}{eH}(\sin z \cos z - z)\sin^2\Theta\right\} J\right\rangle, \tag{46}$$

where

$$J \equiv \left[\Pi^\mu(s/2) + z \sin z\left(ke^{-e\hat{q}Fs}\frac{e\hat{q}F}{eH}\right)\right]\Pi_\mu(-s/2) \quad =$$

$$\cos z \Pi_\perp^2 - E^2 - ieH \sin 2z + 2\sin z\left[\sin 2z (k\Pi)_\perp + \cos 2z\left(k\frac{e\hat{q}F}{eH}\Pi\right)_\perp\right] \tag{47}$$

$$\xi = 2\frac{\omega}{\omega_0}\sin z \tag{48}$$

$$\omega_0 = \frac{eH}{E} \tag{49}$$

and where ω_0 is so called the synchrotron frequency.

Now, we need to evaluate the diagonal matrix element

$$\langle\cdots\rangle \equiv \langle n|\cdots|n\rangle \tag{50}$$

occuring in (46), where $|n\rangle$ is the state characterized by the principal quantum number n. First we calculate the ingredient of $\langle\cdots\rangle$ such as $\langle n|\Pi_\perp^2|n\rangle$, $\langle|(k\Pi)_\perp|\rangle$, ... and then we sum up them.

For a charged particle moving in a constant magnetic field $\mathbf{H}$, the covariant derivatives satisfy the commutation relations

$$[\Pi_1, \Pi_2] = ie\hat{q}H. \tag{51}$$

Introducing the non-Hermitean operators

$$y = \frac{\Pi_+}{(eH)^{1/2}} \equiv \frac{1}{(2eH)^{1/2}}(\Pi_1 + i\Pi_2) \tag{52}$$

$$y^+ = \frac{\Pi_-}{(eH)^{1/2}} \equiv \frac{1}{(2eH)^{1/2}}(\Pi_1 - i\Pi_2), \tag{53}$$

we have for y, y^+

$$[y, y^+] = 1, \tag{54}$$

which is the commutation relation for the harmonic oscillator system and it immediately means that the matrix element of y and y^+ between states $|n\rangle$ and $|n'\rangle$ are

$$\langle n|y|n'\rangle = n'^{1/2}\delta_{n+1,n'} \tag{55}$$

$$\langle n|y^+|n'\rangle = n^{1/2}\delta_{n-1,n'}. \tag{56}$$

Introducing quantities $k_\pm$

$$k_\pm = \sqrt{\frac{1}{2}}(k_1 + ik_2) \tag{57}$$

we can write easily:

$$\Pi_\perp^2 = eH(2y^+y + 1) \tag{58}$$

$$(k\Pi)_\perp = (eH)^{1/2}(k_+y^+ + k_-y) \tag{59}$$

$$\left[k\left(\frac{e\hat{q}F}{eH}\right)\Pi\right]_\perp = i\hat{q}(eH)^{1/2}(k_+y^+ - k_-y). \tag{60}$$

The diagonal matrix element of $\Pi_\perp^2$ then is

$$\langle n|\Pi_\perp^2|n\rangle = \langle n|(2y^+y+1)eH|n\rangle = (2n+1)eH. \tag{61}$$

On the other hand we have for the matrix containing $(k\Pi)_\perp$ the following procedure:

$$\langle n|\exp\left\{\frac{i\xi}{\omega E}(k\Pi)_\perp\right\}|n\rangle = \langle n|\exp\left\{i\chi(k_\perp y^+ + k_- y)\right\}|n\rangle \quad =$$

$$\langle n|\exp\left\{-\frac{1}{2}\chi^2(k_\perp k_-\right\}\exp\left\{i\chi k_+ y^+\right\}\exp\left\{i\chi k_- y\right\}|n\rangle \quad =$$

$$e^{-\kappa/2}\langle n|\sum_{m=0}^{\infty}\frac{(i\chi k_+)^m}{m!}(y^+)^m\sum_{l=0}^{\infty}\frac{(i\chi k_-)^l}{l!}(y)^l|n\rangle \quad =$$

$$e^{-\kappa/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{(\kappa)^m}{m!}\frac{n!}{(n-m)!} = e^{-\kappa/2}L_n(\kappa), \tag{62}$$

where we used the identity

$$e^a e^b = e^{a+b}e^{\frac{1}{2}[a,b]} \tag{63}$$

for a, b, commuting with $[a,b]$, and

$$\chi = \frac{\xi}{\omega E}\sqrt{eH} \tag{64}$$

$$\kappa = 2\left(\frac{\omega^2}{eH}\right)\sin^2\Theta\sin^2 z. \tag{65}$$

The symbol $L_n(\kappa)$ is the Laguerre polynomial $L_n^\alpha(\alpha=0)$ where L_n^α is defined as

$$L_n^\alpha(\kappa) = \sum_{m=0}^{\infty}\frac{(-1)^m}{(n-m)!}\frac{(n+\alpha)!}{(m+\alpha)!}\frac{\kappa^m}{m!}. \tag{66}$$

