Modified Collatz Method

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Abstract In this paper, a modification of Collatz method is going to be presented, It is going to be proved that iterative implementation of the corresponding modified Collatz function on any natural number will eventually lead to the number 1. An alternative definition of Collatz and modified Collatz functions are going to be presented, and some interesting results that are obtained from it are going to be briefly analyzed.

1 Introduction

The Collatz method [1-3] is concerned with the following arithmetic procedure that is applied on natural numbers – if the number is odd, multiply it by 3 and add 1, and if the number is even, divide it by 2. This operation can be expressed by Collatz function:

$$C(x)=\begin{cases} 3\mathrm{x}+1 & \text{if } x\equiv 1\, mod(2), \\ \dfrac{x}{2} & \text{if } x\equiv 0\, mod(2). \end{cases}$$

Collatz conjecture states that iterations of the Collatz function starting from any natural number will eventually lead to the number 1, and then it will cycle between numbers 1, 4, 2, 1 … The conjecture, that is also known under many different names [1-3], is still not proven.

Here, a small modification of the Collatz method is going to be proposed. In this case it is easy to prove that iterations of the corresponding modified Collatz function will eventually lead to the number 1. Then, the Collatz and modified Collatz methods, and corresponding functions are going to be defined alternatively and some consequences of that modified definitions are going to be presented.

2 Modified Collatz function

Modified Collatz function is defined by the following expression:

$$C_m(x)=\begin{cases} 3x\pm 1 & if\ x\equiv \pm 1\ mod(4), \\ \dfrac{x}{2} & if\ x\equiv 0\ mod(2). \end{cases}$$

In other words, if the odd number produces a reminder equal to 1 when it is divided by 4, number is multiplied by 3 and 1 is *added* (like in the original Collatz function), but when it produces reminder 3 when it is divided by 4, then the number is multiplied by 3 and 1 is *subtracted.* (In the case of even numbers, nothing is changed.) This small change will lead toward very simple proof that starting from any natural number, iterations of modified Collatz function will eventually lead to 1. The reason is the following – after implementation of the function C_m on an odd number, the number that is going to be obtained is always even and divisible by 4 (if not some higher power of 2). This means that after one step of C_m applied on an odd natural number $k > 1$, and then several (at least 2) steps are iterated on an even number $3k \pm 1$, a number f is obtained, such that the following holds:

$$f \leqslant \frac{3k\pm 1}{4} < k\ ,$$

and that will obviously, eventually lead to the number 1 (equality sign (in the left inequality) is going to be obtained every time $3k \pm 1$ is divisible by 4, and inequality when it is divisible by 2^n, where $n > 2$).

It is easy to understand that number $3x + 1$ is divisible by the 2^2, if x is in the form $4l + 1$ (l is a natural number), since

$$3(4l + 1) + 1 = 12l + 4= 4(3l + 1).$$

The same holds for the number $3x - 1$, if the x is in the form $4l + 3$, since

$$3(4l + 3) – 1 = 12l + 8 = 4(3l + 2).$$

In the case of modified Collatz function, the upper bound for the number of steps s, that are required to transform an odd number x to number 1, can be, roughly, estimated as

$$s < 3 \cdot floor\left(\log_{\frac{4}{3}}(x)\right).$$

In the most cases, it is a quite conservative estimation. It is interesting to notice that iterative application of $C_m(27)$ will lead toward 1 in only a few steps: $27 \rightarrow 80 \rightarrow 40 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$, comparing to 111 steps that will take in the case of original C function.

In the case of the C function, application of C on an odd number results in an even number divisible by 2^2 on average (here we assume that function C will produce even numbers that will keep the natural structure of even natural numbers, that half of the even numbers is divisible by 4, quarter by 8 and so on), and it is always divisible at least by 2. That is the reason why it is almost certain that iterative application of C function will eventually lead to 1, and why it is difficult to prove it - it is not easy to prove that iterative application of C function will not repeatedly lead to numbers that are divisible by 2 and not some higher power of 2. The major problem is the lack of understanding what factors is going to have number $3x + 1$, even if we know all factors of number x. The difficulty of the problem could easily be understood if we consider the binary representation of the number $x = b_n b_{n-1} \ldots b_1 b_0$, where b_i ($i < n$) takes value 0 or 1, and b_n is 1. Imagine that all b_i are 1. In that case the first n iterations of the function C will produce even numbers that are divisible by 2 and not by any higher power of 2. That means that after n iterations x will be increased by the factor that is larger than $(3/2)^n$, which is much bigger that x, for big n. In the step $n+1$ function C is going to produce an even number that is divisible at least by 4, if not some higher power of 2. However, it is not easy to prove that iterations that follow will not produce the numbers whose binary representations end in multiple ones and create bigger and bigger numbers. Good example for such way of "behaving" are numbers 27 or 109 (among small numbers).

Here we are going to conjecture the following:

An iterative application of C will never produce a number whose binary representation has more than 3 times more bits than the initial number.

3 Some additional results

In this section the alternative definitions of C and C_m functions are going to be presented, and some results that could be obtained from it, are going to be briefly analyzed.

Alternative definition of $C(x)$ is given by the following expression:

$$C_a(x) = 3x + 2^l,$$

where l represents the index of the bit b_l that represents the lowest position bit in the binary representation of number $x = b_n b_{n-1} \ldots b_1 b_0$, that takes value 1 ($b_n$ is 1, and b_i ($i < n$) can have values 1 or 0)). Now, the Collatz conjecture can be stated as the following:

Iterations of the alternative Collatz function C_a starting from any natural number will eventually reach the number 2^c, where c is a natural number, and then it will continue as 2^{c+2}, 2^{c+4} ...