The generalization of eq. (62) can written in the form

$$\langle n|\exp\left[\frac{i\xi}{\omega E}(k\Pi)_\perp\right]f((k\Pi)_\perp)|n\rangle = f\left(\omega E\frac{1}{i}\frac{d}{d\xi}\right)\langle n|\exp\left[\frac{i\xi}{\omega E}(k\Pi)_\perp\right]|n\rangle. \tag{67}$$

Finally, by the procedure similar to (62) we obtain

$$\langle n|\exp\left[\frac{i\xi}{\omega E}(k\Pi)_\perp\right]\left[k\left(\frac{e\hat{q}F}{eH}\right)\Pi\right]_\perp|n\rangle \quad =$$

$$-\frac{1}{2}\omega\omega_0\xi\sin^2\Theta\langle n|\exp\left[\frac{i\xi}{\omega E}(k\Pi)_\perp\right]|n\rangle. \tag{68}$$

Using eqs. (62), (67) and (68), we have

$$\exp\left[\frac{i\xi}{\omega E}(k\Pi)_\perp\right] \to e^{-\frac{\kappa}{2}}L_n(\kappa) \tag{69}$$

$$(k\Pi)_\perp \exp\left[\frac{i\xi}{\omega E}(k\Pi)_\perp\right] \to \omega E\left(\frac{1}{i}\frac{d}{d\xi}\right)\left(e^{-\frac{\kappa}{2}}L_n(\kappa)\right) \tag{70}$$

$$\left(k\frac{e\hat{q}F}{eH}\Pi\right)_\perp \exp\left[\frac{i\xi}{\omega E}(k\Pi)_\perp\right] \to -\omega^2\sin^2\Theta\sin z e^{-\frac{\kappa}{2}}L_n(\kappa) \tag{71}$$

and we get the final compact form for the angular and frequency distribution of the radiation power $P(\omega,\Omega)$ for unpolarized photons:

$$P(\omega,\Omega) = \frac{e^2}{4\pi^3}\frac{\omega^2}{\omega_0^2}\mathrm{Re}\int_0^\infty dz \exp\left[-2i\frac{\omega}{\omega_0}z\right]\exp\left[-i\frac{\omega^2}{eH}(\sin z\cos z - z)\sin^2\Theta\right]$$

$$\left[\beta^2\cos z - 1 - i\frac{eH}{E^2}\sin 2z + \frac{2\omega}{E}\sin^2 z\left(2\sin z\frac{1}{i}\frac{d}{d\xi} - \frac{\omega}{E}\sin^2\Theta\cos 2z\right)\right]e^{-\frac{\kappa}{2}}L_n(\kappa), \tag{72}$$

where ξ is defined by eq. (48) and κ by eq. (65).

Now, we perform the z-integration by parts, using

$$\frac{1}{i}\frac{d}{d\xi} = \left(2i\frac{\omega}{\omega_0}\cos z\right)^{-1}\frac{d}{dz} \tag{73}$$

and noting that the boundary term is purely imaginary, so that it can be discarded, which means, that we can use replacement

$$\sin^2 z\cos z\left(\frac{1}{i}\frac{d}{d\xi}\right) \to \sin^2 z\left(1 - \frac{\omega}{E}\sin^2\Theta\sin^2 z + i\frac{\omega_0}{\omega}\cot z\right). \tag{74}$$

Then, upon substituting eq. (74) into eq. (72), we obtain

$$P(\omega,\Omega) = \frac{e^2}{4\pi^3}\frac{\omega^2}{\omega_0^2}\mathrm{Re}\int_0^\infty dz \exp\left[-2i\frac{\omega}{\omega_0}z\right]\exp\left[-i\frac{\omega^2}{eH}(\sin z\cos z - z)\sin^2\Theta\right] \quad \times$$

$$\left[\beta^2\cos 2z - 1 + \frac{4\omega}{E}\sin^2 z\left(1 - \frac{\omega}{2E}\sin^2\Theta\right) + i\frac{eH}{E^2}\sin 2z\right]e^{-\frac{\kappa}{2}}L_n(\kappa), \tag{75}$$

which is an exact result, valid to the first order in the fine structure constant α for arbitrary strong magnetic field.

2.1 High-energy weak-field limit

The most practical applications comes from the situation with the high-energy particles i.e. for condition $E/m \gg 1$ moving in the weak magnetic field with $eH/m^2 \ll 1$. In this limiting case the principal quantum number is very large. It can be shown that the radiation is concentrated in a very narrow cone of angular range of the order m/E in the forward direction, corresponding to

$$\cos \Theta^2 \sim \left(\frac{m}{E}\right)^2, \tag{76}$$

which implies that only a small part of the electron trajectory is effective in producing the radiation observed in a given direction

$$z = \frac{1}{z}\omega_0 \tau \sim \frac{m}{E} \tag{77}$$

and that very high frequency must be radiated

$$\omega \gg \omega_0. \tag{78}$$

Quantum corrections are modifications to this scenario. After some approximations (Schwinger et al., 1978), it can be shown that

$$P(\omega) = \frac{e^2}{4\pi^2} \frac{1}{3^{1/2}} \omega \frac{m^2}{E^2} \int_\xi^\infty dt K_{5/3}(t), \tag{79}$$

where

$$\xi = \frac{\omega}{\omega_0} \frac{1}{1 - \frac{\omega}{E}} \tag{80}$$

$$\omega_c = \frac{3}{2}(1-\beta^2)^{-3/2}\omega_0 \tag{81}$$

and the total power radiated is

$$P = \int_0^E d\omega P(\omega) = \frac{e^2}{4\pi^2} \frac{m^2}{3^{1/2}} \int_0^E \frac{d\omega}{E} \frac{\omega}{E} \int_\xi^\infty dt K_{5/3}(t) \quad =$$

$$\frac{\alpha}{3^{1/2}\pi} m^2 \mathcal{Y}^2 \int_0^\infty \frac{d\xi \xi}{(1+\mathcal{Y}\xi)^3} \int_0^\infty dt K_{5/3}(t), \tag{82}$$

where

$$\xi = \frac{1}{\mathcal{Y}} \frac{\omega}{E} \left(1 - \frac{\omega}{E}\right)^{-1} \tag{83}$$

$$\mathcal{Y} = \frac{3}{2} \frac{eH}{m^2} \frac{E}{m} = \frac{3}{2} \frac{(e\hbar/mc)H}{mc^2} \frac{E}{mc^2} \tag{84}$$

and for $\mathcal{Y} \ll 1$ we have

$$P \sim \frac{e^2}{4\pi} \frac{2}{3} \omega_0^2 \left(\frac{E}{mc^2}\right)^4 \left(1 - \frac{55(3^{1/2})}{24}\mathcal{Y} + \frac{56}{3}\mathcal{Y}^2 + \cdots\right), \tag{85}$$

from which follows that quantum corrections are controlled by the factor $\mathcal{Y}$ and that there is no evidence for the second-order quantum correction to be more important than the first correction.

3 Discussion

The calculus used by Latal and Erber (2003) involves the complex wave functions and by the long procedures of summing over the particles final states. The main complication enters when one attempts to integrate over the photon angular distribution, because of integration of the square of associated Laguerres polynomials in which both their orders and arguments depend on the angle. The mass operator approach (Schwinger, 1973; 1989) for calculating the synchrotron radiation is avoiding the usage of particle wave functions, the summation of the particles final states, and the integration over the photon angular distribution. However, this method involved too much information from the beginning, so that the needed information (the frequency distribution) has to be projected out explicitly (Schwinger, 1973; 1989) . It also suffers from the appearance of a complicated trigonometric function in the exponential.

We here proposed a proper time method (Schwinger, 1951) to compute this process. We first evaluated the forward Compton scattering process in the external magnetic field, and then applied the optical theorem to obtain the angular and frequency distribution, $P(\omega, \Omega)$. This calculation method has the following attractive characteristics. (1) Only the electron final states are summed over from the beginning, through the use of Greens function; (2) the proper time method enables us to solve the problem in the coordinate representation, and the resulting quantum expression is similar to its classical counterpart, so that the quantum modification can be easily identified; and (3) can be $P(\omega, \Omega)$ evaluated exactly in a one-parameter (proper time) integral form. We find that the quantum correction to the classical spectrum is essentially the same as settled before. There is no indication of a significant second-order quantum correction.

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