This formulation is practically the same as the original one, except that tail zeros in the binary representation of the number obtained by Collatz function are not removed. It can be seen that the same function can be applied on odd as well as even numbers.

Analogously, $C_m(x)$ can be alternatively defined as:

$$C_{ma}(x)=3x+(-1)^{b_l+1}2^l,$$

where l is defined in the same way like in the case of $C_a(x)$ – the index of the lowest bit in the binary representation of x, that has value 1. It is interesting to notice that in this case number 27 is going to be transformed to power of 2 in just 2 steps: 27 → 80 → 256. From previous analysis we know that exists a minimum value for the number of steps s, when any natural number can be transformed to the power of number 2, by successive implementation of $C_a(x)$ or $C_{ma}(x)$ – the only difference is that it can be proved if the case of $C_{ma}(x)$ and not in the case $C_a(x)$.

One interesting fact that follows directly from alternative definition of modified Collatz function, is that it is always possible to create a polynomial in the following form ($n \geq n_{min}$, where n_{min} represent the minimal number of steps to convert an odd number k to power of 2, using C_{ma})

$$k x^n+(-1)^{b_{l_1+1}}x^{n-1}+(-1)^{b_{l_2+1}}2^{l_2}x^{n-2}+\ldots+(-1)^{b_{l_n+1}}2^{l_n}-2^c=0,$$

that has one zero equal to 3, where b_{li} represents the first nonzero bit in the binary representation of the number k after i iterations of C_{ma}, and $l_1=0 < l_2 < l_3 < \ldots < l_n < c$. Analysis of the geometrical position of all zeros of the above defined polynomial is a quite interesting problem, but it will not be done here. Another interesting thing that follows from previous equation, is that it is possible to obtain a representation of an odd number k in the bases 1/3 or 2/3. It is easy to understand that polynomial can be created for even numbers analogously – the only difference is that factor 2^m multiplies the term x^{n-1}, where m is a natural number, and reflects the maximal power of number 2 by which the even number is divisible.

Analogously, it is going to be conjectured that the similar thing can be done in the case of C function. In that case, for any odd number k it is possible to make a polynomial in the following form ($n \geq n_{min}$, where n_{min} represent the minimal number of steps to convert odd number k to power of 2, using C_a):

$$P(k,n)=k x^n+x^{n-1}+2^{l_2}x^{n-2}+\ldots+2^{l_n}-2^c=0,$$

that has 3 as one of the zeros, l_i are defined like in the case of C_{ma}, and where the following holds $l_1=0 < l_2 < l_3 < \ldots < l_n < c$.

For even number k, situation is analogous – the only difference is that term x^{n-1} must be multiplied by 2^{m}, where m is a natural number (like in the case of C_{ma}). Again, it is clear that representations of an natural number k, can be defined in bases 1/3 or 2/3. It is clear that poof that it is always possible to create a polynomial $P(k, n)$ for any odd number k and finite n, would lead to proof of the Collatz conjecture.

Here, it is going to be conjectured that for any odd k, it is possible to find $n_{min} < 2k$, *and/or* $c < 3k$, *such that polynomial P(k, n) can be made, and that it has one zero equal to 3.* This would lead toward proof of the Collatz conjecture.

In the following figure the position of zeros of the above polynomial $P(k, n)$ is depicted for the case (k =1, n = 13).

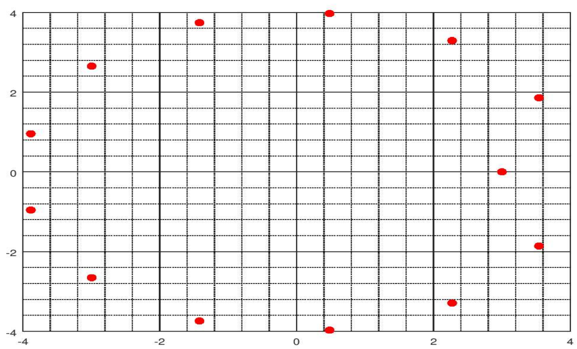

Fig. 1 Position of the zeros of the polynomial P(1, 13)

It can be noticed that zeros are evenly positioned on the circle that has radius 4, with the exception of zero at 3. It will hold for any value of $n > 2$, if $k = 1$. If $k > 1$, geometric position of zeros will deviate from the position of zeros for the case k =1 and same n, and will be more and more evenly distributed for bigger n. This is not going to be discussed here in details.

Conjecture: For the odd numbers k, the following holds:

$$if\ n > n_{min} + 1,\ \sum \frac{\cos(\alpha_i)}{r_i} = \frac{1}{12},$$

where $\underline{\alpha_i}$ and r_i are angle and radius of the polar representation of individual zeros of the polynomial $P(k, n)$.

Generalization of the C_m function can be done for the cases $5x \pm 1$ and $7x \pm 1$, but it is not going to be discussed here.

4 Conclusion

In this paper, a simple modification of Collatz function, C_m, has been proposed. It has been proved that in that case iterative application of function C_m on any natural number will lead toward number 1. Also, an alternative definition of Collatz and modified Collatz functions have been proposed, and several new conjectures have been stated.

References:

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